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Monte-Carlo approximations for 2d Navier-Stokes equations with measure initial data

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Abstract. We are interested in proving Monte-Carlo approximations for 2d Navier-Stokes equations with initial data u_0 belonging to the Lorentz space $L^{2,\infty}$ and such that curl u_0 is a finite measure. Giga, Miyakawa and Osada [7] proved that a solution u exists and that $u = K * \text{curl}\, u$, where K is the Biot-Savart kernel and $v = \text{curl}\, u$ is solution of a nonlinear equation in dimension one, called the vortex equation.

In this paper, we approximate a solution v of this vortex equation by a stochastic interacting particle system and deduce a Monte-Carlo approximation for a solution of the Navier-Stokes equation. That gives in this case a pathwise proof of the vortex algorithm introduced by Chorin and consequently generalizes the works of Marchioro-Pulvirenti [12] and Méléard [15] obtained in the case of a vortex equation with bounded density initial data.

1. Introduction

In this paper, we are interested in proving stochastic approximations for a 2d Navier-Stokes equation with an initial data u_0 belonging to the Lorentz space $L^{2,\infty}$ and such that $\text{curl}(u_0)$ is a finite measure. Giga-Miyakawa-Osada [7] proved that in this case, the equation has a global solution u (not always unique), and $u = K * \text{curl}(u)$, where K is the Biot-Savart kernel and $v = \text{curl}(u)$ is solution of a nonlinear equation in dimension one, called the vortex equation, with a finite measure initial data. This equation appears as a McKean-Vlasov equation, in which the drift term given by K can explode.

Our aim in this paper is to give a pathwise Monte-Carlo approximation for a solution of the Navier-Stokes equation, by using the probabilistic interpretation of the vortex equation.

In a previous work [15], we obtained in the case of a vortex equation with bounded density initial data a pathwise proof of the vortex algorithm introduced by Chorin, i.e. the convergence of the empirical measures of the particle system, considered as probability measures on the path space, to the solution of the equation,

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with a precise rate of convergence. That work generalized a paper of Marchioro-Pulvirenti [12], in which the convergence of the expectation of the empirical measures was obtained for each time, and also the one of Osada [18], where the convergence is only true for large viscosities. However, our result was not completely satisfying since in a probabilitic point of view, the hypothesis of bounded density initial data was ununderstandable.

Here, we obtain a pathwise result, for each viscosity, and for a large class of initial datas. We approach the Biot-Savart kernel by cutoff kernels K_{ε_n} and construct particle systems associated with K_{ε_n} in an asymptotics relying the number of particles n and the level of cutoff ε_n. The difficulties are mainly related to the explosion of the Biot-Savart kernel and to the general form of the initial condition. The good spaces in which the solutions of the vortex equation live are L^q-spaces, with $1 < q < 2$, and that induces new difficulties. We only obtain here a convergence in law for the particle system, instead of a L^1-convergence as in the bounded density case, due to the explosions of all standard estimates.

1.1. Notations

Let $\Omega = C(\mathbb{R}_+, \mathbb{R}^2)$, endowed with the topology of uniform convergence on compact sets and with the corresponding Borel σ-field. For each $T > 0$, Ω_T denotes $C([0, T], \mathbb{R}^2)$ and X the canonical process.

For a Borel space E, $\mathcal{P}(E)$ is the space of probability measures on E endowed with the topology of weak convergence. We denote by M_F the space of finite measures on $\mathbb{R}^2$ normed by the total variation $\|.\|$.

C is a positive real constant which can change from line to line.

2. Existence and uniqueness for the vortex equation with finite measure initial data

Let us consider the velocity flow $u(t, x)$, $t \in \mathbb{R}_+, x \in \mathbb{R}^2$ of a viscous and incompressible fluid in the whole plane. The governing equation of this motion is the **Navier–Stokes equation** given by

$$\begin{aligned} &\frac{\partial u}{\partial t}(t, x) + (u \cdot \nabla)u(t, x) = \nu \Delta u(t, x) - \nabla p \; ; \\ &\nabla \cdot u(t, x) = 0 \; ; \; u(x, t) \to 0 \text{ as } |x| \to +\infty, \qquad \text{for } 0 \leq t < +\infty, \end{aligned} \tag{2.1}$$

where p is the pressure and $\nu > 0$ the viscosity (assumed to be constant).

It is well known that if the initial velocity $u_0 \in L^2$ with a divergence equal to 0 and $\text{curl}(u_0) = v_0 \in L^1 \cap L^\infty$, there is a unique weak solution to (2.1) and the vorticity flow $v(t, x) = \text{curl } u(t, x)$ is weak solution of the nonlinear partial differential equation, called **the vortex equation**

$$\begin{aligned} &\frac{\partial v}{\partial t}(t, x) + (u \cdot \nabla)v(t, x) = \nu \Delta v(t, x) \; ; \; v(x, t) \to 0 \text{ as } |x| \to +\infty, \\ &\quad \text{for} \quad 0 \leq t < +\infty, \end{aligned} \tag{2.2}$$

$$u(t, x) = \int_{\mathbb{R}^2} K(x - y)v(t, y)dy \tag{2.3}$$

with initial data v_0. The Biot-Savart kernel $K(x)$ is obtained from the fundamental solution of the Poisson equation $g(r) = -\frac{1}{2\pi}\ln|r|$ by

$$\forall x \in \mathbb{R}^2, \qquad K(x) = \nabla^{\perp} g(|x|)$$

and $\nabla^{\perp} = (\partial_{x_2}, -\partial_{x_1})$. The computation gives

$$\forall x = (x_1, x_2) \in \mathbb{R}^2, K(x) = \frac{1}{2\pi}\frac{1}{(x_1^2 + x_2^2)}(-x_2, x_1). \tag{2.4}$$

Note that $\nabla \cdot K = 0$.

In [7], Giga–Miyakawa–Osada remark that the convolution of a finite signed measure by the Biot-Savart kernel is a bounded operator in a well chosen Lorentz space and generalize the previous properties with less regular initial velocities. We recall here some of their results. Let us first introduce the Lorentz space $L^{2,\infty}(\mathbb{R}^2)$ (cf. Bergh and Lofstrom [2]).

A measurable function f on $\mathbb{R}^2$ belongs to the space $L^{2,\infty}(\mathbb{R}^2)$ if

$$\|f\|_{2,\infty} = \sup_{\lambda > 0} \lambda (mes\{x; |f(x)| > \lambda\})^{\frac{1}{2}} < +\infty, \tag{2.5}$$

where mes is the Lebesgue measure on $\mathbb{R}^2$. Although $\|f\|_{2,\infty}$ does not satisfy the usual triangle inequality, it is a pseudo-norm on the linear space $L^{2,\infty}(\mathbb{R}^2)$, which is a Banach space with a norm equivalent to $\|f\|_{2,\infty}$.

The following lemma describes the main properties of the Biot-Savart kernel we will use later.

Lemma 2.1. *1) the Biot-Savart kernel K belongs to $L^{2,\infty}(\mathbb{R}^2)$.*

2) For every finite signed measure m_0 on $\mathbb{R}^2$, the function $K * m_0$ belongs to $L^{2,\infty}(\mathbb{R}^2)$ and

$$\|K * m_0\|_{2,\infty} \leq C\|K\|_{2,\infty}\|m_0\|, \tag{2.6}$$

where C is independent of m_0.

Moreover, suppose that $u \in L^{2,\infty}(\mathbb{R}^2)$ with $\nabla \cdot u = 0$ and curl $u \in M_F$. *Then*

$$u = K * (\text{curl } u). \tag{2.7}$$

3) Let $p > 2$ and $1 < q < 2$ such that $\frac{1}{q} = \frac{1}{p} + \frac{1}{2}$, then for $w \in L^q(\mathbb{R}^2)$,

$$\|K * w\|_p \leq C\|K\|_{2,\infty}\|w\|_q. \tag{2.8}$$

Moreover, suppose that $u \in L^p(\mathbb{R}^2)$ with $\nabla \cdot u = 0$ and curl $u \in L^q(\mathbb{R}^2)$. *Then*

$$u = K * (\text{curl } u). \tag{2.9}$$

4) For $1 < q < 2$ and for each function $w(t,x)$ on $\mathbb{R}_+ \times \mathbb{R}^2$ such that $\|w(t,.)\|_1 \leq C$ and $\|w(t,.)\|_{q'} \leq Ct^{-1+\frac{1}{q'}}$ for every $t > 0$, with $\frac{1}{q} + \frac{1}{q'} = 1$,

$$\|K * w(t,.)\|_{L^\infty} \leq C(t^{-\frac{1}{q}} + 1), \tag{2.10}$$

and then the mapping $t \mapsto \|K * w(t,.)\|_{L^\infty}$ is integrable on every time interval $[0,T]$.

