Probab. Theory Relat. Fields 123, 28–40 (2002)
Digital Object Identifier (DOI) 10.1007/s004400100173

Jon Aaronson · Manfred Denker

Group extensions of Gibbs–Markov maps

Received: 20 April 2000 / Revised version: 25 June 2001 /
Published online: 13 May 2002 –

Abstract. Let ϕ be an aperiodic cocycles with values in a locally compact abelian second countable group $\mathbb{G}$ defined on an exact Gibbs–Markov map $T : X \to X$. We show that the group extension $T_\phi(x, g) = (T(x), g + \phi(x))$ $(x \in X; g \in \mathbb{G})$ is exact. Equivalent conditions for exactness are found.

1. Introduction

Let $(X, \mathcal{B}, m, T, \alpha)$ be an exact probability preserving Markov map (as in §4.1 of [A]) where $(X, \mathcal{B}, m)$ denotes a probability space, $T : X \to X$ is a probability preserving transformation and α a generating Markov partition (possibly countable). We can and do assume that X is a topological Markov shift:

$$X = \left\{x = (x_1, x_2, \dots) \in \alpha^{\mathbb{N}} : \; m\left(x_n \cap T^{-1}x_{n+1}\right) > 0 \,\forall\, n \geq 1\right\}$$

endowed with the Polish topology inherited from the product topology on $\alpha^{\mathbb{N}}$.

It follows that T is *locally invertible* with respect to α in the sense that for each $n \geq 1$, $a \in \alpha_0^{n-1}$ the map $T^n : a \to T^n a$ is nonsingular and invertible. The inverse of this map is denoted $v_a : T^n a \to a$ and given by $v_a(x_1, x_2, \dots) = (a, x_1, x_2, \dots)$, where a is identified with an element of $\alpha^{\{1,\cdots,n\}}$. We let v_a' denote the Radon-Nikodym derivative of $m \circ v_a$ with respect to m.

The partition α enables the definition of a Hölder class of metrics $\{d_r : \; 0 < r < 1\}$ on X:
For $n \geq 1$, define $a_n : X \to \alpha_0^{n-1}$ by $x \in a_n(x) \in \alpha_0^{n-1}$.
For $x, y \in X$ define $t(x, y) := \min\{n \geq 1 : \; a_n(x) \neq a_n(y)\}$ $(\leq \infty)$.
For $r \in (0, 1)$ define $d_r : X \times X \to \mathbb{R}$ by $d_r(x, y) := r^{t(x,y)}$.

It is easily seen that the identity : $(X, d_r) \to (X, d_s)$ is Hölder continuous $\forall\, r, s \in (0, 1)$.

J. Aaronson: School of Mathematical Sciences, Tel Aviv University, 69978 Tel Aviv, Israel. e-mail: aaro@math.tau.ac.il. Research supported by Eurandom.

M. Denker: Institut für Mathematische Stochastik, Universität Göttingen, Lotzestr.13, 37083 Göttingen, Germany. e-mail: denker@math.uni-goettingen.de.

Research supported by Eurandom and the Deutsche Forschungsgemeinschaft, Schwerpunkt Ergodentheorie, Analysis und effiziente Simulation dynamischer Systeme.

Mathematics Subject Classification (2000): Primary 28D05, 60B15; Secondary 58F15, 58F19, 58F30

Accordingly, we define the Hölder constants of a function $h : A \to M$ $(A \subset X)$ with values in a metric space (M, ρ) by

$$D_{r,A}(h) := \sup_{x,y\in A} \frac{\rho(h(x), h(y))}{r^{t(x,y)}}.$$

Let $\mathrm{Lip}_r(M) := \{h : X \to M : \sup_{a\in\alpha} D_{r,a}(h) < \infty\}$. In case $M = \mathbb{R}$ we simply write $\mathrm{Lip}_r := \mathrm{Lip}_r(M)$ instead. A function $h : X \to M$ is called uniformly Hölder continuous on states if $h \in \mathrm{Lip}_r(M)$ for some $0 < r < 1$.

Recall (see e.g. [A-D1]) that $(X, \mathcal{B}, m, T, \alpha)$ has the *Gibbs property* if $\exists\, C > 1,\ 0 < r < 1$ such that $\forall\, n \geq 1,\ a \in \alpha_0^{n-1},\ m(a) > 0$: $\left|\frac{v_a'(x)}{v_a'(y)} - 1\right| \leq Cr^{t(x,y)}$ for $m \times m$-a.e. $(x, y) \in T^n a \times T^n a$. It is called a *Gibbs–Markov map* if it has in addition the property

$$\inf_{a\in\alpha} m(Ta) > 0.$$

Recall that any topologically mixing probability preserving Markov map with the Gibbs property is exact (see for example [A-D-U]).

Now let $\mathbb{G}$ be a locally compact, Abelian, second countable group, let $\|\cdot\|$ be a Lipschitz norm on $\mathbb{G}$ (i.e. $\gamma : \mathbb{G} \to S^1$ is $\|\cdot\|$-Lipschitz for every $\gamma \in \widehat{\mathbb{G}}$), and let $\phi : X \to \mathbb{G}$ be measurable. Consider the skew product transformation $T_\phi : X \times \mathbb{G} \to X \times \mathbb{G}$ defined by $T_\phi(x, y) := (Tx, y + \phi(x))$ with respect to the (invariant) product measure $m \times m_{\mathbb{G}}$ where $m_{\mathbb{G}}$ denotes Haar measure. We define $\phi_n = \phi + \phi \circ T + \ldots + \phi \circ T^{n-1}$ and for $x \in X$

$$\mathbb{G}_x = \left\{ t \in \mathbb{G} : \exists\, k_n \to \infty,\ y_n, z_n \in T^{-k_n}\{x\} : \begin{cases} d_r(y_n, z_n) \to 0 \\ \phi_{k_n}(y_n) - \phi_{k_n}(z_n) \to t \end{cases} \right\}.$$

