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Convex duality and the Skorokhod Problem. I*

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Abstract. The solution to the Skorokhod Problem defines a deterministic mapping, referred to as the Skorokhod Map, that takes unconstrained paths to paths that are confined to live within a given domain $G \subset \mathbb{R}^n$. Given a set of allowed constraint directions for each point of ∂G and a path ψ, the solution to the Skorokhod Problem defines the constrained version ϕ of ψ, where the constraining force acts along one of the given boundary directions using the "least effort" required to keep ϕ in G. The Skorokhod Map is one of the main tools used in the analysis and construction of constrained deterministic and stochastic processes. When the Skorokhod Map is sufficiently regular, and in particular when it is Lipschitz continuous on path space, the study of many problems involving these constrained processes is greatly simplified.

We focus on the case where the domain G is a convex polyhedron, with a constant and possibly oblique constraint direction specified on each face of G, and with a corresponding cone of constraint directions at the intersection of faces. The main results to date for problems of this type were obtained by Harrison and Reiman [22] using contraction mapping techniques. In this paper we discuss why such techniques are limited to a class of Skorokhod Problems that is a slight generalization of the class originally considered in [22]. We then consider an alternative approach to proving regularity of the Skorokhod Map developed in [13]. In this approach, Lipschitz continuity of the map is proved by showing the existence of a convex set that satisfies

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a set of conditions defined in terms of the data of the Skorokhod Problem. We first show how the geometric condition of [13] can be reformulated using convex duality. The reformulated condition is much easier to verify and, moreover, allows one to develop a general qualitative theory of the Skorokhod Map. An additional contribution of the paper is a new set of methods for the construction of solutions to the Skorokhod Problem.

These methods are applied in the second part of this paper [17] to particular classes of Skorokhod Problems.

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1. Introduction

1.1. The Skorokhod Problem

The Skorokhod Problem (SP) provides a very useful tool for the construction and analysis of many constrained processes. Given a closed set $G \subset \mathbb{R}^n$, a set of unit vectors $d(x)$ for each point x on the boundary of G, and a path ψ taking values in $\mathbb{R}^n$, the solution to the SP defines a constrained version ϕ of ψ, where the constraint mechanism acts along the directions $d(\cdot)$ using the "least effort" required to keep ϕ in G. Although for historical reasons $d(\cdot)$ has been referred to in the literature as the set of "directions of reflection," we will use the more accurate term *directions of constraint*. A precise definition of the SP is as follows. Let $D([0,\infty) : \mathbb{R}^n)$ denote the set of functions mapping $[0,\infty)$ to $\mathbb{R}^n$ that are right continuous and have limits from the left. For $\eta \in D([0,\infty) : \mathbb{R}^n)$ let $|\eta|(T)$ denote the total variation of η on $[0,T]$ with respect to the Euclidean norm on $\mathbb{R}^n$.

Definition 1.1 (Skorokhod Problem). *Let $\psi \in D([0,\infty) : \mathbb{R}^n)$ with $\psi(0) \in G$ be given. Then (ϕ, η) solves the SP for ψ with respect to G and d if $\phi(0) = \psi(0)$, and if for all $t \in [0,\infty)$*

1. $\phi(t) = \psi(t) + \eta(t)$;
2. $\phi(t) \in G$;
3. $|\eta|(t) < \infty$;
4. $|\eta|(t) = \int_{[0,t]} I_{\{\phi(s)\in\partial G\}} \mathrm{d}|\eta|(s)$;
5. *There exists measurable $\gamma : [0,\infty) \to \mathbb{R}^n$ such that $\gamma(s) \in d(\phi(s))$* ($\mathrm{d}|\eta|$*-almost surely) and*
$$\eta(t) = \int_{[0,t]} \gamma(s)\mathrm{d}|\eta|(s).$$

Note that ϕ is constrained to remain within G, and that η changes only when ϕ is on the boundary ∂G, in which case the change points in one of the directions of $d(\phi)$. We will call the map $\psi \to \phi$ the Skorokhod Map (SM) and use Γ to denote the map wherever it is defined. When considering the existence of solutions to the SP for ψ in any given function space, it is assumed (even if it is not explicitly stated) that $\psi(0) \in G$.

The simplest version of the SP was formulated by Skorokhod in [39]. Skorokhod used regularity properties of the map in order to construct and establish strong uniqueness of solutions to stochastic differential equations on $\mathbb{R}_+ \doteq \{x \in \mathbb{R} : x \geq 0\}$ with a reflecting boundary condition at the origin. The SP has since proved to be convenient in analyzing a variety of processes. Examples include stochastic differential equations with reflection (SDER) [including the important special case of reflecting Brownian motion (RBM)], a related class of constrained ordinary differential equations, constrained stochastic approximation algorithms and related stochastic adaptive algorithms, and certain models of queueing and communication networks [1, 5, 7, 10, 15, 22, 29, 38, 39, 40, 41].

Such processes are ubiquitous in modern applied probability. For example, SDER occur in singular stochastic control and mathematical finance, as diffusion approximations for physical transport processes, and in many other areas. RBM models arise in "heavy-traffic" analysis of queueing, communication, and manufacturing models. Constrained continuous time stochastic processes that are not diffusions arise in current models for ATM-type data networks. A small selection of work on these applications is [5, 6, 21, 23, 24, 25, 26, 28, 31, 32, 33, 34, 36, 42, 43].

In recent years there has also been considerable interest in the related class of constrained ordinary differential equations. These ODEs often appear as law of large numbers limits. For example, under appropriate conditions they provide a characterization of and thus a means for analyzing the so-called "fluid limits" of queueing networks and related processes. These fluid limits are important because of their usefulness in establishing the stability and ergodicity of the prelimit stochastic model [4, 8, 9, 19]. It has also been proposed that they be used as the basis for a simple approach to the design of controls for queueing networks [2]. The same class of constrained ODEs provides the limit differential equation for use in the ODE approach to studying constrained stochastic algorithms [14]. In the setting of economics constrained ODEs define the proper continuous time evolution of prices, inventories, and related quantities in the presence of constraints (e.g., non-negative inventories) [15]. In all of these problems the domain G has corners, and in most the directions of constraint are oblique with respect to ∂G.

We will refer to the SM as "well behaved" if solutions to the SP exist for a broad class of unconstrained paths, and if the mapping from unconstrained to constrained paths is regular (i.e., Lipschitz continuous). It is well known that much of the analysis one would like to perform on the processes mentioned in the last paragraph (e.g., construction, uniqueness, approximation, simulation, weak and large deviation limit theories, etc.) is significantly simplified if the processes can be represented in terms of an appropriate "well behaved" SM. For example, constrained ODEs of the type studied in [13], [15] and [16] have discontinuous right-hand sides, and do not fall into the framework of the classical theory of ODEs or its extension developed by Fillipov [20]. The reformulation of such ODEs in terms of a SP enables one to address questions of existence and uniqueness of solutions for these ODEs. In the case of SDEs with reflection, techniques such as the martingale problem formulation are commonly used to characterize the distribution of the diffusion (see [44] for a specific example). As illustrated in [1, 40], when the SDER can be recast in terms of a SP the analysis is greatly simplified. In addition, strong existence and uniqueness of the diffusion can be established when the corresponding SM is defined on all continuous paths and is Lipschitz continuous [1, 39]. Finally, simple necessary and sufficient conditions for stability can be derived for constrained processes when their weak limits (under a law of large numbers scaling) can be represented in terms of a Lipschitz continuous SM [4].