Proof. The first part consists in a trivial computation giving $\|K\|_{2,\infty} = \frac{1}{2\sqrt{\pi}}$. The proof of the second and third parts can be found in Giga and others [7] Lemma 2.2. The equation (2.8) is obtained by an interpolation theorem for Lorentz spaces. (See [2] Theorem 5.3.4).

For the last part, let us write

$$
\begin{aligned}
|K * w(t,.)| &\le \int_{B(x,1)} |K(x-y)w(t,y)|dy + \int_{B(x,1)^c} |K(x-y)w(t,y)|dy \\
&\le \left(\int_{B(x,1)} |K(x-y)|^q dy\right)^{\frac{1}{q}} \left(\int_{B(x,1)} |w(t,y)|^{q'} dy\right)^{\frac{1}{q'}} \\
&\quad + \sup_{y \in B(0,1)^c} |K(y)| \|w(t,.)\|_{L^1} \quad \text{with } \frac{1}{q} + \frac{1}{q'} = 1, \\
&\le C\left(t^{-1+\frac{1}{q'}} \left(\int_{B(x,1)} |K(x-y)|^q dy\right)^{\frac{1}{q}} + \sup_{y \in B(0,1)^c} |K(y)|\right) \\
&\le C(t^{-\frac{1}{q}} + 1),
\end{aligned}
$$

since the mapping $y \mapsto K(y)^q$, $1 < q < 2$ is integrable at zero (we are in $\mathbb{R}^2$). □

Remark 2.2. *The formulas (2.7) and (2.9) will allow us to deduce approximating results for u from the ones for* curl(u)*, and justify the interest we have in the vortex equation.*

Throughout the paper, v always denotes the vorticity of u.

The probabilistic interpretation will be based on the following theorem, proved by Giga, Miyakawa and Osada ([7] Theorems 4.2 and 4.3).

Theorem 2.3. *Suppose $u_0 \in L^{2,\infty}(\mathbb{R}^2)$, $\nabla \cdot u_0 = 0$ and $\mathrm{curl}(u_0) = m_0$ is a finite signed measure. There is at least one global solution u to the Navier-Stokes equation (2.1) with u_0 as initial data and one solution v to the vortex equation. The function v is bounded and continuous under the weak*-topology, from $[0, +\infty)$ to M_F with initial measure m_0.*

Moreover, for each $t > 0$, the solution $v(t,.)$ is a differentiable function, such that

$$\|v(t,.)\|_r \le Ct^{-1+\frac{1}{r}} \|m_0\|, \quad \forall t > 0, \forall 1 \le r \le \infty\,; \tag{2.11}$$

$$\limsup_{t \mapsto 0} t^{1-\frac{1}{q}} \|v(t,.)\|_q \le C\|(m_0)_{pp}\|, \quad \text{for } 1 < q < 2, \tag{2.12}$$

$$\sup_{[\varepsilon,T]} \|\nabla^k \partial_t^h v(t,.)\|_\infty \le C_\varepsilon, \tag{2.13}$$

where $(m_0)_{pp}$ denotes the purely atomic part of m_0, C and C_ε are real constants and (2.13) is true for each $0 < \varepsilon < T$ and k and h nonnegative integers.

The proof is based on Theorem 2.4 ([17]) concerning generators of generalized divergence form and on Lemma 2.5 ([18]) below.

Theorem 2.4. *Let us consider the parabolic operator* $L_b = \partial_t - \nu\Delta + (b \cdot \nabla)$, *under the following assumptions:*

a) The vector $b = b(x,t)$ *satisfies* $\nabla \cdot b = 0$, *in distribution. (b is not always a function).*

b) There exist $d > 0$ *and functions* $c^{i,j}(x,t)$, $1 \le i, j \le 1$, *with* $b^i = \sum_j \partial_j c^{ij}$, b^i *being the* i^{th} *component of b, and satisfying*

$$\sup_{(x,t)\in\mathbb{R}^2\times[0,T]} \sup_{i,j} |c^{ij}(x,t)| \le d.$$

Then L_b *has a unique fundamental solution* Γ_b, *and there exist constants* C_j *depending only on d and* ν *such that for all* $x, y \in \mathbb{R}^2$ *and* $0 \le s < t \le T$,

$$\frac{C_1}{(t-s)} \exp(-C_2 \frac{|x-y|^2}{(t-s)}) \le \Gamma_b(x,t,y,s) \le \frac{C_3}{(t-s)} \exp(-C_4 \frac{|x-y|^2}{(t-s)}). \tag{2.14}$$

Lemma 2.5. *The function* $K = (K_1, K_2)$ *can be expressed as*

$$K_1 = \partial_1 A_3 + \partial_2 A_1 \; ; \; K_2 = -\partial_1 A_1 - \partial_2 A_2,$$

where $A_1 = -\frac{x_1^2 x_2^2}{\pi|x|^4}$; $A_2 = \frac{-3x_1x_2}{2\pi|x|^2} + \frac{x_1^3 x_2}{\pi|x|^4}$; $A_3 = \frac{-3x_1x_2}{2\pi|x|^2} + \frac{x_1 x_2^3}{\pi|x|^4}$.

The functions A_1, A_2, A_3 *are clearly bounded.*

As an immediate application of (2.10) and (2.11), we have

Corollary 2.6. *Under the same hypotheses as in Theorem 2.3, for every* $t > 0$, *for* $1 < q < 2$,

$$\|K * v(t,.)\|_{L^\infty} \le C\|m_0\|(t^{\frac{-1}{q}} + 1). \tag{2.15}$$

Hence the mapping $t \mapsto \|K * v(t,.)\|_{L^\infty}$ *is integrable on every time interval* $[0,T]$.

We deduce from it the evolutive form of the solution v of (2.2).

Lemma 2.7. *Each weak solution v of the vortex equation (2.2) with a finite measure initial data* m_0 *is solution of the mild equation*

$$v(t,y) = G_t^\nu * m_0(y) + \int_0^t \int_{\mathbb{R}^2} \nabla_x G_{t-s}^\nu(x-y) K * v(s,x) v(s,x) dx ds \tag{2.16}$$

where G_t^ν *is the heat kernel in* $\mathbb{R}^2$ *defined by* $G_t^\nu(x) = \frac{1}{4\nu t\pi} e^{-\frac{|x|^2}{4t\nu}}$.

The proof can be omitted since it consists in a standard mild formulation using

$$\|\nabla_x G_t^\nu\|_{L^1(\mathbb{R}^2)} \le \frac{C}{\sqrt{\nu t}} \tag{2.17}$$

where C is a real constant and so $\int_0^t \|\nabla_x G_{t-s}^\nu\|_{L^1(\mathbb{R}^2)} ds < +\infty$.

Let us now recall the following technical estimates.

Lemma 2.8. *For* $1 \le s \le r \le \infty$, $t > 0$,

$$\|\nabla_x G_t^\nu * f\|_r \le C(\nu t)^{-\frac{1}{2}-(\frac{1}{s}-\frac{1}{r})}\|f\|_s. \tag{2.18}$$

The proof of this lemma can be easily obtained by applying the Young inequality and by using estimates on the function $\nabla_x G_t^\nu$ that one can find for example in Ladyzenskaya-Solonnikov-Ural'ceva [11] p. 274.

Giga and others prove the uniqueness of the solution of the Navier-Stokes equation (2.1), in the case where m_0 is a measure with no atoms. Our probabilistic interpretation is based on the vortex equation, and needs a uniqueness result for this equation which is not directly induced by the one of the Navier-Stokes equation.

Theorem 2.9. *If the finite measure m_0 is a measure with no atoms or if $\|m_0\|$ is sufficiently small, then the solution v of the vortex equation is unique in the space of functions which satisfy (2.11) and (2.12).*

Proof. We consider two solutions v_1 and v_2 of the vortex equation with initial data m_0, and write indifferently $v_{i,s}(x)$ or $v_i(s,x)$. By Lemma 2.7 and for $t \le T$,

$$\begin{aligned}|v_1(t,y) - v_2(t,y)| \le \int_0^t \int_{\mathbb{R}^2} |\nabla_x G_{t-s}^\nu(x-y)|\Big(&|K * v_{1,s}(x)v_1(s,x)\\ &-K * v_{2,s}(x)v_2(s,x)|\Big)dxds \le \int_0^t \int_{\mathbb{R}^2} |\nabla_x G_{t-s}^\nu(x-y)|\\ &\times\Big(|(K * v_{1,s}(x) - K * v_{2,s}(x))v_1(s,x)|\\ &+|K * v_{2,s}(x)(v_1(s,x) - v_2(s,x))|\Big)dxds.\end{aligned}$$

Let us consider a real number q such that $\frac{4}{3} < q < 2$, and r defined by $\frac{1}{r} = \frac{2}{q} - \frac{1}{2}$, then $1 < r < q < 2$ and we denote by q' the real number such that $\frac{1}{r} = \frac{1}{q} + \frac{1}{q'}$.

