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Limit theorems for coupled analytic maps

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Abstract. In [23], H.H. Rugh gave a new and simpler method to prove existence of an invariant measure with nice mixing properties for some weakly coupled analytic maps. He proved in fact a spectral gap property for a well-defined transfer operator. We modify this method to construct generalized transfer operators associated to potentials and preserve the spectral gap for small potentials. This allows us to prove new limit results for these systems: Central Limit Theorem, Moderate Deviations Principle and a partial Large Deviations result. These results are available under wide classes of observables and initial measures.

1. Introduction

Coupled map lattices are models of infinite dimensional dynamical systems. They have been introduced by Kaneko in 1983 as a simple model featuring space-time chaotic behaviour (see [16] for a review).

The first mathematical study was done by Bunimovitch and Sinai. In [8], they considered a lattice model with local expanding maps of the circle as a space-time dynamical system and used a coding to obtain uniqueness of the equilibrium state and decay of correlations. This point of view has been generalized to Anosov systems by Pesin and Sinai in [19]. The last developments in this direction are due to Jiang and Pesin (see [15] and [14]).

Another approach to this problem consists in using transfer operators. The first attempt in this way was [17] where Keller and Künztle constructed transfer operators on spaces of bounded variation functions. The main progress was done by Bricmont and Kupiainen in [5] and [6]. They introduced complex analysis and cluster expansion methods to prove a uniform spectral gap property for finite box restrictions of the map. This implies existence of an SRB measure. This was extended by Baladi et al in [2] where a global transfer operator is constructed.

Using methods of Maes and Van Moffaert [18], Fisher and Rugh found a simpler method, exposed in [12], to prove spectral gap for a transfer operator defined on a well adapted Banach space. In this paper, we use the last improvement done by Rugh in [23]. A modification of the construction of the transfer operator allows us to get more general operators associated to a perturbation. This, together with the

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conservation of the spectral gap property by a perturbation result, gives new limit results: Central Limit Theorem and Moderate Deviations Principle for the evolution of orbits under a large class of observables and initial measures, but also a partial Large Deviations result, which implies in particular exponential convergence to equilibrium.

We define the model in Section 2 and give the results in Section 3. We then prove the probabilistic results in Section 4. The method is similar to the proofs of [21] or [7]. Proof of an intermediate result on the existence of transfer operators and their spectral properties is given in Sections 5 and 6. The key step is the generalization of the combinatorial analysis from [23] to construct the transfer operators. This is presented in Subsection 5.3.

2. Definitions

2.1. Expanding maps

We consider $S^1 = \mathbb{R}/\mathbb{Z}$ as a subset of the complex cylinder $\mathcal{C} = \mathbb{C}/\mathbb{Z}$. This allows us to work with functions not only real-analytic on the circle but holomorphic on a small annulus $A[\rho] = \{z \in \mathcal{C} : |\mathrm{Im}\, z| \leq \rho\}$ for $\rho > 0$. For such functions, we are able to use complex analysis and this is the basis of the method introduced in [12] and [23] to construct transfer operators.

Thus, the single-site functions we will use are real-analytic expanding functions on the circle in the following sense, denoting $\partial A[\rho] = \{z \in \mathcal{C} : |\mathrm{Im}\, z| = \rho\}$ the boundary of $A[\rho]$:

Definition 1. *For $\rho > 0$ and $\lambda > 1$, we say that $f : A[\rho] \to \mathcal{C}$ is a* **real analytic (ρ, λ)-expanding map** *if f is continuous in $A[\rho]$, holomorphic in its interior, $f(S^1) = S^1$ and $f(\partial A[\rho]) \cap A[\lambda\rho] = \emptyset$.*
The set of all such functions is denoted $\mathcal{E}(\rho, \lambda)$.

Remark. Functions of $\mathcal{E}(\rho, \lambda)$ are also λ-expanding in the classical sense, i.e. they verify $|f'| \geq \lambda > 1$ on the circle (see Appendix A in [23]).

2.2. Configuration space

We take Ω an index set and define the configuration space of our dynamical system as the product of circles :

$$S_\Omega = \prod_{p\in\Omega} S^1 \subset A_\Omega = \prod_{p\in\Omega} A[\rho]$$

Ω can be quite general and could even be uncountable. But our main interest will be $\Omega = \mathbb{Z}^d$. For this case, some spatial behaviour can be studied (see [23] or [3] for such applications).

2.3. *Spaces of coupling and observables*

Let $\mathcal{F}$ be the set of finite subsets of Ω, containing the empty set. For all $\Lambda \in \mathcal{F}$, we denote $S_\Lambda = \prod_{p\in\Lambda} S^1 \subset A_\Lambda = \prod_{p\in\Lambda} A[\rho]$. We call E_Λ the set of functions which are continuous in A_Λ and holomorphic in its interior.
For $K \subset \Lambda$, we denote $j_{\Lambda,K} : E_K \to E_\Lambda$ and $j_\Lambda : E_\Lambda \to C(A_\Omega)$ the natural injections, then define $E(A_\Omega)$ as the closure of $\cup_{\Lambda\in\mathcal{F}} j_\Lambda(E_\Lambda)$. $E(A_\Omega)$ is in fact the space of weakly holomorphic continuous functions on A_Ω (see Appendix B of [23]).

We want to control the spatial expansion of the functions which will play the role of coupling and observables. For this, we choose a parameter $0 < \theta \leq 1$ and define:

$$H_\theta = \left\{\phi \in E(A_\Omega) \,:\, \phi = \sum_{\Lambda\in\mathcal{F}} j_\Lambda \phi_\Lambda \text{ with } \phi_\Lambda \in E_\Lambda \text{ and } \sum_{\Lambda\in\mathcal{F}} \theta^{-|\Lambda|}|\phi_\Lambda| < \infty\right\}$$

with, for $\phi \in H_\theta$:

$$|\phi|_\theta = \inf\left\{\sum_{\Lambda\in\mathcal{F}} \theta^{-|\Lambda|}|\phi_\Lambda| \,:\, (\phi_\Lambda)_{\Lambda\in\mathcal{F}} \text{ such that } \phi_\Lambda \in E_\Lambda \text{ and } \phi = \sum_{\Lambda\in\mathcal{F}} j_\Lambda\phi_\Lambda\right\}$$

Then $(H_\theta, |\cdot|_\theta)$ is a θ-penalized inductive limit of the spaces E_Λ. This defines a Banach algebra. If $\theta < 1$, functions of H_θ depend weakly on big sets Λ. For $\theta = 1$, $H_1 = E(A_\Omega)$ and $|\cdot|_\theta = |\cdot|_\infty$. We denote H_θ^r the set of real-analytic maps of H_θ.

2.4. *Coupled maps*

We can now define the class of dynamical systems we want to study:

Definition 2. *For $\rho > 0$, $\lambda > 1$, $0 < \theta \leq 1$ and $0 \leq \kappa < \infty$, we take $(f_p)_{p\in\Omega}$ expanding maps from $\mathcal{E}(\rho, \lambda)$, and $(g_p)_{p\in\Omega}$ coupling maps from H_θ^r such that $|g_p|_\theta < \kappa$.*
We define the associated **coupled analytic map** *as $F_\Omega = (F_p)_{p\in\Omega} : A_\Omega \to \mathcal{C}^\Omega$, where:*

$$F_p(z) = f_p(z_p) + g_p(z) \quad \forall p \in \Omega$$

We denote $CM[\rho, \lambda, \theta, \kappa]$ the space of all such coupled analytic maps.

3. Results

For all observable $b \in C(S_\Omega)$ and all $T \geq 1$, we write:

$$S_T b = \sum_{t=0}^{T-1} b \circ F^t$$

In [23], an ergodic theorem for the random variables $S_T b$ under Lebesgue measure and decay of correlations for the limit measure are proved under the assumption that the coupling is weak enough:

Theorem 1 (Th. 2.1 of [23]). *For all $\rho > 0$, $\lambda > 1$, there exists $\theta_0(\rho, \lambda) \in (0, \frac{1}{3})$ such that for $\theta < \theta_0$ there is $\kappa > 0$ for which the following holds for every $F \in CM[\rho, \lambda, \theta, \kappa]$:*

1. *There exists a natural probability measure ν invariant under F, i.e. $F^*\nu = \nu$,*
2. *For all $b \in C(S_\Omega)$, m^Ω-almost every x (with m the Lebesgue measure on the circle),*

$$\lim_{T\to\infty} \frac{1}{T} S_T b = \int_{S_\Omega} b \, d\nu \tag{1}$$

3. *There exists $\gamma > 1$ and $\theta < \vartheta < 1$ such that for all $b \in H_\theta$, $a \in H_\vartheta$ and $T \geq 1$,*

$$\left| \int_{S_\Omega} b \circ F^T \cdot a \, d\nu - \int_{S_\Omega} b \, d\nu \int_{S_\Omega} a \, d\nu \right| \leq 2|b|_\theta |a|_\vartheta \gamma^{-T} \tag{2}$$

These properties are consequences of a more technical result, the fact that a transfer operator associated to F exists on a well chosen Banach space and has a spectral gap below 1, which is the simple maximal eigenvalue. They are really an infinite dimensional version of classical single site results.

Our method consists in generalizing the construction of this operator to its perturbations by potentials and then extending the spectral gap by perturbation theory (see Theorem 4 and its proof Sections 5 and 6 for more details).