Although there are many uses for a well behaved SM, apart from a few special cases relatively little is known regarding even basic qualitative properties of the map. As mentioned earlier, the one-dimensional case was first analyzed by Skorokhod [39] in order to construct a SDER on the half line. The paper [1] by Anderson and Orey considers the generalization of the one-dimensional SP to the half space $I\!R_+ \times I\!R^{n-1}$ with constant and oblique directions of constraint, and shows that the SP is Lipschitz on $C([0, T] : I\!R^n)$. The Lipschitz property allows a Picard iteration type construction of a strong solution to the SDER for the half space, as well as a simple proof of a large deviation result for the SDER when the diffusion matrix is multiplied by a small parameter. A "localization" argument using suitable coordinate transformations is used in order to solve SDERs for which the domain G is the closure of an open set possessing a smooth boundary. If $n(x)$ denotes the inward normal to ∂G at x, then the required conditions are that $d(x)$ vary smoothly for x in ∂G and that $\inf_{x \in \partial G} \langle d(x), n(x) \rangle > 0$. Other authors that have made use of the SP to analyze and construct SDER include Tanaka [40], Lions and Sznitman [29], and Saisho [38]. However, none of them obtains Lipschitz continuity and none considers the case of nonsmooth domains and oblique directions of constraint in any generality.

In the context of oblique directions of constraint on polyhedral domains, the only general results on Lipschitz continuity to date are [22] and [13]. In [22], the case where $G = I\!R^n_+$ and the directions of constraint satisfy a certain spectral radius condition is considered. A so called "reflection mapping" technique is employed to obtain existence and Lipschitz continuity of Γ on $C([0, T] : I\!R^n)$. (Although the Lipschitz property is not stated explicitly in [22], it follows directly from the technique used to prove continuity.) Chen and Mandelbaum [5] also consider $G = I\!R^n_+$, but with a different set of restrictions on d that is natural for the study of closed, single class queueing networks. They obtain continuity (though not Lipschitz continuity) of Γ on a particular subset of $C([0, T] : I\!R^n)$, which proves to be adequate for their purpose. In both [22] and [5], the properties of Γ are then used to construct RBMs. Dupuis and Ishii [13] consider solutions to the SP on $D([0, \infty) : I\!R^n)$ and for more general domains. They develop a sufficient condition for Lipschitz continuity of the SM which is phrased in terms of the existence of a convex set B satisfying a certain geometric property. Two particular classes of SPs are then considered in detail in [13]. It is proved that the set B exists for any SP on a polyhedral domain for which the constraint directions are the inward normals. It is also shown to exist for a class of SPs, which we refer to as the generalized Harrison–Reiman class, that satisfies a generalization of the algebraic criterion derived in [22]. The set B turns out to have other important applications, including the qualitative (existence and uniqueness) theory for partial differential equations that characterize functionals of constrained processes [11].

1.2. The objectives of this paper

One of the main objectives of this paper is to develop a method for deriving algebraic conditions for regularity (i.e. existence and Lipschitz continuity) of SMs associated with SPs on polyhedral domains with oblique directions of constraint. The two main existing approaches for studying regularity of SPs are the reflection mapping method of [22] and the geometric approach of [13]. We develop techniques that allow us to exactly identify which SPs can be analyzed using the reflection mapping techniques of [22]. More precisely, we show that the techniques of [22] are limited to a class of SPs that is a slight generalization of the one originally considered in [22], thereby motivating the geometric approach to the SP. While [13] was useful because it introduced the geometric approach, it fell short of indicating either how to verify the existence of the set B, or how the structure of the set B was linked to the geometry of the associated SP. It also did not prescribe any method for converting the geometric condition into algebraic conditions, which are in general more easily verified. The present paper makes two important contributions in this direction. First, we reformulate the geometric condition of [13] using convex duality. The conditions required of B are recast as conditions on a set B^* which is the convex dual of B. Although the dual conditions are still geometric in nature, they are easier to verify and characterize in an algebraic form than the original conditions on B. Indeed, the second main contribution is the development of a set of techniques for the conversion of geometric into algebraic characterizations. The methods developed in the paper are applied to three concrete classes of SPs in [17] to prove new general results on regularity of the associated SMs.

A brief outline of the paper is as follows. In Section 2 we state the sufficient geometric condition for Lipschitz continuity of the SM proved in [13]. We then derive a number of interesting properties of the set B which shed light on the connection between the reflection mapping technique of [22] and the geometric approach. We discuss why the reflection mapping approach is limited to the class of SPs we have labeled the generalized Harrison–Reiman class. A rigorous proof of this fact uses convex duality methods developed in this paper and is given in [17, Section 2.2]. The comparison of the two methods motivates the need for a systematic approach to the construction of B as well as the algebraic characterization of conditions for its existence.

Section 3 concentrates on methods for constructing the set B (and thereby establishing Lipschitz continuity of the associated SM) using convex duality. In this approach one does not directly construct B but instead one constructs its convex dual B^*. The conditions originally posed on B are reformulated in Section 3.2 as conditions on the dual set. Several necessary structural properties of the dual set are derived in Section 3.3, and these are used to identify families of convex sets that are appropriate for particular classes of SPs. Section 3.4 broaches the issue of when these geometric conditions can be transformed to algebraic ones, thereby yielding algebraic conditions for Lipschitz continuity of the SM. Section 4 develops a new set of tools to construct solutions to the SP and illustrates how the geometric condition proves useful in analyzing this aspect of the SP as well. The paper concludes with a few remarks in Section 5.

2. Lipschitz continuity of the Skorokhod Map

2.1. Introduction

In this paper we focus on the class of SPs described in Section 2.2 that have polyhedral domains. In Section 2.3 we present a sufficient condition for Lipschitz continuity of the corresponding SMs that was developed in

[13]. The condition is expressed in terms of the existence of a set B that satisfies a certain geometric property (Assumption 2.1). In Section 2.4 we review the main alternative approach to the analysis of SPs of the type we consider, which is due to Harrison and Reiman [22] and uses contraction mapping arguments.

The analysis of the SP in this paper will always require the construction of convex sets that satisfy Assumption 2.1 (or its dual formulation as given in Section 3.2). These sets have a number of interesting features and properties that can be used to simplify the problems of construction and characterization of the sets, and also to understand necessary conditions for the regularity of the SM. Section 2.5 discusses one of the most important characterizations of B: as an invariant set for a family of projection operators that are naturally associated to the given SP. These properties will turn out to be even more significant when we consider the construction of dual sets in Section 3.