Since $r < q$, by (2.8) and the Hölder inequality, we obtain

$$\begin{aligned}\|v_1(t,.) - v_2(t,.)\|_q \le C \int_0^t (t-s)^{-\frac{1}{2}-(\frac{1}{r}-\frac{1}{q})}\Big(&\|(K * v_{1,s} - K * v_{2,s})v_1(s,.)\|_r\\ &+\|K * v_{2,s}(v_1(s,.) - v_2(s,.))\|_r\Big)ds\\ \le C \int_0^t (t-s)^{-\frac{1}{q}}\Big(&\|K * v_{1,s} - K * v_{2,s}\|_{q'}\|v_1(s,.)\|_q +\\ &+\|K * v_{2,s}\|_{q'}\|(v_1(s,.) - v_2(s,.))\|_q\Big)ds\\ \le C\|K\|_{2,\infty} \int_0^t (t-s)^{-\frac{1}{q}}\Big(&\|v_{1,s} - v_{2,s}\|_q\|v_1(s,.)\|_q +\\ &+\|v_{2,s}\|_q\|v_1(s,.) - v_2(s,.)\|_q\Big)ds.\end{aligned}$$

Let us now introduce $\|v\|_{q,t} = \sup_{0<s\le t} s^{1-\frac{1}{q}} \|v(s,.)\|_q$. Then a simple computation gives

$$\|v_1 - v_2\|_{q,T} \le C\|K\|_{2,\infty}\beta(1-\frac{1}{q}, \frac{2}{q}-1)(\|v_1\|_{q,T} + \|v_2\|_{q,T})\|v_1 - v_2\|_{q,T},$$

where β is the beta function for the parameters $1-\frac{1}{q}$ and $\frac{2}{q}-1$.

1) If m_0 is diffuse, the estimate (2.12) implies that there exists T sufficiently small for which $C\|K\|_{2,\infty}\beta(1-\frac{1}{q}, \frac{2}{q}-1)(\|v_1\|_{q,T} + \|v_2\|_{q,T}) < 1$. This yields $v_1 = v_2$ on $[0, T]$. On the interval $[T, \infty)$, both v_1 and v_2 are classical solutions and then equal on $[T, \infty)$ by a standard uniqueness result.

2) If other cases, we use (2.11), and if $2C\|K\|_{2,\infty}\beta(1-\frac{1}{q}, \frac{2}{q}-1)\|m_0\| < 1$, (for the adequate real constant C), we obtain immediately a global uniqueness result. □

3. The nonlinear martingale problem and the SDE associated with the vortex equation

We are in a McKean-Vlasov context, and the interpretation of the vortex equation as a Fokker-Planck equation allows us to define naturally a nonlinear martingale problem (See for example Méléard [14], Jourdain [10]).

We want to take into account any finite initial measure m_0. So let $|m_0|$, $\|m_0\|$, and h denote respectively the absolute value of m_0, the total variation of m_0 and a density of m_0 with respect to the probability measure $\frac{|m_0|}{\|m_0\|}$. The function h is thus a bounded measurable function with values in $[-\|m_0\|; \|m_0\|]$.

For P a probability measure on $C(\mathbb{R}_+, \mathbb{R}^2)$, we denote by $(P_t)_{t\ge 0}$ the flow of time-marginals of P at each time t and define the flow $(\tilde{P}_t)_{t\ge 0}$ of signed measures on $\mathbb{R}^2$ by

$$\forall B \in \mathcal{B}(\mathbb{R}^2), \quad \tilde{P}_t(B) = E^P(\mathbf{1}_B(X_t)h(X_0)), \tag{3.1}$$

where X denotes the canonical process on $C(\mathbb{R}_+, \mathbb{R}^2)$. (One associates with each sample path a signed weight depending on the initial data).

It is easy to prove that for each $t \ge 0$, the signed measure $\tilde{P}_t$ is bounded with a total mass less than $\|m_0\|$, and that if P_t is absolutely continuous with respect to the Lebesgue measure, then it is the same for $\tilde{P}_t$.

The vortex equation, seen as Fokker-Planck equation, leads naturally to the following definition.

Definition 3.1. *The probability measure $P \in \mathcal{P}(C(\mathbb{R}_+, \mathbb{R}^2))$ is solution of the nonlinear martingale problem $(\mathcal{M})$ if for each $\phi \in C_b^2(\mathbb{R}^2)$,*

$$\phi(X_t) - \phi(X_0) - \int_0^t K * \tilde{P}_s(X_s).\nabla\phi(X_s)ds - \nu\int_0^t \Delta\phi(X_s)ds \tag{3.2}$$

is a P-martingale, where X is the canonical process on $C(\mathbb{R}_+, \mathbb{R}^2)$, $P_s = P \circ X_s^{-1}$ and $P_0 = \frac{|m_0|}{\|m_0\|}$.

Let us remark that, if for each $s > 0$, the probability $\tilde{P}_s$ is absolutely continuous with respect to the Lebesgue measure, with density $v(s, .)$, then the function v is a weak solution of the vortex equation (2.2) with initial data m_0.

The nonlinear martingale problem $(\mathcal{M})$ is related to the following nonlinear stochastic differential equation.

Definition 3.2. *Let us consider a random $\mathbb{R}^2$-valued variable X_0 with law $\frac{|m_0|}{\|m_0\|}$ and B a 2-dimensional Brownian motion independent of X_0. We say that $\bar{X} \in C(\mathbb{R}_+, \mathbb{R}^2)$ is solution of the nonlinear SDE satisfies for each $t > 0$,*

$$\bar{X}_t = X_0 + \sqrt{2\nu} B_t + \int_0^t K * \tilde{P}_s(\bar{X}_s) ds,$$
$$P_s \text{ is the marginal at time } s \text{ of the law of } \bar{X}_s. \tag{3.3}$$

We will see in the following that there exist weak solutions of this equation.

Remark 3.3. *We denote by* **P** *the space of probability measures on $C(\mathbb{R}_+, \mathbb{R}^2)$ whose marginals P_s, $s > 0$ are absolutely continuous with respect to the Lebesgue measure. For such probability measures P, there exists a measurable version $(s, x) \to p(s, x)$ ofthedensitiesoftheflowofsignedmeasures$\tilde{P}_s$.(cfMeyer[16]p.194)*

The following theorem gives the probabilistic interpretation of the vortex equation.

Theorem 3.4. *Let us consider a probability measure m_0 satisfying the hypotheses of Theorem 2.9. Then there exists a unique solution $P \in$* **P** *to the nonlinear martingale problem $(\mathcal{M})$ such that $P_0 = \frac{|m_0|}{\|m_0\|}$.*

Moreover, the flow of measurable densities (v_t) of $(\tilde{P}_t)$ is the unique solution of the vortex equation.

Proof. The proof will use a shift argument introduced by Jourdain [8].

1) Uniqueness

Let P and Q be two solutions of $(\mathcal{M})$ belonging to **P**. Then for each $t > 0$, the signed measures $\tilde{P}_t$ and $\tilde{Q}_t$ have densities p_t and q_t. By taking expectations in the martingale problem, we obtain immediately that the flows (p_t) and (q_t) are solutions of the vortex equation (2.2) with the same initial condition m_0. They are both solutions of $\partial_t p - \nu\Delta p + (K * p)\nabla p = 0$, and then by Theorem 2.4 and Lemma 2.5, they satisfy (2.11) and (2.12). They are then equal and equal to the unique solution v. So P and Q are solution of the same martingale problem with the given drift term $K * v_s$. By Theorem 2.3, for each $t > 0$, $v(t, .)$ is in L^1, and is bounded on every interval $[\varepsilon, T]$. Then the function $(s, x) \to K * v_s(x)$ is bounded on $[\varepsilon, T] \times \mathbb{R}^2$, for each $\varepsilon > 0$.

We introduce the shift $y \to D_n(y) = y(\frac{1}{n} + .) \in \Omega$. Let $P^n = P \circ D_n^{-1}$, $Q^n = Q \circ D_n^{-1}$. Both P^n and Q^n solve the martingale problem:

$$R_0 = v(\frac{1}{n}, x)dx \quad ;$$
$$\phi(X_t) - \phi(X_0) - \int_0^t (\nu\Delta\phi(X_s) + K * v_{s+\frac{1}{n}}(X_s).\nabla\phi(X_s))ds \tag{3.4}$$

is a R martingale for any $\phi \in C_b^2(\mathbb{R}^2)$.

Since the mapping $x \to K * v_{s+\frac{1}{n}}(x)$ is bounded uniformly in s, the martingale problem admits a unique solution and $P^n = Q^n$, for each $n \in \mathbb{N}^*$.

As for any $y \in \Omega$, $\lim_{n\to+\infty} D_n(y) = y$, P^n and Q^n converge weakly to P and Q. Therefore, $P = Q$.

2) Existence.