We improve the result of [23] with the following large deviations upper bound, and an associated partial lower bound (see Theorem 5 for a more precise statement):

Theorem 2. *Under the same conditions on the parameters as in Theorem 1, for all $u \in H^r_\theta$, there exists a lower semi-continuous, convex and non-negative function $I_u : \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$, with an unique zero at $\int_{S_\Omega} u \, d\nu$, and there are $a_u < \int_{S_\Omega} u \, d\nu < b_u$ such that:*

1. *For all closed $F \subset \mathbb{R}$:*

$$\limsup_{T\to\infty} \frac{1}{T} \log m^\Omega \left(z : \frac{S_T u(z)}{T} \in F \right) \leq - \inf_{x \in F} I_u(x) \qquad \text{(Upper Bound)}$$

2. *For all $x \in (a_u, b_u)$ and $\delta > 0$:*

$$\liminf_{T\to\infty} \frac{1}{T} \log m^\Omega \left(z : \frac{S_T u(z)}{T} \in B(x, \delta) \right) \geq -I_u(x) \qquad \text{(Lower Bound)}$$

The upper bound implies in particular that the convergence in (1) is exponential, which means that for all $A \in \mathbb{R}$ such that $\int_{S_\Omega} u \, d\nu \notin \bar{A}$:

$$\limsup_{T\to\infty} \frac{1}{T} \log m^\Omega \left\{ z : \frac{S_T u(z)}{T} \in A \right\} < 0 \tag{3}$$

Moreover, we obtain new probabilistic results for the random variables $S_T u$ under Lebesgue measure, namely a Central Limit Theorem and a Moderate Deviations Principle:

Theorem 3. *Suppose the Hypotheses of Theorem 1 are satisfied. For every $u \in H^r_\theta$, we write $m_u = \int_{S_\Omega} u\, d\nu$. Then the limit*

$$\lim_{T\to\infty} \int_{S_\Omega} \left(\frac{S_T u - T m_u}{\sqrt{T}} \right)^2 d\nu,$$

exists and is non negative. We denote it σ_u^2 and have the following condition:

$$\sigma_u^2 = 0 \qquad \textit{iff} \qquad \exists\, v \in L^2(\nu) \ \ \textit{such that} \ \ u = v - v\circ F \ \ \textit{in } L^2(\nu) \tag{4}$$

For u such that $\sigma_u^2 > 0$, we have:

$$\left(\frac{S_T u - T m_u}{\sqrt{T}\sigma_u} \right)^* \left(m^\Omega\right) \xrightarrow{Law} \mathcal{N}(0,1) \tag{CLT}$$

and for all $\frac{1}{2} < \alpha < 1$, $A \subset \mathbb{R}$ Borel set:

$$\begin{aligned} &- \inf_{x\in \overset{\circ}{A}} \frac{x^2}{2\sigma_u^2} \le \liminf_{T\to\infty} \frac{1}{T^{2\alpha-1}} \log m^\Omega \left(z \ : \ \frac{S_T u(z) - T m_u}{T^\alpha} \in A \right) \\ &\quad \le \limsup_{T\to\infty} \frac{1}{T^{2\alpha-1}} \log m^\Omega \left(z \ : \ \frac{S_T u(z) - T m_u}{T^\alpha} \in A \right) \le - \inf_{x\in \bar{A}} \frac{x^2}{2\sigma_u^2} \end{aligned} \tag{MDP}$$

Remark. All results above are given with Lebesgue measure as initial probability. In fact, they remain true taking measures in the Banach space on which our operators act (exactly on the subset of this Banach space which contains probabilities, denoted $\mathcal{M}^p_\vartheta$, see Section 4.1). We will prove our results in this more general context. The same generalization for the ergodic theorem (1) is valid and the proof of [23] adapts in a simple way.

4. Use of the spectral gap

In this section, we will prove Theorems 2 and 3 given an intermediate result (Theorem 4) on the spectral gap for perturbed operators. We use in these proofs the same type of methods as in the papers of J. Rousseau-Egele [21] or A. Broise [7].

4.1. Space of densities

For $K \subset \Lambda$, let $\pi_{K,\Lambda} : E_\Lambda \to E_K$ be the projection defined by:

$$\pi_{K,\Lambda}\phi_\Lambda(z_K) = \int_{S_{\Lambda\setminus K}} \phi_\Lambda(z_\Lambda)\, m^{\Lambda\setminus K}(dz_{\Lambda\setminus K})$$

If $\Lambda = \Omega$, we will note $\pi_K = \pi_{K,\Omega}$.

Following [23], we define now the Banach space on which our operators work. We need to take it sufficiently large, and specifically not included in $L^1(dm^\Omega)$. Indeed, in the uncoupled case (when the couplings g_p are zero), we know that the natural measure will be the infinite product of the SRB measures $h_p dm$ for the single site functions f_p, which will generally not be absolutely continuous with respect to Lebesgue measure. To get a large enough space, we choose a parameter $0 < \theta \leq 1$ and define:

$$\mathcal{M}_\theta = \Big\{ \phi = (\phi_\Lambda)_{\Lambda \in \mathcal{F}} \,:\, \pi_{\Lambda,\Lambda'}\phi_{\Lambda'} = \phi_\Lambda \ \forall \Lambda \subset \Lambda' \text{ and}$$
$$\|\phi\|_\theta = \sup_{\Lambda \in \mathcal{F}} \theta^{|\Lambda|} |\phi_\Lambda| < \infty \Big\}$$

$(\mathcal{M}_\theta, \|\cdot\|_\theta)$ is a Banach space and a H_θ-module: $g = \sum_{\Lambda' \in \mathcal{F}} g_{\Lambda'}$ element of H_θ acts on $\phi = (\phi_\Lambda)_{\Lambda \in \mathcal{F}}$ to get $g * \phi \in \mathcal{M}_\vartheta$ defined by:

$$(g * \phi)_\Lambda = \sum_{\Lambda' \in \mathcal{F}} \pi_{\Lambda,\Lambda\cup\Lambda'} \left(j_{\Lambda\cup\Lambda',\Lambda'}(g_{\Lambda'}) \cdot \phi_{\Lambda\cup\Lambda'} \right)$$

and the following bound holds: $\|g * \phi\|_\theta \leq |g|_\theta \|\phi\|_\theta$.
$\mathcal{M}_\theta$ contains the uncoupled natural measure $\otimes_{p\in\Omega}(h_p dm)$, for $\theta^{-1} > \sup_{p\in\Omega} |h_p|_\infty$. This measure is represented by $\phi = \left(\phi_\Lambda = \prod_{p\in\Lambda} h_p(z_p)\right)_{\Lambda\in\mathcal{F}}$, although it is not absolutely continuous with respect to Lebesgue measure. More generally, if we consider the following subset of $\mathcal{M}_\theta$:

$$\mathcal{M}_\theta^m = \left\{ \phi \in \mathcal{M}_\theta \,:\, \sup_{\Lambda\in\mathcal{F}} \int_{S_\Lambda} |\phi_\Lambda(z_\Lambda)|\, dz_\Lambda < \infty \right\}$$

then every $\phi \in \mathcal{M}_\theta^m$ can be seen as a measure on S_Ω defined by

$$\int_{S_\Omega} g\, d\phi = \phi(g) = \lim_{\Lambda\to\Omega} \int_{S_\Lambda} g_\Lambda \phi_\Lambda dm^\Lambda \quad \forall g \in C(S_\Omega)$$
$$\text{and } g_\Lambda \in C(S_\Lambda) \text{ such that } g_\Lambda \to g$$

All these measures have finite marginals on S_Λ which are absolutely continuous with respect to m^Λ, with density $\phi_\Lambda \in E(A_\Lambda)$. We will denote $\mathcal{M}_\theta^p$ the set of probability measures in $\mathcal{M}_\theta^m$.

4.2. Spectral gap for perturbed transfer operators

We state now the existence and the property of spectral gap for perturbed transfer operators:

Theorem 4. *For $F \in CM[\rho, \Lambda, \theta, \kappa]$, whose parameters satisfy conditions of Theorem 1 and with ϑ, γ and ν as in this result, there exists for all $T \geq 1$ an analytic functional:*

$$\begin{aligned} M^{(T)} \,:\, H_\theta &\longrightarrow L(\mathcal{M}_\vartheta, \mathcal{M}_\theta) \\ u &\longmapsto M_u^{(T)} \end{aligned} \tag{5}$$

satisfying:

- *There exists* $T_0 \geq 1$ *such that* $M_u^{(T)} \in L(\mathcal{M}_\vartheta) \quad \text{if } T \geq T_0$
- $\|M_u^{(T)}\| \leq e^{T|u|_\theta}$ $\qquad$ • $\|M_u^{(T)} - M_0^{(T)}\| \leq e^{T|u|_\theta} - 1$ (6)
- $M_u^{(t)} \circ M_u^{(T)} = M_u^{(t+T)} \quad \text{for } t \geq 1, T \geq T_0$ (7)
- $M_u^{(T)}\left(\mathcal{M}_\vartheta^m\right) \subset \mathcal{M}_\theta^m$ (8)
- $\displaystyle\int_{S_\Omega} b \circ F^T \exp\left(S_T u\right) \, d\phi = \int_{S_\Omega} b \, d\left(M_u^{(T)}\phi\right) \qquad \forall\, b \in C(S_\Omega),\ \phi \in \mathcal{M}_\vartheta^m$ (9)

Moreover, for all $\delta < \frac{1-\gamma^{-T_0}}{3}$*, there exists* $\rho > 0$ *such that if* $|u|_\theta < \rho$*, we can write for* $k \geq 1$*:*

$$M_u^{(kT_0)} = \lambda^{kT_0}(u)\, Q_u + R_u^k \tag{10}$$

with, for $D_\theta(0,\rho)$ *the ball of radius* ρ *around* 0 *in* H_θ*:*

– $\lambda : u \in D_\theta(0,\rho) \longmapsto \lambda(u) \in \mathbb{C}$ *is analytic and satisfies* $\lambda^{T_0}(u) \in D(1,\delta)$ *and* $\lambda(0) = 1$,
– $Q : u \in D_\theta(0,\rho) \longmapsto Q_u \in L(\mathcal{M}_\vartheta)$ *is analytic and satisfies* $Q_u^2 = Q_u$, $Q_0 = \nu\pi_\emptyset$ *and* $\|Q_u - \nu\pi_\emptyset\| \leq \delta^2$,
– $R : u \in D_\theta(0,\rho) \longmapsto R_u \in L(\mathcal{M}_\vartheta)$ *is analytic and satisfies* $Sp(R_u) \subset D(0, \gamma^{-T_0} + \delta)$ *and* $\|R_u^k\| \leq (\gamma^{-T_0} + 2\delta)^k$.

