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Filtration-consistent nonlinear expectations and related g-expectations

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Abstract. From a general definition of nonlinear expectations, viewed as operators preserving monotonicity and constants, we derive, under rather general assumptions, the notions of conditional nonlinear expectation and nonlinear martingale. We prove that any such nonlinear martingale can be represented as the solution of a backward stochastic equation, and in particular admits continuous paths. In other words, it is a g-martingale.

1. Introduction

A (possibly nonlinear) expectation on a probability space $(\Omega, \mathcal{F}, P)$ is a map

$$\mathcal{E} : L^2(\Omega, \mathcal{F}, P) \longmapsto R$$

which satisfies the following properties:

$$\begin{aligned} &\text{if} \quad X_1 \geq X_2 \quad a.s., \quad \mathcal{E}[X_1] \geq \mathcal{E}[X_2], \quad \text{and} \\ &\text{if} \quad X_1 \geq X_2 \quad a.s., \quad \mathcal{E}[X_1] = \mathcal{E}[X_2] \quad \Longleftrightarrow \quad X_1 = X_2 \quad a.s. \end{aligned}$$

$$\mathcal{E}[c] = c, \quad \text{for each constant } c.$$

In particular, if $\mathcal{E}[\cdot]$ is linear, then it becomes a classic expectation under the probability measure defined by $P_{\mathcal{E}}(A) = \mathcal{E}[1_A]$, $A \in \mathcal{F}$. In fact, there is a one-to-one correspondence between the set of linear expectations and that of σ-additive probability measures on $(\Omega, \mathcal{F})$. But in the nonlinear case this one-to-one correspondence no longer holds true: a nonlinear expectation can always induce a, generally non-additive, 'probability measure' by $P(A) = \mathcal{E}[1_A]$. But, in general, a (possibly non-additive) probability measure can not characterize a nonlinear expectation. For example, if E is the classical linear expectation defined by the probability measure

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P, and f denotes a strictly increasing continuous function on $\mathbf{R}$ such that $f(x) = x$ whenever $0 \leq x \leq 1$, $\mathcal{E}^f[X] = f^{-1}(E[f(X)])$ defines a non linear expectation (unless f is a linear mapping). But clearly, any such expectation induces the same probability measure, that is P itself: $P(A) = E[1_A] = \mathcal{E}^f[1_A]$. In fact, for each nonlinear expectation $\mathcal{E}$, $f^{-1}(\mathcal{E}[f(X)])$ defines a different nonlinear expectation associated with the same non-additive probability.

A nonlinear expectation is said to be filtration-consistent under a given filtration $\{\mathcal{F}_t\}_{t\geq 0}$ if, for each $t \geq 0$, the corresponding conditional expectation $\mathcal{E}[X|\mathcal{F}_t]$ of X under $\mathcal{F}_t$, characterized by

$$\mathcal{E}\Big[\mathcal{E}[X|\mathcal{F}_t]1_A\Big] = \mathcal{E}[X1_A], \quad \forall A \in \mathcal{F}_t,$$

exists.

A type of filtration-consistent nonlinear expectations, under a Brownian filtration, was introduced in [11], under the name "g-expectation" (see Section 2 for details). These g-expectations can be considered as a nonlinear extension of the well-known Girsanov transformations. It is a nonlinear mapping, but it preserves almost all other properties of the classical linear expectations. For more detailed views on this topic, we refer to [11], [4], [12], or [1] where some special cases are studied in depth, including the y-independent case, which will turn out to be the natural setting behind the present work. For applications of g-expectations to utility theory in economics, we refer to [3]. Note that the original motivation for studying g-expectations comes from the theory of expected utility, which is fundamentally important in economics. This theory is seriously challenged by the well-known Allais paradox and Ellsberg paradox. The notion of non-additive probability, or capacity, is then introduced to axiomatize the preferences which do not satisfy von Neumann-Morgenstern's axioms. Nonlinear expectations are another useful notion in this setting.

A very interesting problem is: is this notion of g-expectation general enough to represent all "enough regular" filtration-consistent nonlinear expectations? Answering this question is the main objective of the present paper. We will prove in Section 7 that if for a large enough $\mu > 0$, a nonlinear expectation $\mathcal{E}[\cdot]$ is dominated by the '$\mu|z|$-expectation' $\mathcal{E}^{\mu}[\cdot]$ (that is, the g-expectation defined by $g(z) = \mu|z|$), and if $\mathcal{E}[X + \eta|\mathcal{F}_t] = \mathcal{E}[X|\mathcal{F}_t] + \eta$ for all $\mathcal{F}_t$-measurable η, then, there exists a unique g such that $\mathcal{E}[\cdot]$ is the nonlinear expectation defined by g, still according to the definition of [11]. Our main tool will be the decomposition theorem for g-supermartingales proved in [12], developed here along a new version suitable for continuous $\mathcal{E}$-supermartingales, which we prove in Section 6. Basic definitions about g-expectations are given in Section 2. Sections 3 and 4 give the general framework of non-linear expectations, while Section 5 is devoted to martingales defined under non-linear expectations.

2. Basic notations and results about g-expectations

Let $(\Omega, \mathcal{F}, P)$ be a probability space and let $(B_t)_{t\geq 0}$, be a d-dimensional standard Brownian motion on this space such that $B_0 = 0$. Let $(\mathcal{F}_t)_{t\geq 0}$ be the filtration generated by this Brownian motion:

$$\mathcal{F}_t = \sigma\{B_s, s \in [0,t]\} \vee \mathcal{N},$$

where $\mathcal{N}$ is the set of all P-null subsets. Let $T > 0$ be a given number. Without loss of generality, in this paper, we always work in the space $(\Omega, \mathcal{F}_T, P)$, and only consider processes indexed by $t \in [0,T]$.

$L^2_{\mathcal{F}}(0,T;E)$ will denote the space of all E-valued, $(\mathcal{F}_t)_{t\le T}$-adapted processes ϕ such that

$$E\int_0^T |\phi(s)|^2 ds < \infty.$$

We will shorten this notation by putting $L^2_{\mathcal{F}}(0,T)= L^2_{\mathcal{F}}(0,T;\mathbf{R})$.

We first recall the notion of g-expectations, defined in [11], from which most basic material of this section is taken . We are given a function g:

$$g(\omega,t,y,z) : \Omega \times [0,T] \times R \times R^d \longmapsto R$$

satisfying

$$\begin{cases} \text{(i)}\ \ g(\cdot,y,z) \in L^2_{\mathcal{F}}(0,T), \quad \text{for each } y \in R,\ z \in R^d; \\ \text{(ii)}\ \ g(\cdot,y,0) \equiv 0, \quad \text{for each } y \in R; \\ \text{(iii)}\ \ \exists C_0, \mu > 0 \quad \text{such that} \quad \forall y_1,\ y_2 \in R, \quad z_1, z_2 \in R^d, \\ \qquad |g(t,y_1,z_1) - g(t,y_2,z_2)| \le C_0|y_1-y_2| + \mu|z_1-z_2|. \end{cases} \tag{2.1}$$

For each given $X \in L^2(\Omega, \mathcal{F}_T, P)$, let $(y^X(\cdot), z^X(\cdot)) \in L^2_{\mathcal{F}}(0,T;R^1 \times R^d)$ be the unique solution of the following backward stochastic differential equation (BSDE):

$$\begin{aligned} -dy^X(t) &= g(t,y^X(t),z^X(t))dt - z^X(t)dB_t, \\ y^X(T) &= X. \end{aligned}$$

(We refer to [9] for definitions and basic results about BSDEs; it will be enough here to remember that, provided that g satisfies (2.1), there is a unique pair $(y^X(\cdot), z^X(\cdot))$ of adapted processes solving the equation above).

Definition 2.1. (g-expectation) *The g-expectation $\mathcal{E}_g[\cdot] : L^2(\Omega,\mathcal{F},P) \longmapsto R$ is defined by*

$$\mathcal{E}_g[X] = y^X(0).$$

Definition 2.2. (conditional g-expectation) *The conditional g-expectation of X with respect to $\mathcal{F}_t$ is defined by*

$$\mathcal{E}_g[X|\mathcal{F}_t] = y^X(t).$$

If $\tau \le T$ is a stopping time, we define similarly

$$\mathcal{E}_g[X|\mathcal{F}_\tau] = y^X(\tau).$$

g-expectations and conditional g-expectations are in general not linear. However, they meet the following basic properties of usual expectations (see [11] for proofs):

Proposition 2.1.
(i) *(preserving of constants): For each constant c,* $\mathcal{E}_g[c] = c$*;*
(ii) *(monotonicity): If* $X_1 \geq X_2$ *a.s., then* $\mathcal{E}_g[X_1] \geq \mathcal{E}_g[X_2]$*;*
(iii) *(strict monotonicity): If* $X_1 \geq X_2$ *a.s., and* $P(X_1 > X_2) > 0$*, then*

$$\mathcal{E}_g[X_1] > \mathcal{E}_g[X_2].$$

Proposition 2.2.
(i) *If X is* $\mathcal{F}_t$*-measurable, then* $\mathcal{E}_g[X|\mathcal{F}_t] = X$*;*
(ii) *For all stopping times* τ *and* $\sigma \leq T$, $\mathcal{E}_g\Big[\mathcal{E}_g[X|\mathcal{F}_\tau]|\mathcal{F}_\sigma\Big] = \mathcal{E}_g[X|\mathcal{F}_{\tau\wedge\sigma}]$*;*
(iii) *If* $X_1 \geq X_2$ *a.s., then* $\mathcal{E}_g[X_1|\mathcal{F}_t] \geq \mathcal{E}_g[X_2|\mathcal{F}_t]$*; if, moreover,* $P(X_1 > X_2) > 0$*, then* $P\Big(\mathcal{E}_g[X_1|\mathcal{F}_t] > \mathcal{E}_g[X_2|\mathcal{F}_t]\Big) > 0$*;*
(iv) *For each* $B \in \mathcal{F}_t$, $\mathcal{E}_g[1_B X|\mathcal{F}_t] = 1_B\mathcal{E}_g[X|\mathcal{F}_t]$.