The first idea consists in considering the martingale problem with the drift $K * v_s$, where v is the solution of the vortex equation. This drift term is not bounded and not Lipschitz continuous, and we do not have an immediate existence result. So we consider again the solution P^n of the martingale problem (3.4). Since the drift $K * v_{s+\frac{1}{n}}(x)$ is bounded uniformly in s and by Girsanov's theorem, the law P^n belongs to $\mathbf{P}$, and we denote by $q^n(t,x)$ the measurable version of the densities of $(\tilde{P}^n_t)$. Then, multiplying all the terms in (3.4) by $h(X_0)$ and taking expectations, we obtain that for each $t > 0$ and for each $\psi \in C_b^{1,2}([0,t] \times \mathbb{R}^2)$,

$$\int_{\mathbb{R}^2} \psi(t,x) q^n(t,x)dx = \int_{\mathbb{R}^2} \psi(0,x) v(\frac{1}{n},x)dx + \int_{(0,t]\times\mathbb{R}^2} \Bigg(\frac{\partial \psi(s,x)}{\partial s} + \nu \Delta \psi(s,x) + K * v_{s+\frac{1}{n}}(x).\nabla\psi(s,x) \Bigg) q^n(s,x)dxds.$$

Then by choosing for a fixed time t the function $\psi(s,x) = \int_{\mathbb{R}^2} G^\nu_{t-s}(x-y)\phi(y)dy$ for $\phi \in C_b^2(\mathbb{R}^2)$ and applying Fubini's theorem, we obtain that q^n is solution of the evolution equation

$$\forall t > 0, \quad q^n(t,x) = G_t * v(\frac{1}{n},.)(x) - \int_0^t \frac{\partial G_{t-s}}{\partial x} * (q^n(s,.)K * v_{s+\frac{1}{n}}(.))(x)ds.$$

Now, using the weak equation satisfied by v issued from m_0, and taking the same test functions as before between times $\frac{1}{n}$ and t, we obtain in the same way that

$$\forall t > 0, \quad v(\frac{1}{n}+t,x) = G_t * v(\frac{1}{n},.)(x) - \int_0^t \frac{\partial G_{t-s}}{\partial x} * (v(\frac{1}{n}+s,.)K * v_{s+\frac{1}{n}}(.))(x)ds.$$

Then by (2.17), (2.15) and for $1 < q < 2$, $n \in \mathbb{N}^*$,

$$\|q^n(t,.) - v(\frac{1}{n}+t,.)\|_1 \leq \frac{C}{\sqrt{\nu}} \|m_0\| \int_0^t \frac{(s^{-\frac{1}{q}}+1)\|q^n(s,.) - v_{s+\frac{1}{n}}(.)\|_1}{\sqrt{t-s}} ds,$$

with C independent of n. By Gronwall's lemma, we finally get that for each $t > 0$,

$$\|q^n(t,.) - v_{t+\frac{1}{n}}(.)\|_1 = 0$$

Hence, the function $(t,x) \to v_{t+\frac{1}{n}}(x)$ is a measurable version of the densities for $(\tilde{P}^n_t)$.

Let us now denote by Q^n the image measure of P^n by the shift $y \in \Omega \to y((. - \frac{1}{n}) \vee 0) \in \Omega$. We would like to prove that the sequence (Q^n) converges to a solution of $(\mathcal{M})$.

We know by Theorem 2.3 that $Q_0^n = v(\frac{1}{n}, x)dx$ converges weakly to m_0 as n tends to infinity. Moreover by (2.15) the mapping $s \mapsto \|K * v(s, .)\|_\infty$ is integrable on $[0, T]$, for each $T > 0$. Therefore, the sequence of laws (Q^n) is tight. Let Q^∞ be the limit of a subsequence that we will index by n for convenience.

Let $p \in \mathbb{N}^*$, $\phi \in C_b^2(\mathbb{R}^2)$, $g \in C_b(\mathbb{R}^{2p})$, $0 < s_1 \leq ... \leq s_p \leq s < t \leq T$ and $G : \Omega^T \to \mathbb{R}$ defined by

$$G(y) = \left(\phi(y(t)) - \phi(y(s)) - \int_s^t \left(\nu\Delta\phi(y(r)) + K * v_r(y(r)).\nabla\phi(y(r))\right)dr\right)$$
$$g(y(s_1), ..., y(s_p)).$$

Since by Theorem 2.3, the function $x \mapsto K * v_r(x)$ is continuous and bounded uniformly in $s \in [s_1, T]$, the function G is continuous and bounded on Ω^T. Hence, $E^{Q^\infty}(G(X)) = \lim_{n\to+\infty} E^{Q^n}(G(X))$. For $n \geq \frac{1}{s_1}$, $E^{Q^n}(G(X)) = 0$, and then $E^{Q^\infty}(G(X)) = 0$.

Since $s \mapsto \|K * v(s, .)\|_\infty$ is integrable on $[0, T]$ and by Lebesgue's theorem, that always holds for a functional G such that $s_p \mapsto 0$ and $s \mapsto 0$. It implies that

$$\phi(X_t) - \phi(X_0) - \int_0^t \left(\nu\Delta\phi(X_r) + K * v_r(X_r).\nabla\phi(X_r)\right)dr$$

is a Q^∞ martingale.

By construction, for $t > 0$, for $n \geq \frac{1}{t}$, v_t is the density of $\tilde{Q}_t^n = \tilde{P}^n_{t-\frac{1}{n}}$, and then $\tilde{Q}_t^\infty$ is absolutely continuous with respect to the Lebesgue measure with the density v. Moreover, $Q_0^n = v(\frac{1}{n}, x)dx$ converge weakly to m_0 as n tends to infinity. Then Q^∞ is solution of the nonlinear martingale problem $(\mathcal{M})$.

Theorem 3.4 is proved. □

Remark 3.5. *1) We have then obtained a unique weak solution to (3.3), but the function $K * v$ is not Lipschitz continuous (see Proposition 5.5 for details), and we do not have a strong uniqueness result.*

2) If m_0 does not satisfy the hypotheses of Theorem 2.9, then the previous proof shows at least the existence of a solution of the nonlinear martingale problem.

We will see in the next section another proof of the existence of a solution obtained by limit of particle approximations.

4. Stochastics Approximations of a solution of the vortex equation

4.1. *The case of a cutoff kernel*

We follow here the same scheme as in Méléard [15], but we take a different cutoff kernel K_ε. For technical reasons, we consider a convolution regularizing kernel of the form $K_\varepsilon(x) = K * \varphi_\varepsilon(x)$, where $\varphi_\varepsilon(x) = \frac{1}{\varepsilon^2}\varphi(\frac{\cdot}{\varepsilon})$ and φ is a smooth positive function with a mass equal to 1 and with a radial symmetry. We then prove

Lemma 4.1. *$\forall x = (x_1, x_2) \in \mathbb{R}^2$ and $r = |x|$,*

$$K_\varepsilon(x) = \left(-\frac{1}{r^2}\int_0^r \rho\varphi_\varepsilon(\rho)d\rho \right)(-x_2, x_1).$$

Proof. We have seen that $K = \nabla^\perp g$ where $g(x) = g(|x|) = -\frac{1}{2\pi}\ln|x|$. In the same way, $K_\varepsilon = \nabla^\perp g_\varepsilon$, with g_ε defined by $\Delta g_\varepsilon = -\varphi_\varepsilon$. We write the Laplace operator in polar coordinates. Since g_ε has a radial symmetry, we get

$$\frac{\partial}{\partial r}(r\frac{\partial g_\varepsilon}{\partial r}) = -r\varphi_\varepsilon.$$

We deduce that $r\frac{\partial g_\varepsilon}{\partial r} = -\int_0^r \rho\varphi_\varepsilon(\rho)d\rho$ and get the value for K_ε. □

To fix the ideas, we choose a good cutoff function, given by Raviart [19] in a general context of approximations, and proposed by Bossy [3] for a numerical study of the vortex algorithm. We consider

$$\varphi(r) = \frac{2(2-r^2)}{\pi(1+r^2)^4}.$$

The function φ is a C_b^1-function. By Lemma 4.1 we compute

$$K_\varepsilon(x) = \frac{4\varepsilon^4 + (r^2+3\varepsilon^2)r^2}{2\pi(r^2+\varepsilon^2)^3}(-x_2, x_1). \tag{4.1}$$

Since for each fixed $\varepsilon > 0$, the function φ_ε belongs to $L^\infty \cap L^1$ and the kernel K is integrable near 0 and bounded at infinity, the function K_ε is bounded. Moreover, it is Lipschitz continuous since φ_ε is in C_b^1. We denote by M_ε the maximum value of K_ε on $\mathbb{R}^2$ (which behaves as $\frac{1}{\varepsilon^2}$ when $\varepsilon \ll 1$), and by L_ε a Lipschitz constant (which behaves as $\frac{1}{\varepsilon^3}$).