We're interested in the exactness of T_ϕ and prove

Theorem. *Let $\mathbb{G}$ be a LCA, second countable group, let $(X, \mathcal{B}, m, T)$ be an exact probability preserving Gibbs–Markov map and let $\phi : X \to \mathbb{G}$ be uniformly Hölder continuous on states.*

The following are equivalent:

1.) *ϕ is aperiodic in the sense that $\gamma \circ \phi = \frac{zgT}{g}$ has no non-trivial solutions in $\gamma \in \widehat{\mathbb{G}}$, $z \in S^1$ and $g : X \to S^1$ Hölder continuous.*
2.) *T_ϕ is weakly mixing (cf. §2.7 in* [A]*).*
3.) *T_ϕ is exact.*
4.) *For some $A \in \mathcal{B}$, $m(A) > 0$ and for all $x \in A$, the smallest closed subgroup generated by $\mathbb{G}_x$ is $\mathbb{G}$.*
5.) *For every $x \in X$, $\mathbb{G} = \mathbb{G}_x$.*

Remarks. 1. In case α is a finite Markov partition and m a Gibbs measure as in [Bo], Guivarc'h ([G]) has obtained exactness of the group extension with respect to aperiodic, Hölder-continuous, $\mathbb{R}^d$-valued cocycles.

2. Let T be as in the theorem and let $\phi : X \to \mathbb{Z}^d$ be aperiodic, locally Lipschitz and in the domain of attraction of a stable distribution of order $0 < p < 2$. Exactness of T_ϕ follows from section 7 in [A-D1].

3. The assumptions on the cocycle and the dynamics in these results have been weakened in [A-D2]:
For an exact Markov map T with the Renyi property and a cocycle $\phi : X \to \mathbb{R}^d$ which is locally constant (on cylinders in α_0^N for some $N \geq 0$), topological mixing of T_ϕ implies its exactness.

4. Let T be a locally invertible, exact endomorphism with quasicompact Frobenius-Perron operator whose perturbations have a spectral representation à la Nagaev ([N]). As shown in theorem 2 of [A-D2], if $\phi : X \to \mathbb{R}^d$ is aperiodic and for each real number $\lambda > 1$ there is a subsequence n_k such that $\phi + \ldots + \phi \circ T^{n_k} = o(\lambda^{n_k})$ a.e., then T_ϕ is exact.

The proof of the theorem is given in the subsequent sections. The only non-trivial implications are 4.) $\Longrightarrow$ 3.) and 1.) $\Longrightarrow$ 5.). Our proof follows general concepts, like [L-R-W] and [F] for the first implication and [S] for the second. In particular the last section contains a ratio limit theorem of independent interest.

The *Frobenius-Perron operators* $\widehat{R}^n : L_1(m) \to L_1(m)$ of a nonsingular transformation $(X, \mathcal{B}, m, R)$ are defined by

$$\int_X \widehat{R}^n f \cdot g dm = \int_X f \cdot g \circ R^n dm$$

where $f \in L_1(m)$ and $g \in L_\infty(m)$. For a Gibbs–Markov map T these operators have the form

$$\widehat{T}^n f(x) = \sum_{a \in \alpha_0^{n-1}} 1_{T^n a}(x) \cdot v_a'(x) \cdot f(v_a(x)) = \sum_{T^n(z)=x} p_n(x,z) f(z),$$

where $p_n(x,z) = v'_{a_n(z)}(x) 1_{\{T^n(z)\}}(x)$, and for the group extension T_ϕ

$$\widehat{T}_\phi^n f(x, g) = \widehat{T}^n [f(\cdot, g - \phi_n(\cdot))](x).$$

Fix some $r \in (0, 1)$. We define the Banach space L of all L_∞-functions $f : X \to \mathbb{R}$ with

$$D_{r,X}(f) < \infty.$$

Define the norm $\|\cdot\|_L$ by $\|f\|_L := \|f\|_\infty + D_{r,X}(f)$, then $(L, \|\cdot\|_L)$ is a Banach space, and $\|\cdot\|_L$-bounded sets are $\|\cdot\|_\infty$ -precompact (see e.g. §4.7 in [A]).

We may assume that r is chosen so large that $D_\phi = \sup_{a \in \alpha} D_{r,a}(\phi) < \infty$. It is shown in [A-D1] that $\widehat{T}^n : L \to L$ $(n \geq 1)$ has a spectral representation

$$\widehat{T}^n f(x) = \int f dm + O\left(\rho^n \|f\|_L\right)$$

for some $0 < \rho < 1$ independent of $f \in L$.

Proof of 4.) $\Longrightarrow$ 3.).

We begin with the following easy observation: For $\Psi \in L_1(m)$ and $\Gamma \in L_1(\mathbb{G})$ we obtain

$$\begin{aligned}
&\int_X \int_{\mathbb{G}} \left|\hat{T}_\phi^{n+1}(\Psi \otimes \Gamma)(x, g)\right| dg\, m(dx) \\
&\quad \leq \int_X \int_{\mathbb{G}} \sum_{T(z)=x} \left|\hat{T}_\phi^n[\Psi \otimes \Gamma](z, g - \phi(z))\right| p_1(x, z) dg\, m(dx) \\
&\quad = \int_{\mathbb{G}} \int_X \hat{T}\left[\left|\hat{T}_\phi^n[\Psi \otimes \Gamma](\cdot, g - \phi(\cdot))\right|\right](x) m(dx) dg \\
&\quad = \int_X \int_{\mathbb{G}} \left|\hat{T}_\phi^n[\Psi \otimes \Gamma](x, g - \phi(x))\right| dg\, m(dx) \\
&\quad = \int_X \int_{\mathbb{G}} \left|\hat{T}_\phi^n[\Psi \otimes \Gamma](x, g)\right| dg\, m(dx) =: U_n(\Psi \otimes \Gamma).
\end{aligned}$$