Proposition 2.3. $\mathcal{E}_g[X|\mathcal{F}_t]$ *is the unique random variable* η *in* $L^2(\Omega, \mathcal{F}_t, P)$ *such that*

$$\mathcal{E}_g[1_A X] = \mathcal{E}_g[1_A \eta] \quad \text{for all} \quad A \in \mathcal{F}_t. \tag{2.2}$$

Definition 2.3. **(g-martingales)** *A process* $(Y_t)_{0\leq t\leq T}$ *such that* $E[Y_t^2] < \infty$ *for all t is a g-martingale (resp. g-supermartingale, g-submartingale) iff*

$$\mathcal{E}_g[Y_t|\mathcal{F}_s] = Y_s, \quad (\text{resp.} \leq Y_s, \geq Y_s), \quad \forall s \leq t \leq T.$$

In the following proposition, $\|\cdot\|_p$ denotes the norm of $L^p(\Omega, \mathcal{F}_T, P)$.

Proposition 2.4. *Let* $g(\omega, t, y, z) : \Omega \times [0, T] \times R \times R^d \longmapsto R$ *be a given function satisfying (2.1). Then for every* ε *such that* $0 < \varepsilon \leq 1$*, there exists a constant* C_ε *such that, for every X,*

$$|\mathcal{E}_g[X]| \leq C_\varepsilon \|X\|_{1+\varepsilon} \tag{2.3}$$

Proof. This result comes from Girsanov's Theorem. Indeed, if we write $\mathcal{E}_g[X]$ as the initial value of the solution of a BSDE, it comes

$$\begin{aligned}\mathcal{E}_g[X] = Y_0 &= X + \int_0^T g(s, Y_s, Z_s)ds - \int_0^T Z_s dB_s \\ &= X - \int_0^T Z_s dW_s\end{aligned}$$

where we have set

$$W_t = -\int_0^t \frac{g(s, Y_s, Z_s)}{Z_s} ds + B_t. \tag{2.4}$$

(with the convention $0/0 = 0$, (2.4) makes sense thanks to (2.1).)

By Girsanov's Theorem, W is then a Q-Brownian motion, where Q is the probability mesure on $(\Omega, \mathcal{F}_T)$ defined by

$$L := \frac{dQ}{dP} = e^{\int_0^T \frac{g(s,Y_s,Z_s)}{Z_s} dBs - \frac{1}{2}\int_0^T \left|\frac{g(s,Y_s,Z_s)}{Z_s}\right|^2 ds}.$$

Then, as L is in every L^p $(1 \leq p < \infty)$, Hölder's inequality yields

$$\begin{aligned} |\mathcal{E}_g[X]| &= |E_Q(X)| \\ &= |E(LX)| \\ &\leq \|L\|_{\frac{1+\varepsilon}{\varepsilon}} \|X\|_{1+\varepsilon}, \end{aligned}$$

whence the claim. □

We shall often have to assume that

$$g \text{ does not depend on } y. \tag{2.5}$$

The importance of this special setting follows from the following lemma, which is proven in [1], subsection 4.2:

Lemma 2.1. *Let $g(\omega, t, y, z) : \Omega \times [0, T] \times R \times R^d \longmapsto R$ be a given function satisfying (2.1). Then*

$$\mathcal{E}_g[X + \eta|\mathcal{F}_t] = \mathcal{E}_g[X|\mathcal{F}_t] + \eta, \quad \forall \eta \in L^2(\Omega, \mathcal{F}_t, P) \tag{2.6}$$

if and only if g satisfies (2.5)

We will always write in the sequel $\mathcal{E}^\mu[X] \equiv \mathcal{E}_g[X]$ for $g = \mu|z|$ and $\mathcal{E}^{-\mu}[X] = \mathcal{E}_g[X]$ for $g \equiv -\mu|z|$. Note that

$$\forall C > 0, \quad \mathcal{E}^\mu[CX|\mathcal{F}_t] = C\mathcal{E}^\mu[X|\mathcal{F}_t] \tag{2.7}$$

and

$$\forall C < 0, \quad \mathcal{E}^\mu[CX|\mathcal{F}_t] = -C\mathcal{E}^\mu[-X|\mathcal{F}_t].$$

Next lemma will be useful later.

Lemma 2.2. *We have for all $\mu > 0$ and $X \in L^2(\Omega, \mathcal{F}_T, P)$,*

$$E\Big[\mathcal{E}^\mu[X|\mathcal{F}_t]^2\Big] \leq e^{\mu^2(T-t)} E[X^2].$$

Proof. By definition,

$$\mathcal{E}^\mu[X|\mathcal{F}_t] = X + \int_t^T \mu|Z_s|ds - \int_t^T Z_s dB_s.$$

Ito's formula gives

$$\mathcal{E}^\mu[X|\mathcal{F}_t]^2 = X^2 + \int_t^T 2\mu\mathcal{E}^\mu[X|\mathcal{F}_s]|Z_s|ds - 2\int_t^T \mathcal{E}^\mu[X|\mathcal{F}_s]Z_s dB_s - \int_t^T Z_s^2 ds.$$

Taking expectations, we deduce that

$$E\Big[\mathcal{E}^{\mu}[X|\mathcal{F}_t]^2\Big] = E[X^2] + \int_t^T E[2\mu\mathcal{E}^{\mu}[X|\mathcal{F}_s]|Z_s|]ds - \int_t^T E[Z_s^2]ds$$
$$\leq E[X^2] + \mu^2 \int_t^T E\Big[\mathcal{E}^{\mu}[X|\mathcal{F}_s]^2\Big]ds$$

(because of $2ab \leq a^2 + b^2$). The claim follows then immediately from Gronwall's inequality. □

Next Proposition of Doob-Meyer's type is taken from [12].

Proposition 2.5. *Assume that g satisfies (2.1) and (2.5), and that* (Y_t) *is a right-continuous g-supermartingale on* $[0, T]$ *such that* $E[\sup_{t\leq T} Y_t^2] < \infty$. *Then there exists a unique pair* (M, A) *of processes such that*

M is a g-martingale;

A is an increasing càdlàg process;

$$Y_t = M_t - A_t, \quad \forall t \in [0, T].$$

More specifically, Y is the unique solution of the BSDE

$$Y_t = Y_T + \int_t^T g(s, Z_s)ds + (A_T - A_t) - \int_t^T Z_s dB_s, \quad t \in [0, T].$$

We end this Section by giving an appropriate version of a downcrossing inequality given in [5] as Theorem 6.

Proposition 2.6. *Let g satisfy (2.1) and* (Y_t) *be a g-supermartingale on* $[0, T]$. *Let* $0 = t_0 < t_1 < \cdots < t_n = T$, *and* $a < b$ *be two constants. Then the number* $D_a^b[Y, n]$ *of downcrossings of* $[a, b]$ *by* $\{X_{t_j}\}_{0\leq j\leq n}$ *satisfies*

$$\mathcal{E}^{-\mu}\Big[D_a^b[Y, n]\Big] \leq \frac{1}{b-a}\mathcal{E}^{\mu}[Y_0 \wedge b - Y_T \wedge b].$$

Remark 2.1. *Contrarily to Theorem 6 in [5], we need not assume that Y is positive: indeed, as* $g(\cdot, y, 0) = 0$, *one checks easily that the proof given in [5] can be carried over for every g-supermartingale.*

Remark 2.2. *This proposition allows us to prove, by classical means, that a g-supermartingale* (Y_t) *admits a càdlàg modification if and only if the mapping* $t \to \mathcal{E}_g(Y_t)$ *is right-continuous. More details on this topic will be given in Lemma 5.2.*

3. Filtration-consistent nonlinear expectations

We now turn to the main object of this paper.

Definition 3.1. *A nonlinear expectation is a functional:*

$$\mathcal{E}[\cdot] : L^2(\Omega, \mathcal{F}_T, P) \longmapsto R$$

which satisfies the following properties:

(i) Strict monotonicity:

$$\begin{aligned} &\text{if} \quad X_1 \geq X_2 \quad a.s., \quad \mathcal{E}[X_1] \geq \mathcal{E}[X_2], \quad \text{and} \\ &\text{if} \quad X_1 \geq X_2 \quad a.s., \quad \mathcal{E}[X_1] = \mathcal{E}[X_2] \quad \Longleftrightarrow \quad X_1 = X_2 \quad a.s. \end{aligned}$$

(ii) preserving of constants:

$$\mathcal{E}[c] = c, \quad \text{for each constant } c.$$

Lemma 3.1. *Let $t \leq T$ and $\eta_1, \eta_2 \in L^2(\Omega, \mathcal{F}_t, P)$. If*

$$\mathcal{E}[\eta_1 1_A] = \mathcal{E}[\eta_2 1_A], \quad \forall A \in \mathcal{F}_t,$$

then

$$\eta_2 = \eta_1, \quad a.s. \tag{3.1}$$

Proof. We choose $A = \{\eta_1 \geq \eta_2\} \in \mathcal{F}_t$. Since $(\eta_1 - \eta_2)1_A \geq 0$ and $\mathcal{E}[\eta_1 1_A] = \mathcal{E}[\eta_2 1_A]$, it follows that $\eta_1 1_A = \eta_2 1_A$ a.s. Thus $\eta_2 \geq \eta_1$ a.s. With the same argument we can prove that $\eta_1 \geq \eta_2$ a.s. It follows that (3.1) holds. The proof is complete. □

Definition 3.2. *For the given filtration $(\mathcal{F}_t)_{0 \leq t \leq T}$, a nonlinear expectation is called $\mathcal{F}$-consistent expectation(or $\mathcal{F}$-expectation) if for each $X \in L^2(\Omega, \mathcal{F}_T, P)$ and for each $t \in [0, T]$ there exists a random variable $\eta \in L^2(\Omega, \mathcal{F}_t, P)$, such that*

$$\mathcal{E}[X 1_A] = \mathcal{E}[\eta 1_A], \quad \forall A \in \mathcal{F}_t.$$

From Lemma 3.1 above, such an η is uniquely defined. We denote it by $\eta = \mathcal{E}[X|\mathcal{F}_t]$. $\mathcal{E}[X|\mathcal{F}_t]$ is called the conditional $\mathcal{F}$-expectation of X under $\mathcal{F}_t$. It is characterized by

$$\mathcal{E}[X 1_A] = \mathcal{E}\Big[\mathcal{E}[X|\mathcal{F}_t] 1_A\Big], \quad \forall A \in \mathcal{F}_t. \tag{3.2}$$

Remark that, if f is a continuous, strictly increasing function on **R** such that $f(0) = 0$, $\mathcal{E}[X] = f^{-1}\Big(E[f(X)]\Big)$ defines an $\mathcal{F}$-expectation. Indeed, it is readily seen that $\mathcal{E}[X|\mathcal{F}_t] := f^{-1}\Big(E[f(X)|\mathcal{F}_t]\Big)$ satisfies (3.2).