We now define the interacting particle system we are interested in.

Definition 4.2. *Consider a sequence $(B^i)_{i\in\mathbb{N}}$ of independent Brownian motions on $\mathbb{R}^2$ and a sequence of independent variables $(Z_0^i)_{i\in\mathbb{N}}$ with values in $\mathbb{R}^2$ distributed according $\frac{|m_0|}{\|m_0\|}$, and independent of the Brownian motions. For a fixed ε, for each $n \in \mathbb{N}^*$, and $1 \le i \le n$, let us consider the interacting processes defined by*

$$Z_t^{in,\varepsilon} = Z_0^i + \sqrt{2\nu}B_t^i + \int_0^t K_\varepsilon * \tilde{\mu}_s^{n,\varepsilon}(Z_s^{in,\varepsilon})ds \tag{4.2}$$

where $\tilde{\mu}_s^{n,\varepsilon} = \frac{1}{n}\sum_{j=1}^n h(Z_0^j)\delta_{Z_s^{jn,\varepsilon}} \in \mathcal{P}(\mathbb{R}^2)$ is the weighted empirical measure of the system at time s.

We also define the limiting independent processes by

$$\bar{Z}_t^{i,\varepsilon} = Z_0^i + \sqrt{2\nu}B_t^i + \int_0^t K_\varepsilon * \tilde{P}_s^\varepsilon(\bar{Z}_s^{i,\varepsilon})ds, \tag{4.3}$$

where P_s^ε is the law of $\bar{Z}_s^{i,\varepsilon}$, and $\tilde{P}_s^\varepsilon$ is defined from P_s^ε by (3.1).

Proposition 4.3. *1) For each $T > 0$ and for each n, there exits a unique (pathwise) solution to the interacting particle system (4.2) in $C([0,T],\mathbb{R}^{2n})$ and a unique (pathwise) solution to the nonlinear equation (4.3) in $C([0,T],\mathbb{R}^2)$.*

2) For a fixed $\varepsilon > 0$, for each $T > 0$,

$$E(\sup_{t\le T}|Z_t^{in,\varepsilon} - \bar{Z}_t^{i,\varepsilon}|) \le \frac{M_\varepsilon}{\sqrt{n}L_\varepsilon}\exp(\|m_0\|TL_\varepsilon). \tag{4.4}$$

Proof. We are in the well-known case of a McKean-Vlasov equation

$$dY_t = b[Y_t, m_t]dt + \sigma[Y_t, m_t]dB_t\ ;\ m_t = \mathcal{L}(Y_t)$$

with $b[x,\mu] = \int b(x,z)\mu(dz)\ ;\ \sigma[x,\mu] = \int \sigma(x,z)\mu(dz)$. Here, $\sigma(x,z) = \sqrt{2\nu}$ and $b(x,z) = K_\varepsilon(x-z) = \int \varphi_\varepsilon(x-z-y)K(y)dy$. Since φ_ε is in $C_b^\infty \cap L^1(\mathbb{R}^2)$, it is obvious that $b \in C_b^\infty(\mathbb{R}^4)$.

The proof of the first assertion is standard and can be adapted from Sznitman [21] Theorem 1.1 and the second assertion comes from an easy adaptation of the computations in Jourdain-Méléard [9] Proposition 2.3. See also Jourdain [10]. □

4.2. The approximating interacting particle system

We now consider $T > 0$ and a sequence (ε_n) tending to 0 such that

$$\lim_n \frac{M_{\varepsilon_n}}{\sqrt{n}L_{\varepsilon_n}}\exp(\|m_0\|TL_{\varepsilon_n}) = 0. \tag{4.5}$$

Let us now consider for each n the coupling of processes $(Z^{in}, \bar{Y}^{in})_{1\le i\le n}$ driven by the same Brownian motion, where $Z^{in} = Z^{in,\varepsilon_n}$ are defined with the drift K_{ε_n} as in (4.2), and $\bar{Y}^{in} = \bar{Z}^{i,\varepsilon_n}$.

Let us denote by P^n the common law of each $\bar{Y}^{in}$. Then since the drift term is bounded, it turns out from the Girsanov theorem that $\forall s > 0$, the law $\tilde{P}_s^n$ admits a density function p_s^n of class C^∞, which is solution of the equation

$$\frac{\partial p^n}{\partial t} = \nu\Delta p^n - (K_{\varepsilon_n} * p^n \cdot \nabla)p^n\ ;\ p_0^n = m_0. \tag{4.6}$$

(As previously, we get (4.6) by computing $\phi(\bar{Y}_t^{in})$ for a smooth function ϕ by Itô's formula and by taking expectations after having multiplied by $h(Z_0^i)$).

A key point of the paper is the following result.

Theorem 4.4. *1) For each n, there exist a kernel $\Gamma_n(x,t,y,s)$ and constants C_j depending only on ν such that $\forall n, \forall x, y \in \mathbb{R}^2$ and $0 \le s < t \le T$,*

$$p^n(x,t) = \int_{\mathbb{R}^2}\Gamma_n(x,t,y,0)m_0(dy),\ t > 0 \tag{4.7}$$

$$\frac{C_1}{(t-s)}\exp(-C_2\frac{|x-y|^2}{(t-s)}) \le \Gamma_n(x,t,y,s) \le \frac{C_3}{(t-s)}\exp(-C_4\frac{|x-y|^2}{(t-s)}). \tag{4.8}$$

2) We deduce immediately that $\forall r \geq 1$,

$$\sup_n \sup_{t \leq T} t^{1-\frac{1}{r}} \|p_t^n\|_r \leq C\|m_0\|. \tag{4.9}$$

The constant C *depends only on* r *and* ν.

Proof. The function $(s, y) \mapsto K_{\varepsilon_n} * p_s^n(y)$ is clearly continuous and bounded on $[0, T] \times \mathbb{R}^2$, and $y \mapsto K_{\varepsilon_n} * p_s^n(y)$ is Lipschitz continuous, uniformly in $s \in [0, T]$. Then, (4.7) and (4.8) follow from results of Friedman ([6] Theorem 4.5), but the constants C_i depend a priori on n. We apply Theorem 2.4. Let us prove that the functions $K_{\varepsilon_n} * p^n$ satisfy the required assumptions. Since $\nabla \cdot K = 0$, then $\nabla \cdot K_{\varepsilon_n} = 0$, and hence $\nabla \cdot K_{\varepsilon_n} * p^n = 0$.

Moreover, using Lemma 2.5,

$$K_{\varepsilon_n}(x) = (\partial_{x_1}(A_3 * \varphi_{\varepsilon_n})(x) + \partial_{x_2}(A_1 * \varphi_{\varepsilon_n})(x), -\partial_{x_1}(A_1 * \varphi_{\varepsilon_n})(x) - \partial_{x_2}(A_2 * \varphi_{\varepsilon_n})(x)).$$

Since p^n and φ_{ε_n} are densities of probability, the functions $K_{\varepsilon_n} * p^n$ satisfy the assumption (b) of Theorem 2.4, with d independent of ε_n and n.

The proof of Theorem 4.4 is thus immediate. □

Let us now introduce for each n the coupling of processes $(Z^{in}, \bar{Y}^{in}, \bar{X}^i)_{1 \leq i \leq n}$, where $(\bar{X}^i)$ are independent copies of $\bar{X}$ defined as in (3.2) on a certain probability space and $Z^{in}, \bar{Y}^{in}$ are driven, for each i respectively, following the same Brownian motion as $\bar{X}^i$. We will now compare the two processes $\bar{Y}^{in}$ and $\bar{X}^i$. We need to estimate $v - p^n$.

As in the proof of (2.16), and using the boundedness of K_{ε_n}, (2.17) and Fubini's theorem, we can prove that p^n is solution of the evolution equation

$$p_t^n(x) = G_t^\nu * m_0(x) + \int_0^t \nabla_x G_{t-s}^\nu * (p_s^n \cdot K_{\varepsilon_n} * p_s^n)(x) ds. \tag{4.10}$$

Using (2.16), we obtain

$$p_t^n(x) - v_t(x) = \int_0^t \int_{\mathbb{R}^2} \nabla_x G_{t-s}^\nu (x-y). \left(K_{\varepsilon_n} * p_s^n(y) p_s^n(y) - K * v_s(y) v_s(y) \right) dy ds \tag{4.11}$$

We now prove the

Theorem 4.5. *For every* $1 < q < 2$,

$$\sup_{t \leq T} t^{1-\frac{1}{q}} \|p_t^n - v_t\|_q \leq C(\varepsilon_n)^{\frac{2-q}{q}} \tag{4.12}$$

where the constant C *depends only on* T, q, ν *and* $\|m_0\|$.

The proof of this theorem uses some technical lemmas.

Lemma 4.6. *For all* $1 \leq l < 2$, *for all* $n \in \mathbb{N}^*$,

$$\|K_{\varepsilon_n} - K\|_l \leq C(\varepsilon_n)^{\frac{2-l}{l}} \tag{4.13}$$

where the constant C *depends only on* l.

