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On the partition function of a directed polymer in a Gaussian random environment

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Abstract. The purpose of this work is the study of the partition function $Z_n(\beta)$ of a $(d+1)$-dimensional lattice directed polymer in a Gaussian random environment ($\beta > 0$ being the inverse of temperature). In the low-dimensional cases ($d = 1$ or $d = 2$), we prove that for all $\beta > 0$, the renormalized partition function $Z_n(\beta)/\mathbf{E}Z_n(\beta)$ converges to 0 and the correlation $\langle \mathbb{1}_{(S_n^1=S_n^2)}\rangle^{(n)}$ of two independent configurations does not converge to 0. In the high dimensional case ($d \geq 3$), a lower tail of $Z_n(\beta)$ has been obtained for small $\beta > 0$. Furthermore, we express some thermodynamic quantities in terms of the path measure alone.

1. Introduction

Let $(S_n, n \geq 0)$ be a simple nearest-neighbor symmetric random walk on $\mathbb{Z}^d$ starting from 0 defined on the probability space $(\Omega^S, \mathcal{F}, \mathbb{P})$. We consider $(g(k,x), k \geq 1, x \in \mathbb{Z}^d)$ a sequence of i.i.d. gaussian variables $\mathcal{N}(0,1)$ defined on the probability space $(\Omega^g, \mathcal{G}, \mathbf{P})$, independent of $(S_n, n \geq 0)$. Fix $\beta > 0$ (which is often interpreted as the inverse of temperature). Define

$$Z_n \equiv Z_n(\beta) \stackrel{\text{def}}{=} \mathbb{E}\exp\Big(\beta \sum_{i=1}^n g(i, S_i))\Big),$$

where here and in the sequel, $\mathbb{P}_x$ denotes the law of the random walk S starting from x, and $\mathbb{E}_x$ is the expectation under $\mathbb{P}_x$, $\mathbb{E} = \mathbb{E}_0$ and $\mathbb{P} = \mathbb{P}_0$. When we write $\mathbf{P}$ or $\mathbf{E}$, we take the expectation with respect to the environment g.

We are interested in the Gibbs path-measure $\langle\cdot\rangle^{(n)}$, which is defined on $\Omega_n = \{\gamma : [1,n] \to \mathbb{Z}^d, |\gamma_k - \gamma_{k-1}| = 1\}$ as follows: for $f : \Omega_n \to \mathbb{R}$,

$$\langle f\rangle^{(n)} \stackrel{\text{def}}{=} \frac{1}{Z_n(\beta)}\mathbb{E}\Big(f(S)e^{\beta\sum_{k=1}^n g(k,S_k)}\Big).$$

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This model of a length n directed polymer in a random environment is due to Imbrie and Spencer [15]. The limit case $\beta = 0$ corresponds to the usual random walk case, whereas the case $\beta = \infty$ corresponds to the directed first-passage site percolation problem, see Johansson [16] and [17].

Following the pioneer work of Imbrie and Spencer [15], the situation in high dimension ($d \geq 3$) is well understood for small $\beta > 0$. In particular, it has been shown that (S_n) is diffusive under $\langle\cdot\rangle^{(n)}$:

Theorem A (Imbrie and Spencer [15], Bolthausen [2], Sinai [24], Albeverio and Zhou [1], Kifer [19]). *Let $d \geq 3$ and $0 < \beta < \beta_0$ be sufficiently small. Then*

$$\frac{\langle |S_n|^2\rangle^{(n)}}{n} \to 1, \qquad a.s.$$

and there exists some positive r.v. $\widetilde{Z}_\infty(\beta) > 0$ such that

$$Z_n(\beta)e^{-\frac{\beta^2}{2}n} \to \widetilde{Z}_\infty(\beta) > 0, \qquad a.s.$$

The above theorem holds in fact for a large class of random walks S and random environments g, and we refer to the above mentioned references for deeper studies of $\langle\cdot\rangle^{(n)}$ such as the convergence in terms of processes and the speed of convergence (see also Conlon and Olsen [7] and Coyle [8] for a continuous version). Let us mention that the (strict) positivity of the limit variable $\widetilde{Z}_\infty(\beta)$ guarantees that (S_n) cannot be subdiffusive.

However, in dimensions $d = 1, 2$, very little is known. It was conjectured (cf. Imbrie [14]) that

$$\mathbf{E}\langle |S_n|^2\rangle^{(n)} \asymp n^{4/3}, \qquad d = 1.$$

So far, this conjecture remains open. This paper is devoted to the study of the asymptotic behavior of the partition function Z_n, which naturally plays a key role in the study of the polymer. Let $(\mathcal{G}_n, n \geq 1)$ be the filtration defined by

$$\mathcal{G}_n \stackrel{\text{def}}{=} \sigma\{g(k,x), x \in \mathbb{Z}^d, 1 \leq k \leq n\}.$$

It is elementary to check that the process $(\widetilde{Z}_n(\beta) \stackrel{\text{def}}{=} Z_n(\beta)e^{-\frac{\beta^2}{2}n})$ is a $(\mathbf{P}, \mathcal{G}_n)$ positive martingale. Hence $\widetilde{Z}_n(\beta) \to \widetilde{Z}_\infty(\beta) \geq 0$, a.s. by the martingale convergence theorem. In contrast with Theorem A, when $d = 1, 2$, the limit variable $\widetilde{Z}_\infty(\beta)$ vanishes almost surely, even when β is small:

Theorem 1.1. *When $d = 1$ or $d = 2$, we have for all $\beta > 0$,*

$$Z_n(\beta)e^{-\frac{\beta^2}{2}n} \to 0, \qquad \text{a.s.}.$$

Or equivalently, we have

$$\sum_1^\infty \langle \mathbb{1}_{(S_n^1 = S_n^2)}\rangle_2^{(n)} = \infty, \qquad \text{a.s.},$$

where S^1 and S^2 are two independent configurations under $\langle\cdot\rangle_2^{(n)}$.

It turns out that the behavior of the partition function $Z_n(\beta)$ is strongly related to the "correlation" function $\langle \mathbb{1}_{(S_n^1=S_n^2)}\rangle_2^{(n)}$:

$$\langle \mathbb{1}_{(S_n^1=S_n^2)}\rangle_2^{(n)} \stackrel{\text{def}}{=} \frac{1}{Z_n^2(\beta)}\mathbb{E}\Big[\mathbb{1}_{(S_n^1=S_n^2)}e^{\beta\sum_1^n(g(i,S_i^1)+g(i,S_i^2))}\Big],$$

where S^1 and S^2 denote two independent copies of S under $\mathbb{E}$ and the subscript 2 indicates that we are considering two independent configurations. We have obtained a stronger result which implies that the correlation $\langle \mathbb{1}_{(S_n^1=S_n^2)}\rangle_2^{(n)}$ does not converge to 0:

Theorem 1.2. *For $d = 1$ or $d = 2$ and for all $\beta > 0$, there exists some small constant $0 < c_0 = c_0(d, \beta) < 1$ such that*

$$\liminf_{N\to\infty} \frac{\sum_1^N \mathbb{1}_{\{\langle \mathbb{1}_{(S_n^1=S_n^2)}\rangle_2^{(n)}\ge c_0\}}}{\sum_1^N \langle \mathbb{1}_{(S_n^1=S_n^2)}\rangle_2^{(n)}} \ge c_0, \qquad \text{a.s..} \tag{1.1}$$

Consequently,

$$\limsup_{n\to\infty} \langle \mathbb{1}_{(S_n^1=S_n^2)}\rangle_2^{(n)} \ \ge\ c_0, \qquad \text{a.s.,} \tag{1.2}$$

$$\limsup_{n\to\infty}\ \max_{x\in\mathbb{Z}^d} \langle \mathbb{1}_{(S_n=x)}\rangle^{(n)} \ \ge\ c_0, \qquad \text{a.s..} \tag{1.3}$$

Remark 1.3. It is known that in the diffusive case (for $d \ge 3$ and small $\beta > 0$, cf. the forthcoming Theorem 1.5), $\langle \mathbb{1}_{(S_n^1=S_n^2)}\rangle_2^{(n)}$ is of order $n^{-d/2}$, hence the last two limsup are equal to 0.

The above Theorem 1.1 shows in particular that in the low-dimensional cases the median of Z_n is much smaller than its expectation. To understand this phenomenon, consider

$$p_n(\beta) \stackrel{\text{def}}{=} \frac{1}{n}\mathbf{E}\Big(\log Z_n(\beta)\Big).$$

Combining a subadditivity argument and the Gaussian concentration inequality, we obtain:

Proposition 1.4. *For all $d \ge 1, \beta > 0$, there exists some constant $p(\beta) \equiv p(\beta, d) > 0$ such that*

$$\lim_{n\to\infty} p_n(\beta) = p(\beta) = \sup_{n\ge1} p_n(\beta) \le \frac{\beta^2}{2} \wedge (\beta\sqrt{2\log(2d)}). \tag{1.4}$$

The function $p(\cdot)$ is convex nondecreasing. For any $u > 0$, we have

$$\mathbf{P}\Big(\Big|\frac{1}{n}\log Z_n(\beta) - p_n(\beta)\Big| > u\Big) \ \le\ \exp\Big(-\frac{nu^2}{2\beta^2}\Big). \tag{1.5}$$

Consequently,

$$\lim_{n\to\infty} \frac{1}{n}\log Z_n(\beta) = p(\beta), \qquad \text{a.s..}$$

We have also obtained a lower tail of the partition function in the high dimensional case ($d \geq 3$). Consider S^1 and S^2 two independent copies of S, and define

$$q_d \stackrel{\text{def}}{=} \mathbb{P}\Big(\exists n \geq 1 : S_n^1 = S_n^2\Big) < 1, \qquad d \geq 3, \tag{1.6}$$

($q_d < 1$ thanks to the transience of the random walk $S^1 - S^2$). We learned from Talagrand [26] how to obtain the following result:

Theorem 1.5. *Let $d \geq 3$. For all $0 < \beta < \sqrt{\log(1/q_d)}$, there exists some constant $C_1 = C_1(\beta, d) > 1$ such that for all $u > 0$*

$$\mathbf{P}\Big(\log Z_n(\beta) \leq \frac{\beta^2 n}{2} - u\Big) \leq C_1 \exp\Big(-\frac{u^2}{C_1}\Big).$$

As a by-product, we have that

$$p(\beta) = \frac{\beta^2}{2}, \qquad \forall\, 0 < \beta < \sqrt{\log(1/q_d)}.$$

Remark 1.4. Sinai ([24], page 175, formula (1)) computed the value of β_0 in Theorem A for general random environments. In the Gaussian environment case, we find $\beta_0 = \sqrt{\log(1/q_d)}$. It remains an open question to find the critical β for which the conclusion of Theorem A holds.

The above result has an independent interest in the understanding of the directed polymers problem in high dimensional case. For instance, Theorem 1.5 yields that the limit variable $\widetilde{Z}_\infty$ in Theorem A admits all negative (and positive) moments: this answers a question raised in [9].

Although in this paper, we shall exclusively deal with the simple random walk S, and it is not difficult to extend the above results to a general random walk. It is also noteworthy that Theorem 1.1 holds for a large class of random environments, see Remark 5.2. We shall not discuss the exponents related to the mean displacement and to the variance of $\log Z_n(\beta)$, see Piza [21]; these directions are explored in the paper [3].

We close this introduction by mentioning a nice result obtained by Khanin et al. [18]: Considering a model of polymer with Hamiltonian $\sum_{k=1}^n g(S_k)$ instead of $\sum_{k=1}^n g(k, S_k)$, they showed the effect of "traps" and that the model is not diffusive.