The following lemma is obvious:

Lemma 3.2. *Let $g(\omega, t, y, z) : \Omega \times [0, T] \times R \times R^d \longmapsto R$ be a function satisfying (2.1), then the related g-expectation $\mathcal{E}_g[\cdot]$ is an $\mathcal{F}$-expectation.*

Lemma 3.3. *We have, for each* $0 \le s \le t \le T$,

$$\mathcal{E}\Big[\mathcal{E}[X|\mathcal{F}_t]|\mathcal{F}_s\Big] = \mathcal{E}[X|\mathcal{F}_s] \quad a.s. \tag{3.3}$$

In particular,

$$\mathcal{E}\Big[\mathcal{E}[X|\mathcal{F}_t]\Big] = \mathcal{E}[X]. \tag{3.4}$$

Proof. For each $A \in \mathcal{F}_s$ we have $A \in \mathcal{F}_t$. Thus

$$\begin{aligned}\mathcal{E}\Big[\mathcal{E}\big[\mathcal{E}[X|\mathcal{F}_t]|\mathcal{F}_s\big]1_A\Big] &= \mathcal{E}\Big[\mathcal{E}[X|\mathcal{F}_t]1_A\Big]\\ &= \mathcal{E}[X1_A]\\ &= \mathcal{E}\Big[\mathcal{E}[X|\mathcal{F}_s]1_A\Big]\end{aligned}$$

It follows from Lemma 3.1 that (3.3) holds.

(3.4) follows then easily from the fact that $\mathcal{F}_0$ is the trivial σ-algebra (since $B_0 = 0$). □

Lemma 3.4. *We have a.s.*

$$\mathcal{E}[X1_A|\mathcal{F}_t] = \mathcal{E}[X|\mathcal{F}_t]1_A, \quad \forall A \in \mathcal{F}_t. \tag{3.5}$$

Proof. For each $B \in \mathcal{F}_t$, we have

$$\begin{aligned}\mathcal{E}\Big[\mathcal{E}[X1_A|\mathcal{F}_t]1_B\Big] &= \mathcal{E}[X1_A1_B]\\ &= \mathcal{E}\Big[\mathcal{E}[X|\mathcal{F}_t]1_{A\cap B}\Big]\\ &= \mathcal{E}\Big[[\mathcal{E}[X|\mathcal{F}_t]1_A]1_B\Big].\end{aligned}$$

□

Lemma 3.5. *For any* $X, \zeta \in L^2(\Omega, \mathcal{F}_T, P)$ *and for each* $t \in [0, T]$ *and* $A \in \mathcal{F}_t$ *we have*

$$\mathcal{E}[X1_A + \zeta 1_{A^C}|\mathcal{F}_t] = \mathcal{E}[X|\mathcal{F}_t]1_A + \mathcal{E}[\zeta|\mathcal{F}_t]1_{A^C}$$

Proof. According to Lemma 3.4 above,

$$\begin{aligned}\mathcal{E}[X1_A + \zeta 1_{A^C}|\mathcal{F}_t] &= \mathcal{E}[X1_A + \zeta 1_{A^C}|\mathcal{F}_t]1_A + \mathcal{E}[X1_A + \zeta 1_{A^C}|\mathcal{F}_t]1_{A^C}\\ &= \mathcal{E}[(X1_A + \zeta 1_{A^C})1_A|\mathcal{F}_t] + \mathcal{E}[(X1_A + \zeta 1_{A^C})1_{A^C}|\mathcal{F}_t]\\ &= \mathcal{E}[X1_A|\mathcal{F}_t] + \mathcal{E}[\zeta 1_{A^C}|\mathcal{F}_t]\\ &= \mathcal{E}[X|\mathcal{F}_t]1_A + \mathcal{E}[\zeta|\mathcal{F}_t]1_{A^C}.\end{aligned}$$

□

Lemma 3.6. *For any* $X, Y \in L^2(\Omega, \mathcal{F}_T, P)$, *if* $X \le Y$ *a.s., then we have for each* $t \in [0, T]$,

$$\mathcal{E}[X|\mathcal{F}_t] \le \mathcal{E}[Y|\mathcal{F}_t] \quad a.s.$$

If moreover $\mathcal{E}[X|\mathcal{F}_t] = \mathcal{E}[Y|\mathcal{F}_t] \quad a.s.$ *for some* $t \ge 0$, *then* $X = Y \quad a.s.$

Proof. Define $X_t = \mathcal{E}[X|\mathcal{F}_t]$ and $Y_t = \mathcal{E}[Y|\mathcal{F}_t]$, and let $A \in \mathcal{F}_t$. Because of the monotonicity of $\mathcal{E}$, we have

$$\mathcal{E}(X_t 1_A) = \mathcal{E}(X 1_A) \leq \mathcal{E}(Y 1_A) = \mathcal{E}(Y_t 1_A).$$

Now, take $A = \{X_t > Y_t\}$. If $P(A) > 0$, the strict monotonicity of $\mathcal{E}$ implies that

$$\mathcal{E}(X_t 1_A) > \mathcal{E}(Y_t 1_A).$$

Comparing the two above inequalities, we conclude that $P(A) = 0$.

At last, assume that $\mathcal{E}[X|\mathcal{F}_t] = \mathcal{E}[Y|\mathcal{F}_t]$ a.s. for some $t \geq 0$. Then $\mathcal{E}[X] = \mathcal{E}[Y]$ and it follows again from the strict monotonicity of $\mathcal{E}$ that $X = Y$ a.s. □

4. $\mathcal{E}^\mu$-dominated $\mathcal{F}$-expectations

From now on, we will somewhat restrict the scope of our study. Recall that we have defined $\mathcal{E}^\mu[X] = \mathcal{E}_g[X]$ for $g \equiv \mu|z|$ and $\mathcal{E}^{-\mu}[X] = \mathcal{E}_g[X]$ for $g \equiv -\mu|z|$.

We will study now $\mathcal{F}$-expectations dominated by $\mathcal{E}^\mu$, for some large enough $\mu > 0$, according to the following

Definition 4.1. **($\mathcal{E}^\mu$-domination)** *Given $\mu > 0$, we say that an $\mathcal{F}$-expectation $\mathcal{E}$ is dominated by $\mathcal{E}^\mu$ if*

$$\mathcal{E}[X + \eta] - \mathcal{E}[X] \leq \mathcal{E}^\mu[\eta], \ \forall X, \eta \in L^2(\Omega, \mathcal{F}_T, P) \tag{4.1}$$

Remark 4.1. *For any g satisfying (2.1) and (2.5), the associated g-expectation is dominated by $\mathcal{E}^\mu$, where μ is the Lipschitz constant in (2.1).*

Lemma 4.1. *If $\mathcal{E}$ is dominated by $\mathcal{E}^\mu$ for some $\mu > 0$, then*

$$\mathcal{E}^{-\mu}[\eta] \leq \mathcal{E}[X + \eta] - \mathcal{E}[X] \leq \mathcal{E}^\mu[\eta]. \tag{4.2}$$

Proof. It is a simple consequence of

$$\mathcal{E}^{-\mu}[\eta|\mathcal{F}_t] = -\mathcal{E}^\mu[-\eta|\mathcal{F}_t].$$

□

Lemma 4.2. *If $\mathcal{E}$ is dominated by $\mathcal{E}^\mu$ for some $\mu > 0$, then $\mathcal{E}[\cdot]$ is, for all $\varepsilon \in]0, 1]$, a continuous operator on $L^{1+\varepsilon}(\Omega, \mathcal{F}_T, P)$ in the following sense:*

$$\exists C > 0, \quad |\mathcal{E}[\xi_1] - \mathcal{E}[\xi_2]| \leq C \, \|\xi_1 - \xi_2\|_{L^{1+\varepsilon}}, \quad \forall \xi_1, \xi_2 \in L^2(\Omega, \mathcal{F}_T, P). \tag{4.3}$$

Proof. The claim follows easily from Lemma 4.1 above and Proposition 2.4. □

Remark 4.2. *Note that Lemma 4.2 provides easy examples of $\mathcal{F}$-expectations that are not μ-dominated : just take $\mathcal{E}[X] = f^{-1}\Big(E[f(X)]\Big)$ with $f(x) = x^{\frac{5}{3}}$ and $\varepsilon = 1/2$ for instance.*

Until the end of the paper, we will deal with $\mathcal{F}$-expectations $\mathcal{E}[\cdot]$ also satisfying the following condition:

$$\mathcal{E}[X+\eta|\mathcal{F}_t] = \mathcal{E}[X|\mathcal{F}_t]+\eta, \quad \forall X \in L^2(\Omega,\mathcal{F}_T,P) \quad \text{and} \quad \eta \in L^2(\Omega,\mathcal{F}_t,P) \tag{4.4}$$

Recall that, when $\mathcal{E}[\cdot]$ is a g-expectation, (4.4) means that g satisfies (2.5). The meaning of this condition is obvious: the nonlinearity depends only on the risks. We observe also that an expectation $E_Q[\cdot]$ under a Girsanov transformation $\dfrac{dQ}{dP}$ satisfies this assumption.

Our first result connected to (4.4) will consist in deducing '$\mathcal{E}^\mu$-domination at time t' from (4.1). This will be correctly stated and proved in Lemma 4.4, but we need first to introduce some new notation.

For a given $\zeta \in L^2(\Omega,\mathcal{F}_T,P)$, we consider the mapping $\mathcal{E}_\zeta[\cdot]$ defined by

$$\mathcal{E}_\zeta[X] = \mathcal{E}[X+\zeta]-\mathcal{E}[\zeta] : L^2(\Omega,\mathcal{F}_T,P) \longmapsto R. \tag{4.5}$$

Lemma 4.3. *If $\mathcal{E}[\cdot]$ is an $\mathcal{F}$-expectation satisfying (4.1) and (4.4), then the mapping $\mathcal{E}_\zeta[\cdot]$ is also an $\mathcal{F}$-expectation satisfying (4.1) and (4.4). Its conditional expectation under $\mathcal{F}_t$ is*

$$\mathcal{E}_\zeta[X|\mathcal{F}_t] = \mathcal{E}[X+\zeta|\mathcal{F}_t]-\mathcal{E}[\zeta|\mathcal{F}_t]. \tag{4.6}$$

Proof. It is easily seen that $\mathcal{E}_\zeta[\cdot]$ is a nonlinear expectation.

We now prove that the notion $\mathcal{E}_\zeta[X|\mathcal{F}_t]$ defined in (4.6) is actually the conditional $\mathcal{F}$-expectation induced by $\mathcal{E}_\zeta[\cdot]$ under $\mathcal{F}_t$.