Proof. We have

$$K_{\varepsilon_n}(x) - K(x) = \frac{\varepsilon_n^4}{2\pi} \frac{r^2 - \varepsilon_n^2}{(r^2 + \varepsilon_n^2)^3} \frac{1}{r^2}(-x_2, x_1).$$

Then, for $l \geq 1$,

$$\begin{aligned}
\|K_{\varepsilon_n} - K\|_l^l &\leq \frac{(\varepsilon_n)^{4l}}{(2\pi)^{l-1}} \int_0^{+\infty} \frac{(r^2 - \varepsilon_n^2)^l}{(r^2 + \varepsilon_n^2)^{3l} r^{l-1}} dr \\
&\leq (\varepsilon_n)^{2-l} \frac{1}{(2\pi)^{l-1}} \int_0^{+\infty} \frac{(\alpha^2 - 1)^l}{(\alpha^2 + 1)^{3l} \alpha^{l-1}} d\alpha \\
&\leq C(\varepsilon_n)^{2-l}, \text{ for } l < 2.
\end{aligned}$$

□

Corollary 4.7. *1) For all* n, $s \in [0, T]$, $1 < q < 2$,

$$\|K_{\varepsilon_n} * p_s^n - K * p_s^n\|_q \leq C_q(\varepsilon_n)^{\frac{2-q}{q}} \tag{4.14}$$

2) For all n, $s \in [0, T]$, $p > 2$,

$$\begin{aligned}
\|K_{\varepsilon_n} * p_s^n - K * p_s^n\|_p &\leq \|K_{\varepsilon_n} - K\|_q \|p_s^n\|_2 \\
&\leq C_q s^{\frac{-1}{2}} (\varepsilon_n)^{\frac{2-q}{q}} \|m_0\|,
\end{aligned} \tag{4.15}$$

and $q \in]1, 2[$ *is related to* p *by* $\frac{1}{q} = \frac{1}{p} + \frac{1}{2}$, *and* C_q *is a constant depending only on* q *and* ν.

Proof. In the two cases one uses the Young inequality: if $f \in L^q$ and $g \in L^m$, then $f * g \in L^p$, for all $\frac{1}{p} = \frac{1}{q} + \frac{1}{m} - 1$ and $\|f * g\|_p \leq \|f\|_q \|g\|_m$.

1) we take $p = q$ and $m = 1$, and apply the previous lemma.

2) We take $1 < q < 2$ and p with $\frac{1}{q} = \frac{1}{p} + \frac{1}{2}$, and $m = 2$. Then the result follows from (4.9). □

Corollary 4.8. *For all* $1 < q < 2$ *and* $\frac{1}{r} = \frac{2}{q} - \frac{1}{2}$, *for all* $n \in \mathbb{N}^*$ *and* $t \leq T$,

$$\int_0^t (t-s)^{-\frac{1}{q}} \|(K_{\varepsilon_n} * p_s^n - K * p_s^n) p_s^n\|_r ds \leq C(\varepsilon_n)^{\frac{2-q}{q}} \tag{4.16}$$

where C *is a real constant depending only on* $T, q, \nu, \|m_0\|$.

Proof.

$$\int_0^t (t-s)^{-\frac{1}{q}} \|(K_{\varepsilon_n} * p_s^n - K * p_s^n) p_s^n\|_r ds$$
$$\leq \int_0^t (t-s)^{-\frac{1}{q}} \|K_{\varepsilon_n} * p_s^n - K * p_s^n\|_q \|p_s^n\|_{q'} ds \text{ with } \frac{1}{q} + \frac{1}{q'} = \frac{1}{r}$$
$$\leq C(\varepsilon_n)^{\frac{2-q}{q}} \int_0^t (t-s)^{-\frac{1}{q}} s^{-1+\frac{1}{q'}} ds.$$

But $-1 + \frac{1}{q'} = -1 + \frac{1}{r} - \frac{1}{q} = -1 + \frac{1}{q} - \frac{1}{2} = \frac{1}{q} - \frac{3}{2} > -1$.

Then the integral converges and the corollary follows. □

Let us now come back to the proof of Theorem 4.5.

Proof. As in the proof of Theorem 2.9, for $1 < q < 2$ and $\frac{1}{r} = \frac{2}{q} - \frac{1}{2}$, we can write

$$\|p_t^n - v_t\|_q$$
$$\leq C \int_0^t (t-r)^{-\frac{1}{2}-(\frac{1}{r}-\frac{1}{q})} \Big(\|(K_{\varepsilon^n} * p_s^n - K * v_s) p_s^n\|_r + \|K * v_s (p_s^n - v_s)\|_r \Big) ds$$
$$\leq C \int_0^t (t-s)^{-\frac{1}{q}} \Big(\|(K_{\varepsilon_n} * p_s^n - K * p_s^n) p_s^n\|_r + \|K * p_s^n - K * v_s\|_{q'} \|p_s^n\|_q$$
$$+ \|K * v_s\|_{q'} \|p_s^n - v_s\|_q \Big) ds \quad \text{with } \frac{1}{q} + \frac{1}{q'} = \frac{1}{r},\ (q' > 2)$$
$$\leq C(\varepsilon_n)^{\frac{2-q}{q}} + C \int_0^t (t-s)^{-\frac{1}{q}} \|p_s^n - v_s\|_q (\|p_s^n\|_q + \|v_s\|_q) ds$$
by (2.8) and Corollary 4.8
$$\leq C(\varepsilon_n)^{\frac{2-q}{q}} + C \int_0^t (t-s)^{-\frac{1}{q}} s^{-1+\frac{1}{q}} \|p_s^n - v_s\|_q \Big) ds$$

The function $t \mapsto \|p_s^n - v_t\|_q$ is not bounded on $[0, T]$, so we introduce $|||p_t^n - v_t|||_q = t^{1-\frac{1}{q}} \|p_t^n - v_t\|_q$ and we obtain

$$|||p_t^n - v_t|||_q \leq C(\varepsilon_n)^{\frac{2-q}{q}} t^{1-\frac{1}{q}} + C t^{1-\frac{1}{q}} \int_0^t (t-s)^{-\frac{1}{q}} s^{-2+\frac{2}{q}} |||p_s^n - v_s|||_q ds. \tag{4.17}$$

Since the function $s \mapsto (t-s)^{-\frac{1}{q}} s^{\frac{2}{q}-2}$ is integrable in 0 and in t, we can apply Gronwall's lemma and finally deduce (4.12). □

By associating Theorem 4.5 and Corollary 4.7, we deduce

Corollary 4.9. *For any $p > 2, n \in \mathbb{N}^*, t \leq T$,*

$$\|K_{\varepsilon_n} * p_t^n - K * v_t\|_p \leq C(\varepsilon_n)^{\frac{2-q}{q}} (t^{-\frac{1}{2}} + t^{-1+\frac{1}{q}}), \tag{4.18}$$

where C depends only on $p, T, \nu, \|m_0\|$ and $\frac{1}{q} = \frac{1}{p} + \frac{1}{2}$.

Proof.

$$\|K_{\varepsilon_n} * p_t^n - K * v_t\|_p \leq \|K_{\varepsilon_n} * p_t^n - K * p_t^n\|_p + \|K * p_t^n - K * v_t\|_p$$
$$\leq C(\varepsilon_n)^{\frac{2-q}{q}} t^{-\frac{1}{2}} + C\|p_t^n - v_t\|_q$$
$$\leq C(\varepsilon_n)^{\frac{2-q}{q}} t^{-\frac{1}{2}} + C(\varepsilon_n)^{\frac{2-q}{q}} t^{-1+\frac{1}{q}}. \qquad \square$$

5. The convergence theorem

Let us now come back to the processes we have introduced before. We are interested in proving a propagation of chaos result for the interacting particle system (Z^{in}) to the solution P of the nonlinear martingale problem $(\mathcal{M})$ defined in Section 3.

Theorem 5.1. *$\forall T > 0$, the laws of the particle systems $(Z^{in})_{1\leq i\leq n}$, considered as probability measures on the path space $\mathcal{C}([0,T],\mathbb{R}^2)$, are P-chaotic in the sense that for every fixed k,*

$$\mathcal{L}(Z^{1n}, ..., Z^{kn}) \Longrightarrow P^{\otimes k}\ ,\ n \to +\infty. \tag{5.1}$$

This convergence is then a pathwise convergence.

Remark 5.2. *Unhappily, and in contrast to the case with an initial bounded density data, we do not obtain a L^1-type convergence, and we have no rate of convergence. That is due to the degenerated behaviour of the laws of the processes at time* 0.