2.2. Skorokhod Problems on polyhedral domains

In Definition 1.1 we introduced the SP associated with a domain G and directions of constraint $d(x)$. In this paper, we will concentrate on the case when the set G is a convex polyhedron with constant directions of constraint along each face. In this case the domain G takes the form $\cap_{i=1}^{N} G_i$, where each $G_i \doteq \{x : \langle x, n_i\rangle \geq c_i\}$ is a closed half space with inward normal n_i and intercept c_i. If $x \in \partial G$, and $I(x) \doteq \{i : x \in \partial G_i\} = \{j\}$, then $d(x)$ is assumed to be equal to $\{d_j\}$, where $\langle d_j, n_j\rangle > 0$. At all other points of ∂G $d(x)$ is defined by

$$d(x) \doteq \left\{ \gamma = \sum_{i \in I(x)} \alpha_i d_i : \alpha_i \geq 0, \|\gamma\| = 1 \right\}.$$

Thus for such domains the SP is completely specified by the set of triplets $\{(d_i, n_i, c_i), i = 1, \ldots, N\}$, where n_i is the inward normal to the closed half space G_i with intercept c_i, and d_i is the direction of constraint associated with that face. Note that $G \doteq \cap_{i=1}^{N} G_i$ need not always be a minimal description of the domain in the sense that there may exist $j \in \{1, \ldots, N\}$ such that $G = \cap_{i=1, i\neq j}^{N} G_i$. In such cases, G_j acts as a supporting hyperplane to the domain G, and is introduced along with the corresponding direction of constraint d_j in order to enlarge the set of directions of constraint applicable along the face $G_j \cap G$. Such situations are often encountered in examples [17, Section 3]. The setup considered here is the simplest one in which all the difficulties due to corners and nonsmooth directions of constraint are present. Problems in which the directions d_i vary smoothly within each face, or where the faces themselves are smooth but not flat, can often be reduced to the setup we consider via localization techniques [1, 13]. We always adopt the convention that the directions have been normalized so that $\langle d_i, n_i \rangle = 1$.

2.3. A sufficient condition for Lipschitz continuity: the set B

The class of SPs identified in the last subsection was studied in [13], and the main result stated there is a sufficient condition for the Lipschitz continuity (with respect to the sup norm metric) of the SM. The condition, which is stated below as Assumption 2.1, is characterized geometrically in terms of the existence of a convex set B that satisfies certain constraints on its inward normals as a function of angular position. Given a convex set $C \subset I\!R^n$ and $x \in \partial C$, define the set of inward normals to C at x by

$$\nu(x) \doteq \{\gamma : \|\gamma\| = 1, \text{ and } \langle \gamma, x - y \rangle \leq 0 \, \forall y \in C\}. \tag{2.1}$$

A set $C \subset I\!R^n$ is called *symmetric* if $x \in C \Rightarrow -x \in C$. Consider a SP with representation $\{(d_i, n_i, c_i), i = 1, \ldots, N\}$.

Assumption 2.1 (Set B). *There exists a compact, convex, symmetric set B with $0 \in B^\circ$, such that if $\nu(z)$ denotes the set of inward normals to B at $z \in \partial B$, then there exists $\delta > 0$ such that for $i = 1, \ldots, N$,*

$$\left.\begin{cases} z \in \partial B \\ |\langle z, n_i \rangle| < \delta \end{cases}\right\} \Rightarrow \langle \nu, d_i \rangle = 0 \quad \textit{for all } \nu \in \nu(z). \tag{2.2}$$

This assumption differs from the one used in [13] by the additional requirement that B be symmetric. This strengthening is in appearance only, as the assumption without the symmetry condition holds if and only if the assumption holds as stated. Obviously the assumption used in [13] is implied by Assumption 2.1. Conversely, it is easy to check that if B satisfies all the other conditions of Assumption 2.1 save the symmetry, then so do $-B$ and the symmetric set $D \doteq B \cap (-B)$. We impose the additional symmetry requirement since it greatly simplifies the construction of the dual set B^* (which is discussed in Section 3.3).

A two-dimensional SP and the associated set B that satisfies (2.2) for the SP is illustrated in Figure 1.

The following theorem is proved in [13, Theorem 2.2].

Theorem 2.1 (Lipschitz Continuity). *Suppose Assumption 2.1 is satisfied for a given SP $\{(d_i, n_i, c_i), i = 1, \ldots, N\}$. Let ψ_1 and ψ_2 in $D([0, \infty) : I\!R^n)$*

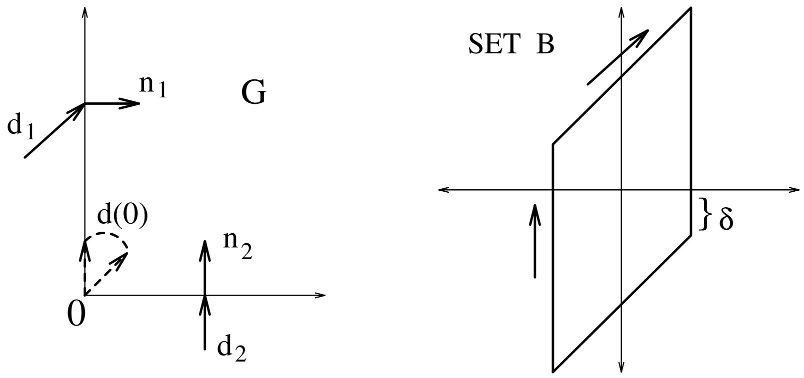

Fig. 1. A Skorokhod problem and the associated set B

be given, and assume (ϕ_1, η_1) *and* (ϕ_2, η_2) *are solutions to this SP for* ψ_1 *and* ψ_2*, respectively. Define*

$$\begin{cases} \Delta\phi \doteq \phi_1 - \phi_2 \\ \Delta\psi \doteq \psi_1 - \psi_2 \\ \Delta\eta \doteq \eta_1 - \eta_2 \ . \end{cases}$$

Then there exists $K < \infty$ *such that*

$$\sup_{t\in[0,\infty)} \|\Delta\eta(t)\| \leq K \sup_{t\in[0,\infty)} \|\Delta\psi(t)\|,$$

$$\sup_{t\in[0,\infty)} \|\Delta\phi(t)\| \leq K \sup_{t\in[0,\infty)} \|\Delta\psi(t)\|.$$

Thus the question of sufficient conditions for Lipschitz continuity of the SM can be reformulated as a static geometric problem.

Remark. Note that Theorem 2.1 guarantees Lipschitz continuity only for paths in $D([0,\infty) : I\!R^n)$ that lie in the domain of the SM, i.e. paths for which solutions to the SP exist. As discussed in greater detail in Section 4.2, there are often cases when the domain of the SP is a strict subset of $D([0,\infty) : I\!R^n)$, for example the SP analyzed in [17, Section 3].

Observe that if a set B satisfies (2.2) for a given value of $\delta > 0$, then for any $c > 0$ cB also satisfies the property but with δ replaced by $c\delta$. Thus if Assumption 2.1 holds then there is a set B_1 which satisfies this assumption and (2.2) with $\delta = 1$. As noted in [13], and as is evident from the heuristic discussion of Theorem 2.1 given below, the diameter of B_1 plus 1 serves as a valid Lipschitz constant K in Theorem 2.1.