The rest of this paper is organized as follows:

- Section 2 is devoted to some Gaussian inequalities.
- Proposition 1.4 is proven in Section 3 by virtue of a subadditivity argument.
- In Section 4, we use Talagrand [25]'s method to obtain Theorem 1.5.
- Theorem 1.1 is proven in Section 5 by using a martingale argument and a time-reversal technique.
- In Section 6, we prove Theorem 1.2 by induction on the Gibbs measure $\langle\cdot\rangle^{(n)}$.
- In Section 7, we express the thermodynamic quantity $p_n'(\beta)$ in terms of a global correction where the gaussian character of the environments plays a key role.

During the preparation of this paper, we have been much inspired from Talagrand's works on spin glasses ([25], [26]). Furthermore, we would like to stress the fact that while in spin glasses the correlation of independent configurations is influenced by the exchangeability of the individuals spins, in the directed polymer model it is influenced by the Markov property of the underlying random walk.

Throughout the whole paper, we write $f(x) \asymp g(x)$ when $x \to x_0 \in [0, \infty]$ if $0 < \liminf_{x\to x_0} f(x)/g(x) \le \limsup_{x\to x_0} f(x)/g(x) < \infty$. The notations S^1 and S^2 means two independent copies of S under $\mathbb{E}_x$ or under $\langle\cdot\rangle^{(n)}$. And $(C_j = C_j(d, \beta) > 0, 2 \le j \le 12)$ denote some positive constants.

2. Preliminaries

We shall need some exponential inequalities. The first one deals with a gaussian family:

Lemma 2.1. *Let Υ be some countable set and $(g(x), x \in \Upsilon)$ be a family of jointly gaussian centered random variables, with common variance $\sigma^2 > 0$ (let us stress the fact that the random variables $(g(x), x \in \Upsilon)$ are not assumed independent). Assume that $(\alpha(x), x \in \Upsilon)$ are nonnegative real numbers such that $\sum_{x\in\Upsilon} \alpha(x) = 1$. For any finite subset $\Theta \subset \Upsilon$ and for $q, \beta > 0$ and $\lambda(x) \in \mathbb{R}$, we have*

$$e^{-\frac{\beta^2\sigma^2}{2}q} \le \mathbf{E}\left(\frac{e^{\beta\sum_{x\in\Theta}\lambda(x)g(x)}}{\left(\sum_{x\in\Upsilon}\alpha(x)e^{\beta g(x)}\right)^q}\right) \le e^{\frac{\beta^2\sigma^2}{2}(q+\sum_{x\in\Theta}|\lambda(x)|)^2}.$$

Proof. Of course, we may assume that $\sigma = 1$. Using the relation:

$$x^{-q} = \int_0^\infty \frac{u^{q-1}}{\Gamma(q)} e^{-ux} du, \qquad x > 0,$$

together with Jensen's inequality, we get

$$\begin{aligned}
&\mathbf{E}\left(\frac{e^{\beta\sum_{x\in\Theta}\lambda(x)g(x)}}{\left(\sum_{x\in\Upsilon}\alpha(x)e^{\beta g(x)}\right)^q}\right)\\
&\quad= \int_0^\infty \frac{u^{q-1}}{\Gamma(q)} du \mathbf{E}\exp\Big(\beta\sum_{x\in\Theta}\lambda(x)g(x) - u\sum_y \alpha(y)e^{\beta g(y)}\Big)\\
&\quad\ge \int_0^\infty \frac{u^{q-1}}{\Gamma(q)} du \exp\Big(\mathbf{E}\big(\beta\sum_{x\in\Theta}\lambda(x)g(x) - u\sum_y \alpha(y)e^{\beta g(y)}\big)\Big)\\
&\quad= \int_0^\infty \frac{u^{q-1}}{\Gamma(q)} du \exp\Big(-u\sum_y \alpha(y)e^{\beta^2/2}\Big)\\
&\quad= e^{-\beta^2 q/2},
\end{aligned}$$

proving the lower bound. To derive the upper bound, we make use of the convexity of the function $x \to x^{-q}$:

$$\begin{aligned}
\mathbf{E}\left(\frac{e^{\beta\sum_{x\in\Theta}\lambda(x)g(x)}}{\left(\sum_{y\in\Upsilon}\alpha(y)e^{\beta g(y)}\right)^q}\right) &\le \sum_y \alpha(y)\,\mathbf{E}\exp\Big(\beta\sum_{x\in\Theta}\lambda(x)g(x) - q\,\beta g(y)\Big)\\
&= \sum_y \alpha(y)\,\exp\Big(\frac{\beta^2}{2}\,\mathbf{E}\big(\sum_{x\in\Theta}\lambda(x)g(x) - qg(y)\big)^2\Big).
\end{aligned}$$

Now, we observe that

$$\begin{aligned}\mathbf{E}\big(\sum_{x\in\Theta}\lambda(x)g(x)-qg(y)\big)^2 &= q^2-2q\sum_{x\in\Theta}\lambda(x)\,\mathbf{E}g(x)g(y)\\ &\quad+\sum_{x_1,x_2\in\Theta}\lambda(x_1)\lambda(x_2)\mathbf{E}g(x_1)g(x_2)\\ &\le (q+\sum_{x\in\Theta}|\lambda(x)|)^2,\end{aligned}$$

since $\mathbf{E}\big(g(x_1)g(x_2)\big)\le 1$. The upper bound follows. □

In the rest of this section, we assume that $(g(x), x\in\mathbb{Z}^d)$ are independent.

Lemma 2.2. *Let $\{g(x), x\in\mathbb{Z}^d\}$ be a sequence of i.i.d. standard gaussian variables and let $\alpha(x)\ge 0$ be nonnegative numbers such that $\sum_x \alpha(x)=1$. For all $\beta>0$, we have*

$$-\frac{e^{4\beta^2}-1}{8}\sum_x\alpha^2(x)\le \mathbf{E}\log\Big(\sum_x\alpha(x)e^{\beta g(x)-\beta^2/2}\Big)\le -\frac{(1-e^{-\beta^2})}{2}\sum_x\alpha^2(x).$$

Fix $\gamma\in(0,1)$. We have

$$\begin{aligned}&-\frac{\gamma(1-\gamma)(e^{8\beta^2}-1)}{16}\sum_x\alpha^2(x)\le\mathbf{E}\Big(\sum_x\alpha(x)e^{\beta g(x)-\beta^2/2}\Big)^\gamma-1\\ &\le-\frac{\gamma(1-\gamma)(1-e^{-\beta^2})}{2}\sum_x\alpha^2(x).\end{aligned}$$

Finally, for any $\mathcal{C}^2$ function $\phi:(0,\infty)\to\mathbb{R}$ such that $\phi(1)=0$ and $|\phi''(x)|\le c_{p,q}\big(x^p+x^{-q}\big)$ for some constants $p,q>0$, there exists a constant $c_{\beta,p,q}>0$ such that

$$\mathbf{E}\phi\Big(\sum_x\alpha(x)e^{\beta g(x)-\beta^2/2}\Big)\le c_{\beta,p,q}\sum_x\alpha^2(x).$$

Proof. Let $\{B_x(t), t\ge 0\}_{x\in\mathbb{Z}^d}$ be a family of independent one-dimensional Brownian motions starting from 0. Define

$$X(t)\overset{\text{def}}{=}\sum_x\alpha(x)e^{\beta B_x(t)-\frac{\beta^2 t}{2}},\qquad t\ge 0.$$

Notice that $X(1)\overset{\text{law}}{=}\sum_x\alpha(x)e^{\beta g(x)-\beta^2/2}$. Applying Itô's formula

$$\begin{aligned}\mathbf{E}\log X(1)&=-\frac{\beta^2}{2}\sum_x\alpha^2(x)\int_0^1 dt\,\mathbf{E}\Big(\frac{e^{2\beta B_x(t)-\beta^2 t}}{X^2(t)}\Big)\\ &=-\frac{\beta^2}{2}\sum_x\alpha^2(x)\int_0^1 dt\,\mathbf{E}\Big(\frac{e^{2\beta\sqrt{t}\,g(x)}}{\big(\sum_y\alpha(y)e^{\beta\sqrt{t}\,g(y)}\big)^2}\Big)\\ &\le-\frac{\beta^2}{2}\sum_x\alpha^2(x)\int_0^1 dt\,e^{-\beta^2 t}\end{aligned}$$

implying the upper bound by means of the lower bound of Lemma 2.1. The lower bound follows in the same way by using the upper bound of Lemma 2.1. To deal with $X(1)^\gamma$, we apply again Itô's formula and obtain

$$\mathbf{E}\Big(X(1)^\gamma\Big) = 1 + \frac{\gamma(\gamma-1)}{2}\int_0^1 dt \sum_x \alpha^2(x)\,\mathbf{E}\Big(\frac{e^{2\beta B_x(t)-\beta^2 t}}{X(t)^{2-\gamma}}\Big).$$

Using Lemma 2.1 with $q = 2-\gamma$, $\sigma = \sqrt{t}$, we obtain

$$\begin{aligned} e^{-\beta^2 t} \le \mathbf{E}\Big(\frac{e^{2\beta B_x(t)-\beta^2 t}}{X(t)^{2-\gamma}}\Big) &= e^{-\frac{\beta^2\gamma t}{2}}\,\mathbf{E}\Big(\frac{e^{2\beta B_x(t)}}{(\sum_x \alpha(x)e^{\beta B_x(t)})^{2-\gamma}}\Big)\\ &\le e^{\beta^2 t((4-\gamma)^2-\gamma)/2} \le e^{8\beta^2 t}. \end{aligned}$$

From this, the desired estimates follow. It remains to show the last assertion. By assumption and using successively Cauchy-Schwarz's inequality and Lemma 2.1, we have

$$\begin{aligned} \mathbf{E}\phi(X(1)) &= \frac{\beta^2}{2}\int_0^1 dt \sum_x \alpha^2(x)\,\mathbf{E}\Big(\phi''(X(t))e^{2\beta B_x(t)-\beta^2 t}\Big)\\ &\le \frac{\beta^2}{2}\int_0^1 dt \sum_x \alpha^2(x)\, c_{p,q}\,\Big(\mathbf{E}(X^{2p}(t)+X^{-2q}(t))\Big)^{1/2}\\ &\quad\times\Big(\mathbf{E}e^{4\beta B_x(t)-2\beta^2 t}\Big)^{1/2} \le c_{\beta,p,q}\sum_x \alpha^2(x), \end{aligned}$$

completing the whole proof. □

With the same assumptions on $(g(x))$ and $(\alpha(x))$ as in Lemma 2.2, we shall estimate

$$I_1 \stackrel{\text{def}}{=} \mathbf{E}\left(\frac{e^{\beta\, g(z_1)+\beta g(z_2)}}{\Big[\sum_x \alpha(x)\, e^{\beta\, g(x)}\Big]^2}\right), \qquad z_1, z_2 \in \mathbb{Z}^d. \tag{2.1}$$

Before giving a more accurate estimate on I_1 than Lemma 2.1, we want to stress the fact that there exist some situations when $I_1 < 1$. For instance, when $\alpha(z_1) = \alpha(z_2) = 1/2$ and $z_1 \ne z_2$, $I_1 = \mathbf{E}\Big(\cosh^{-2}(\beta\mathcal{N}/\sqrt{2})\Big) < 1$.