Indeed, put $G(X,\zeta,\mathcal{F}_t) = \mathcal{E}[X+\zeta|\mathcal{F}_t]-\mathcal{E}[\zeta|\mathcal{F}_t]$. We want to show that, for all $A \in \mathcal{F}_t$, $\mathcal{E}_\zeta(G(X,\zeta,\mathcal{F}_t)1_A) = \mathcal{E}_\zeta(X1_A)$. Computations give:

$$\begin{aligned}
\mathcal{E}_\zeta[G(X,\zeta,\mathcal{F}_t)] &= \mathcal{E}\Big[\mathcal{E}[X+\zeta|\mathcal{F}_t]-\mathcal{E}[\zeta|\mathcal{F}_t]+\zeta\Big|\mathcal{F}_t\Big]-\mathcal{E}[\zeta] \quad \text{(by (3.4))}\\
&= \mathcal{E}\Big[\mathcal{E}[X+\zeta|\mathcal{F}_t]-\mathcal{E}[\zeta|\mathcal{F}_t]+\mathcal{E}[\zeta|\mathcal{F}_t]\Big]-\mathcal{E}[\zeta] \quad \text{(by (4.4))}\\
&= \mathcal{E}\Big[\mathcal{E}[X+\zeta|\mathcal{F}_t]\Big]-\mathcal{E}[\zeta]\\
&= \mathcal{E}[X+\zeta]-\mathcal{E}[\zeta].
\end{aligned}$$

Thus we have

$$\mathcal{E}_\zeta[G(X,\zeta,\mathcal{F}_t)] = \mathcal{E}_\zeta[X], \quad \forall X. \tag{4.7}$$

Now for each $A \in \mathcal{F}_t$, we have,

$$\begin{aligned}
G(X1_A,\zeta,\mathcal{F}_t) &= \mathcal{E}[X1_A+\zeta 1_A+\zeta 1_{A^C}|\mathcal{F}_t]-\mathcal{E}[\zeta|\mathcal{F}_t]\\
&= \mathcal{E}[(X+\zeta)1_A+\zeta 1_{A^C}|\mathcal{F}_t]-\mathcal{E}[\zeta|\mathcal{F}_t]\\
&= \mathcal{E}[X+\zeta|\mathcal{F}_t]1_A+\mathcal{E}[\zeta|\mathcal{F}_t]1_{A^C}-\mathcal{E}[\zeta|\mathcal{F}_t]\\
&= (\mathcal{E}[X+\zeta|\mathcal{F}_t]-\mathcal{E}[\zeta|\mathcal{F}_t])1_A\\
&= G(X,\zeta,\mathcal{F}_t)1_A.
\end{aligned}$$

From this with (4.7) it follows that $\mathcal{E}_\zeta[X|\mathcal{F}_t]$ satisfies (3.2):

$$\mathcal{E}_\zeta[G(X,\zeta,\mathcal{F}_t)1_A] = \mathcal{E}_\zeta[G(X1_A,\zeta,\mathcal{F}_t)] = \mathcal{E}_\zeta[X1_A], \quad \forall A \in \mathcal{F}_t.$$

Thus $\mathcal{E}_\zeta[\cdot]$ is an $\mathcal{F}$-expectation with $\mathcal{E}_\zeta[\cdot|\mathcal{F}_t]$ given by (4.6).

We now check that (4.1) is satisfied. For each $X, \eta \in L^2(\Omega, \mathcal{F}_T, P)$,

$$\begin{aligned}\mathcal{E}_\zeta[X+\eta] - \mathcal{E}_\zeta[X] &= (\mathcal{E}[X+\eta+\zeta] - \mathcal{E}[\zeta]) - (\mathcal{E}[X+\zeta] - \mathcal{E}[\zeta]) \\ &= \mathcal{E}[X+\eta+\zeta] - \mathcal{E}[X+\zeta].\end{aligned}$$

Since $\mathcal{E}[\cdot]$ satisfies (4.1), $\mathcal{E}_\zeta[\cdot]$ satisfies

$$\mathcal{E}_\zeta[X+\eta] - \mathcal{E}_\zeta[X] \le \mathcal{E}^\mu[\eta].$$

Finally, let $\eta \in L^2(\Omega, \mathcal{F}_t, P)$; since $\mathcal{E}[\cdot]$ satisfies property (4.4), thus

$$\begin{aligned}\mathcal{E}_\zeta[X+\eta|\mathcal{F}_t] &= \mathcal{E}[X+\zeta|\mathcal{F}_t] - \mathcal{E}[\zeta|\mathcal{F}_t] + \eta \\ &= \mathcal{E}_\zeta[X|\mathcal{F}_t] + \eta.\end{aligned}$$

Thus $\mathcal{E}_\zeta[\cdot]$ also satisfies property (4.4). The proof is complete. □

Lemma 4.4. *Let $\mathcal{E}[\cdot]$ be an $\mathcal{F}$-expectation satisfying (4.1) and (4.4). Then, for each $t \le T$, we have a.s.*

$$\mathcal{E}^{-\mu}[X|\mathcal{F}_t] \le \mathcal{E}[X|\mathcal{F}_t] \le \mathcal{E}^\mu[X|\mathcal{F}_t], \ \forall X \in L^2(\Omega, \mathcal{F}_T, P).$$

This lemma is a simple consequence of the following one, whose proof is inspired by [1].

Lemma 4.5. *Let $\mathcal{E}_1[\cdot]$ and $\mathcal{E}_2[\cdot]$ be two $\mathcal{F}$-expectations satisfying (4.1) and (4.4). If*

$$\mathcal{E}_1[X] \le \mathcal{E}_2[X], \quad \forall X \in L^2(\Omega, \mathcal{F}_T, P),$$

then a.s. and for all t,

$$\mathcal{E}_1[X|\mathcal{F}_t] \le \mathcal{E}_2[X|\mathcal{F}_t], \quad \forall X \in L^2(\Omega, \mathcal{F}_T, P).$$

Proof. Indeed, for all $Y \in L^2(\mathcal{F}_T)$, we have by (4.4)

$$\begin{aligned}\mathcal{E}_1\Big[Y - \mathcal{E}_1[Y|\mathcal{F}_t]\Big] &= \mathcal{E}_1\Big[\mathcal{E}_1\Big[Y - \mathcal{E}_1[Y|\mathcal{F}_t]|\mathcal{F}_t\Big]\Big] \\ &= \mathcal{E}_1\Big[\mathcal{E}_1[Y|\mathcal{F}_t] - \mathcal{E}_1[Y|\mathcal{F}_t]\Big] \\ &= \mathcal{E}_1[0] = 0.\end{aligned}$$

On the other hand,

$$\begin{aligned}\mathcal{E}_1\Big[Y - \mathcal{E}_1[Y|\mathcal{F}_t]\Big] &\le \mathcal{E}_2\Big[Y - \mathcal{E}_1[Y|\mathcal{F}_t]\Big] \\ &= \mathcal{E}_2\Big[\mathcal{E}_2\big[Y - \mathcal{E}_1[Y|\mathcal{F}_t]|\mathcal{F}_t\big]\Big].\end{aligned}$$

Thus

$$\mathcal{E}_2\Big[\mathcal{E}_2[Y|\mathcal{F}_t] - \mathcal{E}_1[Y|\mathcal{F}_t]\Big] \ge 0, \quad \forall Y \in L^2(\mathcal{F}_T).$$

Now, for a fixed $X \in L^2(\mathcal{F}_T)$, we set $\eta = \mathcal{E}_2[X|\mathcal{F}_t] - \mathcal{E}_1[X|\mathcal{F}_t]$. Since

$$\begin{aligned} \eta 1_{\{\eta<0\}} &= 1_{\{\eta<0\}}\mathcal{E}_2[X|\mathcal{F}_t] - 1_{\{\eta<0\}}\mathcal{E}_1[X|\mathcal{F}_t] \\ &= \mathcal{E}_2[X1_{\{\eta<0\}}|\mathcal{F}_t] - \mathcal{E}_1[X1_{\{\eta<0\}}|\mathcal{F}_t], \end{aligned}$$

we have then

$$\mathcal{E}_2[\eta 1_{\{\eta<0\}}] = 0.$$

But since $\eta 1_{\{\eta<0\}} \leq 0$, it follows from the strict monotonicity of $\mathcal{E}_2[\cdot]$ that $\eta 1_{\{\eta<0\}} = 0$ a.s. Thus

$$\mathcal{E}_2[X|\mathcal{F}_t] - \mathcal{E}_1[X|\mathcal{F}_t] \geq 0 \quad \text{a.s.}$$

The proof is complete. □

Lemma 4.6. *If $\mathcal{E}$ meets (4.1) and (4.4), there exists a positive constant C such that, for all X and η in $L^2(\Omega, \mathcal{F}_T, P)$, and for all $t \geq 0$,*

$$\mathcal{E}\Big[\mathcal{E}[X+\eta|\mathcal{F}_t] - \mathcal{E}[X|\mathcal{F}_t]\Big] \leq C\|\eta\|_{L^2}.$$

Proof. Indeed, Lemmas 4.3 and 4.4 above imply that

$$\begin{aligned} \mathcal{E}\Big[\mathcal{E}[X+\eta|\mathcal{F}_t] - \mathcal{E}[X|\mathcal{F}_t]\Big] &= \mathcal{E}\Big[\mathcal{E}_X[\eta|\mathcal{F}_t]\Big] \\ &\leq \mathcal{E}\Big[\mathcal{E}^\mu[\eta|\mathcal{F}_t]\Big] \\ &\leq \mathcal{E}^\mu\Big[\mathcal{E}^\mu[\eta|\mathcal{F}_t]\Big] = \mathcal{E}^\mu[\eta] \leq C\|\eta\|_{L^2}. \end{aligned}$$

(Last equality coming from Lemma 4.2) □

5. $\mathcal{F}$-Martingales

Henceforth, we will always assume that $\mathcal{E}$ is an $\mathcal{F}$-expectation satisfying (4.1) for some $\mu > 0$, and (4.4) as well.