Sets that satisfy Assumption 2.1 and its dual formulation have a number of interesting properties and interpretations. One useful interpretation is in terms of a related norm on $I\!R^n$. Consider the class of sets described by

$$\mathscr{S} = \left\{ C \subset I\!R^n : C \text{ is compact, convex, symmetric, and } 0 \in C^\circ \right\}, \quad (2.3)$$

where C° denotes the interior of C. There is a one to one correspondence between the elements of $\mathscr{S}$ and norms on $I\!R^n$. The elementary proof of this fact can be found in [18, Lemma A.1]. Common examples of this correspondence are the n-ball with the usual Euclidean norm and the n-cube with the sup norm on $I\!R^n$. In particular, any set B satisfying Assumption 2.1 belongs to $\mathscr{S}$ and consequently defines a norm on $I\!R^n$. We will denote this norm by $\|\cdot\|_B$, where

$$\|x\|_B \doteq \min\{r \geq 0 : x \in rB\}. \quad (2.4)$$

Now consider a particular SP, and assume that B satisfies Assumption 2.1 for this SP with a value $\delta > 0$. The norm defined above in terms of B can be used to intuitively understand the idea behind Theorem 2.1. In [13] this norm is used as a sort of Lyapunov function to prove that if (ϕ_1, η_1) and (ϕ_2, η_2) solve the SP for ψ_1 and ψ_2 respectively and if $a \doteq \sup\{\|\psi_1(t) - \psi_2(t)\| : t \in [0, \infty)\}$ (where $\|\cdot\|$ denotes the usual Euclidean norm), then

$$\sup\{\|\eta_1(t) - \eta_2(t)\|_B : t \in [0, \infty)\} \leq a/\delta.$$

This implies the Lipschitz continuity of the SM. The conditions on the inward normals at points $x \in \partial B$ for which $|\langle x, n_i\rangle| < \delta$ are exactly what are needed in order that the restrictions on the dynamics of $\eta_1 - \eta_2$ imposed by the SP prohibit $\eta_1 - \eta_2$ from ever escaping the level set of $\|\cdot\|_B$ of height a/δ.

Observe that Assumption 2.1 depends on the given SP only through the vectors n_i and d_i, and not at all on the scalars c_i. Thus when trying to verify Lipschitz continuity it will often prove convenient to set $c_i = 0$. This property is also exploited in Section 4 where the existence of the set B is used to establish existence of solutions to the SP.

It should be noted that the description of the SP in terms of a finite set of triplets is not unique. However, it is obviously true that the SM is Lipschitz continuous if Assumption 2.1 holds for any valid representation of the given SP. Section 5.3 of [17] discusses the issue of the choice of representation that is most convenient for the verification of Assumption 2.1.

In the next section we review the main alternative approach to analyzing regularity properties of the SP.

2.4. The reflection map approach

The first paper to treat a SP on a domain with corners and oblique directions of constraint was that of Harrison and Reiman [22], although Harrison and Reiman defined the map $\psi \to \phi$ in terms of what is called a "reflection map" rather than the SP. In [22] the domain G is the positive orthant $\mathbb{R}^n_+ = \{(x_1, x_2, \ldots, x_n) : x_i \geq 0 \text{ for } i = 1, \ldots, n\}$, with one direction of constraint associated to each face. In terms of the notation of Section 2.2, the related SP is $\{(d_i, e_i, 0), i = 1, \ldots, n\}$, where e_i is the unit vector in the ith coordinate direction. Note that exactly n directions of constraint are allowed for an n-dimensional domain.

Given a continuous unconstrained trajectory ψ with $\psi(0) \in G$, ϕ is called the reflected version of ψ if there exist continuous nondecreasing $\theta_i : [0, \infty) \to \mathbb{R}$ with $\theta_i(0) = 0$ such that

$$\phi(t) \doteq \psi(t) + \sum_{i=1}^{n} d_i \theta_i(t) \in G$$

for all $t \in [0, \infty)$, and such that for all $i \in \{1, \ldots, n\}$ θ_i increases only when $\phi_i(t) = 0$:

$$\int_0^\infty I_{\{t:\phi_i(t)>0\}} d\theta_i(t) = 0.$$

One of the main results of [22] is the formulation of an algebraic sufficient condition for the existence of solutions to the reflection map and Lipschitz continuity of the mapping $\psi \to \phi$.

The connection between the reflection map and the SP is straightforward. From the given properties of the θ_i, it is clear that if $\eta(t) \doteq \sum_{i=1}^n d_i \theta_i(t)$ then $(\phi(t), \eta(t))$ solves the SP for $\psi(t)$. It is also easy to check (under the conditions used in [22] and restated below) that if $(\phi(t), \eta(t))$ solves the SP $\{(d_i, e_i, 0), i = 1, \ldots, n\}$, then aside from a measurable selection issue addressed in [19] suitable θ_i's can be constructed from η. Thus [22] also proves regularity properties for a family of SPs.

The algebraic condition used in [22] is the following. Assume that $\langle d_i, e_j \rangle \leq 0$ if $i \neq j$ and $\langle d_i, e_i \rangle = 1$, and define $Q = [q_{ij}]$ by

$$q_{ij} = \begin{cases} -\langle d_i, e_j \rangle & \text{if } i \neq j, \\ 0 & \text{if } i = j. \end{cases} \tag{2.5}$$

If the spectral radius $\sigma(Q)$ of Q is less than one, then a solution to the SP exists for all $\psi \in C([0, \infty) : \mathbb{R}^n)$, and moreover the SM is Lipschitz continuous. (The Lipschitz continuity is not stated explicitly in [22], but follows easily from the method used to prove existence of solutions and continuity of the SM.) We collectively call these conditions the Harrison–Reiman (H–R) condition.

We will argue below that this class of SPs nearly exhausts the set of all SPs that can be analyzed via the contraction mapping techniques used in [22]. However, the following generalization can be proved. For any SP of the form $\{(d_i, n_i, c_i), i = 1, \ldots, n\}$ where $\mathrm{span}(\{d_i, i = 1, \ldots, n\}) = I\!R^n$ and $\langle d_i, n_i \rangle = 1$, define $Q = [q_{ij}]$ by

$$q_{ij} = \begin{cases} |\langle d_i, n_j \rangle| & \text{if } i \neq j, \\ 0 & \text{if } i = j. \end{cases}$$

We will say that the generalized Harrison–Reiman (gH–R) condition holds if $\sigma(Q) < 1$. Under this condition, the SM is defined and Lipschitz continuous on all of $D([0, \infty) : I\!R^n)$. Note that when compared with the conditions of the previous paragraph (where $n_i = e_i$ and $G = I\!R^n_+$) this new condition is equivalent to dropping the assumption that $\langle d_i, e_j \rangle \leq 0$ for $i \neq j$ and defining Q by $q_{ij} = |\langle d_i, e_j \rangle|$ when $i \neq j$ and $q_{ij} = 0$ otherwise. Thus the gH–R condition is indeed a generalization of the original H–R condition.

We now describe the technique used in [22]. Let $I([0, \infty) : I\!R^n)$ be the set of all $\theta \in D([0, \infty) : I\!R^n)$ such that each component θ_i is nondecreasing and $\theta_i(0) = 0$. Let $x \vee 0$ denote $\max(x, 0)$. Given any path $\psi \in D([0, \infty) : I\!R^n)$, one first defines the map Θ_ψ on $I([0, \infty) : I\!R^n)$ by

$$[\Theta_\psi(\theta)]_j(t) \doteq \sup_{s \in [0,t]} \left[- \sum_{i=1, i \neq j}^{n} \langle d_i, n_j \rangle \theta_i(s) - \langle \psi(s), n_j \rangle \right] \vee 0. \quad (2.6)$$

Note that for the H–R class of SPs this map reduces to that given in [22]:

$$[\Theta_\psi(\theta)]_j(t) \doteq \sup_{s \in [0,t]} \left[\sum_{i=1, i \neq j}^{n} q_{ij} \theta_i(s) - \psi_j(s) \right] \vee 0,$$

where the q_{ij} are as defined in (2.5). The map Θ_ψ has the following two properties.