Lemma 2.3. *Let* $\{g(x), x \in \mathbb{Z}^d\}$ *be a sequence of i.i.d. standard gaussian variables and* $\alpha(x) \ge 0$ *are nonnegative numbers such that* $\sum_x \alpha(x) = 1$. *When* $z_1 \ne z_2$, *we have*

$$\begin{aligned} I_1 &\le 1 - \frac{4}{3}(1-e^{-3\beta^2/2})\big(\alpha(z_1)+\alpha(z_2)\big) + \frac{3}{16}(e^{16\beta^2}-1)\sum_x \alpha^2(x),\\ I_1 &\ge 1 - \frac{2}{9}(e^{9\beta^2}-1)\big(\alpha(z_1)+\alpha(z_2)\big) + \frac{3}{2}(1-e^{-2\beta^2})\sum_x \alpha^2(x). \end{aligned}$$

When $z_1 = z_2$, we have

$$I_1 \le e^{\beta^2}\Big(1 - \frac{8}{3}(1-e^{-3\beta^2/2})\,\alpha(z_1) + \frac{3}{16}(e^{16\beta^2}-1)\sum_x \alpha^2(x)\Big),$$
$$I_1 \ge e^{\beta^2}\Big(1 - \frac{4}{9}(e^{9\beta^2}-1)\,\alpha(z_1) + \frac{3}{2}(1-e^{-2\beta^2})\sum_x \alpha^2(x)\Big).$$

Proof. Keeping the notations $X(t)$ and $(B_x(t))$ introduced in the proof of the previous lemma. For $z_1 \neq z_2$, we define

$$\widetilde{X}(t) \stackrel{\text{def}}{=} \exp\Big(\beta(B_{z_1}(t) + B_{z_2}(t)) - \beta^2 t\Big), \qquad t \ge 0.$$

Observe that

$$I_1 = \mathbf{E}\Big(\frac{\widetilde{X}(1)}{X^2(1)}\Big) = 1 + \mathbf{E}\int_0^1 d\Big(\frac{\widetilde{X}(t)}{X^2(t)}\Big).$$

Furthermore, Itô's formula gives

$$d\Big(\frac{\widetilde{X}(t)}{X^2(t)}\Big) = -\frac{2\beta^2\widetilde{X}(t)}{X^3(t)}\Big(\alpha(z_1)e^{\beta B_{z_1}(t)-\beta^2 t/2} + \alpha(z_2)e^{\beta B_{z_2}(t)-\beta^2 t/2}\Big)dt$$
$$+\frac{3\beta^2\,\widetilde{X}(t)}{X^4(t)}\sum_x \alpha^2(x)e^{2\beta B_x(t)-\beta^2 t}dt + \text{l.m.},$$

where "l.m." denotes the local martingale part which in this case is a true martingale. Using the scaling property of the Brownian motion, we have for example

$$\mathbf{E}\Big(\frac{e^{\beta B_{z_1}(t)-\beta^2 t/2}\,\widetilde{X}(t)}{X^3(t)}\Big) = \mathbf{E}\Big(\frac{e^{\beta\sqrt{t}\,\left(2g(z_1)+g(z_2)\right)}}{\left(\sum_x \alpha(x)e^{\beta\sqrt{t}\,g(x)}\right)^3}\Big),$$

which in view of Lemma 2.1 lives in $[e^{-3\beta^2 t/2}, e^{18\beta^2 t}]$. Similarly,

$$e^{-2\beta^2 t} \;\le\; \mathbf{E}\Big(\frac{\widetilde{X}(t)}{X^4(t)}e^{2\beta B_x(t)-\beta^2 t}\Big) \;\le\; e^{32\beta^2 t}.$$

From these, the estimates on I_1 follow.

When $z_1 = z_2$, we put

$$\widehat{X}(t) \stackrel{\text{def}}{=} \exp\Big(2\beta B_{z_1}(t) - 2\beta^2 t\Big), \qquad t \ge 0,$$

so that $\widehat{X}$ remains a martingale. Notice that $I_1 = e^{\beta^2}\,\mathbf{E}\Big(\frac{\widehat{X}(1)}{X^2(1)}\Big)$. The rest of the proof can be done in the same way as above, and the details are omitted. □

We end this section by a simple observation on the positive moments of Z_n:

Lemma 2.4. *Let $m \geq 1$, we have*

$$\mathbf{E}\Big(Z_n^m(\beta)\Big) = \mathbb{E}\exp\Big(\beta^2 \sum_{1\leq i<j\leq m} L_n(S^i - S^j) + \frac{m\,n}{2}\beta^2\Big),$$

where $(S^i, 1 \leq i \leq m)$ denote m independent copies of the random walk S and $L_n(S^i - S^j) \stackrel{\text{def}}{=} \sum_{k=1}^n \mathbb{1}_{(S_k^i = S_k^j)}$.

Remark 2.5. In the recurrent case ($d = 1$ or $d = 2$), $\frac{L_n(S^1-S^2)}{n}$ satisfies a large deviation principle with speed function n and rate function $\psi_d(\cdot) > 0$ (see Gantert and Zeitouni [12]). According to Varadhan's lemma, we have that for all $\beta > 0$

$$\frac{\mathbf{E}\Big(Z_n^2(\beta)\Big)}{\big(\mathbf{E}Z_n(\beta)\big)^2} = \mathbb{E}\exp\Big(\beta^2 L_n(S^1 - S^2)\Big) = \exp\Big(n(\psi_d^*(\beta^2) + o(1))\Big), \qquad n \to \infty, \tag{2.2}$$

with $\psi_d^*(\lambda) = \sup_{x\geq 0}\big(\lambda x - \psi_d(x)\big)$.

Proof of Lemma 2.4. Consider $S^1, ..., S^m$ m independent copies of the random walk S. We have

$$\begin{aligned}\mathbf{E}\Big(Z_n(\beta)\Big)^m &= \mathbb{E}_{S^1,...,S^m}\,\mathbf{E}\exp\Big(\sum_{i=1}^n\sum_{j=1}^m g(i, S_i^j)\Big)\\ &= \mathbb{E}_{S^1,...,S^m}\prod_{i=1}^n \mathbf{E}\exp\Big(\sum_{j=1}^m g(i, S_i^j)\Big).\end{aligned}$$

Now, it suffices to remark that for $g(x) \equiv g(i, x), x \in \mathbb{Z}^d$, and for any $x_1, ..., x_m \in \mathbb{Z}^d$, we have

$$\mathbf{E}\exp\Big(\beta\sum_1^m g(x_i) - \frac{m}{2}\beta^2\Big) = \exp\Big(\beta^2\sum_{1\leq i<j\leq m}\mathbb{1}_{(x_i=x_j)}\Big).$$

In fact, this formula is obvious by computing the variance of $\sum_1^m g(x_i)$: $\mathrm{Var}(\sum_1^m g(x_i)) = \sum_{1\leq i,j\leq m}\mathbf{E}\Big(g(x_i)g(x_j)\Big) = m + 2\sum_{1\leq i<j\leq m}\mathbb{1}_{(x_i=x_j)}$. □

3. A subadditivity argument

For the sake of readability, we shall omit in the next sections the dependence of Z_n on β. We shall need to consider varying starting points, and therefore we introduce the notations:

$$Z_n(x) \equiv Z_n(x; g) \stackrel{\text{def}}{=} \mathbb{E}_x e^{\beta\sum_1^n g(k,S_k)}, \qquad x \in \mathbb{Z}^d. \tag{3.1}$$

$$Z_n(x, y) \equiv Z_n(x, y; g) \stackrel{\text{def}}{=} \mathbb{E}_x\Big(\mathbb{1}_{(S_n=y)}e^{\beta\sum_1^n g(k,S_k)}\Big), \qquad x, y \in \mathbb{Z}^d. \tag{3.2}$$

where $\mathbb{E}_x$ means that the random walk S starts from x (So $Z_n = Z_n(0)$ according to the above notation). We have the following simple consequence of the Markov property:

Lemma 3.1. *For all $n, m \geq 1$,*

$$Z_{n+m}(x) = \sum_{y\in\mathbb{Z}^d} Z_n(x, y; g)\, Z_m(y; \theta_n g)$$
$$= Z_n(x; g) \sum_y \langle \mathbb{1}_{(S_n=y)} \rangle^{(n,x)} Z_m(y; \theta_n g), \qquad x, y \in \mathbb{Z}^d. \quad (3.3)$$

where $\langle \mathbb{1}_{(S_n=y)} \rangle^{(n,x)} \stackrel{\text{def}}{=} \frac{Z_n(x,y)}{Z_n(x)}$ and (θ_n) denotes the shift operator: $\theta_n g(k, x) = g(n + k, x)$ for all k, n, x.

We prove in this section Proposition 1.4 by using a subadditivity argument.

Proof of Proposition 1.4. Applying Lemma 3.1 with the starting point $x = 0$, we have

$$\begin{aligned}
\mathbf{E}\Big(\log Z_{n+k}(0)\Big) &= \mathbf{E}\Big(\log \sum_y \langle \mathbb{1}_{(S_n=y)} \rangle^{(n)} Z_k(y; \theta_n g)\Big) + \mathbf{E}\Big(\log Z_n(0)\Big)\\
&\geq \mathbf{E} \sum_y \Big(\langle \mathbb{1}_{(S_n=y)} \rangle^{(n)} \log Z_k(y; \theta_n g)\Big) + \mathbf{E}\Big(\log Z_n(0)\Big)\\
&= \sum_y \mathbf{E}\Big(\langle \mathbb{1}_{(S_n=y)} \rangle^{(n)}\Big)\, \mathbf{E}\Big(\log Z_k(y; \theta_n g)\Big) + \mathbf{E}\Big(\log Z_n(0)\Big)\\
&= \mathbf{E}\Big(\log Z_k(0)\Big) + \mathbf{E}\Big(\log Z_n(0)\Big),
\end{aligned}$$

where the inequality is due to the concavity of the function log and the second equality follows from the fact that $Z_k(y; \theta_n g)$ only depends on $\{g(k + n, x), k \geq 1, x \in \mathbb{Z}^d\}$, hence is independent of $\langle \mathbb{1}_{(S_n=x)} \rangle^{(n)}$. This superadditivity implies that

$$\lim_{n\to\infty} p_n(\beta) = \sup_{n\geq 1} p_n(\beta) \stackrel{\text{def}}{=} p(\beta).$$

Thanks to Jensen's inequality, the function $\beta (\in \mathbb{R}_+) \to p_n(\beta)$ is convex and nondecreasing, hence the same is for $p(\cdot)$ (see also Lemma 7.1 for an expression of $p_n'(\beta)$). The rest of the proof of Proposition 1.4 can be completed by applying Talagrand's method ([26]). In fact, define $F : \mathbb{R}^m \to \mathbb{R}$ by ($m = n(2n + 1)$):

$$F(\mathbf{z}) \stackrel{\text{def}}{=} \frac{1}{n} \log \mathbb{E} e^{\beta \sum_{x\in\mathbb{Z}^d} \sum_1^n \mathbf{z}_{i,x} \mathbb{1}_{(S_i=x)}}, \qquad \mathbf{z} = \big(\mathbf{z}_{i,x}, 1 \leq i \leq n, x \in \mathbb{Z}^d, |x| \leq n\big).$$

By the Cauchy-Schwarz inequality,

$$\Big|\sum_x \sum_i z_{i,x} \mathbb{1}_{(S_i=x)} - \sum_x \sum_i z'_{i,x} \mathbb{1}_{(S_i=x)}\Big| \leq n^{1/2} \Big(\sum_x \sum_i (z_{i,x} - z'_{i,x})^2\Big)^{1/2}.$$

Hence F is a Lipschitz function:

$$|F(\mathbf{z^1}) - F(\mathbf{z^2})| \leq \frac{\beta}{\sqrt{n}} |\mathbf{z^1} - \mathbf{z^2}|.$$

Since $Z_n(\beta) = F(\mathbf{g})$ ($\mathbf{g} = (g(i,x), 1 \le i \le n, x \in \mathbb{Z}^d, |x| \le n)$), the estimate (1.5) exactly follows from the Gaussian concentration inequality for the Lipschitz function of a gaussian vector (cf. [13]).