Remark. The important fact in these estimates is that they imply for such u:

$$\lim_{k\to\infty} \frac{\|R_u^k\|}{|\lambda^{kT_0}(u)|} = 0$$

so that $\lambda(u)$ will give the main term in asymptotic estimates.

4.3. Identification of the derivatives of $\lambda(u)$

Analyticity in the previous result is understood in the general sense given for example in Definition 3.17.2 of [13]: namely a map is **analytic** when it is expandable around each point as a convergent series of homogeneous terms with increasing degree. For λ, an analytic function of u on $D_\theta(0,\rho)$, we can write its expansion around 0:

$$\lambda(u) = \sum_{n\geq 0} \frac{1}{n!} \partial^n \lambda(0; u)$$

where in fact $\partial^0\lambda(0;u) = \lambda(0) = 1$ and $\partial^n\lambda(0;u) = \frac{\partial^n}{\partial z^n}\Big|_{z=0} \lambda(zu)$.

The key of our probabilistic study is the identification of the first two derivatives of λ in real-analytic directions with statistical estimates of the system.

Proposition 1. *For every $u \in H^r_\theta$, we have the two following identities:*

$$\begin{aligned}\partial^1\lambda(0;u) &= \int_{S_\Omega} u\,d\nu = m_u \\ \partial^2\lambda(0;u) &= \lim_{T\to\infty}\int_{S_\Omega}\left(\frac{S_T u - T m_u}{\sqrt{T}}\right)^2 d\nu = \sigma_u^2 \geq 0\end{aligned} \tag{11}$$

Remark. The identifications of $\partial^1\lambda(0;u)$ and $\partial^2\lambda(0;u)$ with the mean and the asymptotic variance of u under the equilibrium state ν are natural results in view of classical thermodynamic formalism results (see [22]): $\log\lambda(u)$, in the domain where it is defined, really plays the role of a topological pressure.

Proof. (a) *Identification of* $\partial^1\lambda(0;u)$. We will decompose each $T \geq 1$ as $T = kT_0 + \tilde{T}$, with $0 \leq \tilde{T} < T_0$, and write:

$$\int_{S_\Omega} \exp\left(\frac{1}{T}S_T u\right) d\nu = \int_{S_\Omega} \exp\left(S_{kT_0}\left(\frac{u}{T}\right) + \frac{1}{T}S_{\tilde{T}}\left(u \circ F^{kT_0}\right)\right) d\nu$$

We have then a uniform estimate for the term with $\tilde{T}$:

$$\exp\left(-\frac{T_0}{T}|u|_\infty\right) \leq \exp\left(\frac{1}{T}S_{\tilde{T}}\left(u\circ F^{kT_0}\right)\right) \leq \exp\left(\frac{T_0}{T}|u|_\infty\right) \tag{12}$$

For the remaining term, if $T > \frac{|u|_\theta}{\rho}$, we apply the identity (9) and the spectral decomposition (10) to $M^{(kT_0)}_{\frac{u}{T}}$ to get:

$$\begin{aligned}\int_{S_\Omega} \exp\left(S_{kT_0}\left(\frac{u}{T}\right)\right) d\nu &= \pi_\emptyset\left(M^{(kT_0)}_{\frac{u}{T}}(\nu)\right) \\ &= \lambda^{kT_0}\left(\frac{u}{T}\right)\pi_\emptyset\left(Q_{\frac{u}{T}}(\nu)\right) + \pi_\emptyset\left(R^k_{\frac{u}{T}}(\nu)\right)\end{aligned} \tag{13}$$

We can now evaluate the limit as T tends to infinity of each term in this expression:

$$\lambda^{kT_0}\left(\frac{u}{T}\right) = \left(1 + \frac{1}{T}\partial^1\lambda(0;u) + o\left(\frac{1}{T}\right)\right)^{kT_0} \longrightarrow \exp\left(\partial^1\lambda(0;u)\right)$$

because the derivatives $\partial^n\lambda(0;u)$ are n-homogeneous and $\frac{kT_0}{T} \to 1$. It will be the main term in our estimation.

We control the two others:

$$\left|\pi_\emptyset\left(Q_{\frac{u}{T}}(\nu)\right) - 1\right| \leq \left\|Q_{\frac{u}{T}} - Q_0\right\| \|\nu\|_\vartheta \longrightarrow 0$$

by continuity of Q_u, and:

$$\left|\pi_\emptyset\left(R^k_{\frac{u}{T}}(\nu)\right)\right| \leq \left\|R_{\frac{u}{T}}\right\|^k \|\nu\|_\vartheta \leq \left(\gamma^{-T_0} + 2\delta\right)^k \|\nu\|_\vartheta \longrightarrow 0$$

We get, using estimate (12) and inserting previous limits in (13):

$$\lim_{T\to\infty}\int_{S_\Omega}\exp\left(\frac{1}{T}S_T u\right)d\nu = \lim_{T\to\infty}\pi_\emptyset\left(M^{(kT_0)}_{\frac{u}{T}}(\nu)\right) = \exp\left(\partial^1\lambda(0;u)\right) \quad (14)$$

On the other hand, (2) implies that ν is mixing, hence ergodic, which gives the limit, because u is bounded:

$$\lim_{T\to\infty}\int_{S_\Omega}\exp\left(\frac{1}{T}S_T u\right)d\nu = \exp\left(\int_{S_\Omega}u\,d\nu\right) \quad (15)$$

And we can identify both RHS in (14) and (15) to get:

$$\partial^1\lambda(0;u) = \int_{S_\Omega}u\,d\nu$$

(b) *Identification of* $\partial^2\lambda(0;u)$. It is enough to show that for $u\in H^r_\theta$ such that $\partial^1\lambda(0;u) = \int_{S_\Omega}u\,d\nu = 0$, we have:

$$\lim_{T\to\infty}\int_{S_\Omega}\left(\frac{S_T u}{\sqrt{T}}\right)^2 d\nu = \partial^2\lambda(0;u)$$

And for this, u being bounded, we know that we can write:

$$\int_{S_\Omega}\left(\frac{S_T u}{\sqrt{T}}\right)^2 d\nu = \left.\frac{\partial^2}{\partial t^2}\right|_{t=0}\int_{S_\Omega}\exp\left(\frac{t}{\sqrt{T}}S_T u\right)d\nu = \left.\frac{\partial^2}{\partial t^2}\right|_{t=0}\pi_\emptyset\left(M^{(T)}_{\frac{tu}{\sqrt{T}}}(\nu)\right) \quad (16)$$

For $T > \left(\frac{|t||u|_\theta}{\rho}\right)^2$, we write again $T = kT_0 + \tilde{T}$ with $0 \le \tilde{T} < T_0$ and use the composition rule (7) and the spectral decomposition (10) to get:

$$\pi_\emptyset\left(M^{(T)}_{\frac{tu}{\sqrt{T}}}(\nu)\right) = \lambda^{kT_0}\left(\frac{tu}{\sqrt{T}}\right)\pi_\emptyset\left(M^{(\tilde{T})}_{\frac{tu}{\sqrt{T}}}\circ Q_{\frac{tu}{\sqrt{T}}}(\nu)\right) + \pi_\emptyset\left(M^{(\tilde{T})}_{\frac{tu}{\sqrt{T}}}\circ R^k_{\frac{tu}{\sqrt{T}}}(\nu)\right)$$

We compute then the second derivative of this expression:

$$\begin{aligned}\left.\frac{\partial^2}{\partial t^2}\right|_{t=0}\left(\lambda^{kT_0}\left(\frac{tu}{\sqrt{T}}\right)\pi_\emptyset\left(M^{(\tilde{T})}_{\frac{tu}{\sqrt{T}}}\circ Q_{\frac{tu}{\sqrt{T}}}(\nu)\right)\right) &= (vw)''(0)\\ &= v''(0)w(0) + 2v'(0)w'(0)\\ &\quad + v(0)w''(0)\end{aligned}$$

with $v(t) = \lambda^{kT_0}\left(\frac{tu}{\sqrt{T}}\right)$, so that $v(0) = 1$, $v'(0) = 0$, and $v''(0) = \frac{kT_0}{T}\partial^2\lambda(0;u)$, and $w(t) = \pi_\emptyset\left(M^{(\tilde{T})}_{\frac{tu}{\sqrt{T}}}\circ Q_{\frac{tu}{\sqrt{T}}}(\nu)\right)$, so that $w(0) = 1$, and

$$\begin{aligned}w''(0) = \frac{1}{T}\pi_\emptyset\Big(&M_0^{(\tilde{T})}\circ\partial^2 Q(0;u) + 2\partial^1 M^{(\tilde{T})}(0;u)\circ\partial^1 Q(0;u)\\ &+\partial^2 M^{(\tilde{T})}(0;u)\circ Q_0\Big)(\nu)\end{aligned}$$

which goes to zero when T goes to infinity.

In the same way

$$\frac{\partial^2}{\partial t^2}\bigg|_{t=0}\left(\pi_\emptyset\left(M^{(\tilde{T})}_{\frac{tu}{\sqrt{T}}}\circ R^k_{\frac{tu}{\sqrt{T}}}(\nu)\right)\right)=\frac{1}{T}\pi_\emptyset\Big(M_0^{(\tilde{T})}\circ\partial^2R^k(0;u)$$
$$+\ 2\partial^1M^{(\tilde{T})}(0;u)\circ\partial^1R^k(0;u)$$
$$+\,\partial^2M^{(\tilde{T})}(0;u)\circ R_0^k\Big)(\nu)$$

which goes to zero when T goes to infinity since $\lim_{k\to\infty}R^k=0$.

Combining all these results, we get

$$\lim_{T\to\infty}\frac{\partial^2}{\partial t^2}\bigg|_{t=0}\pi_\emptyset\left(M^{(T)}_{\frac{tu}{\sqrt{T}}}(\nu)\right)=\lim_{T\to\infty}\frac{kT_0}{T}\partial^2\lambda(0;u)=\partial^2\lambda(0;u)$$

This, together with equation (16) implies the desired equality:

$$\partial^2\lambda(0;u)=\lim_{T\to\infty}\int_{S_\Omega}\left(\frac{S_Tu}{\sqrt{T}}\right)^2d\nu=\sigma_u^2\geq 0$$

and gives also the existence of the limit. □

4.4. *Condition for positivity of* σ_u^2

It is straightforward that $u=v-v\circ F$ implies $\sigma_u^2=0$ because in this case $S_Tu=v-v\circ F^T$.