1. Θ_ψ maps $I([0, \infty) : I\!R^n)$ into itself.
2. ϕ, ψ and θ solve the reflection map [and hence $(\phi, \eta = \Sigma_j d_j \theta_j)$ solve the SP for ψ] if and only if $\theta = \Theta_\psi(\theta)$.

The first property above is a direct consequence of the definition (2.6), while the second was proved in [22,Theorem 1] for the case $n_j = e_j$. Equivalence for the general case can be proved in an analogous fashion.

One then constructs a norm $\|\cdot\|_H$ on $\mathbb{R}^n$ whose level sets are multiples of a convex set H of the form $\cap_{i=1,\ldots,n}\{x : \langle x, e_i\rangle \leq \alpha_i\}$, where $\alpha_i > 0, i = 1, \ldots, n$. (Recall the equivalence between norms and sets in $\mathscr{S}$ discussed in Section 2.3, and note that $H \in \mathscr{S}$.) The norm $\|\cdot\|_H$ depends only on the directions $\{d_i\}$, and is chosen so that when $I([0,\infty) : \mathbb{R}^n)$ is endowed with the norm $\|\cdot\|_{H,D}$, defined to be the sup norm on path space with respect to the $\|\cdot\|_H$ norm on $\mathbb{R}^n$, then the following statements hold.

1. Θ_ψ is a contraction on $I([0,\infty) : \mathbb{R}^n)$ uniformly in ψ. In other words there exists $\alpha \in (0,1)$ such that for any $\psi \in D([0,\infty) : \mathbb{R}^n)$ and θ_1, $\theta_2 \in I([0,\infty) : \mathbb{R}^n)$,
$$||\Theta_\psi(\theta_1) - \Theta_\psi(\theta_2)||_{H,D} \leq \alpha||\theta_1 - \theta_2||_{H,D}. \tag{2.7}$$
2. The unique fixed point of the mapping $\Theta_\psi(\cdot)$ is Lipschitz continuous in ψ.

Since the space $I([0,\infty) : \mathbb{R}^n)$ is complete with respect to the norm $\|\cdot\|_{H,D}$, the first property above implies that for every $\psi \in D([0,\infty) : \mathbb{R}^n)$ $\Theta_\psi(\cdot)$ has a unique fixed point, which we denote by x_ψ. Suppose $\alpha \in (0,1)$ is the contraction parameter defined in (2.7). Then for $\psi, \zeta \in D([0,\infty) : \mathbb{R}^n)$,

$$\begin{aligned}
||x_\psi - x_\zeta||_{H,D} &= ||\Theta_\psi(x_\psi) - \Theta_\zeta(x_\zeta)||_{H,D}\\
&\leq ||\Theta_\psi(x_\psi) - \Theta_\psi(x_\zeta)||_{H,D} + ||\Theta_\psi(x_\zeta) - \Theta_\zeta(x_\zeta)||_{H,D}\\
&\leq \alpha||x_\psi - x_\zeta||_{H,D} + C||\psi - \zeta||_{H,D},
\end{aligned}$$

where the last inequality follows from (2.7), (2.6), the form of H and the fact that there exists $C < \infty$ such that $||\hat{\psi} - \hat{\zeta}||_{H,D} \leq C||\psi - \zeta||_{H,D}$, where $\hat{\psi} \doteq \sum_j \langle \psi, n_j\rangle e_j$. Rearranging terms in the last display we obtain

$$||x_\psi - x_\zeta||_{H,D} \leq \frac{C}{1-\alpha}||\psi - \zeta||_{H,D},$$

which establishes the Lipschitz continuity of the map $\psi \to x_\psi$. Two immediate consequences of the properties derived above are the existence of solutions to the SP and Lipschitz continuity of the SM.

In contrast to the reflection mapping technique, the SP formulation works directly with the constraining process η, and not with the individual local times or components θ_j along the directions d_j. In order to establish a connection between the two techniques, we first observe that since η has a unique decomposition $\eta = \Sigma_{j=1}^n d_j\theta_j$, the mapping Θ_ψ on $I([0,\infty) : \mathbb{R}^n)$ induces a mapping $\tilde{\Theta}_\psi$ on $D([0,\infty) : \mathbb{R}^n)$. More precisely, let A be the unique transformation such that $Ae_j = d_j$, $j = 1, \ldots, n$. Let $||\cdot||_H$ be the norm used above to make Θ_ψ a contraction, and define the mapping $\tilde{\Theta}_\psi : D([0,\infty) : \mathbb{R}^n) \to D([0,\infty) : \mathbb{R}^n)$ by

$$\tilde{\Theta}_{\psi}(\eta) \doteq A(\Theta_{\psi}(A^{-1}\eta)). \tag{2.8}$$

Define $B \doteq AH$ and let $||\cdot||_{B,D}$ be the sup norm on path space with respect to the $||\cdot||_B$ norm on $\mathbb{R}^n$, whose level sets are scalar multiples of the set B. It is easy to see that properties 1 and 2 in the last paragraph hold if and only if they hold with Θ_ψ, $||\cdot||_{H,D}$ and $I([0,\infty):\mathbb{R}^n)$ replaced by $\tilde{\Theta}_\psi$, $||\cdot||_{B,D}$ and $D([0,\infty):\mathbb{R}^n)$. As we will see below, the set B forms the link between the Harrison–Reiman approach and the geometric approach towards SPs of this type adopted in the present paper. Note that since all norms on $\mathbb{R}^n$ are equivalent, Lipschitz continuity would hold irrespective of the underlying norm. However, the proof relies on the contraction property which does depend on the particular choice of norm.

Although the technique of [22] described above is very natural, the class of SPs to which it can be applied is limited by two factors. For one, the technique requires that the constraining term η for the SP have a unique decomposition in terms of the directions $\{d_j\}$ with the coefficients θ_j's taking values in $I([0,\infty):\mathbb{R}^n)$. This dictates that the set of directions $\{d_j\}$ be a basis for $\mathbb{R}^n$. Thus the class of SPs to which the reflection mapping techniques apply is restricted to those that can be described using n faces in $\mathbb{R}^n$, and that satisfy $\text{span}\{d_j\} = \mathbb{R}^n$. For SPs with more than n faces in n dimensions, non-uniqueness of the coefficients that play the role of θ_j's is expected, and any argument that would imply such uniqueness cannot work. Secondly, there are a number of SPs of interest (for example the processor sharing model introduced in [17,Section 3]) for which a solution is only possible on a strict subset of $D([0,\infty):\mathbb{R}^n)$ (e.g., paths of bounded variation). In such a case one cannot expect the argument of [22] to work, since it would necessarily define a solution to the SP for all ψ in $D([0,\infty):\mathbb{R}^n)$ because the contraction norm is independent of ψ.

Comparing the approaches of [22] and [13] for the class of SPs for which the reflection mapping technique can be applied, we note that the set $B \doteq AH$ (where H defines the norm on path space used in [22] to make Θ_ψ a contraction, and A is the transformation defined above) satisfies Assumption 2.1. In particular, for the case where $\Sigma_{i=1}^n q_{ij} < 1$ for all j, the sup norm, which corresponds to H being the unit hypercube, was used in [22]. The corresponding set AH does in fact satisfy Assumption 2.1 for this class of SPs [17, Theorem 2.1]. The other cases treated in [22] are reduced to this case by a change of variable. The same change of variable can be directly applied to the unit hypercube to produce a set H such that (2.7) holds. One can also prove a converse: given a set B in $\mathbb{R}^n$ with $2n$ faces that satisfies Assumption 2.1 for a SP, the norm $||\cdot||_{H,D}$ associated with the set $H \doteq A^{-1}B$ makes Θ_ψ a contraction. Thus, if one were to restrict the geometric approach to only sets B with exactly $2n$ faces in dimension

n, then these two approaches give the same results for this class of SPs. However, as is shown in [17,Section 2.4], for $n > 2$ there are SPs that fall within the class that can be described by both the SP and the reflection map, and yet require a set B with strictly more than $2n$ faces, and in this sense the geometric approach is more general.