Therefore $C(\Psi \otimes \Gamma)$ is well defined by

$$U_n(\Psi \otimes \Gamma) \downarrow C(\Psi \otimes \Gamma) \geq 0. \tag{1}$$

We define the operators $M_t : L_1(\mathbb{G}) \to L_1(\mathbb{G})$ by $M_t\Gamma(g) = \Gamma(g + t)$. Let $\Psi \in L_1(X)$ be fixed and let the measures $\{\mu_{n,x} : n \geq 1\}$ on $\mathbb{G}$ be defined by

$$\mu_{n,x} = \sum_{T^n(z)=x} \Psi(z) p_n(x, z) \delta_{\phi_n(z)}.$$

Note that

$$\mu_{n,x} \star \Gamma(g) = \widehat{T}_\phi^n(\Psi \otimes \Gamma)(x, g)$$

hence $\|\mu_{n,x} \star \Gamma\|_{L_1(\mathbb{G})} \leq \widehat{T}^n|\Psi|(x)\ \|\Gamma\|_{L_1(\mathbb{G})}$ and $t \mapsto \|\mu_{n,x} \star M_t\Gamma\|_{L_1(\mathbb{G})}$ is continuous with modulus of continuity bounded by $\widehat{T}^n|\Psi|(x)\|\Gamma - M_\delta\Gamma\|_{L_1(\mathbb{G})}$.

Remark. Following [L-R-W], p. 287, a family of signed random measures $\{\mu_{n,x} : n \geq 1, x \in X\}$ on $\mathbb{G}$ is called *completely mixing in* $L_1(m)$ if for every $\Gamma \in L_1(\mathbb{G})$ with integral $\int_{\mathbb{G}} \Gamma(g)dg = 0$ we have

$$\|\mu_{n,\cdot} \star \Gamma\|_{L_1(\mathbb{G})} \to 0$$

in $L_1(m)$. We'll show in Proposition 1 and Lemma 2 below that the random signed measures $\{\mu_{n,x} : n \geq 1\}$ are completely mixing in $L_1(m)$.

Proposition 1. *For every* $\Gamma \in L_1(\mathbb{G})$ *the random sequence*

$$\|\mu_{n,\cdot} \star \Gamma\|_{L_1(\mathbb{G})}$$

converges in $L_1(m)$ *to* $C(\Psi \otimes \Gamma)$*. In addition,*

$$C(\Psi \otimes \Gamma) \leq \|\Psi\|_{L_1(m)}\|\Gamma\|_{L_1(\mathbb{G})}.$$

Proof. Since $\hat{T}_\phi^n(\Psi \otimes \Gamma)(x, g) = \hat{T}^n[\Psi(\cdot)\Gamma(g - \phi_n(\cdot))](x)$ for $\Psi \in L_1(X)$ and $\Gamma \in L_1(\mathbb{G})$, it suffices to show the theorem for a subclass of pairs (Ψ, Γ) which generates a dense subspace in $L_1(X) \times L_1(\mathbb{G})$. Here we take the class of all functions $\Psi \otimes \Gamma$ where Ψ belongs to the space L and Γ is an integrable and Lipschitz continuous function on $\mathbb{G}$.

By definition

$$\begin{aligned}\mu_{n+1,x} \star \Gamma(g) &= \int_{\mathbb{G}} \Gamma(g-h)\mu_{n+1,x}(dh)\\ &= \sum_{T^{n+1}(z)=x} \Psi(z)p_{n+1}(x,z)\Gamma(g-\phi_{n+1}(z))\\ &= \sum_{T(z)=x} p_1(x,z)\hat{T}_\phi^n[\Psi\otimes\Gamma](z,g-\phi(z))\end{aligned}$$

whence as before,

$$\begin{aligned}&\|\mu_{n+1,x} \star \Gamma\|_{L_1(\mathbb{G})}\\ &\quad\le \int_{\mathbb{G}} \sum_{T(z)=x} p_1(x,z)\left|\hat{T}_\phi^n[\Psi\otimes\Gamma](z,g-\phi(z))\right| dg\\ &\quad= \sum_{T(z)=x} p_1(z,x)\int_{\mathbb{G}}\left|\hat{T}_\phi^n[\Psi\otimes\Gamma](z,g)\right| dg\\ &\quad= \hat{T}\left[\|\mu_{n,\cdot} \star \Gamma\|_{L_1(\mathbb{G})}\right](x).\end{aligned}$$

By induction it is easily seen that for n fixed and $k \ge 1$

$$\|\mu_{n+k,x} \star \Gamma\|_{L_1(\mathbb{G})} \le \hat{T}^k\left[\|\mu_{n,\cdot} \star \Gamma\|_{L_1(\mathbb{G})}\right](x).$$

Since the function

$$x \to \|\mu_{n,x} \star \Gamma\|_{L_1(\mathbb{G})}$$

is of class L it follows from the spectral representation of $\widehat{T}$ (mentioned above) that $\forall\, n \ge 1$, as $k \to \infty$

$$\begin{aligned}\hat{T}^k\left[\|\mu_{n,\cdot} \star \Gamma\|_{L_1(\mathbb{G})}\right] &= \int_X \|\mu_{n,x} \star \Gamma\|_{L_1(\mathbb{G})} m(dx) + O(\rho^k)\\ &\to U_n(\Psi\otimes\Gamma) \downarrow C(\Psi\otimes\Gamma),\end{aligned}$$

whence

$$\limsup_{n\to\infty} \|\mu_{n,x} \star \Gamma\|_{L_1(\mathbb{G})} \le C(\Psi\otimes\Gamma). \tag{2}$$

By (1) and (2), given $\epsilon > 0$, we can choose n_0 so large that for $n \ge n_0$

$$\int_{\{x:\|\mu_{n,x}\star\Gamma\|_{L_1(\mathbb{G})}-C(\Psi\otimes\Gamma)>0\}} \left[\|\mu_{n,x} \star \Gamma\|_{L_1(\mathbb{G})} - C(\Psi\otimes\Gamma)\right] m(dx) \le \epsilon$$

and

$$\int_X \|\mu_{n,x} \star \Gamma\|_{L_1(\mathbb{G})} m(dx) - C(\Psi\otimes\Gamma) \ge 0.$$