For the necessary condition in (4), we have to introduce the adjoint of the composition by F, $P:L^2(\nu)\to L^2(\nu)$ defined by

$$\int_{S_\Omega}\varphi\circ F\cdot\psi\,d\nu=\int_{S_\Omega}\varphi\cdot(P\psi)\,d\nu\qquad\forall\varphi,\psi\in L^2(\nu)$$

and we note that if $u\in C(S_\Omega)$ and $g\in H_\theta$, then

$$\int_{S_\Omega}u\cdot P^Tg\,d\nu=\int_{S_\Omega}u\,d\left(M_0^{(T)}(g\star\nu)\right)=\left(\int_{S_\Omega}u\,d\nu\right)\cdot\left(\int_{S_\Omega}g\,d\nu\right)$$
$$+\int_{S_\Omega}u\,d\left(R_0^T(g\star\nu)\right)$$

We can then use the spectral gap property of M_0 (see Theorem 8) to get, when $m_u=0$ and $T\geq T_0$:

$$\left|\int_{S_\Omega}u\cdot P^Tg\,d\nu\right|\leq|u|_\theta\gamma^{-T}|g|_\theta\|\nu\|_\vartheta\tag{17}$$

This estimate allows to give another expression for σ_u^2. We write:

$$\frac{1}{T}\int_{S_\Omega}(S_T u)^2\,d\nu = \int_{S_\Omega} u^2\,d\nu + 2\sum_{k=1}^{T-1}\left(1-\frac{k}{T}\right)\int_{S_\Omega} u\cdot P^k u\,d\nu$$
$$= -\int_{S_\Omega} u^2\,d\nu + 2\sum_{k=0}^{T-1}\left(1-\frac{k}{T}\right)\int_{S_\Omega} u\cdot P^k u\,d\nu$$

and (17) implies the existence of

$$\sigma_u^2 = \lim_{T\to\infty}\frac{1}{T}\int_{S_\Omega}(S_T u)^2\,d\nu = -\int_{S_\Omega} u^2\,d\nu + 2\sum_{k\geq 0}\int_{S_\Omega} u\cdot P^k u\,d\nu$$

If $\sigma_u^2 = 0$, then $\int_{S_\Omega}(S_T u)^2\,d\nu = -2T\sum_{k\geq T}\int_{S_\Omega} u\cdot P^k u\,d\nu - 2\sum_{k=0}^{T-1} k\int_{S_\Omega} u\cdot P^k u\,d\nu$, hence $S_T u$ is bounded in $L^2(\nu)$ by estimate (17): $\|S_T u\|_{L^2(\nu)} \leq C$ for all $T \geq 0$.

Again with estimate (17), for $g \in H_\theta$:

$$l(g) = \lim_{T\to\infty}\int_{S_\Omega} S_T u\cdot g\,d\nu = \sum_{k\geq 0}\int_{S_\Omega} u\cdot P^k g\,d\nu$$

defines a $L^2(\nu)$-bounded linear functional on H_θ, because $|l(g)| \leq C\|g\|_{L^2(\nu)}$. This functional extends then to $L^2(\nu)$ by density of H_θ, and there is $S_u \in L^2(\nu)$ such that:

$$l(g) = \lim_{T\to\infty}\int_{S_\Omega} S_T u\cdot g\,d\nu = \int_{S_\Omega} S_u\cdot g\,d\nu$$

In the same way, $\lim_{T\to\infty}\int_{S_\Omega} u\circ F^T\cdot g\,d\nu = 0$ for every $g \in L^2(\nu)$.

We get then for every $g \in L^2(\nu)$:

$$\int_{S_\Omega} u\cdot g\,d\nu = \int_{S_\Omega} S_T u\cdot g\,d\nu - \int_{S_\Omega} S_T u\circ F\cdot g\,d\nu + \int_{S_\Omega} u\circ F^T\cdot g\,d\nu$$
$$= \int_{S_\Omega} S_T u\cdot g\,d\nu - \int_{S_\Omega} S_T u\cdot Pg\,d\nu + \int_{S_\Omega} u\circ F^T\cdot g\,d\nu$$
$$\longrightarrow \int_{S_\Omega} S_u\cdot g\,d\nu - \int_{S_\Omega} S_u\cdot Pg\,d\nu = \int_{S_\Omega}(S_u - S_u\circ F)\cdot g\,d\nu$$

as T goes to infinity. This proves the desired identity $u = S_u - S_u\circ F$ in $L^2(\nu)$.

4.5. Proof of the Central Limit Theorem

To show that $\frac{S_T u - T m_u}{\sqrt{T}\sigma_u}$ converges in law under any initial probability $\phi \in \mathcal{M}_\vartheta^p$ to the standard normal law, it is enough to show the convergence of its Laplace transform. We treat only the centered case and note that this result is valid even if $\sigma_u^2 = 0$.

Proposition 2. *For all $u \in H_\theta^r$ such that $m_u = 0$ and for all $\phi \in \mathcal{M}_\vartheta^p$, we have:*

$$\lim_{T\to\infty} \int_{S\Omega} \exp\left(\frac{t}{\sqrt{T}} S_T u\right) d\phi = \exp\left(\frac{t^2}{2}\sigma_u^2\right) \qquad \forall t \in \mathbb{R} \tag{18}$$

If $\sigma_u^2 > 0$, this implies the central limit theorem (CLT) by Lévy's Theorem (see for example Theorem 2.5.1 in [4]). The case $\sigma_u^2 = 0$ corresponds to a faster convergence to the limit.

Proof. We proceed as in the proof of the first part of Proposition 1, but replace $\frac{1}{T}$ by $\frac{t}{\sqrt{T}}$. We can then use the decomposition (10) as soon as $T > \left(\frac{t|u|_\theta}{\rho}\right)^2$.

As we take $m_u = \partial^1\lambda(0; u) = 0$, the main term in $\lambda^{kT_0}\left(\frac{tu}{\sqrt{T}}\right)$ will be the second derivative:

$$\lambda^{kT_0}\left(\frac{tu}{\sqrt{T}}\right) = \left(1 + \frac{t^2}{2T}\partial^2\lambda(0; u) + o\left(\frac{1}{T}\right)\right)^{kT_0} \longrightarrow \exp\left(\frac{t^2}{2}\partial^2\lambda(0; u)\right)$$

when $T \to \infty$ □

4.6. *Proof of the Moderate Deviations Principle*

A Moderate Deviations Principle with parameter $\frac{1}{2} < \alpha < 1$ is in fact a Large Deviations result for the law of the random variables $\frac{S_T u}{T^\alpha}$. For these the exponential scale of probabilities is known to be of the order of $T^{2\alpha-1}$. This will then be the speed of the Large Deviations result (see Theorem 3.7.1 in [9]).

It is hence sufficient to evaluate the appropriate log-Laplace transform:

$$\begin{aligned}\Lambda_\alpha(\beta) &= \lim_{T\to\infty} \frac{1}{T^{2\alpha-1}} \log \int_{S\Omega} \exp\left(\beta T^{2\alpha-1}\frac{S_T u}{T^\alpha}\right) d\phi \\ &= \lim_{T\to\infty} \frac{1}{T^{2\alpha-1}} \log \int_{S\Omega} \exp\left(\beta \frac{S_T u}{T^{1-\alpha}}\right) d\phi\end{aligned}$$

Proposition 3. *For all fixed $\frac{1}{2} < \alpha < 1$, for all $\phi \in \mathcal{M}_\vartheta^p$ and all $u \in H_\theta^r$ such that $m_u = 0$, we have:*

$$\Lambda_\alpha(\beta) = \frac{\beta^2}{2}\sigma_u^2 \tag{19}$$

The analyticity of $\Lambda_\alpha(\beta)$ allows to apply Gärtner-Ellis Theorem (see Theorem II.6.1 in [11]). The latter says that $\frac{S_T u}{T^\alpha}$ satisfies a Large Deviations Principle with speed $T^{2\alpha-1}$ and rate function given by the Legendre transform of Λ_α:

$$I_\alpha(x) = \Lambda_\alpha^*(x) = \sup_{\beta\in\mathbb{R}} (\beta x - \Lambda_\alpha(\beta)) = \frac{x^2}{2\sigma_u^2},$$

which is independent of α, if $\sigma_u^2 > 0$. This result is exactly the property (MDP) of Theorem 3.

If $\sigma_u^2 = 0$, then $I_\alpha(0) = 0$ and $I_\alpha(x) = +\infty$ for all $x \neq 0$, which corresponds to a trivial case.

Proof. We can proceed as for the central limit theorem because $T^{1-\alpha} \to \infty$ as $T \to \infty$ so that, for T great enough, we are again in the domain where Theorem 4 can be applied. The main difference is that $\int_{S_\Omega} \exp\left(\beta \frac{S_T u}{T^{1-\alpha}}\right) d\phi$ diverges exponentially fast so that we need the factor $\frac{1}{T^{2\alpha-1}}$ to rescale it.