In Section 3 we will consider the general problem of constructing B for classes of SPs. We end this section with a discussion of a number of qualitative properties of sets that satisfy Assumption 2.1. These properties are interesting in their own right, and moreover their dual formulations will be used in Section 3.2 to simplify the construction of B.

2.5. Interpretation of the set B as an invariant set

The main results of this subsection are Theorem 2.4, which gives a useful necessary condition for the existence of B, and Lemma 2.5, which concerns transformations.

Associated in a natural way with the SP $\{(d_i, n_i, c_i), i = 1, \ldots, N\}$ is a collection of oblique projection operators $\{L_i, i = 1, \ldots, N\}$, where L_i projects along the direction d_i onto the hyperplane $\{x : \langle x, n_i \rangle = c_i\}$. Note that since the constants c_i do not play any role in the existence of the set B, we set $c_i = 0$ so that the projection operators can without loss of generality be assumed to project onto the hyperplanes $\{x : \langle x, n_i \rangle = 0\}$. The oblique projection operators L_i are then given by

$$L_i x = x - \langle x, n_i \rangle d_i, \tag{2.9}$$

where we recall the normalization condition $\langle n_i, d_i \rangle = 1$.

The existence of solutions to a given SP is shown in Section 4 to depend on the existence of a discrete projection π that maps points onto ∂G along the directions of constraint $d(x)$. The existence of π, which is highly non-trivial, in fact automatically guarantees the existence of solutions to the SP for the class of pure jump functions $F_G([0, \infty) : I\!R^n)$ defined in (4.1). Given $\psi \in F_G([0, \infty) : I\!R^n)$, ϕ can be defined in a simple manner by repeated application of π. For points outside G that are near the relative interiors of the faces of G one can imagine that the discrete projection π for the SP $\{(d_i, n_i, 0), i = 1, \ldots, N\}$ behaves like L_i, and in general it can be loosely thought to be some convex combination of the L_i. Thus one might expect that bounds on the projections L_i that are independent of i would be closely related to continuity properties of the SM. This is indeed the case, and in fact the satisfaction of Assumption 2.1 implies a strong boundedness property of the associated projections. The precise statement of this property is given in Theorem 2.4.

Given any SP, let

$$\mathscr{L} \doteq \{L_i : i = 1, \ldots, N\}$$

denote the collection of projection operators associated with the SP. We next introduce the concept of an invariant set for a collection of linear operators.

Definition 2.2 (Invariant Set). *For a given collection of linear operators*

$$\mathscr{M} \doteq \left\{M_i : \mathbb{R}^n \to \mathbb{R}^n, i = 1, \ldots, N\right\},$$

we say that C is an invariant set with respect to $\mathscr{M}$ if $C \in \mathscr{S}$ and

$$M_i C \subset C$$

for $i = 1, \ldots, N$, where $M_i C = \{M_i x : x \in C\}$.

Given a collection of linear operators $\mathscr{M}$, let $\mathscr{P}$ denote the set of all possible products of elements of $\mathscr{M}$:

$$\mathscr{P} \doteq \left\{\prod_{j=1}^{k} \bar{M}_j : \bar{M}_j \in \mathscr{M}, k < \infty\right\}.$$

Theorem 2.3 shows that the existence of an invariant set for $\mathscr{M}$ guarantees the uniform boundedness of arbitrary products of the projection operators M_i. This uniform boundedness property is sometimes referred to as the absolute stability of the collection $\mathscr{M}$. The existence of an invariant set is equivalent to the stability of a related dynamical system [12, 27]. For any choice of $x_0 \in \mathbb{R}^n$ and any choice $\bar{M}_j \in \mathscr{M}$, the sequence $\{x_j\}$ defined by $x_{j+1} = \bar{M}_j x_j$, $j \geq 0$ remains within a bounded set which depends only on x_0. The theorem also establishes an equivalence between uniform boundedness of the collection of operators and the existence of a norm with respect to which the operator norms are bounded by 1. The proof of this theorem can be found in [18, Appendix A.1].

Theorem 2.3. *The following are equivalent*:

1. *There exists an invariant set for the collection $\mathscr{M}$.*
2. *The elements of $\mathscr{P}$ are uniformly bounded.*
3. *There exists a norm $\|\cdot\|$ on $\mathbb{R}^n$ such that for all $M \in \mathscr{M}$ the operator norm of M with respect to this norm,*

   $$\|M\| \doteq \sup_{x \neq 0} \|Mx\|/\|x\|,$$

   is bounded by 1.

Assume that the set $B \in \mathscr{S}$ satisfies (2.2) for a given SP $\{(d_i, n_i, c_i), i = 1, \ldots, N\}$. In Theorem 2.4 we show that with respect to the norm $\|\cdot\|_B$ all the projection operators in $\mathscr{L}$ have norm less than or equal to 1, or in other words that B is an invariant set for $\mathscr{L}$. It should be noted that general conditions which guarantee the existence of such a norm are far from obvious. Indeed, if $Lx = x - \langle x, n\rangle d$ with $n \neq d$ and $\langle n, d\rangle = 1$, then even the single operator L has norm strictly greater than 1 with respect to the usual Euclidean norm. The analogue of this theorem for a set that is dual to B will prove to be of consequence in deriving a dual condition for Lipschitz continuity that implies Assumption 2.1.

Theorem 2.4. *Given the SP* $\{(d_i, n_i, c_i), i = 1, \ldots, N\}$*, suppose that a set* $B \in \mathscr{S}$ *satisfies* (2.2)*. Let* $L_i, i = 1, \ldots, N$ *be the associated projection operators defined in* (2.9)*. Then the norm of every operator* L_i *is bounded by* 1 *with respect to the norm* $\|\cdot\|_B$ *defined in* (2.4)*.*

Proof. Fix $i \in \{1, \ldots, N\}$, $x \in \mathbb{R}^n$ and let

$$\begin{aligned} c &\doteq \|L_i x\|_B \\ &= \|x - \langle x, n_i\rangle d_i\|_B \\ &= \min\{r \geq 0 : x - \langle x, n_i\rangle d_i \in rB\}. \end{aligned}$$

We must show that $\|L_i x\|_B \leq \|x\|_B$. If $x = 0$ or if $c = 0$, then this inequality is automatically satisfied. For the rest of the proof we will assume that $x \neq 0$ and $c \neq 0$.