Definition 5.1. *A process $(X_t)_{t\in[0,T]} \in L^2_{\mathcal{F}}(0,T)$ is called an $\mathcal{E}$-martingale (resp. $\mathcal{E}$-supermartingale, -submartingale) if for each $0 \leq s \leq t \leq T$*

$$X_s = \mathcal{E}[X_t|\mathcal{F}_s], \ (\textit{resp.} \ \geq \mathcal{E}[X_t|\mathcal{F}_s], \ \leq \mathcal{E}[X_t|\mathcal{F}_s]).$$

Lemma 5.1. *An $\mathcal{E}^\mu$-supermartingale (ξ_t) is both an $\mathcal{E}$-supermartingale and $\mathcal{E}^{-\mu}$-supermartingale. An $\mathcal{E}^{-\mu}$-submartingale (ξ_t) is both an $\mathcal{E}$- and $\mathcal{E}^\mu$-submartingale. An $\mathcal{E}$-martingale (ξ_t) is an $\mathcal{E}^{-\mu}$-supermartingale and an $\mathcal{E}^\mu$-submartingale.*

Proof. It comes simply from the fact that, for each $0 \leq s \leq t \leq T$,

$$\mathcal{E}^{-\mu}[\xi_t|\mathcal{F}_s] \leq \mathcal{E}[\xi_t|\mathcal{F}_s] \leq \mathcal{E}^\mu[\xi_t|\mathcal{F}_s].$$

□

We will now prove through two lemmas that every $\mathcal{E}$-martingale admits continuous paths.

Lemma 5.2. *For each $X \in L^2(\Omega, \mathcal{F}_T, P)$ the process $\mathcal{E}[X|\mathcal{F}_t]$, $t \in [0, T]$ admits a unique modification with a.s. càdlàg paths.*

Proof. We can deduce from Lemma 5.1 that the process $\mathcal{E}[X|\mathcal{F}_t]$, $t \in [0, T]$, is an $\mathcal{E}^{-\mu}$-supermartingale. Hence we can apply the downcrossing inequality recalled in Proposition 2.6.

This dowcrossing equality tells us that $\mathcal{E}[X|\mathcal{F}_t]$, $t \in [0, T]$ has P-a.s. finitely many downcrossings of every interval $[a, b]$ with rational $a < b$. By classical methods, this imply the almost sure existence of left and right limits for the paths of $\mathcal{E}[X|\mathcal{F}_\cdot]$.

Define now $Y_t = \lim_{\substack{s \searrow t \\ s \in \mathbf{Q} \cap [0,T]}} \mathcal{E}[X|\mathcal{F}_s]$, whose existence a.s. has just been proved. Taking A in $\mathcal{F}_t$, we have that

$$Y_t 1_A = \lim_{\substack{s \searrow t \\ s \in \mathbf{Q} \cap [0,T]}} \mathcal{E}[X|\mathcal{F}_s] 1_A,$$

the above limit being taken in L^2. From Lemma 4.2, it follows that

$$\mathcal{E}[Y_t 1_A] = \lim_{\substack{s \searrow t \\ s \in \mathbf{Q} \cap [0,T]}} \mathcal{E}\Big[\mathcal{E}[X|\mathcal{F}_s] 1_A\Big].$$

But

$$\begin{aligned} \mathcal{E}\Big[\mathcal{E}[X|\mathcal{F}_s] 1_A\Big] &= \mathcal{E}\Big[\mathcal{E}\Big[\mathcal{E}[X|\mathcal{F}_s] 1_A\Big]\Big|\mathcal{F}_t\Big] \\ &= \mathcal{E}\Big[1_A \mathcal{E}\Big[\mathcal{E}[X|\mathcal{F}_s]\Big|\mathcal{F}_t\Big]\Big] \\ &= \mathcal{E}\Big[1_A \mathcal{E}[X|\mathcal{F}_t]\Big]. \end{aligned}$$

It follows that a.s. $Y_t = \mathcal{E}[X|\mathcal{F}_t]$.

Now it is again classical to prove, using the existence of left and right limits, that the process Y defined above is a càdlàg modification of $\mathcal{E}[X|\mathcal{F}_t]$, $t \in [0, T]$, and the lemma is proved. □

While this result shows that any $\mathcal{E}$-martingale admits càdlàg paths, next one will show that these paths are indeed continuous ones.

Lemma 5.3. *For each $X \in L^2(\Omega, \mathcal{F}_T, P)$, let*

$$y(t) = \mathcal{E}[X|\mathcal{F}_t].$$

Then there exists a pair $(g(\cdot), z(\cdot)) \in L^2_{\mathcal{F}}(0, T; R \times R^d)$ with

$$|g(t)| \leq \mu |z(t)| \tag{5.1}$$

such that

$$y(t) = X + \int_t^T g(s) ds - \int_t^T z(s) dB_s. \tag{5.2}$$

In particular, y admits a.s. continuous paths.

Furthermore, take $X' \in L^2(\Omega, \mathcal{F}_T, P)$*, put* $y'(t) = \mathcal{E}[X'|\mathcal{F}_t]$*, and let* $(g'(\cdot), z'(\cdot)) \in L^2_{\mathcal{F}}(0, T; R \times R^d)$ *be the corresponding pair. Then we have*

$$|g(t) - g'(t)| \leq \mu|z(t) - z'(t)| \tag{5.3}$$

Proof. Since

$$y(t) = \mathcal{E}[X|\mathcal{F}_t], \quad 0 \leq t \leq T,$$

is an $\mathcal{E}$-martingale, and since it is càdlàg, it is a right-continous $\mathcal{E}^\mu$-submartingale (resp. $\mathcal{E}^{-\mu}$-supermartingale) and we know from the g-supermartingale decomposition theorem (Proposition 2.5) that there exist (z^μ, A^μ) and $(z^{-\mu}, A^{-\mu})$ in $L^2_{\mathcal{F}}([0, T]; R \times R^d)$ with A^μ and $A^{-\mu}$ càdlàg and increasing such that $A^\mu(0) = 0$, $A^{-\mu}(0) = 0$ and

$$y(t) = y(T) + \int_t^T \mu|z^\mu(s)|ds - A^\mu(T) + A^\mu(t) - \int_t^T z^\mu(s)dB_s.$$

$$y(t) = y(T) - \int_t^T \mu|z^{-\mu}(s)|ds + A^{-\mu}(T) - A^{-\mu}(t) - \int_t^T z^{-\mu}(s)dB_s.$$

Hence,

$$z^\mu(t) \equiv z^{-\mu}(t),$$
$$-\mu|z^\mu(t)|dt + dA^\mu(t) \equiv \mu|z^\mu(t)|dt - dA^{-\mu}(t),$$

whence

$$2\mu|z^\mu(t)|dt \equiv dA^\mu(t) + dA^{-\mu}(t).$$

It follows that A^μ and $A^{-\mu}$ are both absolutely continuous and we can write:

$$dA^\mu(t) = a^\mu(t)dt, \quad dA^{-\mu}(t) = a^{-\mu}(t)dt$$

with

$$0 \leq a^\mu(t), \quad 0 \leq a^{-\mu}(t).$$

We also have

$$a^\mu(t) + a^{-\mu}(t) \equiv 2\mu|z^\mu(t)|,$$

so, if we define

$$z(t) = z^\mu(t)$$
$$g(t) = \mu|z(t)| - a^\mu(t),$$

we get (5.2) and (5.1).

Now, we prove (5.3). We have

$$\begin{aligned} y(t) - y'(t) &= \mathcal{E}[X|\mathcal{F}_t] - \mathcal{E}[X'|\mathcal{F}_t] \\ &= \mathcal{E}[X - X' + X'|\mathcal{F}_t] - \mathcal{E}[X'|\mathcal{F}_t] \\ &= \mathcal{E}_{X'}[X - X'|\mathcal{F}_t] \end{aligned}$$

Recall (Lemma 4.3 in Section 4) that $\mathcal{E}_{X'}[\cdot]$ is another $\mathcal{F}$-expectation satisfying (4.1) and (4.4). Thus there also exists a pair $(\tilde{g}(\cdot), \tilde{z}(\cdot)) \in L^2_{\mathcal{F}}([0, T]; R \times R^d)$ with

$$|\tilde{g}(t)| \leq \mu|\tilde{z}(t)| \tag{5.4}$$

such that the $\mathcal{E}_{X'}$-martingale $y(t) - y'(t)$ satisfies

$$y(t) - y'(t) = X - X' + \int_t^T \tilde{g}(s)ds - \int_t^T \tilde{z}(s)dB_s.$$

On the other hand, we have

$$y(t) - y'(t) = X - X' + \int_t^T [g(s) - g'(s)]ds - \int_t^T [z(s) - z'(s)]dB_s.$$

It follows then that

$$\tilde{g}(t) \equiv g(t) - g'(t), \quad \text{and} \quad \tilde{z}(t) \equiv z(t) - z'(t).$$

This with (5.4) yields (5.3). The proof is complete. □

Let us note the following easy consequence of Lemma 5.3 :

Lemma 5.4. *Let $\mathcal{E}[\cdot]$ be an $\mathcal{F}$-expectation satisfying (4.1) and (4.4). Then for each $X \in L^2(\Omega, \mathcal{F}_T, P)$ and $g \in L^2_{\mathcal{F}}(0, T)$ the process $\mathcal{E}[X + \int_t^T g(s)ds|\mathcal{F}_t]$, $t \in [0, T]$ is a.s. continuous.*

Proof. Indeed, we can write

$$\begin{aligned}\mathcal{E}[X + \int_t^T g(s)ds|\mathcal{F}_t] &= \mathcal{E}[X + \int_0^T g(s)ds - \int_0^t g(s)ds|\mathcal{F}_t] \\ &= \mathcal{E}[X + \int_0^T g(s)ds|\mathcal{F}_t] - \int_0^t g(s)ds\end{aligned}$$

because of (4.4). The claim follows then easily from Lemma 5.3. □

To end this section, it is useful to remark that, by the same way as in Lemma 5.2, we can prove the following optimal sampling theorem for $\mathcal{E}$-martingales (resp. supermartingales, submartingales):

Lemma 5.5. *Let the process X be an $\mathcal{E}$-martingales (resp. supermartingale, submartingale), and let σ and τ be two stopping times such that $\sigma \leq \tau$ a.s. Then*

$$\mathcal{E}[X_\tau|\mathcal{F}_\sigma] = X_\sigma \quad (\textit{resp.} \leq, \geq).$$

6. $\mathcal{E}$-Supermartingale decompositions

Here again, $\mathcal{E}$ denotes an $\mathcal{F}$-expectation satisfying (4.1) for some $\mu > 0$, and (4.4) as well.