It follows that

$$\begin{aligned}
&\int_X \left| \|\mu_{n,x} \star \Gamma\|_{L_1(\mathbb{G})} - C(\Psi \otimes \Gamma) \right| m(dx) \\
&\quad = 2 \int_{\{x : \|\mu_{n,x} \star \Gamma\|_{L_1(\mathbb{G})} - C(\Psi\otimes\Gamma) > 0\}} \left[\|\mu_{n,x} \star \Gamma\|_{L_1(\mathbb{G})} - C(\Psi \otimes \Gamma) \right] m(dx) \\
&\quad\quad - \int_X \left[\|\mu_{n,x} \star \Gamma\|_{L_1(\mathbb{G})} - C(\Psi \otimes \Gamma) \right] m(dx) \\
&\quad \le 2\epsilon.
\end{aligned}$$

The additional claim follows from

$$C(\Psi \otimes \Gamma) \leftarrow_{L_1(m)} \|\mu_{n,x} \star \Gamma\|_{L_1(\mathbb{G})} \le \widehat{T}^n |\Psi|(x) \|\Gamma\|_{L_1(\mathbb{G})} \to \|\Psi\|_{L_1(m)} \|\Gamma\|_{L_1(\mathbb{G})}.$$

□

Let $(Y, \mathcal{A}, \mu, R)$ and $(Z, \mathcal{C}, \nu, S)$ be nonsingular transformations of probability spaces. The factor map $\pi : Y \to Z$ is called *relatively exact* if for $f \in L_1(\mu)$

$$E(f | \pi^{-1}\mathcal{C}) = 0 \Longrightarrow \widehat{R}^n f \to 0$$

in $L_1(\mu)$. By [G], see alternatively [A-D2], R is exact if the factor map $\pi : Y \to Z$ is relatively exact and the factor S is exact. In the present situation T_ϕ is exact if the factor map $(x, g) \mapsto x =: \Pi(x, g)$ $(X \times \mathbb{G} \to X)$ is relatively exact. To establish relative exactness of T_ϕ, it suffices to show

$$\int_X \int_{\mathbb{G}} \left| \hat{T}_\phi^n [\Psi \otimes \Gamma](x, g) \right| m_{\mathbb{G}}(dg) m(dx) \to 0$$

for all $\Psi \in L_1(m)$ and $\Gamma \in L_1(\mathbb{G})$ satisfying $\int_{\mathbb{G}} \Gamma dg = 0$ (see [G], [A-D2]).

It is left to prove the following

Lemma 2. *If* $\int_{\mathbb{G}} \Gamma(g) dg = 0$*, then*

$$C(\Psi \otimes \Gamma) = 0.$$

Proof. The proof of this statement follows from a series of claims. For the first 4 claims we assume that $\Gamma \in L_1(\mathbb{G})$ is Lipschitz continuous and has compact support. These claims are needed for the proof of the statement of the lemma in claim 5.

Define the measures $\nu_{n,x} = \sum_{T^n(z)=x} p_n(x, z) \delta_z$ on X.

Claim 1. *Let* $k \ge 0$ *be fixed. For any subsequence* $\{n_l : l \in \mathbb{N}\} \subset \mathbb{N}$ *there exists a further subsequence* $\{m_j : j \ge 1\}$ *such that for a.e.* $x \in X$ *and for every* $B \in \mathcal{B}$

$$\lim_{j\to\infty} \frac{1}{\nu_{k,x}(B)} \int_{\mathbb{G}} \left| \int_B \left(\mu_{m_j,y} \star M_{\phi_k(y)} \Gamma \right)(g) \nu_{k,x}(dy) \right| dg = C(\Psi \otimes \Gamma). \qquad (3)$$

In order to see this claim, let n_l be any subsequence and choose m_j so that

$$\|\mu_{m_j,x} \star \Gamma\|_{L_1(\mathbb{G})}, \|\mu_{m_j+k,x} \star \Gamma\|_{L_1(\mathbb{G})} \to C(\Psi \otimes \Gamma) \tag{4}$$

for $x \in \Omega$ where Ω is a T-invariant set of full measure (cf. Proposition 1). On the one hand it follows from this that for every B fixed

$$\begin{aligned}&\frac{1}{\nu_{k,x}(B)} \int_{\mathbb{G}} \left| \int_B \mu_{m_j,y} \star M_{\phi_k(y)}\Gamma\ \nu_{k,x}(dy) \right| dg \\ &\quad \le \frac{1}{\nu_{k,x}(B)} \int_B \|\mu_{m_j,y} \star \Gamma\|_{L_1(\mathbb{G})} \nu_{k,x}(dy) \to C(\Psi \otimes \Gamma), \end{aligned} \tag{5}$$

because the integrand is uniformly bounded and pointwise convergent by (4).

On the other hand, for $x \in \Omega$,

$$\begin{aligned} C(\Psi \otimes \Gamma) &= \lim_{j\to\infty} \|\mu_{m_j+k,x} \star \Gamma\|_{L_1(\mathbb{G})} \\ &= \lim_{j\to\infty} \int_{\mathbb{G}} \left| \sum_{T^k(y)=x} p_k(x,y) \hat{T}_\phi^{m_j}[\Psi \otimes \Gamma](y, g - \phi_k(y)) \right| dg \\ &\le \lim_{j\to\infty} \int_{\mathbb{G}} \left| \int_B \mu_{m_j,y} \star M_{\phi_k(y)}\Gamma(g)\nu_{k,x}(dy) \right| \\ &\quad + \left| \int_{B^c} \mu_{m_j,y} \star M_{\phi_k(y)}\Gamma(g)\nu_{k,x}(dy) \right| dg \\ &\le C(\Psi \otimes \Gamma) \end{aligned}$$

by (5), hence for $x \in \Omega$

$$\lim_{j\to\infty} \frac{1}{\nu_{k,x}(B)} \int_{\mathbb{G}} \left| \int_B \mu_{m_j,y} \star M_{\phi_k(y)}\Gamma\nu_{k,x}(dy) \right| dg = C(\Psi \otimes \Gamma),$$

proving claim 1.