The properties $\langle L_i x, n_i\rangle = 0$ and $L_i x \in \partial(cB)$ follow from the definition of L_i and c respectively. Since B satisfies (2.2), so does cB and thus $\langle \nu, d_i\rangle = 0$ for all $\nu \in \nu_c(L_i x)$, where $\nu_c(x) = \nu(x/c)$ is the set of inward normals to cB at x. Thus $L_i x + \alpha d_i$ lies in a supporting hyperplane to cB at $L_i x$ for all $\alpha \in \mathbb{R}$. The convexity of B then implies $L_i x + \alpha d_i \notin (cB)^\circ$ for all $\alpha \in \mathbb{R}$. In particular, choosing $\alpha = \langle x, n_i\rangle$, we find that $L_i x + \alpha d_i = x - \langle x, n_i\rangle d_i + \langle x, n_i\rangle d_i = x \notin (cB)^\circ$. Thus from the definition of the norm $\|x\|_B \geq c = \|L_i x\|_B$, and hence the norm of the operator is bounded by 1 with respect to this norm. □

We conclude this section with a simple but extremely useful observation on the relation between the invariant sets for two collections of operators that are obtained as similarity transforms of each other.

Lemma 2.5. *Suppose that the collection* $\mathscr{M}$ *possesses an invariant set* C*. Let* A *be invertible, and define* $\tilde{\mathscr{M}} = \{AMA^{-1} : M \in \mathscr{M}\}$*. Then* $\tilde{C} = AC \doteq \{Ax : x \in C\}$ *is an invariant set for* $\tilde{\mathscr{M}}$*.*

Proof. Choose any operator $\tilde{M}$ in $\tilde{\mathscr{M}}$. By definition it can be expressed as $\tilde{M} = AMA^{-1}$ for some $M \in \mathscr{M}$. Then since C is invariant for $\mathscr{M}$,

$$\tilde{M}\tilde{C} = AMA^{-1}AC = AMC \subset AC = \tilde{C}.$$

Moreover, $\tilde{C} \in \mathscr{S}$ since $C \in \mathscr{S}$ and A is invertible. Thus $\tilde{C}$ is invariant for $\tilde{\mathscr{M}}$. □

3. Construction of the set *B*

3.1. Introduction

As discussed in Section 2.5, the construction of a set B that satisfies Assumption 2.1 for a given SP is a highly non-trivial task. Here we use convex duality to obtain a dual characterization of the geometric condition for Lipschitz continuity stated in Assumption 2.1. This dual condition, which is introduced in Section 3.2, is phrased in terms of the existence of a set B^* which is the convex dual to a set B that satisfies Assumption 2.1. Although the dual property is also geometric in nature, it is generally much easier to verify and often lends itself to algebraic characterizations. Indeed the aim of this section is to outline how one can obtain an algebraic criterion for the satisfaction of the dual condition for SPs with a given structure.

Section 3.3 develops guidelines for the construction of dual sets B^* that have the form $\text{conv}[\pm a_k w_k, k = 1, \ldots, K]$, where the w_k are known unit vectors and the $a_k > 0$ are unknown constants that must be found to satisfy the dual condition. We note that the number $2K$ is an interesting quantity because it provides an upper bound on the number of faces of the corresponding dual set B, and can be regarded as a measure of the complexity of the SP. If B^* can be expressed as an intersection of a finite number of half spaces then the property required of the dual set can be rephrased as a finite number of linear inequalities. The derivation of such a *finite* external representation for the dual set is usually simpler if the directions of constraint have some degree of symmetry. Transformation techniques like those developed in Section 3.4 can then be used to generalize algebraic conditions derived for symmetric directions of constraint to arbitrary directions of constraint. In [17] the techniques derived here are applied to concrete classes of SPs, and it is seen that the complexity of the set B depends strongly on the structure of the particular SP.

3.2. A dual sufficient condition for Lipschitz continuity: the Set B^*

Given a convex set $C \subset \mathbb{R}^n$, we define the dual closed convex set by

$$C^* = \left\{ y : \sup_{x \in C} \langle y, x \rangle \leq 1 \right\}. \tag{3.1}$$

C^* is sometimes referred to as the polar of the set C. A standard result from elementary convex analysis is that $(C^*)^* = \mathrm{cl}(C)$, where $\mathrm{cl}(C)$ denotes the closure of C. The following properties can be verified without difficulty: if the origin is in C° then C^* is bounded; if C is bounded then the origin is in $(C^*)^\circ$; if C is symmetric then so is C^*. Thus $C \in \mathscr{S}$ implies $C^* \in \mathscr{S}$.

We introduce operators L_i^* that are adjoint (with respect to the usual inner product on $\mathbb{R}^n$) to the oblique projection operators L_i defined in (2.9):

$$L_i^* x = x - \langle x, d_i \rangle n_i. \tag{3.2}$$

Let

$$H_i \doteq \{x : \langle x, d_i \rangle = 0\}$$

be the hyperplane normal to d_i. $L_i^* x = x$ for $x \in H_i$, and thus H_i is an invariant linear sub-space for L_i^*. Since $\langle n_i, d_i \rangle = 1$, L_i^* is an oblique projection operator onto H_i. The collection of adjoint projection operators is denoted by

$$\mathscr{L}^* \doteq \{L_i^*, i = 1, \ldots, N\}.$$

Let M^* denote the adjoint of an operator M and let $\mathscr{M}^* = \{M^* : M \in \mathscr{M}\}$. The following lemma shows that the dual of an invariant set for $\mathscr{M}$ is an invariant set for $\mathscr{M}^*$. Note that all matrices considered here are real and therefore the transpose of a matrix is equal to its adjoint.

Lemma 3.1. *If C is an invariant set for a collection of linear operators $\mathscr{M}$, then*

$$C^* = \left\{ y : \sup_{x \in C} \langle x, y \rangle \le 1 \right\}$$

is an invariant set for $\mathscr{M}^*$, and $C^* \in \mathscr{S}$.

Proof. As observed above, $C \in \mathscr{S}$ implies that $C^* \in \mathscr{S}$. If $y \in C^*$, then for any $M^* \in \mathscr{M}^*$

$$\begin{aligned}
\sup_{x \in C} \langle x, M^* y \rangle &= \sup_{x \in C} \langle Mx, y \rangle \\
&= \sup_{x \in MC} \langle x, y \rangle \\
&\le \sup_{x \in C} \langle x, y \rangle \\
&\le 1.
\end{aligned}$$

This implies $M^* y \in C^*$, and so C^* is an invariant set for $\mathscr{M}^*$. $\square$

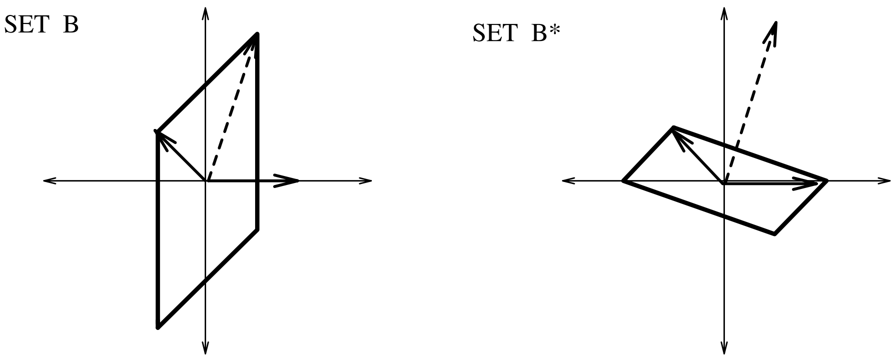

Fig. 2. The set B and its dual B^*

Given a SP $\{(d_i, n_i, c_i), i = 1, \ldots, N\}$, we now propose an alternate geometric condition for Lipschitz continuity of the SM, expressed in terms of the collection of adjoint operators $\mathscr{L}^*$. If the convex hull of a finite set of points has interior then it is called a *polytope*. A set B^* that satisfies (3.3) for the SP given in Figure 1 is presented in Figure 2 Note that the vertex directions of B correspond to normals of the faces of the dual B^* and vice versa. This is a generic property of sets and their duals which is stated precisely in Lemma 3.3.