For $T = kT_0 + \tilde{T}$ with $T > \left(\frac{|\beta||u|_\theta}{\rho}\right)^{\frac{1}{1-\alpha}}$, we denote $u_T = \frac{\beta u}{T^{1-\alpha}}$. Then:

$$\exp\left(-\frac{\beta T_0}{T^{1-\alpha}}|u|_\infty\right) \le \exp\left(\beta\frac{S_{\tilde{T}}u \circ F^{kT_0}}{T^{1-\alpha}}\right) \le \exp\left(\frac{\beta T_0}{T^{1-\alpha}}|u|_\infty\right)$$

and:

$$\int_{S_\Omega} \exp\left(\beta\frac{S_{kT_0}u}{T^{1-\alpha}}\right) d\phi = \pi_\emptyset\left(M_{u_T}^{(kT_0)}\phi\right) = \lambda^{kT_0}(u_T)\,\pi_\emptyset\left(Q_{u_T}\phi\right) + \pi_\emptyset\left(R_{u_T}^k\phi\right)$$

with:

$$\begin{aligned}\frac{1}{T^{2\alpha-1}}\log\left(\lambda^{kT_0}(u_T)\right) &= \frac{kT_0}{T}T^{2-2\alpha}\log\left(1+\frac{\beta^2}{2T^{2-2\alpha}}\partial^2\lambda(0;u)+o\left(\frac{1}{T^{2-2\alpha}}\right)\right)\\ &\longrightarrow \frac{\beta^2}{2}\partial^2\lambda(0;u) = \frac{\beta^2}{2}\sigma_u^2 \quad \text{as } T\to\infty\end{aligned}$$

And we have for the remaining term:

$$\frac{1}{T^{2\alpha-1}}\log\left(\pi_\emptyset\left(Q_{u_T}\phi\right) + \frac{\pi_\emptyset\left(R_{u_T}^k\phi\right)}{\lambda^{kT_0}(u_T)}\right)$$

which tends to zero when T goes to infinity since

$$\pi_\emptyset\left(Q_{u_T}\phi\right) \longrightarrow 1 \qquad \text{and} \qquad \left|\frac{\pi_\emptyset\left(R_{u_T}^k\phi\right)}{\lambda^{kT_0}(u_T)}\right| \le \left(\frac{\gamma^{-T_0}+2\delta}{1-\delta}\right)^k \cdot \|\phi\|_\vartheta \longrightarrow 0$$

We get in conclusion that

$$\Lambda_\alpha(\beta) = \lim_{T\to\infty}\frac{1}{T^{2\alpha-1}}\log\int_{S_\Omega}\exp\left(\frac{\beta}{T^{1-\alpha}S_T u}\right) d\phi = \frac{\beta^2}{2}\sigma_u^2 \qquad \square$$

4.7. Proof of the Large Deviations result

We cannot prove a complete Large Deviations Principle because the existence of the spectral gap for $M_u^{(T_0)}$ in Theorem 4 is valid only for small u and the scaling taken to compute the log-Laplace transform is not the same as for Moderate Deviations (it corresponds to the case $\alpha = 1$). What we can obtain is for every $u \in H_\theta^r$ an upper bound and a partial lower bound controlled by a rate function with an unique minimum.

For $u \in H_\theta^r$ and $\phi \in \mathcal{M}_\vartheta^p$ such that $\|\phi\|_\vartheta < \delta^{-2}$ (this is a technical assumption which is not very important: it is satisfied by Lebesgue measure, and we can modify δ in Theorem 4 such that it is satisfied by any fixed ϕ), we write:

$$\Lambda_u(\beta) = \limsup_{T\to\infty} \frac{1}{T} \log \int_{S_\Omega} \exp(\beta S_T u)\, d\phi$$

the limsup of log-Laplace transforms of $S_T u$.

Proposition 4. *For $|\beta| < \frac{\rho}{|u|_\theta}$, the map*

$$\beta \longmapsto \Lambda_u(\beta) = \lim_{T\to\infty} \frac{1}{T} \log \int_{S_\Omega} \exp(\beta S_T u)\, d\phi = \log(\lambda(\beta u)) \tag{20}$$

is analytic.

Proof. We proceed exactly as in the proof of Proposition 3, with $|\beta| < \frac{\rho}{|u|_\theta}$ such that:

$$\pi_\emptyset\left(M_{\beta u}^{(kT_0)}\right) = \lambda^{kT_0}(\beta u)\, \pi_\emptyset\left(Q_{\beta u}(\phi)\right) + \pi_\emptyset\left(R_{\beta u}^k(\phi)\right)$$

with

$$\frac{1}{T} \log \lambda^{kT_0}(\beta u) \longrightarrow \log \lambda(\beta u) \quad \text{when } T \to \infty$$

and

$$\frac{1}{T} \log \left(\pi_\emptyset\left(Q_{\beta u}(\phi)\right) + \frac{\pi_\emptyset\left(R_{\beta u}^k(\phi)\right)}{\lambda^{kT_0}(\beta u)} \right)$$

tends to zero when T goes to infinity, since

$$|\pi_\emptyset(Q_{\beta u}\phi) - 1| < \delta^2 \|\phi\|_\vartheta \qquad \text{and} \qquad \left| \frac{\pi_\emptyset\left(R_{\beta u}^k(\phi)\right)}{\lambda^{kT_0}(\beta u)} \right| \longrightarrow 0 \qquad \square$$

This local differentiability implies the following partial large deviations result:

Theorem 5. *For all $u \in H_\theta^r$ and $\phi \in \mathcal{M}_\vartheta^p$ such that $\|\phi\|_\vartheta \le \delta^{-2}$, we define:*

$$I_u(x) = \sup_{\beta\in\mathbb{R}} (\beta x - \Lambda_u(\beta))$$

Then:

1. *I_u is convex and lower semi-continuous, $I_u(x) = +\infty$ if $|x| > |u|_\infty$, $I_u(x) \ge 0$ and:*

$$I_u(x) = 0 \quad \textit{if and only if} \quad x = m_u \tag{21}$$

2. *For all closed $F \subset \mathbb{R}$:*

$$\limsup_{T\to\infty} \frac{1}{T} \log \phi\left(z : \frac{S_T u(z)}{T} \in F\right) \le -\inf_{x\in F} I_u(x) \qquad \text{(Upper Bound)}$$

3. *For all* $x \in \Lambda'_u\left(-\frac{\rho}{|u|_\theta}, \frac{\rho}{|u|_\theta}\right)$ *and* $\delta > 0$*:*

$$\liminf_{T\to\infty} \frac{1}{T} \log \phi\left(z : \frac{S_T u(z)}{T} \in B(x,\delta)\right) \geq -I_u(x) \qquad \text{(Lower Bound)}$$

Proof. 1) is a classical result of Large Deviations theory (see for example Lemma 2.2.5 in [9]). Equivalence (21) is a direct consequence of Theorem II.6.3 in [11]: m_u is the differential of Λ_u at point 0, hence the unique minimizer of I_u.

Both bounds of Large Deviations are obtained by applying the Theorem of Gärtner-Ellis (see Theorem 2.3.6 and Lemma 2.3.9 in [9]). □

5. Perturbed transfer operators

We prove in this section the part of Theorem 4 on the existence of perturbed transfer operators. We modify for this the method introduced in [23]. This allows us to construct a wide class of operators (in fact more than those of Theorem 4) as stated in Theorem 7.

5.1. Finite box operators

The construction of the transfer operators is well understood by looking at restrictions of the coupled map to finite boxes: we fix some boundary condition $\xi \in S_\Omega$ and define for all $\Lambda \in \mathcal{F}$:

$$\begin{aligned} F_\Lambda \ &: A_\Lambda \to \mathcal{C}^\Lambda \\ &z_\Lambda \mapsto F(z_\Lambda \vee \xi_{\Lambda^C})|_\Lambda \end{aligned}$$

where $z_\Lambda \vee \xi_{\Lambda^C}$ denotes the point $w \in S_\Omega$ such that $w_i = z_i$ for all $i \in \Lambda$ and $w_i = \xi_i$ for all $i \in \Lambda^C$.
This function F_Λ is expanding as soon as $\kappa < (\lambda - 1)\rho$ and we can define the associated transfer operator $L_\Lambda : E_\Lambda \to E_\Lambda$ as follows:

$$\int_{S_\Lambda} \varphi \circ F_\Lambda \cdot \psi \, dm^\Lambda = \int_{S_\Lambda} \varphi \cdot L_\Lambda(\psi)\, dm^\Lambda \quad \forall \varphi, \psi \in E_\Lambda \tag{22}$$

This is a classical tool to study asymptotic properties of such dynamical systems (see [1] for an extended study of this domain).

In the same way, for $u \in E_\Lambda$, we can define a perturbed operator by:

$$\int_{S_\Lambda} \varphi \circ F_\Lambda \cdot e^u \cdot \psi \, dm^\Lambda = \int_{S_\Lambda} \varphi \cdot M_{\Lambda,u}(\psi)\, dm^\Lambda \quad \forall \varphi, \psi \in E_\Lambda$$

Or, equivalently:

$$M_{\Lambda,u}(\psi) = L_\Lambda\left(e^u \cdot \psi\right) \tag{23}$$

The interest of $M_{\Lambda,u}$ comes from the formula:

$$\int_{S_\Lambda} \exp\left(\sum_{t=0}^{T-1} u \circ F_\Lambda^t\right) \cdot \psi \, dm^\Lambda = \int_{S_\Lambda} M_{\Lambda,u}^T(\psi) \, dm^\Lambda$$

which identifies the Laplace transform of $\sum u \circ F_\Lambda^t$ with some spectral characteristic of $M_{\Lambda,u}$. We thus need an infinite dimensional equivalent of these operators, as described in Theorem 4.