Remark 5.3. *Since the laws $\mathcal{L}(Z^{1n}, ..., Z^{nn})$ are exchangeable, the propagation of chaos for the system is equivalent to the convergence in probability of its empirical measures to P, as probability measures on the path space (cf. [21]). That implies the convergence in probability of the flow of weighted empirical measures $(\tilde{\mu}_t^{n,\varepsilon_n})_{0\leq t\leq T}$, where*

$$\tilde{\mu}_t^{n,\varepsilon_n} = \frac{1}{n}\sum_{i=1}^n h(Z_0^i)\delta_{Z_t^{in}},$$

to the flow $(v_t(x)dx)_{0\leq t\leq T}$ in the space $C([0,T],M_F)$. Indeed, Theorem 5.1 implies immediatly the convergence of the flow of the empirical measures $(\mu_t^{n,\varepsilon_n})_{0\leq t\leq T}$ to $(P_t)_{t\geq 0}$. The weight function h is not necessarily continuous and we approximate it by a sequence of continuous functions h_k bounded as h by $\|m_0\|$, in the sense that $\frac{|m_0|}{\|m_0\|}(\{h_k \neq h\}) \leq \frac{1}{k}$. Then if F is a continuous and bounded function on $\mathbb{R}^2$,

$$E|\langle\tilde{\mu}_t^{n,\varepsilon_n}, F\rangle - \int F(x)v_t(x)dx|$$
$$\leq E|\frac{1}{n}\sum_{i=1}^n h(Z_0^i)F(Z_t^{in}) - \langle P, h(X_0)F(X_t)\rangle|$$
$$\leq E|\frac{1}{n}\sum_{i=1}^n (h(Z_0^i) - h_k(Z_0^i))F(Z_t^{in})| + E|\frac{1}{n}\sum_{i=1}^n h_k(Z_0^i)F(Z_t^{in})$$
$$-\langle P, h_k(X_0)F(X_t)\rangle| + |\langle P, (h_k(X_0) - h(X_0))F(X_t)\rangle|.$$

The first and third terms tend to 0 *as* k *tends to infinity, and if now* k *is fixed, the second term tends to* 0 *as* n *tends to infinity.*

Let us now prove some preliminaries for the proof of Theorem 5.1.

Proposition 5.4. *For each* $1 < q < 2$*, for each* $1 \leq i \leq n$*, we have*

$$E(\sup_{t\leq T}\int_0^t |K_{\varepsilon_n} * p_s^n(\bar{Y}_s^{in}) - K * v_s(\bar{Y}_s^{in})|ds) \leq C(\varepsilon_n)^{\frac{2-q}{q}}, \tag{5.2}$$

where the constant C *depends only on* T, ν, q *and* $\|m_0\|$.

Proof. We have seen that for each $s > 0$ and $i \in \mathbb{N}^*$, the variable $\bar{Y}_s^{in}$ has the law $p_s^n(x)dx$. Then,

$$\begin{aligned}
&E(\sup_{t\leq T}\int_0^t |K_{\varepsilon_n} * p_s^n(\bar{Y}_s^{in}) - K * v_s(\bar{Y}_s^{in})|ds)\\
&\leq \int_0^T (\int |K_{\varepsilon_n} * p_s^n(x) - K * v_s(x)|p_s^n(x)dx)ds\\
&\leq \int_0^T \|K_{\varepsilon_n} * p_s^n - K * v_s\|_p \|p_s^n\|_{p'} ds \quad \text{for } p > 2 \text{ and } \frac{1}{p'} = 1 - \frac{1}{p}\\
&\leq C(\varepsilon_n)^{\frac{2-q}{q}} \int_0^T (s^{-\frac{1}{2}} + s^{-1+\frac{1}{q}}) s^{\frac{1}{p'}-1} ds \quad \text{by Corollary 4.9, with} \frac{1}{q} = \frac{1}{p} + \frac{1}{2}\\
&\leq C(\varepsilon_n)^{\frac{2-q}{q}} \int_0^T (s^{-\frac{3}{2}+\frac{1}{p'}} + s^{-2+\frac{1}{q}+\frac{1}{p'}}) ds\\
&\leq C(\varepsilon_n)^{\frac{2-q}{q}},
\end{aligned}$$

since $\frac{1}{p'} = \frac{3}{2} - \frac{1}{q}$ and then the integral is convergent. □

Proposition 5.5. *For each* $x, z \in \mathbb{R}^2$*, for each* $1 < q < 2$*,*

$$|K * v_s(x) - K * v_s(z)| \leq \phi_{s,q}(|x - z|) \tag{5.3}$$

where for each $r > 0$*,* $\phi_{s,q}(r) = C(r + s^{-\frac{1}{q}} r^{\frac{2}{q}-1})$ *if* $0 < r < 1$ *and* $\phi_{s,q}(r) = 1 + s^{-\frac{1}{q}}$ *if* $r \geq 1$.

Proof. The proof follows approximately the same steps than the one of Marchiro-Pulvirenti ([12] Lemma 3.1), but with the additionnal difficulty that the functions v_s are not in L^∞. Let us denote $r = |x - z|$ and $A = \{y \in \mathbb{R}^2, |x - y| \leq 2r\}$. Then

$$\begin{aligned}
|K * v_s(x) - K * v_s(z)| &\leq \int_A |K(x - y) - K(z - y)|v_s(y)dy\\
&+ \int_{A^c} |K(x - y) - K(z - y)|v_s(y)dy.
\end{aligned}$$

Now,

$$\int_A |K(x-y) - K(z-y)| v_s(y) dy$$
$$\leq \int_{|x-y|\leq 2r} \frac{1}{|x-y|} v_s(y) dy + \int_{|z-y|\leq 3r} \frac{1}{|z-y|} v_s(y) dy$$
$$\leq \left(\left(\int_{|y|\leq 2r} \frac{1}{|y|^q} dy \right)^{\frac{1}{q}} + \left(\int_{|y|\leq 3r} \frac{1}{|y|^q} dy \right)^{\frac{1}{q}} \right) \|v_s\|_{q'} \quad \text{with } \frac{1}{q} + \frac{1}{q'} = 1$$
$$\leq C s^{-\frac{1}{q}} r^{\frac{2}{q}-1}.$$

By a Taylor expansion, we obtain

$$\int_{A^c} |K(x-y) - K(z-y)| v_s(y) dy \leq r \int_{A^C} \frac{1}{|x''-y|^2} v_s(y) dy$$

where $x'' \in [x, z]$. We remark $|x''-y| > \frac{1}{2}|x-y|$ if $y \in A^c$. Then, for $r < 1$,

$$\begin{aligned} \int_{A^c} |K(x-y) - K(z-y)| v_s(y) dy &\leq Cr \int_{A^c} \frac{1}{|x-y|^2} v_s(y) dy \\ &\leq Cr \int_{2r<|x-y|<2} \frac{1}{|x-y|^2} v_s(y) dy \\ &\quad + Cr \int_{2<|x-y|} \frac{1}{|x-y|^2} v_s(y) dy. \end{aligned}$$

The second term is trivially upperbounded by Cr since $v_s \in L^1$, uniformly in s. Now

$$\begin{aligned} r \int_{2r<|x-y|<2} \frac{1}{|x-y|^2} v_s(y) dy &\leq Cr \left(\int_{2r}^{2} \frac{\rho}{(\rho)^{2q}} d\rho \right)^{\frac{1}{q}} \|v_s\|_{q'} \\ &\leq C s^{-\frac{1}{q}} r \left(r^{2-2q} - 2^{2-2q} \right)^{\frac{1}{q}} \\ &\leq C s^{-\frac{1}{q}} r^{\frac{2}{q}-1} \end{aligned}$$

Then

$$\int_{A^c} |K(x-y) - K(z-y)| v_s(y) dy \leq Cr + C s^{-\frac{1}{q}} r^{\frac{2}{q}-1}.$$

If $r \geq 1$,

$$\begin{aligned} |K * v_s(x) - K * v_s(z)| &\leq |K * v_s(x)| + |K * v_s(z)| \leq 2\|K * v_s\|_\infty \\ &\leq C(1 + s^{-\frac{1}{q}}). \end{aligned}$$

□

Proposition 5.6. $\forall T > 0$, *the sequence of processes* $(\bar{Y}^{1n})_{n\geq 1}$ *converges in law in* $C([0,T], \mathbb{R}^2)$ *to the process* $(\bar{X}^1)$ *as* n *tends to infinity.*

Proof. The proof has two steps. Firstly we prove the uniform tightness of the laws P^n of $(\bar{Y}^{1n})$ and secondly we identify the limiting process.

1) Exactly as in (2.10) and (2.15), we can prove that

$$\|K_{\varepsilon_n} * p_s^n\|_\infty \leq C(s^{-\frac{1}{q}} + 1)$$

where C depends only on $q, \nu, \|m_0\|$, for any $1 < q < 2$. Then the function $t \mapsto \int_0^t \|K_{\varepsilon_n} * p_s^n\|_\infty ds$ is continuous and then uniformly continuous on $[0, T]$. So the Aldous criterion (cf. Aldous [1]) is satisfied for the laws of $(\bar{Y}^{1n})$. The initial laws being all equal to $\frac{|m_0|}{\|m_0\|}$, the laws of $(\bar{Y}^{1n})$ are uniformly tight.