Claim 2. *Let $k \ge 0$ be fixed. For any subsequence $\{n_l : l \in \mathbb{N}\} \subset \mathbb{N}$ there exists a further subsequence $\{m_j : j \ge 1\}$ such that for a.e. $x \in X$ and for every disjoint sets $A, B \in \mathcal{B}$*

$$\begin{aligned} \lim_{j\to\infty} \int_{\mathbb{G}} &\left| \frac{1}{\nu_{k,x}(A)} \int_A \mu_{m_j,y} \star M_{\phi_k(y)}\Gamma(g)\nu_{k,x}(dy) \right. \\ &\left. + \frac{1}{\nu_{k,x}(B)} \int_B \mu_{m_j,y} \star M_{\phi_k(y)}\Gamma(g)\nu_{k,x}(dy) \right| dg = 2C(\Psi \otimes \Gamma) \end{aligned} \tag{6}$$

Choose the subsequence and Ω as in (4). It follows that for $x \in \Omega$ by (3)

$$\begin{aligned}
&\int_{\mathbb{G}} \Bigg| \frac{1}{\nu_{k,x}(A)} \int_A \mu_{m_j,y} \star M_{\phi_k(y)} \Gamma \nu_{k,x}(dy) \\
&\qquad + \frac{1}{\nu_{k,x}(B)} \int_B \mu_{m_j,y} \star M_{\phi_k(y)} \Gamma \nu_{k,x}(dy) \Bigg| dg \\
&\leq \frac{1}{\nu_{k,x}(A)} \int_{\mathbb{G}} \Bigg| \int_A \mu_{m_j,y} \star M_{\phi_k(y)} \Gamma(g) \nu_{k,x}(dy) \Bigg| dg \\
&\qquad + \frac{1}{\nu_{k,x}(B)} \int_{\mathbb{G}} \Bigg| \int_B \mu_{m_j,y} \star M_{\phi_k(y)} \Gamma(g) \nu_{k,x}(dy) \Bigg| dg \\
&\to 2C(\Psi \otimes \Gamma)
\end{aligned} \tag{7}$$

and, since $A \cap B = \emptyset$ (and w.l.o.g. assume that $\nu_{k,x}(A) \leq \nu_{k,x}(B)$),

$$\begin{aligned}
&\frac{1}{\nu_{k,x}(A)} \int_{\mathbb{G}} \Bigg| \int_A \mu_{m_j,y} \star M_{\phi_k(y)} \Gamma \nu_{k,x}(dy) \\
&\qquad + \frac{\nu_{k,x}(A)}{\nu_{k,x}(B)} \int_B \mu_{m_j,y} \star M_{\phi_k(y)} \Gamma \nu_{k,x}(dy) \Bigg| dg \\
&\geq \frac{1}{\nu_{k,x}(A)} \Bigg(\int_{\mathbb{G}} \Bigg| \int_{A \cup B} \mu_{m_j,y} \star M_{\phi_k(y)} \Gamma(g) \nu_{k,x}(dy) \Bigg| dg \\
&\qquad - \left(1 - \frac{\nu_{k,x}(A)}{\nu_{k,x}(B)}\right) \int_{\mathbb{G}} \Bigg| \int_B \mu_{m_j,y} \star M_{\phi_k(y)} \Gamma(g) \nu_{k,x}(dy) \Bigg| dg \Bigg) \\
&\to 2C(\Psi \otimes \Gamma).
\end{aligned} \tag{8}$$

Claim 2 follows from (7) and (8).

Claim 3. *Let $A, B \in \alpha_0^{k-1}$ be images of inverse branches v_A and v_B of T^k, where k is still fixed. Let $\epsilon = d_r(A, B)$ and let Γ be Lipschitz continuous with compact support K; then there exist constants $C_0, C_1 > 0$ such that for every $n \geq 1$*

$$\begin{aligned}
&\int_{\mathbb{G}} \left| \mu_{n,v_A(x)} \star M_{\phi_k(v_A(x))} \Gamma(g) - \mu_{n,v_B(x)} \star M_{\phi_k(v_A(x))} \Gamma(g) \right| dg \\
&\qquad \leq \left[C_1 \|\Gamma\|_{L_1(\mathbb{G})} + D_\Gamma C_0 D_\phi m_{\mathbb{G}}(B(K, C_0 D_\phi \epsilon)) \right] \epsilon,
\end{aligned} \tag{9}$$

where D_Γ denotes the Lipschitz constant of Γ.

Let $x \in X$, $v = v_A(x)$ and $w = v_B(x)$. We may assume that $d_r(A, B) < r$ so that $A \cup B$ is contained in some atom from α. By the Lipschitz property of ϕ and by the expanding property of T, we have for any inverse branch $v_a : A \cup B \to a \in (\alpha)_0^{n-1}$ of T^n that

$$\begin{aligned}
|\phi_n(v_a(v)) - \phi_n(v_a(w))| &\leq D_\phi \sum_{l=0}^{n-1} d_r(T^l(v_a(v)), T^l(v_a(w))) \\
&\leq C_0' D_\phi d_r(v, w) \leq C_0' D_\phi \epsilon,
\end{aligned}$$

where C_0' denotes some constant. Since Γ has compact support

$$\|\Gamma(g) - \Gamma(g + \phi_n(v_a(v)) - \phi_n(v_a(w)))\| \le D_\Gamma C_0' D_\phi \epsilon 1_{B(K, C_0' D_\phi \epsilon)}(g).$$