The method to construct them is based on the following perturbative expansion, derived from Theorem 3.2 and Lemma 3.4 of [23]:

Theorem 6. *If $\kappa < (\lambda - 1)\rho$, L_Λ has the integral representation:*

$$L_\Lambda(\psi)(\omega_\Lambda) = \pm \int_{\Gamma_\Lambda} \prod_{p\in\Lambda} \left(k(\omega_p, f_p(z_p)) + \sum_{V\in\mathcal{F}} \beta_{p,V}(\omega_p, z_{V\cap\Lambda} \vee \xi_{V\cap\Lambda^c}) \right) \times \psi(z_\Lambda)\, \mu^\Lambda(dz_\Lambda) \tag{24}$$

where:

- $\Gamma_\Lambda = \prod_{p\in\Lambda} \partial A[\rho]$ *and* μ^Λ *is the unique holomorphic differential form on* $\prod_{p\in\Lambda} \mathcal{C}$ *which extends* m_Λ.
- k *is the periodic Cauchy kernel:*

$$k(\omega, z) = \frac{1}{2i} \cot(z - \omega) = \frac{1}{2\pi i} \sum_{n\in\mathbb{Z}} \frac{1}{z - \omega + n}$$

- $\beta(p, V)$ *are weakly holomorphic functions on* $D_{p,V} = A_p \times \Gamma_p \times \prod_{q\in V\setminus\{p\}} A_q$ *(i.e. continuous in all variables and holomorphic in* $w_p \in Int(A_p)$ *and* $z_{V\setminus\{p\}} \in Int(A_{V\setminus\{p\}})$*, see appendix B of [23]) such that:*

$$\sum_{V\in\mathcal{F}} \theta^{-|V|} |\beta_{p,V}| \le C_\beta = \frac{e^{2\pi\kappa}}{e^{2\pi(\lambda-1)\rho} - e^{2\pi\kappa}} - \frac{1}{e^{2\pi(\lambda-1)\rho} - 1}$$

and

$$\int_{S_p} \beta_{p,V}(\omega_p, z_{V\cup\{p\}})\, d\omega_p = 0 \quad \forall z_p \in \Gamma_p, z_{V\setminus\{p\}} \in A_{V\setminus\{p\}}$$

5.2. Existence of the operators

We can write a similar integral representation for L_Λ^T and expand it by interchange of products and sums. As in Section 4 of [23], we associate to each term in the corresponding infinite expansion (each configuration) a configurational operator. To define a general operator, we have to control the sum of these configurational operators, hence the norm of each of them. We do this, following [23], with an estimation by trees, where the spectral gap result for the single site map and the weak coupling are extensively used. The main difference in our case is that we expand also the perturbation terms. This disturbs the analysis but we overcome this

difficulty: we associate several trees to each configuration and control their weights by the choice of perturbation terms from the space H_θ (see Subsection 5.3 for the detailed construction).
This allows to construct the general operators described in the following result:

Theorem 7. *Suppose the hypotheses of Theorem 1 are verified, then there is $\theta < \vartheta < 1$ such that for all $T \geq 1$ we have a multilinear functional:*

$$\begin{aligned} \mathcal{L}^{(T)}: \qquad\qquad H_\theta^T &\longrightarrow L(\mathcal{M}_\vartheta, \mathcal{M}_\theta) \\ (U_0, \dots, U_{T-1}) &\longmapsto L^{(T)}_{[U_0,\dots,U_{T-1}]} \end{aligned}$$

with the following properties:

- *There exists $T_0 \geq 1$ such that $L^{(T)}_{[U_0,\dots,U_{T-1}]} \in L(\mathcal{M}_\vartheta) \quad \forall\, T \geq T_0$*
- $\left\| L^{(T)}_{[U_0,\dots,U_{T-1}]} \right\| \leq \prod_{t=0}^{T-1} |U_t|_\theta$ (25)
- $L^{(t+T)}_{[V_0,\dots,V_{t-1},U_0,\dots,U_{T-1}]} = L^{(t)}_{[V_0,\dots,V_{t-1}]} \circ L^{(T)}_{[U_0,\dots,U_{T-1}]}$ *if $T \geq T_0$* (26)
- $L^{(T)}(\mathcal{M}_\vartheta^m) \subset \mathcal{M}_\theta^m$ (27)
- $\int_{S_\Omega} b \circ F^T \cdot \prod_{t=0}^{T-1} U_t \circ F^t \, d\phi = \int_{S_\Omega} b\, d\left(L^{(T)}_{[U_0,\dots,U_{T-1}]} \phi \right)$
  $\forall\, b \in C(S_\Omega),\ \phi \in \mathcal{M}_\vartheta^m$ (28)

Operators of Theorem 4 are a particular case of the general operators constructed in Theorem 7. For $u \in H_\theta$, we take

$$M_u^{(T)} = L^{(T)}_{[e^u,\dots,e^u]}$$

We can already obtain some properties of these operators:

Proof of Theorem 4 (first part). Since it is the composition of the analytic function $u \mapsto e^u$ and of the multilinear map $\mathcal{L}^{(T)}$, $M_u^{(T)}$ is analytic.

We can write explicitly the series expansion of $M_u^{(T)}$ around a point u:

$$M_{u+h}^{(T)} = \sum_{n \geq 0} \frac{1}{n!} \partial^n M^{(T)}(u; h),$$

where

$$\partial^n M^{(T)}(u; h) = \sum_{\substack{n_0,\dots,n_{T-1} \geq 0 \\ \sum_{t=0}^{T-1} n_t = n}} \frac{n!}{n_0! \cdots n_{T-1}!} L^T_{[h^{n_0} e^u, \dots, h^{n_{T-1}} e^u]},$$

which is an element of $L(\mathcal{M}_\vartheta, \mathcal{M}_\theta)$ (or $L(\mathcal{M}_\vartheta)$ if $T \geq T_0$), is homogeneous of degree n and satisfies the bound

$$\left\| \partial^n M^{(T)}(u;h) \right\| \le \left(T|h|_\theta\right)^n e^{T|u|_\theta}$$

hence we control the difference between two operators by:

$$\left\| M^{(T)}_{u+h} - M^{(T)}_u \right\| \le \left(e^{T|h|_\theta} - 1\right) e^{T|u|_\theta}$$

Estimate (6) is the particular case of this inequality around $u = 0$. Formulas (7), (8) and (9) are easily deduced from (26), (27) and (28). □

5.3. Proof of Theorem 7

This Subsection contains a sketch of the construction of the transfer operator done in [23] and presents also the main modifications which have to be done to extend it to perturbed operators, and obtain the results stated in Theorem 7.

5.3.1. Single site operators

For $f_p \in \mathcal{E}(\rho, \lambda)$ an expanding map on A_p, the associated transfer operator $L_{f_p} : E_p \longrightarrow E_p$ can be written (this is a particular case of identity (24)):

$$L_{f_p}\phi(\omega_p) = \int_{\Gamma_p} k(\omega_p, f_p(z_p))\phi(z_p)\, \mu^p(dz_p)$$

It satisfies $l_p \circ L_{f_p} = l_p$ with $l_p(\phi) = \int_{S_p} \phi(z_p)\, dz_p$ and enjoys a spectral gap property with the following estimates, uniformly in $f_p \in \mathcal{E}(\rho, \lambda)$:

$$\left\| L^T_{f_p} \right\| \le c_h \quad \text{and} \quad \left\| L^T_{f_p}\Big|_{\mathrm{Ker}\, l_p} \right\| \le c_r \eta^T \tag{29}$$

where $c_h \ge 1$, $c_r > 0$ and $\eta < 1$. A proof of these results can be found in Appendix A of [23].

5.3.2. Configurations

We define what a branching, the main element to define the configurations, is:

Definition 3. *A* **branching pair** (S, V) *is composed by a subset* $S \in \mathcal{F}$ *and a function* $V : S \to \mathcal{F}$. *We denote* $V[S] = S \cup \left(\cup_{p\in S} V(p)\right)$.

Given $K \in \mathcal{F}$, (S, V) a branching pair such that $S \subset K$, $U \in H_\theta$ and $W \in \mathcal{F}$, we define $H = K \cup V[S] \cup W$ and the operator $L_{K,(S,V),(U,W)} : E_H \to E_K$ by:

$$L_{K,(S,V),(U,W)}(\varphi_H)(\omega_K) = \pm \int_{S_{H\setminus K}} m^{H\setminus K}(dz_{H\setminus K}) \int_{\Gamma_K} \prod_{p\in S} \beta_{p,V(p)}(\omega_p, z_{V(p)\cup p}) \times \prod_{p\in K\setminus S} k(\omega_p, f_p(z_p)) U_W(z_W)\varphi_H(z_H)\, \mu^K(dz_K)$$

We then have some compatibility properties for these operators:

Lemma 1. *We have:*

$$\pi_{K\setminus\{p\},K} L_{K,(S,V),(U,W)} = \begin{cases} 0 & \text{if } p \in S \\ L_{K\setminus\{p\},(S,V),(U,W)} & \text{if } p \in (V[S] \cup W)\setminus S \\ L_{K\setminus\{p\},(S,V),(U,W)} \circ \pi_{H\setminus\{p\},H} & \text{if } p \in K\setminus (V[S] \cup W) \end{cases} \tag{30}$$

and the sum

$$\sum_{W\in\mathcal{F}} L_{K,(S,V),(U,W)} \circ \pi_H(\phi) = L_{K,(S,V),(1,\emptyset)} \circ \pi_{K\cup V[S]}(U \star \phi)$$

is independent of the decomposition $U = \sum_{W\in\mathcal{F}} U_W$ of $U \in H_\theta$.

Proof. For the first part, we proceed as for Lemma 4.2 in [23]. For the second part, we commute sum and integral, obtaining

$$\sum_{W\in\mathcal{F}} L_{K,(S,V),(U,W)} \circ \pi_H(\phi) = L_{K,(S,V),(1,\emptyset)} \circ \left(\sum_{W\in\mathcal{F}} \pi_{K\cup V[S],K\cup V[S]\cup W}(U_W \phi_H) \right)$$

and use then the projectivity of ϕ and the definition of the module product on $\mathcal{M}_\theta$. □

Given $T \geq 1$, $U_0, \ldots, U_{T-1} \in H_\theta$, we want to construct for any $K \in \mathcal{F}$ an operator $L^{(T)}_{K,[U_0,\ldots,U_{T-1}]} : \mathcal{M}_\vartheta \to E_K$ and control its norm.
We introduce configurations and associated configurational operators:

Definition 4. *A* **configuration** *on $K \in \mathcal{F}$ at time $T \geq 1$ is the choice of :*

- *$W_{T-1}, \ldots W_0 \in \mathcal{F}$, for the expansion of the perturbative terms U,*
- *$(S_{T-1}, V_{T-1}), \ldots, (S_0, V_0)$, branching pairs for the expansion of the $\beta_{p,V}$,*
- *$I \in \mathcal{F}$ an* **initial state***,*

such that if K is expanded by $K_T = K$ and $K_t = K_{t+1} \cup V_t[S_t] \cup W_t$ for $0 \leq t < T$, the following conditions are satisfied:

$$S_t \subset K_{t+1} \text{ for } 0 \leq t < T \qquad \text{and} \qquad I \subset K_0$$

We denote $\mathcal{C}[K, T]$ the set of all these configurations.