Let a function f be given

$$f(\omega, t, y) : \Omega \times [0, T] \times R \longmapsto R$$

satisfying, for some constant $C_1 > 0$,

$$\begin{cases} \text{(i)} \ \ f(\cdot, y) \in L^2_{\mathcal{F}}(0, T), \quad \text{for each } y \in R; \\ \text{(ii)} \ \ |f(t, y_1) - f(t, y_2)| \le C_1|y_1 - y_2|, \quad \forall y_1, \ y_2 \in R. \end{cases} \tag{6.1}$$

For a given terminal data $X \in L^2(\Omega, \mathcal{F}_T, P)$, we consider the following type of equation:

$$Y(t) = \mathcal{E}[X + \int_t^T f(s, Y(s))ds|\mathcal{F}_t] \tag{6.2}$$

Theorem 6.1. *We assume (6.1). Then there exists a unique process $Y(\cdot)$ solution of (6.2). Moreover, $Y(\cdot)$ admits continuous paths.*

The proof of this theorem is based on the following lemma.

Lemma 6.1. *Define a mapping $\Phi(y(\cdot)) : L^2_{\mathcal{F}}(0, T) \longmapsto L^2_{\mathcal{F}}(0, T)$ by*

$$\Phi(y(\cdot))(t) = \mathcal{E}\Big[X + \int_t^T f(s, y(s))ds|\mathcal{F}_t\Big].$$

Then we have for all t:

$$E\Big[|\Phi(y_1(\cdot))(t) - \Phi(y_2(\cdot))(t)|^2\Big] \le C_1^2 e^{\mu^2 T}(T - t)E\Big[\int_t^T |y_1(s) - y_2(s)|^2 ds\Big].$$

Proof. Let $Y_1(t) = \Phi(y_1(\cdot))(t)$, $Y_2(t) = \Phi(y_2(\cdot))(t)$: then

$$Y_1(t) - Y_2(t) = \mathcal{E}\Big[X + \int_t^T f(s, y_1(s))ds|\mathcal{F}_t\Big] - \mathcal{E}\Big[X + \int_t^T f(s, y_2(s))ds|\mathcal{F}_t\Big].$$

Using Lemma 4.4, basic properties of $\mathcal{E}^\mu$ (including its monotonicity) and (2.7), we get

$$\begin{aligned} |Y_1(t) - Y_2(t)| &\le \Big|\mathcal{E}^\mu\Big[\int_t^T [f(s, y_1(s)) - f(s, y_2(s))]ds|\mathcal{F}_t\Big]\Big| \\ &\quad \vee \Big|\mathcal{E}^\mu\Big[-\int_t^T [f(s, y_1(s)) - f(s, y_2(s))]ds|\mathcal{F}_t\Big]\Big| \\ &\le \mathcal{E}^\mu\Big[\Big|\int_t^T [f(s, y_1(s)) - f(s, y_2(s))]ds\Big||\mathcal{F}_t\Big] \\ &\le \mathcal{E}^\mu\Big[\int_t^T |[f(s, y_1(s)) - f(s, y_2(s))]|ds|\mathcal{F}_t\Big] \\ &\le C_1\mathcal{E}^\mu\Big[\int_t^T |y_1(s) - y_2(s)|ds|\mathcal{F}_t\Big] \quad \text{because of (2.7)} \end{aligned}$$

Using Lemma 2.2, it follows that

$$E\Big[|Y_1(t)-Y_2(t)|^2\Big] \le C_1^2 E\Big[\mathcal{E}^{\mu}[\int_t^T |y_1(s)-y_2(s)|ds|\mathcal{F}_t]^2\Big]$$
$$\le C_1^2 e^{\mu^2(T-t)} E\Big[\int_t^T |y_1(s)-y_2(s)|ds\Big]^2$$
$$\le C_1^2 e^{\mu^2 T}(T-t) E\Big[\int_t^T |y_1(s)-y_2(s)|^2 ds\Big].$$

This concludes the proof of the Lemma. □

Back to the proof of the Theorem, we deduce from the previous lemma that

$$E\int_t^T \Big[|Y_1(s)-Y_2(s)|^2\Big]ds \le C_1^2 e^{\mu^2 T}(T-t)^2 E\Big[\int_t^T |y_1(s)-y_2(s)|^2 ds\Big].$$

Now, choose $\eta > 0$ such that $C_1^2 e^{\mu^2 T}\eta^2 < 1$: Φ induces a contraction mapping from $L^2_{\mathcal{F}}(T-\eta, T)$ into itself, which therefore admits a fixed point. This fixed point is a solution of (6.2) on $[T-\eta, T]$.

Let us denote it by Y.

Now, for $t \le T-\eta$, we define the mapping

$$\Psi(y(\cdot))(t) = \mathcal{E}\Big[X + \int_{T-\eta}^T f(s, Y(s))ds + \int_t^{T-\eta} f(s, y(s))ds|\mathcal{F}_t\Big].$$

Putting $X' = X + \int_{T-\eta}^T f(s, Y(s))ds$, we can write the same computations as in Lemma 6.1 to deduce

$$E\Big[|\Psi(y_1(\cdot))(t)-\Psi(y_2(\cdot))(t)|^2\Big] \le C_1^2 e^{\mu^2 T}(T-\eta-t) E\Big[\int_t^{T-\eta} |y_1(s)-y_2(s)|^2 ds\Big].$$

It suffices then to write down again the above reasonning to conclude the existence of a solution of (6.2) on $t \in [T-2\eta, T-\eta]$.

Since η is fixed, by iterating this method we conclude the existence of a solution of (6.2) on the whole interval $[0, T]$.

We just have now to prove the uniqueness of the solution of (6.2). So, let Y_1 and Y_2 be two solutions. Lemma 6.1 gives then

$$E\Big[|Y_1(t)-Y_2(t)|^2\Big] \le C_1^2 e^{\mu^2 T} T E\Big[\int_t^T |Y_1(s)-Y_2(s)|^2 ds\Big]$$

and Gronwall's inequality shows then that $Y_1 = Y_2$ a.s.

At last, Lemma 5.4 proves that the solution of (6.2) admits continuous paths, and the proof is complete. □

Theorem 6.2. (Comparison Theorem). *Let Y be the solution of (6.2) and let Y' be the solution of*

$$Y'(t) = \mathcal{E}[X' + \int_t^T [f(s, Y'(s)) + \phi(s)]ds|\mathcal{F}_t]$$

where $X' \in L^2(\Omega, \mathcal{F}_T, P)$ and $\phi \in L^2_{\mathcal{F}}(0, T)$. If

$$X' \geq X, \quad \phi(t) \geq 0, \quad dP \times dt\text{-a.e.,} \tag{6.3}$$

then we have

$$Y'(t) \geq Y(t), \quad dP \times dt\text{-a.e.} \tag{6.4}$$

(6.4) becomes equality if and only if (6.3) become equalities.

Proof. We begin with the case $\phi(t) \equiv 0$. For each $\delta > 0$, we define

$$\tau_1^\delta = \inf\{t \geq 0; Y'(t) \leq Y(t) - \delta\} \wedge T.$$

It is clear that if, for all $\delta > 0$, $\tau_1^\delta = T$ a.s., then (6.4) holds. Now if for some $\delta > 0$ we have

$$P(A^\delta) > 0,$$

where

$$A^\delta = \{\tau_1^\delta < T\} \in \mathcal{F}_{\tau_1^\delta},$$

we then can define

$$\tau_2 = \inf\{t \geq \tau_1^\delta; Y'(t) \geq Y(t)\}.$$

Since $Y'(T) = X' \geq X = Y(T)$, thus $\tau_2 \leq T$ and $1_{A^\delta} Y'(\tau_2) = 1_{A^\delta} Y(\tau_2)$. It follows that, for $t \in [\tau_1^\delta, \tau_2]$, we have

$$1_{A^\delta} Y(t) = \mathcal{E}[1_{A^\delta} Y(\tau_2) + \int_t^{\tau_2} 1_{A^\delta} f(s, 1_{A^\delta} Y(s))ds|\mathcal{F}_t]$$

and

$$1_{A^\delta} Y'(t) = \mathcal{E}[1_{A^\delta} Y(\tau_2) + \int_t^{\tau_2} 1_{A^\delta} f(s, 1_{A^\delta} Y'(s))ds|\mathcal{F}_t]$$

But, according to Theorem 6.1, the solutions of the above equations must coincide. This implies that $Y'(\tau_1^\delta) 1_{A^\delta} = Y(\tau_1^\delta) 1_{A^\delta}$ a.s., which contradicts the assumption $P(A^\delta) > 0$.

In order to prove the general case $\phi(s) \geq 0$, we define for $n = 1, 2, 3, \cdots$, $Y^{(n)}(\cdot)$ to be the solution of

$$Y^{(n)}(t) = \mathcal{E}\left[[X' + \int_{\frac{iT}{n}}^T \phi(s)ds] + \int_t^T f(s, Y^{(n)}(s))ds|\mathcal{F}_t\right],$$
$$\text{for } t \in \left[\frac{iT}{n}, \frac{(i+1)T}{n}\right[,$$

and $Y^{(n)}(T) = X'$.

Note that, due to (4.4), we can write

$$Y^{(n)}(t) = \mathcal{E}\left[[Y^{(n)}\Big(\frac{(i+1)T}{n}\Big) + \int_{\frac{iT}{n}}^{\frac{(i+1)T}{n}} \phi(s)ds] + \int_t^{\frac{(i+1)T}{n}} f(s, Y^{(n)}(s))ds|\mathcal{F}_t\right],$$

for $t \in \Big[\frac{iT}{n}, \frac{(i+1)T}{n}\Big[$.

For $t \in [(n-1)T/n, T]$, $Y^{(n)}(t)$ coincides with the solution of (6.2) with terminal data $X' + \int_{\frac{(n-1)T}{n}}^{T} \phi(s)ds$ and generator f. Since $X' + \int_{\frac{(n-1)T}{n}}^{T} \phi(s)ds \geq X' \geq X$, it follows from the first part of the proof that $Y^{(n)}(t) \geq Y(t)$ as soon as $t \in [(n-1)T/n, T[$. In particular, $Y^{(n)}\Big(\frac{(n-1)T}{n}\Big) \geq Y\Big(\frac{(n-1)T}{n}\Big)$. Then following the same way as above, we prove that $Y^{(n)}(t) \geq Y(t)$ as soon as $t \in [(n-2)T/n, (n-1)T/n[$, and an obvious iteration gives $Y^{(n)}(t) \geq Y(t)$ for all $t \in [0, T]$.