Similarly, there exists a constant C_1' (also depending on the Lipschitz constant of Ψ) so that (see [A-D1])

$$|p_n(v, v_a(v))\Psi(v_a(v)) - p_n(w, v_a(w))\Psi(v_a(w))| \le C_1' p_n(v, v_a(v)) d_r(v, w).$$

Therefore

$$\begin{aligned}
&\int_{\mathbb{G}} \left|\mu_{n, v_A(x)} \star M_{\phi_k(v_A(x))}\Gamma(g) - \mu_{n, v_B(x)} \star M_{\phi_k(v_A(x))}\Gamma(g)\right| dg \\
&= \int_{\mathbb{G}} \Bigg| \sum_a p_n(v, v_a(v))\Psi(v_a(v))\Gamma(g - \phi_k(v) - \phi_n(v_a(v))) \\
&\quad - \sum_a p_n(w, v_a(w))\Psi(v_a(w))\Gamma(g - \phi_k(v) - \phi_n(v_a(w))) \Bigg| dg \\
&\le \int_{\mathbb{G}} \Bigg| \sum_a [p_n(v, v_a(v))\Psi(v_a(v)) - p_n(w, v_a(w))\Psi(v_a(w))] \\
&\quad \times \Gamma(g - \phi_k(v) - \phi_n(v_a(v))) \Bigg| dg \\
&\quad + \int_{\mathbb{G}} \Bigg| \sum_a p_n(w, v_a(w))\Psi(v_a(w)) \\
&\quad \times [\Gamma(g - \phi_k(v) - \phi_n(v_a(v))) - \Gamma(g - \phi_k(v) - \phi_n(v_a(w)))] \Bigg| dg \\
&\le \left(C_1' \|\Gamma\|_{L_1(\mathbb{G})} + D_\Gamma C_0' D_\phi \|\Psi\|_\infty m_{\mathbb{G}}(B(K, C_0' D_\phi \epsilon))\right) \|\hat{T}^n 1\|_\infty \epsilon,
\end{aligned}$$

where $\sum_a$ extends over all $a \in \alpha_0^{n-1}$ satisfying $T^n a \supset A \cup B$. The claim follows setting $C_i = 1 \vee C_i' \sup_{n \ge 1} \|\widehat{T}^n\|_\infty$ for $i = 0, 1$.

Claim 4. *There exists a set Ω of measure 1 and a constant $C > 0$ with the following property:*
If $x \in \Omega$, $k \ge 1$ and $v, w \in T^{-k}(\{x\})$, then

$$\left|2C(\Psi \otimes \Gamma) - C(\Psi \otimes (I + M_{\phi_k(v) - \phi_k(w)})\Gamma)\right| < C d_r(v, w).$$

By claims 1–3 there exists a subsequence $\{m_j : j \ge 1\} \subset \mathbb{N}$ and a subset Ω so that (6) and (9) hold for any $x \in \Omega$, $k \ge 1$ and $v, w \in T^{-k}(\{x\})$, $A = a_k(v)$, $B = a_k(w)$. Therefore

$$\begin{aligned}
&\int_{\mathbb{G}} \left| \frac{1}{\nu_{k,x}(A)} \int_A \mu_{m_j,y} \star M_{\phi_k(y)}\Gamma(g)\nu_{k,x}(dy) \right.\\
&\qquad \left. + \frac{1}{\nu_{k,x}(B)} \int_B \mu_{m_j,y} \star M_{\phi_k(y)}\Gamma(g)\nu_{k,x}(dy) \right| dg \\
&= \int_{\mathbb{G}} \left|\mu_{m_j,v} \star M_{\phi_k(v)}\Gamma(g) + \mu_{m_j,w} \star M_{\phi_k(w)}\Gamma(g)\right| dg \\
&\le \int_{\mathbb{G}} \left|\mu_{m_j,v} \star M_{\phi_k(v)}\Gamma(g) - \mu_{m_j,w} \star M_{\phi_k(v)}\Gamma(g)\right| dg \\
&\qquad + \int_{\mathbb{G}} \left|\mu_{m_j,w} \star M_{\phi_k(w)}\Gamma(g) + \mu_{m_j,w} \star M_{\phi_k(v)}\Gamma(g)\right| dg \\
&\le \int_{\mathbb{G}} \left|\mu_{m_j,w} \star (I + M_{\phi_k(v)-\phi_k(w)})\Gamma(g)\right| dg + Cd_r(v, w),
\end{aligned}$$

where $C = C_1\|\Gamma\|_{L_1(\mathbb{G})} + D_\Gamma C_0 D_\phi \|\Psi\|_\infty m_{\mathbb{G}}(B(K, C_0 D_\phi))$. The lower bound is shown similarly, proving claim 4.

Claim 5. *Let* $\Psi \in L$*, then for all* $\Gamma \in L_1(\mathbb{G})$*,*

$$C(\Psi \otimes (\Gamma - M_t\Gamma)) = 0.$$

First observe that by Proposition 1 the set of $t \in \mathbb{G}$ satisfying the claim is a group.

Hence it suffices to prove the claim for t in a generating set G_0. Moreover, it suffices to prove the claim for Γ Lipschitz continuous with compact support, since $\Gamma \mapsto C(\Psi \otimes \Gamma)$ is $L_1(\mathbb{G})$-norm continuous.