2) Let us now prove that there is a unique limiting law equal to P. Let us denote by Q a limit value of the sequence (P^n). We have to prove that Q satisfies the nonlinear martingale problem $(\mathcal{M})$ defined in Section 3.

If as usual X denotes the canonical process on $C([0, T], \mathbb{R}^2)$, let us define, for any smooth enough function ϕ, for bounded continuous functions $g_1, g_2, ..., g_k$, for $0 < s_1 < ... < s_k \leq s < t \leq T$, the function

$$G_n(X) = \left(\phi(X_t) - \phi(X_s) - \int_s^t \nu\Delta\phi(X_u)du - \int_s^t K_{\varepsilon_n} * p_u^n(X_u).\nabla\phi(X_u)du\right)g_1(X_{s_1})...g_k(X_{s_k}). \qquad (5.4)$$

Then the law Q^n is characterized by $E^{P^n}(G_n(X)) = 0$. Now if we define the function G by the same formula as (5.4) in which we have replaced $K_{\varepsilon_n} * p_u^n$ by $K * v_u$, the distribution law P is characterized by $E^P(G(X)) = 0$.

Let us prove that $E^Q(G(X)) = 0$.

$$E^Q(G(X)) = E^Q(G(X)) - E^{P^n}(G(X)) + E^{P^n}(G(X) - G^n(X))$$

Proposition 5.5 implies that $x \mapsto K * v_s(x)$ is a continuous function, and we have seen otherwise that $s \mapsto \|K * v_s\|_\infty$ is integrable on $[0, T]$. So the function $X \mapsto G(X)$ is a continuous function on the path space, and the first term of the right hand side of the previous expression tends to 0 as n tends to infinity.

In another hand,

$$\begin{aligned} E^{P^n}|G^n(X) - G(X)| &\leq E^{P^n}\left(\int_0^t |(K_{\varepsilon_n} * p_s^n(X_s) - K * v_s(X_s)).\nabla\phi(X_s)|ds\right) \\ &\quad |g_1(X_{s_1})...g_k(X_{s_k})| \\ &\leq CE\left(\int_0^t |K_{\varepsilon_n} * p_s^n(\bar{Y}_s^{1n}) - K * v_s(\bar{Y}_s^{1n})|ds\right) \\ &\leq C(\varepsilon_n)^{\frac{2-q}{q}} \quad \text{by Proposition 5.4 .} \end{aligned}$$

Thus, $E^Q(G(X)) = 0$, and $Q = P$. □

Proof of Theorem 5.1. Since the processes $(\bar{Y}^{in})_i$ are independent, and the same for $(\bar{X}^i)_i$, Proposition 5.6 implies that for every fixed k, the law of $(\bar{Y}^{1n}, ..., \bar{Y}^{kn})$ converges to the law of $(\bar{X}^1, ..., \bar{X}^k)$.

Otherwise, by adding the results obtained in Proposition 4.3 and in (4.5), we obtain that for each $i \in \{1, ..., n\}$,

$$E\left(\sup_{t \leq T} |Z_t^{in} - \bar{Y}_t^{in}|\right) \leq \frac{M_{\varepsilon_n}}{\sqrt{n} L_{\varepsilon_n}} \exp(\|m_0\| T L_{\varepsilon_n})$$

and then tends to 0 as n tends to infinity. Now, if we endow $\mathcal{P}(C([0, T], \mathbb{R}^{2k}))$ with the metric

$$\rho(P, Q) = \inf \left\{ \int_{C([0,T],\mathbb{R}^{2k}) \times C([0,T],\mathbb{R}^{2k})} \sup_{0 \leq t \leq T} |x_t - y_t| R(dx, dy); \right. \\ \left. R \text{ has marginals } P \text{ and } Q \right\}$$

We have:

$$\begin{aligned} &\rho(\mathcal{L}(Z^{1n}, ..., Z^{kn}), \mathcal{L}(\bar{X}^1, ..., \bar{X}^k)) \\ &\leq \rho(\mathcal{L}(Z^{1n}, ..., Z^{kn}), \mathcal{L}(\bar{Y}^{1n}, ..., \bar{Y}^{kn})) + \rho(\mathcal{L}(\bar{Y}^{1n}, ..., \bar{Y}^{kn}), \mathcal{L}(\bar{X}^1, ..., \bar{X}^k)) \\ &\leq k \frac{M_{\varepsilon_n}}{\sqrt{n} L_{\varepsilon_n}} \exp(\|m_0\| T L_{\varepsilon_n}) + \rho(\mathcal{L}(\bar{Y}^{1n}, ..., \bar{Y}^{kn}), \mathcal{L}(\bar{X}^1, ..., \bar{X}^k)). \end{aligned}$$

Since the two terms of the sum tend to 0, then $\rho(\mathcal{L}(Z^{1n}, ..., Z^{kn}), \mathcal{L}(\bar{X}^1, ..., \bar{X}^k))$ converges to 0, and we get the propagation of chaos result we wished. □

We deduce now from Theorem 5.1 a theorical justification for Monte-Carlo approximations for the equation (2.1).

Theorem 5.7. *For each $t \in [0, T]$ and $x \in \mathbb{R}^2$, the random variable $\frac{1}{n} \sum_{i=1}^n h(Z_0^i) K_{\varepsilon_n}(x - Z_t^{in})$ converges in law and then in probability to $u(t, x)$. The convergence is uniform for $t \in [\eta, T]$, for each $0 < \eta \leq T$.*

Proof. Since $\sup_{t \leq T} \rho(\tilde{\mu}_t^{n,\varepsilon_n}, \tilde{P}_t)$ tends to 0 as n tends to infinity (cf. Remark 5.3), $\frac{1}{n} \sum_{i=1}^n h(Z_0^i) F(Z_t^{in})$ converges in law and then in probability to $\int F(x) v_t(x) dx$ uniformly in $t \in [0, T]$ and for each bounded and continuous function F on $\mathbb{R}^2$. Let us fix $q \in]1, 2[$, let $\alpha > 0$ and consider n_0 defined thanks to (4.13) such that

$$\|K_{\varepsilon_{n_0}} - K\|_q \leq \alpha.$$

For each fixed $x \in \mathbb{R}^2$, the function $y \mapsto K_{\varepsilon_{n_0}}(x - y)$ is bounded and continuous, so $\frac{1}{n} \sum_{i=1}^n h(Z_0^i) K_{\varepsilon_{n_0}}(x - Z_t^{in})$ converges in law and then in probability to

$\int K_{\varepsilon_{n_0}}(x-y)v_t(y)dy$ uniformly in $t \in [0, T]$. But,

$$\int |K_{\varepsilon_n}(x-y) - K_{\varepsilon_{n_0}}(x-y)|v_t(y)dy \leq \|K_n - K_{n_0}\|_q \|v_t\|_{q'}$$

$$\text{with } 1 \leq q < 2 \text{ and } \frac{1}{q} + \frac{1}{q'} = 1$$

$$\leq (\|K_n - K\|_q + \|K - K_{n_0}\|_q)\|v_t\|_{q'}$$

$$\leq C((\varepsilon_n)^{\frac{2-q}{q}} + \alpha)t^{-\frac{1}{q}}$$

where the constant C depends only on q, $\|m_0\|$ and ν. Finally we estimate the quantity

$$E|\frac{1}{n}\sum_{i=1}^{n} h(Z_0^i)(K_{\varepsilon_n} - K_{\varepsilon_{n_0}})(x - Z_t^{in})| \leq \|m_0\| E|(K_{\varepsilon_n} - K_{\varepsilon_{n_0}})(x - Z_t^{1n})|.$$

We remark, following Osada [17] p.602 and using Lemma 2.5, that the generator of the particle system $(Z^{1n}, ..., Z^{nn})$ is of generalized divergence form. Then as already seen before (see Theorem 4.4), for each $t > 0$, the law of the random variable Z_t^{1n} has a density of probability w_t^n satisfying (2.11), with a constant depending only on ν and r. Now,

$$E|(K_{\varepsilon_n} - K_{\varepsilon_{n_0}})(x - Z_t^{1n})| = \int |(K_{\varepsilon_n} - K_{\varepsilon_{n_0}})(x-y)w_t^n(y)dy|$$

$$\leq \|K_{\varepsilon_n} - K_{\varepsilon_{n_0}}\|_q \|w_t^n\|_{q'}$$

$$\leq C((\varepsilon_n)^{\frac{2-q}{q}} + \alpha)t^{-\frac{1}{q}} \quad (5.5)$$

which tends to 0 as n tends to infinity, and Theorem 5.7 is proved. □

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