In order to prove that $Y'(t) \geq Y(t)$, it is now sufficient to show the convergence of the sequence $(Y^{(n)})$ to Y'. A computation analogous to the proof of Lemma 6.1 shows that, for fixed $t \in \Big[\frac{iT}{n}, \frac{(i+1)T}{n}\Big[$ and an appropriate constant C,

$$E\Big[|Y^{(n)}(t) - Y'(t)|^2\Big] \leq CE\Big[\int_{\frac{iT}{n}}^{t} |\phi(s)|ds + C_1 \int_t^T |Y^{(n)}(s) - Y'(s)|ds\Big]^2.$$

But

$$\begin{aligned}\Big[\int_{\frac{iT}{n}}^{t} &|\phi(s)|ds + C_1 \int_t^T |Y^{(n)}(s) - Y'(s)|ds\Big]^2 \\ &\leq 2\Big[\int_{\frac{iT}{n}}^{t} |\phi(s)|ds\Big]^2 + 2C_1^2(T-t)\int_t^T |Y^{(n)}(s) - Y'(s)|^2 ds.\end{aligned}$$

Using now Schwarz's inequality we deduce that, for all $t \in [0, T[$,

$$\begin{aligned}E\Big[|Y^{(n)}(t) - Y'(t)|^2\Big] &\leq 2C\frac{T}{n}E\Big[\int_0^T |\phi(s)|^2 ds\Big] \\ &\quad + 2CC_1^2 T E\Big[\int_t^T |Y^{(n)}(s) - Y'(s)|^2 ds\Big]. \qquad (6.5)\end{aligned}$$

Gronwall's Lemma applied to (6.5) shows then that $E\Big[|Y^{(n)}(t) - Y'(t)|^2\Big] \to 0$, and finally $Y'(t) \geq Y(t)$.

Finally, we investigate possible equality in (6.4).

If $Y(t) \equiv Y'(t)$, the continuity of both Y and Y' shows that $X = X'$ a.s. Then from $Y(0) = Y'(0)$, i.e.

$$\mathcal{E}\Big[X + \int_0^T f(s, Y(s))ds\Big] = \mathcal{E}\Big[X + \int_0^T f(s, Y(s))ds + \int_0^T \phi(s)ds\Big]$$

it follows from the strict monotonicity of $\mathcal{E}$ that $\int_0^T \phi(s)ds = 0$, whence $\phi = 0$ $dt \times dP$ a.e. and the end of the proof. □

Our next result generalizes the decomposition theorem for g-supermartingales proved in [12] to continuous $\mathcal{E}$-supermartingales. The proof uses mainly arguments from [12].

Theorem 6.3. (Decomposition theorem for $\mathcal{E}$-supermartingales) *Let $\mathcal{E}[\cdot]$ be an $\mathcal{F}$-expectation satisfying (4.1) and (4.4), and let (Y_t) be a related continuous $\mathcal{E}$-supermartingale with*

$$E[\sup_{t\in[0,T]} |Y(t)|^2] < \infty.$$

Then there exists an $A(\cdot) \in L^2_{\mathcal{F}}(0,T;R)$ such that $A(\cdot)$ is continuous and increasing with $A(0) = 0$, and such that $Y(t) + A(t)$ is an $\mathcal{E}$-martingale.

Proof. For $n \geq 1$, we define $y^{(n)}(\cdot)$, solution of the following BSDE:

$$y^{(n)}(t) = \mathcal{E}[Y(T) + \int_t^T n(Y(s) - y^{(n)}(s))ds|\mathcal{F}_t]$$

We have then the following

Lemma 6.2. *We have, for each t and $n \geq 1$,*

$$Y(t) \geq y^{(n)}(t).$$

Proof. For $\delta > 0$ and a given integer n, let us define

$$\sigma^{n,\delta} := \inf\{t;\ \ y^{(n)}(t) \geq Y(t) + \delta\} \wedge T.$$

If $P(\sigma^{n,\delta} < T) = 0$ for all n and δ, then the proof is done. If it is not the case, then there exist $\delta > 0$ and a positive integer n such that $P(\sigma^{n,\delta} < T) > 0$. We can then define the following stopping times

$$\tau^{n,\delta} := \inf\{t \geq \sigma^{n,\delta};\ \ y^{(n)}(t) \leq Y(t)\}.$$

It is clear that $\sigma^{n,\delta} \leq \tau^{n,\delta} \leq T$. Because of Theorem 6.1, $Y(t) - y^{(n)}(t)$ is continuous, hence we have

$$\begin{aligned} &\text{(i) } y^{(n)}(\sigma^{n,\delta}) \geq Y(\sigma^{n,\delta}) + \delta \text{ on } \{\sigma^{n,\delta} < T\}; \\ &\text{(ii) } y^{(n)}(\tau^{n,\delta}) \leq Y(\tau^{n,\delta}) \end{aligned} \tag{6.6}$$

But since $(Y(s) - y^{(n)}(s)) \leq 0$ on $[\sigma^{n,\delta}, \tau^{n,\delta}]$, by monotonicity of $\mathcal{E}$,

$$\begin{aligned} y^{(n)}(\sigma^{n,\delta}) &= \mathcal{E}[y^{(n)}(\tau^{n,\delta}) + \int_{\sigma^{n,\delta}}^{\tau^{n,\delta}} n(Y(s) - y^{(n)}(s))ds|\mathcal{F}_{\sigma^{n,\delta}}] \\ &\leq \mathcal{E}[y^{(n)}(\tau^{n,\delta})|\mathcal{F}_{\sigma^{n,\delta}}] \\ &\leq \mathcal{E}[Y(\tau^{n,\delta})|\mathcal{F}_{\sigma^{n,\delta}}] \end{aligned}$$

Finally, since Y is an $\mathcal{E}$-supermartingale, Lemma 5.5 gives us that

$$Y(\sigma^{n,\delta}) \geq y^{(n)}(\sigma^{n,\delta}).$$

But as $P(\sigma^{n,\delta} < T) > 0$, this is contrary to (6.6). The proof is complete. $\square$

Lemma 6.2 with Theorem 6.2 above imply that $y^{(n)}(\cdot)$ monotonically converges to some $Y^0(\cdot) \leq Y(\cdot)$. Indeed, writing $\phi(t) = Y(t) - y^{(n+1)}(t) \geq 0$ shows that $(y^{(n)}(\cdot))$ is an increasing sequence of functions.

Observe then that $y^{(n)}(t) + \int_0^t n(Y(s) - y^{(n)}(s))ds$ is an $\mathcal{E}$-martingale. By Lemma 5.3, there exists $(g^{(n)}, z^{(n)}) \in L^2_{\mathcal{F}}(0, T; R \times R^d)$ with

$$|g^{(n)}(s)| \leq \mu |z^{(n)}(s)|, \quad n = 1, 2, \cdots, \tag{6.7}$$

such that

$$\begin{aligned} y^{(n)}(t) + \int_0^t n(Y(s) - y^{(n)}(s))ds = y^{(n)}(T) + \int_0^T n(Y(s) - y^{(n)}(s))ds \\ + \int_t^T g^{(n)}(s)ds - \int_t^T z^{(n)}(s)dB_s, \end{aligned}$$

hence, as $y^{(n)}(T) = Y(T)$,

$$y^{(n)}(t) = Y(T) + \int_t^T [g^{(n)}(s) + n(Y(s) - y^{(n)}(s))]ds - \int_t^T z^{(n)}(s)dB_s. \tag{6.8}$$

(5.3) also tells us that

$$|g^{(n)}(s) - g^{(m)}(s)| \leq \mu |z^{(n)}(s) - z^{(m)}(s)|, \quad n, m = 1, 2, \cdots \tag{6.9}$$

Let us denote, for each $n = 1, 2, \cdots$,

$$A^{(n)}(t) = n \int_0^t (Y(s) - y^{(n)}(s))ds$$

$A^{(n)}$ is a continuous increasing process such that $A^{(n)}(0) = 0$.

We are now going to identify the limit of $y^{(n)}(\cdot)$. To this end, we shall use the following lemma:

Lemma 6.3. *There exists a constant C which is independent of n such that*

$$(i) \quad E\int_0^T |z^{(n)}(s)|^2 ds \leq C; \quad (ii) \quad E[(A_T^{(n)})^2] \leq C. \tag{6.10}$$

Proof. From (6.8) and (6.7), we have

$$A^{(n)}(T) = y^{(n)}(0) - y^{(n)}(T) - \int_0^T g^{(n)}(s)ds + \int_0^T z^{(n)}(s)dB_s$$
$$\leq |y^{(n)}(0)| + |y^{(n)}(T)| + \int_0^T \mu|z^{(n)}(s)|ds + |\int_0^T z^{(n)}(s)dB_s|.$$

Since $y^{(1)}(t) \leq y^{(n)}(t) \leq Y(t)$ for all t, we have $|y^{(n)}(t)| \leq |y^{(1)}(t)| + |Y(t)|$. Thus there exists a constant C, independent of n, such that

$$E[\sup_{0\leq t\leq T} |y^{(n)}(t)|^2] \leq C. \tag{6.11}$$

It follows readily that there exist two constants C_1 and C_2, independent of n, such that

$$E|A^{(n)}(T)|^2 \leq C_1 + C_2 E\int_0^T |z^{(n)}(s)|^2 ds. \tag{6.12}$$

On the other hand, Itô's formula applied to $|y^{(n)}(\cdot)|^2$ gives:

$$\begin{aligned} E[|y^{(n)}(0)|^2] &= E|Y(T)|^2 + E\int_0^T [2y^{(n)}(s)g^{(n)}(s) - |z^{(n)}(s)|^2]ds \\ &\quad + 2E\int_0^T y^{(n)}(s)dA^{(n)}(s) \\ &\leq E|Y(T)|^2 + E\int_0^T [2\mu|y^{(n)}(s)||z^{(n)}(s)| - |z^{(n)}(s)|^2]ds \\ &\quad + 2E[A^{(n)}(T)\sup_{0\leq s\leq T}|y^{(n)}(s)|], \end{aligned}$$

whence, using that, for positive a, b and ε, $2ab \leq \varepsilon a^2 + b^2/\varepsilon$ (noting also that $E[|y^{(n)}(0)|^2] \geq 0$!), we get

$$\begin{aligned} E\int_0^T |z^{(n)}(s)|^2 ds &\leq E|Y(T)|^2 + E\int_0^T \Big[2\mu^2|y^{(n)}(s)|^2 + \frac{1}{2}|z^{(n)}(s)|^2\Big]ds \\ &\quad + 2[E\sup_{0\leq s\leq T}|y^{(n)}(s)|^2]^{1/2}[E|A^{(n)}(T)|^2]^{1/2}, \end{aligned}$$

and using the same inequality with $\varepsilon = 4C_2$,

$$\begin{aligned} E\int_0^T |z^{(n)}(s)|^2 ds &\leq 2E|Y(T)|^2 + 4\mu^2 E\int_0^T |y^{(n)}(s)|^2 ds \\ &\quad + 8C_2[E\sup_{0\leq s\leq T}|y^{(n)}(s)|^2] + \frac{1}{2C_2}[E|A^{(n)}(T)|^2] \\ &\leq 2E|Y(T)|^2 + 4\mu^2 E\int_0^T |y^{(n)}(s)|^2 ds \\ &\quad + 8C_2[E\sup_{0\leq s\leq T}|y^{(n)}(s)|^2] + \frac{C_1}{2C_2} + \frac{1}{2}E\int_0^T |z^{(n)}(s)|^2 ds, \end{aligned}$$

because of (6.12).