Fix such a Γ. By assumption, and by claim 4 there is a measurable set $A \in \mathcal{B}$ of positive measure and a constant $C > 0$ satisfying:
For $x \in A$ there is a subset $G_0 \subset \mathbb{G}$ generating a dense subgroup of $\mathbb{G}$ such that for all $v, w \in T^{-k}(x)$

$$\left|2C(\Psi \otimes \Gamma) - C(\Psi \otimes (I + M_{\phi_k(v)-\phi_k(w)})\Gamma)\right| < Cd_r(v, w), \tag{10}$$

and

$$\begin{aligned}
&\forall\, t \in G_0\ \exists\, k_n \ge 1, v_n, w_n \in T^{-k_n}(x) \\
&\quad \text{such that } \phi_{k_n}(v_n) - \phi_{k_n}(w_n) \to t \ \&\ d_r(v_n, w_n) \to 0.
\end{aligned} \tag{11}$$

Since $t \to C(\Psi \otimes M_t\Gamma)$ is continuous, it follows from properties (10) and (11) that

$$2C(\Psi \otimes \Gamma) = C(\Psi \otimes (I + M_t)\Gamma) \quad (t \in G_0). \tag{12}$$

It follows that (12) holds for all Lipschitz continuous Γ with compact support. Because of continuity, this equation holds for any $\Gamma \in L_1(\mathbb{G})$. Hence, replacing Γ by $(I - M_t)\Gamma$ and repeating this argument for each $(I + M_t)^k(I - M_t)\Gamma$, $k \ge 0$, we obtain

$$C(\Psi \otimes (I - M_t)\Gamma) = 2^{-k} C(\Psi \otimes (I + M_t)^k (I - M_t)\Gamma)$$

for every $k \geq 0$ and $t \in G_0$. From this we deduce $C(\Psi \otimes \Gamma) = 0$ as in [F].

The lemma follows now from the well known fact (see [L], [**?**]) that

$$\overline{\bigcup_{t \in \mathbb{G}} (I - M_t) L_1(\mathbb{G})} = \{f \in L_1(\mathbb{G}) : \int f(g)dg = 0\}.$$

□

Proof of 1.) $\Longrightarrow$ 5.)

Ratio limit theorem for symmetric cocycles. *Suppose that $\phi : X \to \mathbb{G}$ is Hölder continuous, aperiodic and symmetric in the sense that there exists a probability preserving invertible transformation $S : X \to X$ such that $ST = TS$ and $\phi \circ S = -\phi$, then there exists $u_n > 0$ such that*

$$\frac{\widehat{T}_\phi^n (h \otimes f)(x, y)}{u_n} \to \int_{X \times \mathbb{G}} h \otimes f dm \times m_{\mathbb{G}}$$

for all $h \in L,\ f \in C_c(\mathbb{G}),\ x \in X,\ y \in \mathbb{G}$.

Proof. First let (as in[A-D1]) $P_\gamma : L \to L$ $(\gamma \in \widehat{\mathbb{G}})$ be defined by

$$P_\gamma h := \widehat{T}(\gamma \circ \phi \cdot h).$$

As shown in [A-D1], $\gamma \mapsto P_\gamma$ is continuous ($\widehat{\mathbb{G}} \to \mathrm{Hom}\,(L, L)$), and $\exists\ \epsilon > 0,\ 0 \leq \theta < 1$ and continuous functions

$$\lambda : B_{\widehat{\mathbb{G}}}(0, \epsilon) \to B_{\mathbb{C}}(0, 1),\ N : B_{\widehat{\mathbb{G}}}(0, \epsilon) \to \mathrm{Hom}(L, L) \text{ and } g : B_{\widehat{\mathbb{G}}}(0, \epsilon) \to L,$$

such that

$$\begin{aligned} &\lambda(0) = 1,\ g(0) \equiv 1,\ \int_X g(\gamma)dm \equiv 1, \\ &|\lambda(\gamma)| \leq 1 \text{ with equality iff } \gamma = 0, \\ &P_\gamma h = \lambda h \Longrightarrow |\lambda| \leq |\lambda(\gamma)| \quad (\gamma \in B_{\widehat{\mathbb{G}}}(0, \epsilon)), \\ &P_\gamma h = \lambda(\gamma) h \iff h \in \mathbb{R} \cdot g(\gamma) \quad (\gamma \in B_{\widehat{\mathbb{G}}}(0, \epsilon)), \\ &P_\gamma^n h = \lambda(\gamma)^n N(\gamma) h\ g(\gamma) + O(\theta^n) \quad (\gamma \in B_{\widehat{\mathbb{G}}}(0, \epsilon) \end{aligned}$$

and (as is easily shown)

$$g(-\gamma) = \overline{g(\gamma)},\ \lambda(-\gamma) = \overline{\lambda(\gamma)}.$$

Since $TS = ST$ and $\phi \circ S = -\phi$, $P_\gamma h(x) = [P_\gamma h \circ S^{-1}](Sx)$. It follows that $P_{-\gamma}[g(\gamma) \circ S](x) = \lambda(\gamma) g(\gamma) \circ S(x)$ whence

$$g(-\gamma) = g(\gamma) \circ S,\ \text{ and }\ \lambda(\gamma) \in \mathbb{R}.$$

Next, for $0 < \eta \leq \epsilon$ set $u_n(\eta) := \int_{B(0,\eta)} \lambda(\gamma)^n d\gamma$. For η small enough (so that $\lambda > 0$ on $B(0, \eta)$), $u_n(\eta) > 0$. Choose one such $\eta_0 > 0$ and define $u_n := u_n(\eta_0)$.

Note that $\rho^n = o(u_n)\ \forall\, \rho < 1$ since $\exists\, \eta < \eta_0$ such that $\min_{|\gamma|<\eta} |\lambda(\gamma)| = r > \rho$ whence

$$\frac{u_n}{\rho^n} \geq \frac{u_n(\eta)}{\rho^n} \geq \frac{r^n}{\rho^n} \cdot m(B(0,\eta)) \to \infty.$$

Also, for $0 < \eta < \eta'$,

$$u_n(\eta) = u_n(\eta') \pm O(\rho(\eta)^n)$$

where $\rho(\eta) := \sup_{\eta \leq |\gamma| \leq \epsilon} |\lambda(\gamma)| < 1$. Thus

$$u_n(\eta) \sim u_n \ \text{ as } \ n \to \infty \quad \forall\, 0 < \eta \leq \epsilon.$$