To each configuration $C \in \mathcal{C}[K, T]$, we associate a **configurational operator**:

$$L_{K,[U_0,\dots,U_{T-1}]}[C] = L^{(T-1)} \circ \cdots \circ L^{(0)} \circ Q_I^{K_0} \circ \pi_{K_0} : \mathcal{M}_\vartheta \to E_K$$

where

$$L^{(t)} = L_{K_{t+1},(S_t,V_t),(U_t,W_t)} \quad \text{and} \quad Q_I^{K_0} = \prod_{p\in I}(1-l_p) \prod_{p\in K_0\setminus I} l_p$$

The following equivalent of Proposition 4.3 in [23] remains valid and will be useful to construct the global operator on the projective Banach space $\mathcal{M}_\vartheta$:

Proposition 5. *If $\alpha \subset K \in \mathcal{F}$, then:*

$$\pi_{\alpha,K} L_{K,[U_0,\dots,U_{T-1}]}[C] = \begin{cases} L_{\alpha,[U_0,\dots,U_{T-1}]}[C] & \text{if } C \in \mathcal{C}[\alpha, T] \\ 0 & \text{otherwise} \end{cases} \tag{31}$$

Remark. The initial set I, introduced in [23] to prove the spectral gap, is not necessary here. We keep it however to verify that in the case where $U_t = 1$ for all $0 \le t < T$, we really get the operator constructed in this paper.

For $C \in \mathcal{C}[K, T]$ a given configuration, with (S_t, V_t) the branching pairs, W_t the perturbative expansions and $K = K_T \subset K_{T-1} \subset \cdots \subset K_0$ the expansion of K, we call the points $(q, t) \in \cup_{t=0}^{T} K_t \times \{t\}$ **points of the configuration** and classify them, calling (q, t):

- an **inner point** if $q \in W_t$,
- a **vertex point** if $q \in V_t[S_t] \setminus W_t$,
- an **apex point** if $t \ge 1$ and $q \in S_{t-1} \setminus (V_t[S_t] \cup W_t)$,
- a **free point** otherwise.

A **chain** is a maximal sequence of points of the configuration $\gamma = (q, t)_{t_1 \le t \le t_2}$ such that $q \notin S_{t_2-1}$ and $(q, t)_{t_1<t<t_2}$ are free points. t_1 is called the **starting time** of the chain and $|\gamma| = t_2 - t_1$ its **length**. Such a chain is called:

- an **apex chain** if (q, t_1) is an apex point,
- an **initial chain** if $t_1 = 0$, $(q, 0)$ is a free point and $q \in I$,
- an **end chain** otherwise.

This analysis allows to separate the contributions of chains in $L_{K,[U_0,\dots,U_{T-1}]}[C]$. If $\mathrm{ch}(t)$ denotes all the chains starting at time t, we obtain by interchange of the terms in the integral:

$$\begin{aligned} L_{K,[U_0,\dots,U_{T-1}]}[C] = \tilde{L}^{(T-1)} \circ \tilde{U}^{(T-1)} \circ \tilde{L}^{(T-2)} \circ \cdots \\ \tilde{U}^{(1)} \circ \tilde{L}^{(0)} \circ \tilde{U}^{(0)} \circ j_{K_0,I} \circ \prod_{p\in I}(1-l_p) \circ \pi_I \end{aligned}$$

where $\tilde{U}^{(t)} : E_{K_t} \to E_{K_t}$ is defined by $\tilde{U}^{(t)}(\phi) = (j_{K_t,W_t} U_{t,W_t})\phi$ and $\tilde{L}^{(t)} : E_{K_t} \to E_{K_{t+1}}$ by:

$$\tilde{L}^{(t)} = \left(\prod_{\gamma \in \mathrm{ch}(t)} L_\gamma \right) \pi_{K_{t+1},K_t} \prod_{p\in S_t} M_{K_t,\beta_{p,V_t(p)}}$$

with $L_\gamma = (L_{f_p})^{|\gamma|}$ and:

$$M_{K_t,\beta_{p,V_t(p)}}\phi(\omega_p, z_{K_t\setminus\{p\}}) = \pm \int_{\Gamma_p} \beta_{p,V_t(p)}(\omega_p, z_{V_t(p)\cup\{p\}})\phi(z_{K_t})\, \mu^p(dz_p)$$

Using this last expression, we can bound the norm of each $L_{K,[U_0,\dots,U_{T-1}]}[C]$ by the product of the estimation for each term in its expression, using the following estimates:

$$\|Q_I^{K_0} \circ \pi_{K_0}\| \leq \left(1 + \frac{1}{\vartheta}\right)^{|I|} \tag{32}$$

since $Q_I^{K_0} \circ \pi_{K_0} = \sum_{J\subset I}(-1)^{|J|} j_{K_0,J}\pi_J$ and $\|\pi_J\| \leq \vartheta^{-|J|}$.

$$\|M_{\Lambda,\beta_{p,V}}\| \leq 2|\beta_{p,V}| \quad \text{and} \quad \|\tilde{U}^{(t)}\| \leq |U_{t,W_t}| \tag{33}$$

$$\|L_\gamma\| \leq c_h \quad \text{or} \quad \|L_\gamma\| \leq c_r\eta^{|\gamma|} \quad \text{if } \gamma \text{ is an initial or apex chain} \tag{34}$$

This last fundamental estimate comes from the spectral gap result for the single site operator L_{f_p} (see estimates (29)).

5.3.3. Tree structures

We will now associate to each configuration a tree structure in an injective way. This will allow us to bound the norms of configurational operators by some more computable estimates. The set of trees is exactly the same as in [23], but we will keep more of them to describe a configuration.

Definition 5. *For $T \geq 0$ and $p \in \Omega$, the collection of* **trees** $\mathcal{Y}[p,T]$ *is defined recursively on T:*

- $\mathcal{Y}[p,0]$ *contains two elements: an* **initial leaf** *and an* **end leaf**
- *for $t \geq 1$, $\mathcal{Y}[p,t]$ is constituted of the following trees:*
  - *an end leaf*
  - *an initial chain of length t followed by an initial leaf*
  - *an apex chain of length $0 \leq k < t$ followed by a branching over a set V; at each $q \in V \cup \{p\}$, we attach a tree $y_q^{t-k-1} \in \mathcal{Y}[q, t-k-1]$*

We associate now to each $C \in \mathcal{C}[K,T]$ a collection of trees, in fact one $y_{p,T} \in \mathcal{Y}[p,T]$ for each $(p,T)_{p\in K}$ and one $y_{p,t} \in \mathcal{Y}[p,t]$ for each (p,t) inner point, i.e. such that $0 \leq t < T$ and $p \in W_t$. We do this recursively on t, giving us a total ordering of Ω to go through the points associated to a given time (see Figure 1 for an illustration of this construction):
For $t = 0$, we associate to each $p \in K_0$ a tree $y_{p,0}$ which is an initial leaf if $p \in I$ and an end leaf otherwise.

Then, for $1 \leq t \leq T$:

- we go through the $p \in S_{t-1} \subset K_t$ and we take $y(p,t)$ a branching over the set $V_{t-1}(p)$, and we attach at each $q \in V_{t-1}(p) \cup \{p\}$:

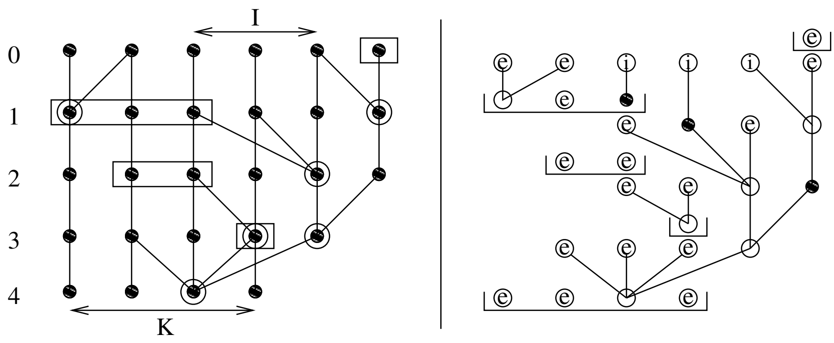

Fig. 1. An example of configuration (left) with its associated tree structure (right). Each circle represents the basis of a branching pair, each rectangle represents the perturbative term U_t. To each of these inner points and to the bottom points are associated independent trees.

  - $y(q, t-1)$ if $q \notin W_{t-1}$ and $y(q, t-1)$ has not yet been attached to another tree
  - an end leaf otherwise
- for $p \in K_t \cap [(V_{t-1}[S_{t-1}] \cup W_{t-1}) \setminus S_{t-1}]$, we forget the tree $y(p, t-1)$ (already attached to another tree or kept until the end) and take for $y(p, t)$ an end leaf
- for $p \in K_t \setminus (V_{t-1}[S_{t-1}] \cup W_{t-1})$:
  - $y(p, t)$ is an end leaf if $y(p, t-1)$ was already one (we forget the length of end chains because it is useless in the estimates)
  - otherwise, $y(p, t)$ is a chain of length 1 attached to $y(p, t-1)$

It should be noted that all $y(p, t)$ for (p, t) inner points are never attached to other ones: we keep them in our description of the configuration in term of trees. In fact, the terms U_{t,W_t}, for U_t chosen in H_θ, will exactly compensate the weights of these trees (see Proposition 8).
We define the weight of a tree as the product of the bounds of its components, and for those, we take the bounds (33) for the branchings and (34) for the chains. We estimate the other terms by:

$$\| \text{ end leaf } \| = c_h \quad \text{ and } \| \text{ initial leaf } \| = 1 + \frac{1}{\vartheta}$$

Proposition 6. *The map which to every $C \in \mathcal{C}[K, T]$ associates the family of trees ($y(p, t)$ where $t = T$ and $p \in K$, or (p, t) is an inner point) is injective and we have the bound:*

$$\left\| L_{K,[U_0,\dots,U_{T-1}]}[C] \right\|_{\mathcal{M}_\vartheta \to E_K} \le \prod_{p \in K} \|y(p, T)\| \prod_{t=0}^{T-1} \left[|U_{t,W_t}| \prod_{p \in W_t} \|y(p, t)\| \right] \tag{35}$$