Finally, it comes

$$E\int_0^T |z^{(n)}(s)|^2 ds \ \le 4E|Y(T)|^2 + 8\mu^2 E\int_0^T |y^{(n)}(s)|^2 ds$$
$$+16C_2[E \sup_{0\le s\le T} |y^{(n)}(s)|^2] + \frac{C_1}{C_2},$$

and it is sufficient to note that, thanks to (6.11), the constant

$$\sup_n \left\{ 4E|Y(T)|^2 + 8\mu^2 E\int_0^T |y^{(n)}(s)|^2 ds + 16C_2[E \sup_{0\le s\le T} |y^{(n)}(s)|^2] + \frac{C_1}{C_2} \right\} < \infty$$

to conclude that (6.10)–(i) and then (using (6.12)), (6.10)–(ii) hold true. The lemma is proved. □

With the help of Lemma 6.3 above we can now end the proof of the Decomposition Theorem.

Note first that (6.10)–(i) with (6.7) also implies

$$\mathbf{E}\int_0^T |g^{(n)}(s)|^2 ds \le \mu^2 C.$$

(6.10)–(ii) obviously implies that

$$y^{(n)}(\cdot) \nearrow Y(\cdot), \quad \text{a.e., a.s.}$$

From Theorem 2.1 in [12], it follows that we can write Y under the form

$$Y(t) = Y(T) + \int_t^T g(s)ds + A(T) - A(t) - \int_t^T z(s)dB_s$$

for some $(g, z) \in L^2_{\mathcal{F}}(0, T; R \times R^d)$ and an increasing process A. From the result in the first part of the same Theorem 2.1 in [12], we have that moreover

$$z^{(n)}(\cdot) \to z(\cdot), \quad \text{strongly in } L^2_{\mathcal{F}}(0, T).$$

But (6.9) gives then

$$g^{(n)}(\cdot) \to g(\cdot), \quad \text{strongly in } L^2_{\mathcal{F}}(0, T).$$

And finally, (6.8) gives

$$A^{(n)}(t) \longmapsto A(t), \quad \text{strongly in } L^2(\Omega, \mathcal{F}_T, P).$$

Thanks to Lemma 4.6, we can pass to the L^2-limit in both sides of

$$y^{(n)}(t) = \mathcal{E}[Y(T) + A^{(n)}(T) - A^{(n)}(t)|\mathcal{F}_t].$$

It follows that

$$Y(t) = \mathcal{E}[Y(T) + A(T) - A(t)|\mathcal{F}_t].$$

Thus $Y(t) + A(t) = \mathcal{E}[Y(T) + A(T)|\mathcal{F}_t]$ is an $\mathcal{E}$-martingale (because of (4.4)). Since A is increasing, the theorem is proved. □

7. Inverse problem: an $\mathcal{F}$-expectation is a g-expectation

We are now ready to state our main result, that is to identify any $\mathcal{F}$-expectation to a g-expectation, provided that (4.1) and (4.4) hold.

Theorem 7.1. *We assume that an $\mathcal{F}$-expectation $\mathcal{E}[\cdot]$ satisfies (4.1) and (4.4) for some $\mu > 0$. Then there exists a unique function $g = g(t, z) : \Omega \times [0, T] \times R^d$ satisfying (2.1) and (2.5) such that*

$$\mathcal{E}[X] = \mathcal{E}_g[X], \quad \forall X \in L^2(\Omega, \mathcal{F}_T, P).$$

In particular, every $\mathcal{E}$-martingale is continuous a.s.

Moreover, we have $|g(t, z)| \le \mu|z|$ for all $t \in [0, T]$.

Proof. For each given $z \in R^d$, we consider the following forward equation

$$\begin{cases} dY^z(t) = -\mu|z|dt + zdB_t, \\ Y^z(0) = 0. \end{cases}$$

We have $E[\sup_{t\in[0,T]} |Y^z(t)|^2] < \infty$. It is also clear that Y^z is an $\mathcal{E}^\mu$-martingale, thus an $\mathcal{E}[\cdot]$-supermartingale. Indeed, we can write $Y^z(t) = \mathcal{E}^\mu[Y^z(T)|\mathcal{F}_t]$. From Theorem 6.3, we know the existence of an increasing process $A^z(\cdot)$ with $A^z(0) = 0$ and $E[A^z(T)^2] < \infty$, such that

$$Y^z(t) = \mathcal{E}[Y^z(T) + A^z(T) - A^z(t)|\mathcal{F}_t].$$

Or

$$Y^z(t) + A^z(t) = \mathcal{E}[Y^z(T) + A^z(T)|\mathcal{F}_t], \quad t \in [0, T].$$

Then, from Lemma 5.3, there exists $(g(z, \cdot), Z^z(\cdot)) \in L^2_{\mathcal{F}}(0, T; R \times R^d)$ with $|g(z, t)| \le \mu|Z^z(t)|$ such that

$$Y^z(t) + A^z(t) = Y^z(T) + A^z(T) + \int_t^T g(z, s)ds - \int_t^T Z^z(s)dB_s.$$

We also have

$$|g(z, t) - g(z', t)| \le \mu|Z^z(t) - Z^{z'}(t)|. \tag{7.1}$$

But on the other hand, since

$$Y^z(t) = Y^z(T) + \int_t^T \mu|z|ds - \int_t^T zdB_s,$$

it follows that

$$A^z(t) \equiv \mu|z|t - \int_0^t g(z, s)ds$$
$$Z^z(t) \equiv z$$

In particular, (7.1) becomes

$$|g(z, t) - g(z', t)| \le \mu|z - z'|. \tag{7.2}$$

Moreover,

$$Y^z(t) + A^z(t) = Y^z(r) + A^z(r) - \int_r^t g(z,s)ds + \int_r^t zdB_s, \quad 0 \le r \le t \le T,$$

and $Y^z(t) + A^z(t)$ is an $\mathcal{E}$-martingale. But with the assumption (4.4) one has, for each $z \in R^d$ and $r \le t$

$$\mathcal{E}[-\int_r^t g(z,s)ds + \int_r^t zdB_s|\mathcal{F}_r] = \mathcal{E}[Y^z(t) + A^z(t) - (Y^z(r) + A^z(r))|\mathcal{F}_r],$$

i.e.

$$\mathcal{E}[-\int_r^t g(z,s)ds + \int_r^t zdB_s|\mathcal{F}_r] = 0 \quad 0 \le r \le t \le T \tag{7.3}$$

Now let $\{A_i\}_{i=1}^N$ be a $\mathcal{F}_r$-measurable partition of Ω (i.e., A_i are disjoint, $\mathcal{F}_r$-measurable and $\cup A_i = \Omega$) and let $z_i \in R^d$, $i = 1, 2, \cdots, N$. From Lemma 3.5, and the fact that $g(0,s) \equiv 0$, it follows that

$$\begin{aligned}
&\mathcal{E}[-\int_r^t g(\sum_{i=1}^N z_i 1_{A_i}, s)ds + \int_r^t \sum_{i=1}^N z_i 1_{A_i} dB_s|\mathcal{F}_r] \\
&= \mathcal{E}[\sum_{i=1}^N 1_{A_i}\left(-\int_r^t g(z_i,s)ds + \int_r^t z_i dB_s\right)|\mathcal{F}_r] \\
&= \sum_{i=1}^N 1_{A_i}\mathcal{E}[-\int_r^t g(z_i,s)ds + \int_r^t z_i dB_s|\mathcal{F}_r] \\
&= 0
\end{aligned}$$

(because of (7.3)). In other words, for each simple function $\eta \in L^2(\Omega, \mathcal{F}_r, P)$,

$$\mathcal{E}[-\int_r^t g(\eta,s)ds + \int_r^t \eta dB_s|\mathcal{F}_r] = 0.$$

From this, the continuity of $\mathcal{E}[\cdot]$ in L^2 given by (4.3) and the fact that g is Lipschitz in z, it follows that the above equality holds for $\eta(\cdot) \in L^2_{\mathcal{F}}(0,T;R^d)$:

$$\mathcal{E}\Big[-\int_r^t g(\eta(s),s)ds + \int_r^t \eta(s)dB_s|\mathcal{F}_r\Big] = 0. \tag{7.4}$$

We just have to prove now that

$$\mathcal{E}_g[X] = \mathcal{E}[X], \quad \forall X \in L^2(\Omega, \mathcal{F}_T, P).$$

To this end we first solve the following BSDE

$$\begin{aligned}
-dy(s) &= g(t, z(s))ds - z(s)dB_s, \\
y(T) &= X.
\end{aligned}$$

Since g is Lipschitz in z, there exists a unique solution $(y(\cdot), z(\cdot)) \in L^2_{\mathcal{F}}(0, T; R \times R^d)$. By the definition of g-expectation,

$$\mathcal{E}_g[X] = y(0).$$

On the other hand, using (7.4), one finds

$$\begin{aligned}\mathcal{E}[X] &= \mathcal{E}[y(0) - \int_0^T g(z(s), s)ds + \int_0^T z(s)dB_s] \\ &= y(0) + \mathcal{E}[-\int_0^T g(z(s), s)ds + \int_0^T z(s)dB_s] \\ &= y(0) = \mathcal{E}_g[X].\end{aligned}$$

It follows that this g-expectation $\mathcal{E}_g[\cdot]$ coincides with $\mathcal{E}[\cdot]$. As the uniqueness of g readily follows from [2], the proof is complete. □

Remark 7.1. *In this paper we have limited ourselves to treat the situation where the filtration is generated by a Brownian motion. A natural question is whether our nonlinear supermartingale decomposition approach can be applied to more general situations. A general positive answer seems unlikely, due to the lack of comparison theorem for BSDE's driven by discontinuous processes. However some partial positive answers may be possible, but anyway some further efforts and techniques will be required to overcome new difficulties due to non linearity and jumps.*

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