It remains to establish the inequality in (1.4). Observe that by Jensen's inequality

$$p_n(\beta) = \frac{1}{n}\mathbf{E}\log Z_n(\beta) \le \frac{1}{n}\log \mathbf{E} Z_n(\beta) = \frac{\beta^2}{2}.$$

Differentiating with respect to β, we get

$$p_n'(\beta) = \frac{1}{n}\mathbf{E}\left(\frac{\mathbb{E}\Big(H_n(g,S)e^{\beta H_n(g,S)}\Big)}{Z_n(\beta)}\right) = \frac{1}{n}\mathbf{E}\Big(\langle H_n(g,S)\rangle^{(n)}\Big), \qquad (3.4)$$

with $H_n(g,S) \overset{\text{def}}{=} \sum_{i=1}^n g(i,S_i)$. We bound $H_n(g,S)$ by $\max_{\gamma\in\Omega_n} H(g,\gamma)$, and obtain

$$\begin{aligned} p_n'(\beta) &\le \frac{1}{\sqrt{n}}\mathbf{E}\max_{\gamma\in\Omega_n}\frac{H_n(g,\gamma)}{\sqrt{n}} \\ &\le \frac{1}{\sqrt{n}}\sqrt{2\log(2d)^n} \\ &= \sqrt{2\log(2d)}, \end{aligned}$$

where in the second inequality we have used the following fact: Let $(g(i), i \ge 1)$ be any sequence of $\mathcal{N}(0,1)$ gaussian variables. Then

$$\mathbf{E}\max_{1\le i\le m} g(i) \le \sqrt{2\log m}, \qquad \forall m \ge 1, \qquad (3.5)$$

see e.g. Talagrand [26]. This implies that $p_n(\beta) \le \beta\sqrt{2\log(2d)}$, and ends the proof of Proposition 1.4. □

4. Talagrand's method: Proof of Theorem 1.5

We follow exactly the same method presented by Talagrand [25] and [26]. Although the following result can be found in [26], we include its proof for the sake of completeness.

Lemma 4.1. *Let* $\mathbf{g}$ *be a* $\mathbb{R}^m$*-valued, centered gaussian vector* $\mathcal{N}(0, \mathbf{Id_m})$ *with covariance matrix* $\mathbf{Id_m} = (\mathbb{1}_{(i=j)})_{1\le i,j\le m}$. *For any measurable set* $\mathbf{A} \subset \mathbb{R}^m$, *if*

$$\mathbf{P}\Big(\mathbf{g}\in\mathbf{A}\Big) \ge p > 0,$$

then for any $u > 0$

$$\mathbf{P}\Big(d(\mathbf{g},\mathbf{A}) > u + \sqrt{2\log(1/p)}\Big) \le \exp\Big(-\frac{u^2}{2}\Big),$$

where $d(x,y) \overset{\text{def}}{=} \sqrt{\sum_{i=1}^m (x_i - y_i)^2}$ *for* $x = (x_i)_{1\le i\le m} \in \mathbb{R}^m$ *and* $y = (y_i)_{1\le i\le m} \in \mathbb{R}^m$.

Proof. The application $x \to d(x, \mathbf{A})$ is a Lipschitz function with coefficient 1. According to the concentration inequality for Gaussian measure, we have that for any $u > 0$,

$$\mathbf{P}\Big(\Big|d(\mathbf{g}, \mathbf{A}) - \mathbf{E}d(\mathbf{g}, \mathbf{A})\Big| > u\Big) \le e^{-u^2/2}.$$

If $u < \mathbf{E}d(\mathbf{g}, \mathbf{A})$, we have

$$p \le \mathbf{P}\Big(\mathbf{g} \in \mathbf{A}\Big) \le \mathbf{P}\Big(\Big|d(\mathbf{g}, \mathbf{A}) - \mathbf{E}d(\mathbf{g}, \mathbf{A})\Big| > u\Big) \le e^{-u^2/2}$$

which implies that

$$\mathbf{E}d(\mathbf{g}, \mathbf{A}) \le \sqrt{2\log(1/p)}.$$

The desired conclusion follows. □

Proof of Theorem 1.5. Let $d \ge 3$ and $0 < \beta < \sqrt{\log(1/q_d)}$. Recall that $L_n(S^1 - S^2) \overset{\text{def}}{=} \sum_{i=1}^n \mathbb{1}_{(S_i^1 = S_i^2)}$. Since the random walk $S^1 - S^2$ is transient, $L_\infty(S^1 - S^2)$ is finite and has geometric distribution:

$$\mathbb{P}\Big(L_\infty(S^1 - S^2) \ge k\Big) = q_d^k, \qquad k \ge 1,$$

where $q_d < 1$ is defined by (1.6). This together with Lemma 2.4 imply that

$$\frac{\mathbf{E}Z_n^2}{\Big(\mathbf{E}Z_n\Big)^2} = \mathbb{E}e^{\beta^2 L_n(S^1 - S^2)} \le \mathbb{E}e^{\beta^2 L_\infty(S^1 - S^2)} = C_2 < \infty.$$

Applying Paley-Zygmund' inequality, we obtain:

$$\mathbf{P}\Big(Z_n \ge \frac{1}{2}\mathbf{E}Z_n\Big) \ge \frac{1}{4}\frac{\Big(\mathbf{E}Z_n\Big)^2}{\mathbf{E}Z_n^2} \ge \frac{1}{4C_2}.$$

We claim that there exists some constant $C_3 > 1$ such that for every $n \ge 1$,

$$\mathbf{P}\Big(Z_n \ge \frac{1}{2}\mathbf{E}Z_n;\ \langle L_n(S^1 - S^2)\rangle_2^{(n)} \le C_3\Big) \ge \frac{1}{C_3}, \tag{4.1}$$

where the notation $\langle\cdot\rangle_2^{(n)}$ means that we have two independent configurations S^1 and S^2:

$$\langle L_n(S^1 - S^2)\rangle_2^{(n)} = \frac{1}{Z_n^2}\mathbb{E}_{S^1,S^2}\Big(L_n(S^1 - S^2)e^{\beta\sum_1^n(g(i,S_i^1)+g(i,S_i^2))}\Big).$$

To prove (4.1), we observe that

$$\begin{aligned}
&\mathbf{P}\Big(Z_n \ge \frac{1}{2}\mathbf{E}Z_n;\ \langle L_n(S^1 - S^2)\rangle_2^{(n)} \le C_3\Big)\\
&\quad \ge \mathbf{P}\Big(Z_n \ge \frac{1}{2}\mathbf{E}Z_n;\ \mathbb{E}_{S^1,S^2}\big[L_n(S^1 - S^2)e^{\beta\sum_1^n(g(j,S_j^1)+g(j,S_j^2))}\big] \le \frac{C_3}{4}\Big(\mathbf{E}Z_n\Big)^2\Big)\\
&\quad \ge \mathbf{P}\Big(Z_n \ge \frac{1}{2}\mathbf{E}Z_n\Big) + \mathbf{P}\Big(\mathbb{E}_{S^1,S^2}\big[L_n(S^1 - S^2)e^{\beta\sum_1^n(g(j,S_j^1)+g(j,S_j^2))}\big]
\end{aligned}$$

$$\leq \frac{C_3}{4}\big(\mathbf{E}Z_n\big)^2\Big) - 1$$

$$\geq \frac{1}{4C_2} - \mathbf{P}\Big(\mathbb{E}_{S^1,S^2}\big[L_n(S^1-S^2)e^{\beta\sum_1^n(g(j,S^1_j)+g(j,S^2_j))}\big] > \frac{C_3}{4}\big(\mathbf{E}Z_n\big)^2\Big)$$

$$\geq \frac{1}{4C_2} - \frac{4}{C_3}\mathbb{E}\Big(L_n(S^1-S^2)e^{\beta^2 L_n(S^1-S^2)}\Big), \tag{4.2}$$

where in the last inequality, we have used Chebychev's inequality and

$$\mathbf{E}\Big(\mathbb{E}_{S^1,S^2}\big(L_n(S^1-S^2)e^{\beta\sum_1^n(g(j,S^1_j)+g(j,S^2_j))}\big)\Big) = \big(\mathbf{E}Z_n\big)^2\,\mathbb{E}\Big(L_n(S^1-S^2)e^{\beta^2 L_n(S^1-S^2)}\Big).$$

Notice that

$$\mathbb{E}\Big(L_n(S^1-S^2)e^{\beta^2 L_n(S^1-S^2)}\Big) \leq \mathbb{E}\Big(L_\infty(S^1-S^2)e^{\beta^2 L_\infty(S^1-S^2)}\Big) < \infty.$$

Therefore if C_3 is large enough, then (4.1) holds. Fix such a constant C_3 and let

$$\mathbf{A} \stackrel{\text{def}}{=} \Big\{\mathbf{a} = (a(i,x), i \geq 1, x \in \mathbb{Z}^d, |x| \leq n) : Z_n(\mathbf{a}) \geq \frac{1}{2}\mathbf{E}Z_n;\ \langle L_n(S^1-S^2)\rangle_{\mathbf{a}}^{(n)} \leq C_3\Big\},$$

where $\langle L_n(S^1-S^2)\rangle_{\mathbf{a}}^{(n)} \stackrel{\text{def}}{=} \frac{1}{Z_n^2(\mathbf{a})}\mathbb{E}_{S^1,S^2}\big(L_n(S^1-S^2)e^{\beta\sum_1^n(a(i,S^1_i)+a(i,S^2_i))}\big)$ and $Z_n(\mathbf{a})$ is the corresponding renormalization constant. We have proven that for the gaussian vector $\mathbf{g} = (g(i,x), i \geq 1, x \in \mathbb{Z}^d, |x| \leq n)$

$$\mathbf{P}\Big(\mathbf{g} \in \mathbf{A}\Big) \geq \frac{1}{C_3} > 0.$$

Taking the distance $d(\mathbf{z},\mathbf{z}') = \sqrt{\sum_{i,x}(z_{i,x}-z'_{i,x})^2}$. Applying Lemma 4.1, we obtain that

$$\mathbf{P}\Big(d(\mathbf{g},\mathbf{A}) > u + C_4\Big) \leq \exp\Big(-\frac{u^2}{2}\Big), \qquad \forall u > 0, \tag{4.3}$$

with $C_4 = \sqrt{2\log C_3} > 0$. For any $\mathbf{a}' \in \mathbf{A}$, we have

$$\begin{aligned} Z_n(\mathbf{a}) &= \mathbb{E}_S e^{\beta\sum_1^n(a(i,S_i)-a'(i,S_i))}e^{\beta\sum_1^n a'(i,S_i)} \\ &= Z_n(\mathbf{a}')\,\langle e^{\beta\sum_1^n(a(i,S_i)-a'(i,S_i))}\rangle_{\mathbf{a}'} \\ &\geq Z_n(\mathbf{a}')\,\exp\Big(\beta\langle\sum_1^n(a(i,S_i)-a'(i,S_i))\rangle_{\mathbf{a}'}\Big), \end{aligned}$$

by Jensen's inequality and where $\langle\cdot\rangle_{\mathbf{a}'}$ indicates that we consider the Gibbs measure under $\mathbf{a}'$ (the dependence on n being omitted). Observe that

$$\begin{aligned} \big|\langle\sum_1^n(a(i,S_i)-a'(i,S_i))\rangle_{\mathbf{a}'}\big| &= \big|\sum_{i=1}^n\sum_x(a(i,x)-a'(i,x))\langle 1\!\!1_{(S_i=x)}\rangle_{\mathbf{a}'}\big| \\ &\leq d(\mathbf{a},\mathbf{a}')\sqrt{\langle L_n(S^1-S^2)\rangle_{\mathbf{a}'}}, \end{aligned}$$

by Cauchy-Schwarz inequality. Recall that for $\mathbf{a}' \in \mathbf{A}$, $\langle L_n(S^1 - S^2)\rangle_{\mathbf{a}'} \le C_3$, then we obtain that for $\mathbf{a}' \in \mathbf{A}$ and any $\mathbf{a}$, we have

$$\log Z_n(\mathbf{a}) \ge \log Z_n(\mathbf{a}') - \sqrt{C_3} d(\mathbf{a}, \mathbf{a}') \ge \log \mathbf{E} Z_n - \log 2 - \sqrt{C_3} d(\mathbf{a}, \mathbf{a}').$$