Our trees are exactly the same as in [23]. We can then use its bounds for the weights of trees under the condition (TR) of [23]. We don't write this condition here, but just notice that for ρ and λ given, there exists $\theta_0(\rho, \lambda) \in (0, 1/3)$ such that for all $\theta < \theta_0$ we can find $\gamma < \eta^{-1}$ (η is the gap of the single site operator) and κ such that any $F \in CM[\rho, \lambda, \theta, \kappa]$ satisfies (TR) with this γ. We write below the results of Lemmas 4.20 and 4.21 of [23], which give these bounds:

Proposition 7. *The size of a tree is the sum of the length of its chains added to the number of its branchings. Define:*

$$u_p^T(s) = \sum_{y \in \mathcal{Y}[p,T]} \|y\| s^{size\,(y)}$$

If condition (TR) is satisfied with $\gamma \in (1, \eta^{-1})$, then there exists $\vartheta \in (\theta, 1)$ and $T_0 \geq 1$ such that:

$$\begin{aligned} u_p^T(\gamma) &\leq \theta^{-1} \\ u_p^T(\gamma) &\leq \vartheta^{-1} \quad \textit{if } T \geq T_0 \end{aligned} \tag{36}$$

5.3.4. Global estimates

We deduce from Propositions 6 and 7 above that:

Proposition 8.

$$\begin{aligned} \sum_{C \in \mathcal{C}[K,T]} \left\| L_{K,[U_0,\dots,U_{T-1}]}[C] \right\|_{\mathcal{M}_\vartheta \to E_K} &\leq \theta^{-|K|} \prod_{t=0}^{T-1} |U_t|_\theta \\ &\leq \vartheta^{-|K|} \prod_{t=0}^{T-1} |U_t|_\theta \quad \textit{if } T \geq T_0 \end{aligned} \tag{37}$$

Proof. Because of the injectivity of the description by trees, we have:

$$\begin{aligned} &\sum_{C \in \mathcal{C}[K,T]} \left\| L_{K,[U_0,\dots,U_{T-1}]}[C] \right\|_{\mathcal{M}_\vartheta \to E_K} \\ &\quad \leq \sum_{C \in \mathcal{C}[K,T]} \prod_{p \in K} \|y(p,T)\| \prod_{t=0}^{T-1} \left[|U_{t,W_t}| \prod_{p \in W_t} \|y(p,t)\| \right] \\ &\quad \leq \sum_{W_0,\dots,W_{T-1}} \prod_{p \in K} u_p^T(1) \prod_{t=0}^{T-1} |U_{t,W_t}| \prod_{p \in W_t} u_p^t(1) \\ &\quad \leq \prod_{p \in K} u_p^T(1) \prod_{t=0}^{T-1} \left(\sum_{W \in \mathcal{F}} |U_{t,W}| \prod_{p \in W} u_p^t(1) \right) \end{aligned}$$

and we can conclude with estimates (36). □

These bounds, together with the first part of Proposition 5 (which assures compatibility of the operators constructed for different subsets K) make it possible to define:

$$L^{(T)}_{[U_0,\dots,U_{T-1}]} = \left(\sum_{C \in \mathcal{C}[K,T]} L_{K,[U_0,\dots,U_{T-1}]}[C] \right)_{K \in \mathcal{F}}$$

as an operator from $\mathcal{M}_\vartheta$ to $\mathcal{M}_\theta$, or to $\mathcal{M}_\vartheta$ when $T \geq T_0$, satisfying the announced bound (25).
We see also that this operator is independent of the decompositions $U_t = \sum_{W \in \mathcal{F}} U_{t,W}$ writing, with $\hat{L}^{(t)} = L_{K_{T+1},(S_t,V_t),(1,\emptyset)}$ and K_t the corresponding expansion:

$$\begin{aligned} &\left(\sum_{C \in \mathcal{C}[K,T]} L_{K,[U_0,\dots,U_{T-1}]}[C] \right)(\phi) \\ &= \sum_{(S_{T-1},V_{T_1})} \hat{L}^{(T-1)} \left[\left(U_{T-1} \star \left[\sum_{(S_{T-2},V_{T_2})} \hat{L}^{(T-2)} \cdots \hat{L}^{(0)} \left[(U_0 \star \phi)_{K_0} \right] \right] \right)_{K_{T-1}} \right] \end{aligned} \tag{38}$$

For this, we use inductively the second part of Proposition 5 and the fact that any intermediate operator defines a projective family.

The multilinearity in the perturbation terms $U_0, \dots, U_{T-1}$ is clear from this last expression. We also can remark that $L^{(T)}_{[1,\dots,1]} = L^{(T)}$ is exactly the Perron Frobenius operator of [23], because in this case, all configurational operators with a $W_t \neq \emptyset$ are null.
All other properties of these operators are straightforward adaptations of equivalent results in [23].

6. Preservation of the spectral gap property

The central result in [23] is a spectral gap property for $M_0^{(T)} = L^{(T)}_{[1,\dots,1]}$. It is a direct consequence of his Lemma 4.25 and can be stated as:

Theorem 8. *Under the same assumptions on the parameters as in Theorem 7, $M_0^{(T)}$ can be written for all $T \geq T_0$:*

$$M_0^{(T)}(\phi) = \left(\int_{S_\Omega} \phi \, dm^\Omega \right) \nu + R^T(\phi)$$

with $\nu \in \mathcal{M}_\vartheta$, $M_0^{(T)}(\nu) = \nu$ and $\|R^T(\phi)\| \leq \gamma^{-T}$ (so that $Sp\,(R^T) \subset D(0, \gamma^{-T})$).

We cannot generalize this result to all our perturbed operators, but only extend it to small u as stated in the second part of Theorem 4. The proof uses an adaptation to our case of the Theorem of Kato-Rellich (see Theorem XII.8 in [20] or Theorem VII.6.9 of [10] for a more general result). We recall below the main steps of its proof and specify some estimates:

Proof of Theorem 4 (second part). For $M \in L(\mathcal{M}_\vartheta)$, let $\mathrm{Res}(M) = \mathbb{C} \setminus \mathrm{Sp}(M)$ denote the resolvent set, and for $\lambda \in \mathrm{Res}(M)$

$$R(\lambda, M) = (\lambda \mathrm{Id} - M)^{-1}$$

the associated resolvent function.

We denote $M_u = M_u^{(T_0)} \in L(\mathcal{M}_\vartheta)$. Then, by Lemmas VII.6.3 and VII.6.4 of [10], we get that for any fixed $\delta < \frac{1-\gamma^{-T_0}}{3}$, there exists $\varepsilon > 0$ such that $\|M_u - M_0\| < \varepsilon$ implies:

1. $\{\lambda \,:\, d\,(\lambda, \mathrm{Sp}(M_0)) \geq \delta\} \subset \mathrm{Res}(M_u)$
2. $\|R\,(\lambda, M_u) - R\,(\lambda, M_0)\| < \delta$ $\qquad \forall\lambda$ s.t. $d\,(\lambda, \mathrm{Sp}(M_0)) \geq \delta$
3. $u \to R(\lambda, M_u)$ is analytic $\qquad \forall\lambda$ s.t. $d\,(\lambda, \mathrm{Sp}(M_0)) > \delta$

The last statement is a straightforward generalization of the proof of Lemma VII.6.4: the set of analytic functions in our sense is stable by the same operations as for classical analytical functions.

Then $\mathrm{Sp}(M_u) \subset D(1, \delta) \cup D\left(0, \gamma^{-T_0} + \delta\right)$ and if we denote

$$Q_u = -\frac{1}{2\pi i} \int_{|\lambda - 1| = \delta} R(\lambda, M_u)\, d\lambda = -\frac{1}{2\pi i} \int_{|\lambda - 1| = 2\delta} R(\lambda, M_u)\, d\lambda$$

the projection associated to the spectrum of M_u included in $D(1, \delta)$, we get that Q_u is an analytic function of u and

$$\|Q_u - Q_0\| \leq \delta \int_0^1 \|R(1 + \delta e^{i2\pi\theta}, M_u) - R(1 + \delta e^{i2\pi\theta}, L^{(T_0)})\| d\theta \leq \delta^2 < 1 \tag{39}$$

This, with Lemma VII.6.7 of [10] and the fact that $\mathrm{Sp}(L(T_0)) \cap D(1, \delta) = \{1\}$ where 1 is a simple eigenvalue, implies that $\mathrm{Sp}(M_u) \cap D(1, \delta) = \{\lambda^{T_0}\,(u)\}$, where

$$\lambda^{T_0}\,(u) = \frac{M_u \circ Q_u(1)}{Q_u(1)}$$

is a simple eigenvalue and an analytic function of u.

Now, setting

$$R_u = M_u - \lambda^{T_0}\,(u)\, Q_u = M_u \circ \left(-\frac{1}{2\pi i} \int_{\{|\lambda| = \gamma^{-T_0} + \delta\}} R(\lambda, M_u)\, d\lambda\right),$$

which is the projection on the rest of the spectrum, we get:

$$M_u^{(kT_0)} = \lambda^{kT_0}\,(u)\, Q_u + R_u^k, \quad \text{with } \mathrm{Sp}(R_u) \subset D\left(0, \gamma^{-T_0} + \delta\right)$$

and

$$\begin{aligned}\|R_u - R_0\| &\le \|M_u\| \left\| -\frac{1}{2\pi i} \int (R(\lambda, M_u) - R(\lambda, M_0))\, d\lambda \right\| \\ &\qquad\qquad + \|M_u - L^{(T_0)}\| \left\| -\frac{1}{2\pi i} \int R(\lambda, M_0)\, d\lambda \right\| \\ &\le (1+\varepsilon)\left(\gamma^{-T_0} + \delta\right)\delta + \varepsilon \\ &\le 2\delta \qquad \text{taking } \varepsilon \text{ smaller if necessary}\end{aligned}$$

so that $\|R_u^k\| \le \left(\gamma^{-T_0} + 2\delta\right)^k$ for every $k \ge 1$, $u \in D_\theta(0, \rho)$. □

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