Now fix $h \in L$ and $f \in L^1(\mathbb{G})$ with $\hat{f} \in C_c(\widehat{\mathbb{G}})$, then $\forall\, x \in X,\ y \in \mathbb{G}$,

$$\begin{aligned}\widehat{T}_\phi^n(h \otimes f)(x,y) &= \int_{\widehat{\mathbb{G}}} \hat{f}(\gamma)\overline{\gamma(y)} P_\gamma^n h(x) d\gamma \\ &= \int_X h dm \int_{B(0,\eta_0)} \lambda(\gamma)^n \Re(\overline{\gamma(y)}\hat{f}(\gamma) g(\gamma)(x)) d\gamma + O(\theta^n)\end{aligned}$$

(by reality of $\lambda(\gamma)$, for some $0 < \theta < 1$). Since $\Re(\hat{f}(\gamma)\overline{\gamma(y)} g(\gamma)(x)) \to \int_{\mathbb{G}} f dm_{\mathbb{G}}$ as $\gamma \to 0$, it follows that

$$\widehat{T}_\phi^n(h \otimes f)(x,y) \sim u_n \int_X h dm \int_{\mathbb{G}} f dm_{\mathbb{G}}.$$

By the method of Breiman ([Brei], Theorem 10.7),

$$\widehat{T}_\phi^n(h \otimes f)(x,y) \sim u_n \int_X h dm \int_{\mathbb{G}} f dm_{\mathbb{G}} \quad \forall\, h \in L,\ f \in C_c(\mathbb{G}).$$

□

Corollary. *Under the same assumptions,* $\forall\, x, y \in X,\ t \in \mathbb{G},\ \epsilon > 0,\ \exists\, n_0$ *such that* $\forall\, n \geq n_0\ \exists\, z \in T^{-n}\{x\}$ *such that* $d(y,z) < \epsilon$ *and* $\|t - \phi_n(z)\| < \epsilon$.

Proof. Let $a = [a_1, \ldots, a_N] = B(y,\epsilon),\ h = 1_a \in L$ and let $f \in C(\mathbb{G}),\ f \geq 0,\ [f > 0] \subset B(0,\epsilon)$, then

$$\frac{\widehat{T}_\phi^n(h \otimes f)(x,t)}{u_n} \to \int_{X \times \mathbb{G}} h \otimes f dm \times m_{\mathbb{G}}$$

and $\exists\, n_0$ such that $\forall\, n \geq n_0$,

$$0 < \widehat{T}_\phi^n(h \otimes f)(x,t) = \sum_{T^n z = x,\ d(y,z) < \epsilon} p_n(x,z) f(t - \phi_n(z))$$

and in particular $\exists\, z \in T^{-n}\{x\}$ such that $d(y,z) < \epsilon$ and $\|t - \phi_n(z)\| < \epsilon$. □

Exactness lemma. *Suppose that* $\phi : X \to \mathbb{G}$ *is Hölder continuous, aperiodic, then* $\forall\, x \in X,\ t \in \mathbb{G},\ \epsilon > 0,\ \exists\, n_0$ *such that* $\forall\, n \geq n_0\ \exists\, y, z \in T^{-n}\{x\}$ *such that* $d(y,z) < \epsilon$ *and* $\|t + \phi_n(y) - \phi_n(z)\| < \epsilon$.

Proof. Consider the mixing Gibbs–Markov map $(X \times X, \mathcal{B}(X \times X), T \times T, m \times m, \alpha \times \alpha)$ equipped with the cocycle $\tilde{\phi} : X \times X \to \mathbb{G}$ defined by $\tilde{\phi}(x, x') := \phi(x) - \phi(x')$.

The cocycle $\tilde{\phi} : X \times X \to \mathbb{G}$ is also Hölder continuous, aperiodic, but also symmetric: $\tilde{\phi} \circ S = -\tilde{\phi}$ where $S(x, x') := (x', x)$ (evidently $S(T \times T) = (T \times T)S$). Thus the conclusion of the corollary holds and this is the lemma. $\square$

References

[A] Aaronson, J.: *An introduction to infinite ergodic theory*, Mathematical surveys and monographs 50, American Mathematical Society, Providence, R.I, U.S., 1997

[A-D1] Aaronson, J., Denker, M.: *Local limit theorems for partial sums of stationary sequences generated by Gibbs–Markov maps*, Stochastics and Dynamics **1**(2), 193–238 (2001)

[A-D2] Aaronson, J., Denker, M.: *On exact group extensions*, Sankhya ser. A **62**(3), 339–349 (2000)

[A-D-U] Aaronson, J., Denker, M., Urbański, M.: *Ergodic theory for Markov fibred systems and parabolic rational maps*, Trans. Amer. Math. Soc. **337**, 495–548 (1993)

[Bo] Bowen, R.: *Equilibrium states and the ergodic theory for Anosov diffeomorphisms*, Lect. Notes in Math. 470, Springer Verlag, Berlin, Heidelberg, New York, 1975

[Brei] Breiman, L.: *Probability*, Addison-Wesley, Reading, Mass. US, 1968

[F] Foguel, S.R.: *On iterates of convolutions*, Proc. Amer. Math. Soc. **47**, 368–370 (1975)

[G] Guivarc'h, Y.: *Propriétés ergodiques, en mesure infinie, de certains systèmes dynamiques fibrés*, Ergod. Th. and Dynam. Sys. **9**, 433–453 (1989)

[L] Lin, M.: *Mixing for Markov operators*, Z. Wahrscheinlichkeitstheorie und Verw. Gebiete **19**, 231–242 (1971)

[L-R-W] Lin, M., Rubshtein, B.-Z., Wittmann, R.: *Limit theorems for random walks with dynamical random transitions*, Probab. Theory Relat. Fields **100**, 285–300 (1994)

[N] Nagaev, S.V.: *Some limit theorems for stationary Markov chains*, Theor. Probab. Appl. **2**, 378–406 (1957)

[S] Stone, C.: *Ratio limit theorems for random walks on groups*, Trans. Amer. Math. Soc. **125**, 86–100 (1966)