This together with (4.3) implies that for any $u > 0$, the following event holds with probability larger than $1 - e^{-u^2/2}$:

$$\log Z_n(\mathbf{g}) \ge \log \mathbf{E} Z_n - \log 2 - \sqrt{C_3} d(\mathbf{g}, \mathbf{A}) \ge \log \mathbf{E} Z_n - \log 2 - \sqrt{C_3}(u + C_4),$$

which yields Theorem 1.5. □

5. Proof of Theorem 1.1

We adopt the following notation:

$$\widetilde{Z}_n(x) \equiv \widetilde{Z}_n(x; g) \stackrel{\text{def}}{=} \mathbb{E}_x \exp\Big(\beta \sum_1^n g(i, S_i) - \frac{\beta^2}{2} n\Big) = Z_n(x; g) e^{-\frac{\beta^2}{2} n}, \quad x \in \mathbb{Z}^d.$$

It follows from the independence of the vector $(g(k, x), x \in \mathbb{Z}^d)_{k \ge 1}$ that the process $n \to \widetilde{Z}_n(x; g)$ is a positive $(\mathcal{G}_n)$-martingale, hence we can define $\widetilde{Z}_\infty(x; g)$ as the almost sure limit of $\widetilde{Z}_n(x; g)$. The zero-one law (cf. [2]) says that $\mathbf{P}\Big(\widetilde{Z}_\infty(x; g) = 0\Big) = 0$ or 1. Since the sequence $(\widetilde{Z}_n(x; g), n \ge 1)$ is strictly stationary on $x \in \mathbb{Z}^d$, we can consider any arbitrary starting point x. The following result follows from a martingale argument:

Proposition 5.1. *The following three assertions are equivalent:*

$$\widetilde{Z}_\infty(x; g) > 0, \qquad \text{a.s.}; \tag{5.1}$$

$$\widetilde{Z}_n(x; g) \to \widetilde{Z}_\infty(x; g), \qquad \textit{in } L^1; \tag{5.2}$$

$$\sum_1^\infty \langle \mathbb{1}_{(S_n^1 = S_n^2)} \rangle_2^{(n)} < \infty, \qquad \textit{a.s.} \tag{5.3}$$

Finally, if $\widetilde{Z}_\infty(x; g) > 0$ a.s., then there exists some constant $C_5 > 0$ such that for any $x \in \mathbb{Z}^d, \gamma < 2$ and any deterministic sequence $\alpha(y) \ge 0$ such that $\sum_y \alpha(y) = 1$, we have

$$\mathbf{E}\left(\frac{\widetilde{Z}_\infty^2(x; g)}{\Big(\sum_y \alpha(y) \widetilde{Z}_\infty(y; g)\Big)^{2-\gamma}}\right) \ge C_5\, 4^{\gamma - 2}. \tag{5.4}$$

Proof. We shall prove (5.1) $\iff$ (5.2) and (5.1) $\iff$ (5.3). Obviously, (5.2) implies that $\mathbb{P}\big(\widetilde{Z}_\infty(x; g) > 0\big) > 0$ hence (5.1) by the zero-one law. Assume (5.1). Since $\widetilde{Z}_n(x; g)$ converges almost surely to $\widetilde{Z}_\infty(x; g)$ and the sequence $\widetilde{Z}_n(x; g)$ is positive, it suffices to show that $\mathbf{E}\widetilde{Z}_\infty(x;g) = 1$. By Markov property (3.3), we have for a fixed j and all $n > j$,

$$\widetilde{Z}_n(x;g) = \widetilde{Z}_j(x;g) \sum_{y:|y-x|\le j} \langle \mathbb{1}_{(S_j=y)}\rangle^{(j,x)}\, \widetilde{Z}_{n-j}(y; g\circ\theta_j).$$

Observe that in the above sum on y, there are only finite terms (x and j being fixed), hence we can let $n\to\infty$ and get that

$$\frac{\widetilde{Z}_\infty(x;g)}{\widetilde{Z}_j(x;g)} = \sum_{y:|y-x|\le j} \langle \mathbb{1}_{(S_j=y)}\rangle^{(j,x)}\ \widetilde{Z}_\infty(y; g\circ\theta_j). \tag{5.5}$$

Let $c = \mathbf{E}\widetilde{Z}_\infty(x;g) > 0$ by the assumption (5.1). It follows from Fatou's lemma that

$$c = \mathbf{E}\lim_{n\to\infty} \widetilde{Z}_n(x;g) \le \liminf_{n\to\infty} \mathbf{E}\widetilde{Z}_n(x;g) = 1.$$

On the other hand, we again apply Fatou's lemma to (5.5) and obtain that

$$1 \le \liminf_{j\to\infty} \mathbf{E}\frac{\widetilde{Z}_\infty(g,x)}{\widetilde{Z}_j(g,x)} = \liminf_{j\to\infty} \mathbf{E} \sum_{y:|y-x|\le j} \langle \mathbb{1}_{(S_j=y)}\rangle^{(j,x)}\, \widetilde{Z}_\infty(y; g\circ\theta_j) = c,$$

since $\widetilde{Z}_\infty(y; g\circ\theta_j)$ is independent of $\mathcal{G}_j$ hence independent of $\langle \mathbb{1}_{(S_j=y)}\rangle^{(j,x)}$ and $\mathbf{E}\widetilde{Z}_\infty(y; g\circ\theta_j) = c$. This shows the equivalence between (5.1) and (5.2).

To prove (5.1) $\Longleftrightarrow$ (5.3), we consider the starting point $x = 0$ and the supermartingale

$$\log \widetilde{Z}_n(0;g) = M_n - A_n,$$

with M the martingale part and the non-increasing process $(-A_n)$ given by

$$\begin{aligned} A_n - A_{n-1} &\stackrel{\text{def}}{=} -\mathbf{E}\Big(\log \frac{\widetilde{Z}_n(0;g)}{\widetilde{Z}_{n-1}(0;g)} \,\Big|\, \mathcal{G}_{n-1}\Big) \\ &= -\mathbf{E}\Big(\log \sum_y \alpha_{n-1}(y) e^{\beta g(n,y) - \frac{\beta^2}{2}} \,\Big|\, \mathcal{G}_{n-1}\Big), \end{aligned} \tag{5.6}$$

where $\alpha_{n-1}(y)$ is $\mathcal{G}_{n-1}$-measurable and defined as follows:

$$\alpha_k(y) \stackrel{\text{def}}{=} \frac{1}{2d} \sum_{x:|x-y|=1} \langle \mathbb{1}_{(S_k=x)}\rangle^{(k)}, \qquad k\ge 1,\ y\in\mathbb{Z}^d. \tag{5.7}$$

We remark that by Cauchy-Schwarz' inequality:

$$V_k \stackrel{\text{def}}{=} \sum_y \alpha_k^2(y) \le \sum_y \frac{1}{2d} \sum_{x:|x-y|=1} \Big(\langle \mathbb{1}_{(S_k=x)}\rangle^{(k)}\Big)^2 = \langle \mathbb{1}_{(S_k^1=S_k^2)}\rangle_2^{(k)} \tag{5.8}$$

On the other hand, we have

$$V_k = \sum_y \alpha_k^2(y) \ge \sum_y \frac{1}{(2d)^2} \sum_{x:|x-y|=1} \Big(\langle \mathbb{1}_{(S_k=x)}\rangle^{(k)}\Big)^2 = \frac{\langle \mathbb{1}_{(S_k^1=S_k^2)}\rangle_2^{(k)}}{2d}. \tag{5.9}$$

Since $g(n, \cdot)$ is independent of $\mathcal{G}_{n-1}$, we apply Lemma 2.2 to (5.6) and obtain that

$$A_n - A_{n-1} \asymp \sum_y \alpha_{n-1}^2(y) = V_{n-1} \asymp \langle \mathbb{1}_{(S_{n-1}^1 = S_{n-1}^2)} \rangle_2^{(n-1)}, \qquad (5.10)$$

in view of (5.8) and (5.9). Hence (5.3) $\iff A_\infty = \infty$, a.s.. Let us estimate the increasing process of M:

$$\begin{aligned} [M, M]_n - [M, M]_{n-1} &= \mathbf{E}\Big((M_n - M_{n-1})^2 \big| \mathcal{G}_{n-1} \Big) \\ &\le 2\mathbf{E}\Big(\log^2 \frac{\widetilde{Z}_n(0; g)}{\widetilde{Z}_{n-1}(0; g)} \,\big|\, \mathcal{G}_{n-1} \Big) + 2\Big(A_n - A_{n-1} \Big)^2 \\ &= 2\mathbf{E}\Big(\log^2 \sum_y \alpha_{n-1}(y) e^{\beta g(n,y) - \frac{\beta^2}{2}} \,\big|\, \mathcal{G}_{n-1} \Big) \\ &\quad + 2\Big(A_n - A_{n-1} \Big)^2 \\ &\le C_6(\beta) V_{n-1}, \end{aligned} \qquad (5.11)$$

where the last inequality follows from Lemma 2.2 by taking $\phi(x) = \log^2 x$. Hence we have shown that

$$[M, M]_n \le C_6 A_n.$$

According to the strong law of large numbers for martingales (cf. [23], Theorem VII.4), we obtain that if $A_\infty = \infty$ a.s. (which is equivalent to $\sum_n V_n = \infty$ a.s.), then

$$\frac{M_n}{A_n} \to 0, \qquad a.s.$$

Hence $\log \widetilde{Z}_n(0; g) \to -\infty$, a.s. and $\widetilde{Z}_\infty(0; g) = 0$, almost surely. Whereas if $\mathbb{P}\Big(A_\infty < \infty \Big) > 0$, then $\mathbb{P}\Big(\widetilde{Z}_\infty(0; g) > 0 \Big) > 0$ which in fact equals 1. This implies the equivalence (5.1) $\iff$ (5.3).

Finally, (5.4) is easy: In fact, since $\mathbf{E}\widetilde{Z}_\infty(y; g) = 1$ we have

$$\mathbf{P}\Big(\sum_y \alpha(y) \widetilde{Z}_\infty(y; g) > 4 \Big) \le \frac{1}{4}.$$

Since $\widetilde{Z}_\infty(x; g) > 0$ a.s. and its distribution is independent of x, there exists some (small) constant $C_7 = C_7(d, \beta) > 0$ such that

$$\mathbf{P}\Big(\widetilde{Z}_\infty(x; g) < C_7 \Big) \le \frac{1}{4}.$$

It follows that the event $\{\widetilde{Z}_\infty(x; g) \ge C_7\} \cap \{\sum_y \alpha(y) \widetilde{Z}_\infty(y; g) \le 4\}$ has probability larger than $1/2$, and on this event

$$\frac{\widetilde{Z}_\infty^2(x; g)}{\Big(\sum_y \alpha(y) \widetilde{Z}_\infty(y; g) \Big)^{2-\gamma}} \ge C_7^2 4^{\gamma - 2},$$

which implies (5.4) by choosing $C_5 = \frac{1}{2} C_7^2$. $\square$

We shall make use of the following analogue of $\widetilde{Z}_n(x; g)$ defined by time-reversal:

$$\widehat{Z}_n(x) \equiv \widehat{Z}_n(x; g) \stackrel{\text{def}}{=} \mathbb{E}_x e^{\beta \sum_{j=0}^{n-1} g(n-j, S_j) - \frac{\beta^2}{2} n}, \qquad x \in \mathbb{Z}^d, \ n \geq 1. \tag{5.12}$$

Since $(g(j, \cdot), 1 \leq j \leq n) \stackrel{\text{law}}{=} (g(n-j, \cdot), 0 \leq j \leq n-1)$, it follows that

$$\Big(\widetilde{Z}_n(x), x \in \mathbb{Z}^d\Big) \stackrel{\text{law}}{=} \Big(P_1 \widehat{Z}_n(x), x \in \mathbb{Z}^d\Big), \tag{5.13}$$

where here and in the sequel, $(P_k, k \geq 0)$ denotes the semigroup of the random walk S: $P_k f(x) \stackrel{\text{def}}{=} \mathbb{E}_x f(S_k)$. We remark that $\widehat{Z}_n$ appears in the discrete form of Feynman-Kac's formula, and is related to a time-dependent random Schrödinger operator, see [4]. Observe that

$$\begin{aligned} \widehat{Z}_{n+1}(x) &= e^{\beta g(n+1,x) - \frac{\beta^2}{2}} \, \mathbb{E}_x \exp\Big(\beta \sum_{j=1}^{n} g(n+1-j, S_j) - \frac{\beta^2}{2} n\Big) \\ &= e^{\beta g(n+1,x) - \frac{\beta^2}{2}} \, P_1 \widehat{Z}_n(x), \end{aligned}$$

for $n \geq 0$ and $x \in \mathbb{Z}^d$. Now we can give the proof of Theorem 1.1:

Proof of Theorem 1.1. Fix $\gamma \in (0, 1)$ and $x_0 \in \mathbb{Z}^d$. In view of (5.13), it suffices to show that when $d \leq 2$,

$$\mathbf{E}\Big(P_1 \widehat{Z}_n(x_0)\Big)^{\gamma} \to 0, \qquad n \to \infty. \tag{5.14}$$

Indeed, by Fatou's lemma this implies that

$$\mathbf{E}\Big(\widetilde{Z}_\infty^{\gamma}\Big) \leq \liminf_{n\to\infty} \mathbf{E}\Big(\widetilde{Z}_n^{\gamma}\Big) = 0.$$

For $1 \leq k \leq n$, we have

$$\begin{aligned} \mathbf{E}\Big(P_k \widehat{Z}_{n-k+1}(x_0)\Big)^{\gamma} &= \mathbf{E}\Big(\sum_{y \in \mathbb{Z}^d} P_k(x_0, y) \widehat{Z}_{n-k+1}(y)\Big)^{\gamma} \\ &= \mathbf{E}\Big(\sum_{y \in \mathbb{Z}^d} P_k(x_0, y) (P_1 \widehat{Z}_{n-k})(y) e^{\beta g(n-k+1, y) - \frac{\beta^2}{2}}\Big)^{\gamma} \\ &= \mathbf{E}\Big(\big(P_{k+1} \widehat{Z}_{n-k}\big)(x_0) \sum_{y \in \mathbb{Z}^d} \mu_k(y) \, e^{\beta g(n-k+1, y) - \frac{\beta^2}{2}}\Big)^{\gamma}, \end{aligned}$$

with

$$\mu_k(y) \stackrel{\text{def}}{=} \frac{P_k(x_0, y) \, (P_1 \widehat{Z}_{n-k})(y)}{\big(P_{k+1} \widehat{Z}_{n-k}\big)(x_0)}, \qquad y \in \mathbb{Z}^d.$$

Observe that the $\mu_k(\cdot)$ are $\mathcal{G}_{n-k}$-measurable, hence independent of $g(n-k+1,\cdot)$, and $\sum_y \mu_k(y) = 1$. By conditioning on $\mathcal{G}_{n-k}$, we apply Lemma 2.2 with $\alpha(x) = \mu_k(x)$. It follows that (with $C_8 \stackrel{\text{def}}{=} \gamma(1-\gamma)(1-e^{-\beta^2})/2$)

$$\begin{aligned}
&\mathbf{E}\Big(P_k\widehat{Z}_{n-k+1}(x_0)\Big)^\gamma \\
&\quad \le \mathbf{E}\Big(\big(P_{k+1}\widehat{Z}_{n-k}(x_0)\big)^\gamma \big(1 - C_8\sum_y \mu_k^2(y)\big)\Big) \\
&\quad = \mathbf{E}\Big(P_{k+1}\widehat{Z}_{n-k}(x_0)\Big)^\gamma - C_8\,\mathbf{E}\frac{\sum_y \Big(P_k(x_0,y)\,(P_1\widehat{Z}_{n-k})(y)\Big)^2}{\big(P_{k+1}\widehat{Z}_{n-k}(x_0)\big)^{2-\gamma}} \\
&\quad \le \mathbf{E}\Big(P_{k+1}\widehat{Z}_{n-k}(x_0)\Big)^\gamma \exp\left(-C_8\,\mathbf{E}\frac{\sum_y \Big(P_k(x_0,y)\,(P_1\widehat{Z}_{n-k})(y)\Big)^2}{\big(P_{k+1}\widehat{Z}_{n-k}(x_0)\big)^{2-\gamma}}\right),
\end{aligned}$$

where we have used the fact that $\mathbf{E}\Big(P_{k+1}\widehat{Z}_{n-k}(x_0)\Big)^\gamma \le \Big(\mathbf{E}P_{k+1}\widehat{Z}_{n-k}(x_0)\Big)^\gamma = 1$ and the inequality: $e^{-u} \ge 1-u$ for $u \in \mathbb{R}$. By considering $k = 1, 2, ..., n$ in the above inequality ($\widehat{Z}_0 \equiv 1$), we obtain:

$$\mathbf{E}\Big(P_1\widehat{Z}_n(x_0)\Big)^\gamma \le \exp\left(-C_8\sum_{k=1}^n \mathbf{E}\,\frac{\sum_y \Big(P_k(x_0,y)\,(P_1\widehat{Z}_{n-k})(y)\Big)^2}{\big(P_{k+1}\widehat{Z}_{n-k}(x_0)\big)^{2-\gamma}}\right). \tag{5.15}$$

Let us prove (5.14) by reduction to absurdity; Assume (5.14) is false. There exists some constant $0 < c < 1$ such that (noticing that the sequence $\mathbf{E}\Big(P_1\widehat{Z}_n(x_0)\Big)^\gamma$ is always decreasing on n by supermartingale property),

$$\mathbf{E}\Big(P_1\widehat{Z}_n(x_0)\Big)^\gamma \ge c, \qquad n \ge 1,$$

which in view of (5.15) imply that

$$\sum_{k=1}^n \sum_y \mathbf{E}\,\frac{\Big(P_k(x_0,y)\,(P_1\widehat{Z}_{n-k})(y)\Big)^2}{\big(P_{k+1}\widehat{Z}_{n-k}(x_0)\big)^{2-\gamma}} \le \frac{\log(1/c)}{C_8}, \qquad \forall n \ge 1.$$

Since $\widetilde{Z}_j(\cdot;g) \stackrel{\text{law}}{=} P_1\widehat{Z}_j(\cdot;g)$, we get that for any large but fixed n_1,

$$\sum_{k=1}^{n_1} \sum_y \mathbf{E}\,\frac{\Big(P_k(x_0,y)\,\widetilde{Z}_{n-k}(y;g)\Big)^2}{\big(P_k\widetilde{Z}_{n-k}(x_0;g)\big)^{2-\gamma}} \le \frac{\log(1/c)}{C_8}, \qquad \forall n \ge n_1.$$

Since $\widetilde{Z}_n(x;g) \to \widetilde{Z}_\infty(x;g)$ a.s. and $\widetilde{Z}_\infty(x;g) > 0$ by hypothesis, Fatou's lemma implies that

$$\sum_{k=1}^{n_1} \sum_y P_k^2(x_0,y)\,\mathbf{E}\,\frac{\widetilde{Z}^2_\infty(y;g)}{\big(P_k\widetilde{Z}_\infty(x_0;g)\big)^{2-\gamma}} \le \frac{\log(1/c)}{C_8}, \qquad \forall n_1 \ge 1.$$

Now, by using (5.4), we obtain that for some constant $c' = \frac{\log(1/c)4^{2-\gamma}}{C_5C_8}$,

$$\sum_{k=1}^{n_1}\sum_y \Big(P_k(x_0,y)\Big)^2 \le c', \qquad \forall n_1 \ge 1,$$

which is absurd because for $d = 1, 2$, $\sum_y \Big(P_k(x_0,y)\Big)^2 = \mathbb{P}\Big(S_k^1 = S_k^2\Big) \asymp k^{-d/2}$ whose sum on k does not converge. Therefore we have proven (5.14) and hence Theorem 1.1. □

Remark 5.2. In fact, Theorem 1.1 holds for more general random environments: Consider the environment $(1+h(j,x))_{j\ge 1, x\in\mathbb{Z}^d}$ in lieu of $(e^{\beta g(j,x)-\beta^2/2})_{j\ge 1, x\in\mathbb{Z}^d}$ where $(h(j,x))$ are i.i.d. centered variables with compact support in $(-1,1)$. Let $0<\gamma<1$. By using the elementary inequality: $(1+u)^\gamma \le 1+\gamma u - c_\gamma u^2$ for small $|u|$, we obtain that for any sequence of nonnegative numbers $(\alpha(y))_{y\in\mathbb{Z}^d}$ such that $\sum_y \alpha(y) = 1$,

$$\mathbf{E}\Big(\sum_y \alpha(y)(1+h(j,y))\Big)^\gamma \le 1 - c_\gamma\, \mathbf{E}h^2(1,0)\sum_y \alpha^2(y).$$

Then by using the same arguments in the above proof, we arrive at

$$\mathbb{E}\prod_{i=1}^n (1+h(i,S_i)) \xrightarrow{\text{a.s.}} 0, \qquad d = 1,\ 2. \tag{5.16}$$

6. Proof of Theorem 1.2

The proof of Theorem 1.2 is based on an induction on the size n of the Gibbs measure $\langle\cdot\rangle^{(n)}$. This induction argument is inspired by the cavity method of spin glasses (see e.g. Talagrand [26]), where you can view the n-particle system as the $n+1$-particle system with a hole in it.

Firstly, let us introduce some notations (cf. (5.8) and (5.9)):

$$U_n(x) \stackrel{\text{def}}{=} \langle \mathbb{1}_{(S_n=x)}\rangle^{(n)}, \qquad x\in\mathbb{Z}^d,$$

$$V_n \stackrel{\text{def}}{=} \sum_x \alpha_n^2(x) = \sum_x \Big(\frac{1}{2d}\sum_{|e|=1} U_n(x+e)\Big)^2, \qquad n\ge 1,$$

$$\frac{1}{2d}\langle \mathbb{1}_{(S_n^1=S_n^2)}\rangle_2^{(n)} \ \le\ V_n\ \le\ \langle \mathbb{1}_{(S_n^1=S_n^2)}\rangle_2^{(n)}. \tag{6.1}$$

Hence it suffices to prove Theorem 1.2 for V_n in lieu of $\langle \mathbb{1}_{(S_n^1=S_n^2)}\rangle_2^{(n)}$. Let $k\ge 1$. Denote by $\mathbb{E}_{S^1,S^2}$ the expectation with respect to two simple independent random walks S^1 and S^2. We have

$$\mathbf{E}\Big(U_{k+1}(x_1)U_{k+1}(x_2)\big|\mathcal{G}_k\Big)$$

$$= \mathbf{E}\Big(\langle \mathbb{1}_{(S_{k+1}^1=x_1, S_{k+1}^2=x_2)}\rangle_2^{(n+1)}\big|\mathcal{G}_k\Big)$$

$$
\begin{aligned}
&= \mathbf{E}\Big(\mathbb{E}_{S^1,S^2}\,\frac{1}{Z_{k+1}^2}\,e^{\beta\sum_1^k(g(j,S_j^1)+g(j,S_j^2))}\\
&\qquad\times\ \mathbb{1}_{(S_{k+1}^1=x_1,S_{k+1}^2=x_2)}\,e^{\beta(g(k+1,x_1)+\beta g(k+1,x_2)}\big|\mathcal{G}_k\Big)\\
&= \frac{1}{Z_k^2}\,\mathbb{E}_{S^1,S^2}\Big[e^{\beta\sum_1^k(g(j,S_j^1)+g(j,S_j^2))}\,\mathbb{1}_{(S_{k+1}^1=x_1,S_{k+1}^2=x_2)}\Big]\\
&\qquad\times\mathbf{E}\Big(\frac{e^{\beta(g(k+1,x_1)+\beta g(k+1,x_2)}}{\big(\sum_x\alpha_k(x)e^{\beta g(k+1,x)}\big)^2}\big|\mathcal{G}_k\Big)\\
&\overset{\text{def}}{=} \frac{1}{Z_k^2}\,\mathbb{E}_{S^1,S^2}\Big[e^{\beta\sum_1^k(g(j,S_j^1)+g(j,S_j^2))}\,\mathbb{1}_{(S_{k+1}^1=x_1,S_{k+1}^2=x_2)}\Big]\,J_1(k,x_1,x_2),\\
&= \frac{1}{(2d)^2}\sum_{|e_1|=|e_2|=1}\langle\mathbb{1}_{(S_k^1=x_1+e_1,S_k^2=x_2+e_2)}\rangle_2^{(n)}\,J_1(k,x_1,x_2)\\
&= J_1(k,x_1,x_2)\,\alpha_k(x_1)\alpha_k(x_2). \qquad (6.2)
\end{aligned}
$$

Applying Lemma 2.3 to J_1, we obtain that

$$
\begin{aligned}
J_1(k,x_1,x_2) &\overset{\text{def}}{=} \mathbf{E}\Big(\frac{e^{\beta(g(k+1,x_1)+\beta g(k+1,x_2)}}{\big(\sum_x\alpha_k(x)e^{\beta g(k+1,x)}\big)^2}\big|\mathcal{G}_k\Big)\\
&\geq 1+(e^{\beta^2}-1)\mathbb{1}_{(x_1=x_2)}-C_9(\alpha_k(x_1)+\alpha_k(x_2))+C_{10}V_k, \qquad (6.3)\\
J_1(k,x_1,x_2) &\leq 1+(e^{\beta^2}-1)\mathbb{1}_{(x_1=x_2)}-C_{11}(\alpha_k(x_1)+\alpha_k(x_2))+C_{12}V_k, \qquad (6.4)
\end{aligned}
$$

with four positive constants $C_9>0$, ..., $C_{12}>0$ only depending on β.

Recall that $(P_n, n\geq 0)$ is the semigroup of the random walk S: $P_nf(x)=\mathbb{E}_xf(S_n),\ x\in\mathbb{Z}^d,\ n\geq 0$. Noticing that $\alpha_n(x)=P_1U_n(x)$. For $k=n, n-2, \dots, 1$, we have from (6.2) and (6.3) that

$$
\begin{aligned}
&\mathbf{E}\Big(\sum_x\big(P_{n-k}\alpha_{k+1}(x)\big)^2\big|\mathcal{G}_k\Big)\\
&\quad=\mathbf{E}\Big(\sum_x\big(P_{n-k+1}U_{k+1}(x)\big)^2\big|\mathcal{G}_k\Big)\\
&\quad=\sum_x\mathbb{E}_{S^1,S^2}\mathbf{E}\Big(U_{k+1}(x+S_{n-k+1}^1)\,U_{k+1}(x+S_{n-k+1}^2)\big|\mathcal{G}_k\Big)\\
&\quad=\sum_x\mathbb{E}_{S^1,S^2}\alpha_k(x+S_{n-k+1}^1)\alpha_k(x+S_{n-k+1}^2)J_1(k,x+S_{n-k+1}^1,x+S_{n-k+1}^2)\\
&\quad\geq\mathbb{E}_{S^1,S^2}\sum_x\alpha_k(x+S_{n-k+1}^1)\alpha_k(x+S_{n-k+1}^2)\\
&\qquad\quad\times\Big(1+(e^{\beta^2}-1)\mathbb{1}_{(S_{n-k+1}^1=S_{n-k+1}^2)}-C_9(\alpha_k(x+S_{n-k+1}^1)\\
&\qquad\quad+\alpha_k(x+S_{n-k+1}^2))+C_{10}V_k\Big)\\
&\quad=\sum_x\Big(P_{n-k+1}\alpha_k(x)\Big)^2+(e^{\beta^2}-1)\,q_{n-k+1}\,V_k
\end{aligned}
$$

$$-C_9 \sum_x (P_{n-k+1}\alpha_k(x))(P_{n-k+1}\alpha_k^2(x)) + C_{10} V_k \sum_x (P_{n-k+1}\alpha_k(x))^2$$

$$\geq \sum_x \Big(P_{n-k+1}\alpha_k(x)\Big)^2 + \beta^2\, q_{n-k+1}\, V_k - C_9 \Big(V_k\Big)^{3/2}, \tag{6.5}$$

where in the last inequality, we have used the facts that $e^{\beta^2} - 1 \geq \beta^2$ and that $\sum_x P_{n-k+1}\alpha_k^2(x) = V_k$ which implies that $\max_x P_{n-k+1}\alpha_k(x) \leq \sqrt{V_k}$ hence $\sum_x (P_{n-k+1}\alpha_k(x))(P_{n-k+1}\alpha_k^2(x)) \leq \big(V_k\big)^{3/2}$, and the sequence (q_j) only depends on d (see e.g. [20] for the asymptotic behavior):

$$q_j \stackrel{\text{def}}{=} \mathbb{P}\Big(S_j^1 = S_j^2\Big) \asymp (1+j)^{-d/2}, \qquad j \geq 0. \tag{6.6}$$

Applying (6.4) with $k = n$, we have a rough bound

$$J_1(n, x_1, x_2) \leq e^{\beta^2} + C_{12}, \qquad x_1, x_2 \in \mathbb{Z}^d,$$

which implies that

$$\begin{aligned}
\mathbf{E}\Big(V_{n+1} \,\Big|\, \mathcal{G}_n\Big) &= \mathbf{E}\Big(\sum_x \alpha_{n+1}^2(x) \,\Big|\, \mathcal{G}_n\Big) \\
&= \frac{1}{(2d)^2} \sum_x \sum_{|e_1|=|e_2|=1} \mathbf{E}\Big(U_{n+1}(x+e_1)U_{n+1}(x+e_2) \,\Big|\, \mathcal{G}_n\Big) \\
&= \frac{1}{(2d)^2} \sum_x \sum_{|e_1|=|e_2|=1} \alpha_n(x+e_1)\alpha_n(x+e_2) J_1(n, x+e_1, x+e_2) \\
&\leq (e^{\beta^2} + C_{12}) \frac{1}{(2d)^2} \sum_x \sum_{|e_1|=|e_2|=1} \alpha_n(x+e_1)\alpha_n(x+e_2) \\
&\leq (e^{\beta^2} + C_{12}) V_n,
\end{aligned} \tag{6.7}$$

since by Jensen's inequality $\frac{1}{(2d)^2} \sum_x \sum_{|e_1|=|e_2|=1} \alpha_n(x_1)\alpha_n(x_2) = \sum_x \big(P_1\alpha_n(x)\big)^2 \leq \sum_x P_1\alpha_n^2(x) = V_n$. The following lemma will be used in the proof of Theorem 1.2:

Lemma 6.1. *Let $T_n \stackrel{\text{def}}{=} \sum_{j=1}^{n-1} V_j$ for $n \geq 2$. For any fixed $j \geq 0$, we consider the martingale $Y_j(\cdot)$ defined by*

$$Y_j(m) \stackrel{\text{def}}{=} \sum_{l=1}^m \left(\sum_x \big(P_j\alpha_l(x)\big)^2 - \mathbf{E}\Big(\sum_x \big(P_j\alpha_l(x)\big)^2 \,\Big|\, \mathcal{G}_{l-1}\Big)\right), \; m \geq 1.$$

Assume that $T_\infty = \infty$, a.s.. *Then the strong law of large numbers holds:*

$$\frac{Y_j(m)}{T_m} \to 0, \qquad m \to \infty, \text{ a.s..}$$

Proof of Lemma 6.1. It suffices to compute the increasing process $([Y_j, Y_j]_m)$ of the martingale Y_j. To this end, observe that $\sum_x (P_j\alpha_m(x))^2 \le \sum_x P_j\alpha_m^2(x) = V_m$ by Cauchy-Schwarz inequality, we have

$$\begin{aligned}(Y_j(m) - Y_j(m-1))^2 &= 2\Big(\sum_x \big(P_j\alpha_m(x)\big)^2\Big)^2 \\ &\quad + 2\Big(\mathbf{E}\Big(\sum_x \big(P_j\alpha_m(x)\big)^2 \,\Big|\, \mathcal{G}_{m-1}\Big)\Big)^2 \\ &\le 2V_m^2 + 2\Big(\mathbf{E}\Big(V_m \,\Big|\, \mathcal{G}_{m-1}\Big)\Big)^2 \\ &\le 2V_m^2 + 2\mathbf{E}\Big(V_m^2 \,\Big|\, \mathcal{G}_{m-1}\Big),\end{aligned}$$

this together with the fact that $V_m \le \sum_x \alpha_m(x) = 1$ imply that

$$\begin{aligned}[Y_j, Y_j]_m - [Y_j, Y_j]_{m-1} &= \mathbf{E}\Big((Y_j(m) - Y_j(m-1))^2 \,\Big|\, \mathcal{G}_{m-1}\Big) \\ &\le 4\,\mathbf{E}\Big(V_m \,\Big|\, \mathcal{G}_{m-1}\Big) \\ &\le 4(e^{\beta^2} + C_{12})\, V_{m-1}, \end{aligned} \tag{6.8}$$

where the last inequality follows from (6.7). Hence we have shown that

$$[Y_j, Y_j]_m \le 4(e^{\beta^2} + C_{12})\, T_m,$$

This in view of the standard law of large numbers for square-integrable martingale (cf. [23], Theorem VII.4) implies the desired lemma. □

Now we can give the following proof:

Proof of Theorem 1.2. Recall (6.6). When $d = 1$ or $d = 2$, the random walk is recurrent $\sum_{k\ge 1} q_k = \infty$, hence we may choose a large but fixed $n_0 = n_0(d, \beta)$ such that

$$\sum_{j=1}^{n_0-1} q_j \ge \frac{8}{\beta^2}.$$

Let

$$\epsilon_0 = \epsilon_0(\beta, d) \overset{\text{def}}{=} \frac{\beta^4}{4C_9^2} \min_{j\le n_0} q_j^2,$$

where the constant $C_9 = C_9(\beta) > 0$ was given by (6.3). Consider $n \ge n_0 + 1$. Taking the sum of the inequalities (6.5) with $k = n, n-1, ..., n-n_0$, we obtain:

$$\begin{aligned}&\sum_{k=n-n_0}^{n} \Big(\mathbf{E}\Big(\sum_x \big(P_{n-k}\alpha_{k+1}(x)\big)^2 \,\Big|\, \mathcal{G}_k\Big) - \sum_x \big(P_{n-k+1}\alpha_k(x)\big)^2\Big) \\ &\quad \ge \beta^2 \sum_{k=n-n_0}^{n} q_{n-k+1} V_k - C_9 \sum_{k=n-n_0}^{n} V_k^{3/2} \\ &\quad \ge \frac{\beta^2}{2} \sum_{k=n-n_0}^{n} q_{n-k+1} V_k - C_9 \sum_{k=n-n_0}^{n} \mathbb{1}_{(V_k \ge \epsilon_0)}, \end{aligned} \tag{6.9}$$

where the last inequality is due to the fact that if $V_k \le \epsilon_0$, then by the definition of ϵ_0, $C_9 V_k^{3/2} \le \frac{\beta^2}{2} q_{n-k+1} V_k$ for all $n - n_0 \le k \le n$; otherwise we bound V_k by 1. Remark that

$$\begin{aligned}
\sum_{k=n-n_0}^{n} & \left(\mathbf{E}\Big(\sum_x \big(P_{n-k}\alpha_{k+1}(x)\big)^2 \,\Big|\, \mathcal{G}_k \Big) - \sum_x \big(P_{n-k+1}\alpha_k(x)\big)^2 \right) \\
&= \sum_{k=n-n_0}^{n} \left(\mathbf{E}\Big(\sum_x \big(P_{n-k}\alpha_{k+1}(x)\big)^2 \,\Big|\, \mathcal{G}_k \Big) - \sum_x \big(P_{n-k}\alpha_{k+1}(x)\big)^2 \right) \\
&\qquad + V_{n+1} - \sum_x \big(P_{n_0+1}\alpha_{n-n_0}(x)\big)^2 \\
&\le \sum_{j=0}^{n_0} \left(\mathbf{E}\Big(\sum_x \big(P_j\alpha_{n-j+1}(x)\big)^2 \,\Big|\, \mathcal{G}_{n-j} \Big) - \sum_x \big(P_j\alpha_{n-j+1}(x)\big)^2 \right) + V_{n+1}.
\end{aligned} \tag{6.10}$$

Pick up a large $N \gg n_0$. Taking the sum of the inequalities (6.9) with $n = n_0 + 1, n_0 + 2, ..., N$, we get in view of (6.10) that

$$\sum_{n=n_0+1}^{N} V_{n+1} \ge \frac{\beta^2}{2} \sum_{n=n_0+1}^{N} \sum_{j=0}^{n_0} q_{j+1} V_{n-j} - C_9 \sum_{n=n_0+1}^{N} \sum_{j=0}^{n_0} \mathbb{1}_{(V_{n-j} \ge \epsilon_0)} + \xi(N), \tag{6.11}$$

where

$$\begin{aligned}
\xi(N) &\stackrel{\text{def}}{=} \sum_{n=n_0+1}^{N} \sum_{j=0}^{n_0} \left(\sum_x \big(P_j\alpha_{n-j+1}(x)\big)^2 - \mathbf{E}\Big(\sum_x \big(P_j\alpha_{n-j+1}(x)\big)^2 \,\Big|\, \mathcal{G}_{n-j} \Big) \right) \\
&= \sum_{j=0}^{n_0} \sum_{l=n_0-j+2}^{N-j+1} \left(\sum_x \big(P_j\alpha_l(x)\big)^2 - \mathbf{E}\Big(\sum_x \big(P_j\alpha_l(x)\big)^2 \,\Big|\, \mathcal{G}_{l-1} \Big) \right) \\
&\stackrel{\text{def}}{=} \sum_{j=0}^{n_0} \Big(Y_j(N - j + 1) - Y_j(n_0 - j + 1) \Big),
\end{aligned}$$

with definition of $Y_j(\cdot)$ from Lemma 6.1. For $d = 1$ or $d = 2$, we have from Theorem 1.1 and Proposition 5.1 that

$$T_N \stackrel{\text{def}}{=} \sum_{n=1}^{N-1} V_n \to \infty, \qquad N \to \infty, \text{ a.s..}$$

Therefore we can apply Lemma 6.1 and obtain that almost surely, for all large $N \ge N_1(g)$ (n_0 being fixed),

$$\big|\xi(N)\big| \le \frac{1}{2} \sum_{n=n_0+1}^{N+1} V_n \le \sum_{l=n_0+1}^{N-n_0} V_l,$$

which in view of (6.11) yield that

$$\begin{aligned}\frac{3}{2}\sum_{n=n_0+1}^{N+1} V_n &\geq \sum_{n=n_0+1}^{N+1} V_n + \left|\xi(N)\right| \\ &\geq \frac{\beta^2}{2}\sum_{j=0}^{n_0} q_{j+1} \sum_{l=n_0+1}^{N-n_0} V_l - C_9(n_0+1)\sum_{l=1}^{N} \mathbb{1}_{(V_l\geq\epsilon_0)} \\ &\geq \frac{\beta^2}{4}\sum_{j=0}^{n_0} q_{j+1} \sum_{l=n_0+1}^{N+1} V_l - C_9(n_0+1)\sum_{l=1}^{N} \mathbb{1}_{(V_l\geq\epsilon_0)} \\ &\geq 2\sum_{n=n_0+1}^{N+1} V_n - C_9(n_0+1)\sum_{l=1}^{N} \mathbb{1}_{(V_l\geq\epsilon_0)},\end{aligned}$$

by our choice of n_0. The above ω-by-ω argument shows that almost surely, for all large N,

$$\sum_{l=1}^{N} \mathbb{1}_{(V_l\geq\epsilon_0)} \geq \frac{1}{2C_9(n_0+1)}\sum_{n=n_0}^{N} V_n,$$

which combined with (6.1) imply (1.1) by taking $c_0 = \min(\epsilon_0, \frac{1}{4dC_9(n_0+1)})$. Recall from Theorem 1.1 that $\sum_{n=1}^{\infty}\langle \mathbb{1}_{(S_n^1=S_n^2)}\rangle_2^{(n)} = \infty$, a.s., we obtain (1.2). Finally, remark that

$$\langle \mathbb{1}_{(S_n^1=S_n^2)}\rangle_2^{(n)} = \sum_x \left(\langle \mathbb{1}_{(S_n=x)}\rangle^{(n)}\right)^2 \leq \max_x \langle \mathbb{1}_{(S_n=x)}\rangle^{(n)},$$

which yields (1.3) and completes the proof of Theorem 1.2. □

7. Integration by parts

This section is devoted to a formula which relates $p_n'(\beta)$ to the global correlation $\langle L_n(S^1-S^2)\rangle_2^{(n)}$ of two independent configurations S^1 and S^2:

Lemma 7.1. *For all $d \geq 1$ and $\beta > 0$, we have*

$$p_n'(\beta) = \beta - \frac{\beta}{n}\mathbf{E}\Big(\langle L_n(S^1-S^2)\rangle_2^{(n)}\Big) \in (0,\beta),$$

where $\langle\cdot\rangle_2^{(n)}$ denotes the Gibbs measure with respect to two independent configurations S^1 and S^2 and $L_n(S^1-S^2) \stackrel{\text{def}}{=} \sum_{i=1}^n \mathbb{1}_{(S_i^1=S_i^2)}$.

Proof. The idea of the use of Itô's formula goes back at least to Comets and Neveu [5]. Fix n. Consider a family of i.i.d. $\{B_{i,x}(t), t\geq 0\}_{i\geq1,x\in\mathbb{Z}^d}$ of one-dimensional Brownian motions. Define

$$N_t \stackrel{\text{def}}{=} \mathbb{E}\Big(e^{\sum_{i=1}^n B_{i,S_i}(t)-\frac{nt}{2}}\Big), \qquad t\geq 0.$$

It turns out that (N_t) is a positive continuous martingale and can be written as (see e.g. [22] Proposition VIII.1.6)

$$N_t = \exp\Big(R_t - \frac{1}{2}[R, R]_t\Big), \qquad t \geq 0,$$

where (R_t) is a continuous martingale given by

$$R_t = \int_0^t \frac{dN_s}{N_s}, \qquad t \geq 0,$$

and $[R, R]_\cdot = \int_0^\cdot \frac{d[N,N]_s}{N_s^2}$ is the continuous increasing process of R (which usually is denoted by $\langle R, R\rangle_t$, we adopt the notation $[R, R]$ to avoid confusion with the Gibbs measure). It follows that

$$\begin{aligned}\frac{d}{dt}\mathbf{E}\Big(\log N_t\Big) &= -\frac{1}{2}\frac{d}{dt}\mathbf{E}\Big([R, R]_t\Big)\\ &= -\frac{1}{2}\mathbf{E}\left(\frac{\mathbb{E}_{S^1,S^2}\Big(L_n(S^1-S^2)e^{\sum_1^n(B_{i,S_i^1}(t)+B_{i,S_i^2}(t))-nt}\Big)}{\Big(N_t\Big)^2}\right), \qquad (7.1)\end{aligned}$$

by noticing that $dN_t = \mathbb{E}\Big(e^{\sum_{i=1}^n B_{i,S_i}(t)-\frac{nt}{2}}d(\sum_1^n B_{i,S_i}(t))\Big)$. Remark that

$$p_n(\beta) = \frac{1}{n}\mathbf{E}\Big(\log N_{\beta^2}\Big) + \frac{\beta^2}{2},$$

which in view of (7.1) yield that

$$\begin{aligned}p_n'(\beta) &= -\frac{\beta}{n}\mathbf{E}\left(\frac{\mathbb{E}_{S^1,S^2}\Big(L_n(S^1-S^2)e^{\sum_1^n(B_{i,S_i^1}(\beta^2)+B_{i,S_i^2}(\beta^2))-n\beta^2}\Big)}{\Big(N_{\beta^2}\Big)^2}\right) + \beta\\ &= -\frac{\beta}{n}\mathbf{E}\langle L_n(S^1-S^2)\rangle_2^{(n)} + \beta,\end{aligned}$$

as desired. □

Finally, we end the whole paper with a formula for $p_n''(\beta)$ which can be proven in the same spirit as in the previous proof and the (tedious) details of the proof are omitted:

Remark 7.2. For all $d \geq 1$, we have

$$p_n''(\beta) = 1 - \mathbf{E}\langle\frac{L_n(S^1-S^2)^{(n)}}{n}\rangle_2 + 2\,\beta^2\, n\, J_n(\beta),$$

with

$$J_n(\beta) \stackrel{\text{def}}{=} \mathbf{E}\Big\langle - \Big(\frac{L_n(S^1-S^2)}{n}\Big)^2 + 4\frac{L_n(S^1-S^2)}{n}\frac{L_n(S^1-S^3)}{n} - 3\frac{L_n(S^1-S^3)}{n}\frac{L_n(S^2-S^4)}{n}\Big\rangle_4^{(n)},$$

and $\langle\cdot\rangle_4^{(n)}$ denotes the Gibbs measure with four independent configurations $S^1, ..., S^4$.

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