Assumption 3.1 (set B*). *There exists a finite set of vertices $v_1, \ldots, v_J$ with* $\mathrm{span}(\{v_j, j = 1, \ldots, J\}) = I\!R^n$*, such that if* $B^* \doteq \mathrm{conv}[\pm v_j, j = 1, \ldots, J]$ *then for every $i \in \{1, \ldots, N\}$ and $j \in \{1, \ldots, J\}$*

$$\text{either } L_i^* v_j = v_j \text{ or } L_i^* v_j \in (B^*)^\circ. \tag{3.3}$$

Note that from the definition it is clear that a set B^* that satisfies Assumption 3.1 must lie in $\mathscr{S}$, and that $B \doteq (B^*)^* \in \mathscr{S}$ if and only if $B^* \in \mathscr{S}$.

As illustrated in Figure 2, a polytope B that satisfies (2.2) for the SP is dual to a set B^* that satisfies (3.3). Theorem 3.2 shows that the dual of any set in $\mathscr{S}$ that satisfies (3.3) satisfies (2.2). Thus Assumption 3.1 implies Assumption 2.1, and hence serves as a more easily verifiable sufficient condition for Lipschitz continuity of the SM. We conjecture that the reverse statement that Assumption 2.1 implies Assumption 3.1 also holds. Note that any set $B^* \in \mathscr{S}$ satisfying (3.3) is invariant with respect to each adjoint operator in $\mathscr{L}^*$. It then follows from Lemma 3.1 that the dual $B \doteq (B^*)^*$ is an invariant set for the projection operators $\mathscr{L} \doteq \{L_i, i = 1, \ldots, N\}$. The additional requirement in (3.3) that each vertex v_j be mapped strictly into the interior of B^* when it is not left unchanged is the strengthening needed so that the invariant set B satisfies (2.2) for some $\delta > 0$. In fact, it is the dual formulation of the strengthening of conditions required to go from the uniqueness of [27] to the Lipschitz continuity of [13].

Theorem 3.2 (Dual Formulation). *Given a SP* $\{(d_i, n_i, c_i), i = 1, \dots, N\}$, *Assumption* 3.1 *implies Assumption* 2.1. *Moreover, if Assumption* 2.1 *is satisfied by a polytope*, *then Assumption* 3.1 *holds.*

Proof. First suppose that Assumption 3.1 holds. For V of the form $\{\pm v_j, j = 1, \dots, J\}$, assume that $B^* \doteq \mathrm{conv}[V]$ satisfies (3.3). We will establish Assumption 2.1 by showing that $(B^*)^*$ satisfies (2.2). Let

$$B \doteq \left\{ x : \sup_{y \in B^*} \langle x, y \rangle \leq 1 \right\} = (B^*)^*.$$

Since Assumption 3.1 implies $B^* \in \mathscr{S}$, we automatically obtain $B \in \mathscr{S}$. If $z \in \partial B$ and if there exists $\tau > 0$ such that $|\alpha| \leq \tau$ implies $z + \alpha d_i \in B$, then $\langle \nu, d_i \rangle = 0$ for all $\nu \in \nu(z)$. Indeed, if $\nu \in \nu(z)$ and $z + \alpha d_i \in B$ then the definition of the inward normal implies $\langle z, \nu \rangle \leq \langle z + \alpha d_i, \nu \rangle$, which in turn shows that $\alpha \langle d_i, \nu \rangle \geq 0$ for all $0 \leq |\alpha| \leq \tau$. Clearly this is satisfied if and only if $\langle \nu, d_i \rangle = 0$.

Claim. There exists $\tau > 0$ such that if $i \in \{1, \dots, N\}$ and if $z \in \partial B$ satisfies $\langle z, n_i \rangle = 0$, then $|\alpha| \leq \tau$ implies $z + \alpha d_i \in B$.

Let us suppose that the claim is true. In this case, the discussion preceding the claim shows that B satisfies (2.2) for $\delta = 0$. However, because B is a polytope (2.2) automatically holds for some $\delta > 0$. The proof is as follows. For each $z \in \{z : \langle z, n_i \rangle = 0\}$, it is possible to choose $\varepsilon_z > 0$ small enough so that $\nu(x) \subset \nu(z)$ for every x in the neighbourhood $\{z : \|z - x\| < \varepsilon_z\}$. This obviously implies $\langle \nu, d_i \rangle = 0$ for every $\nu \in \nu(x)$ when $\|z - x\| < \varepsilon_z$. Since the set $D_i \doteq \{z \in \partial B : \langle z, n_i \rangle = 0\}$ is compact, we can select a finite subset $\{z_j, j = 1, \dots, J\} \subset D_i$ so that

$$D_i \subset \tilde{D}_i \doteq \cup_{j=1}^{J} \{x : \|x - z_j\| < \varepsilon_{z_j}\}.$$

Because $\tilde{D}_i$ is open and D_i is compact, there exist $\delta_i > 0$ such that $z \in \partial B$ and $|\langle z, n_i \rangle| < \delta_i$ imply $z \in \tilde{D}_i$. Thus B satisfies (2.2) with $\delta \doteq \min_{i=1,\dots,N} \delta_i > 0$, and therefore Assumption 2.1 is satisfied.

We now prove the claim. Fix $i \in \{1, \dots, N\}$ and $z \in \partial B$ satisfying $\langle z, n_i \rangle = 0$. For any j such that $\langle v_j, d_i \rangle = 0$ we have

$$\langle z + \alpha d_i, v_j \rangle = \langle z, v_j \rangle \leq 1, \tag{3.4}$$

where the last inequality follows from the fact that $\langle z, v \rangle \leq 1$ for any $z \in B$ and any $v \in B^*$. Since B^* satisfies (3.3), for any j such that $\langle v_j, d_i \rangle \neq 0$,

$\left(L_i^* v_j\right) \in (B^*)^\circ$. For such a value of j and any $y \in B$ we have $\langle y, L_i^* v_j \rangle < 1$, which is the same as $\langle y, v_j - \langle v_j, d_i \rangle n_i \rangle < 1$. If we choose $y = z$ and recall that $\langle z, n_i \rangle = 0$, then the last inequality becomes $\langle z, v_j \rangle < 1$. Since there are only finitely many j with $\langle v_j, d_i \rangle \neq 0$, there exists $\sigma_i(z) > 0$ such that if $\alpha \in I\!R$ satisfies $|\alpha| < \sigma_i(z)$, then

$$\max_{j:\langle v_j, d_i \rangle \neq 0} \langle z + \alpha d_i, v_j \rangle < 1. \tag{3.5}$$

The inequalities (3.4), (3.5), and the definition of B^* as the convex hull of V imply for the given i and z that

$$\sup_{x \in B^*} \langle z + \alpha d_i, x \rangle \leq 1$$

whenever $|\alpha| < \sigma_i(z)$. Since $B = (B^*)^*$, this implies $z + \alpha d_i \in B$ whenever $|\alpha| < \sigma_i(z)$. Let $\tau_i(z)$ be the maximum value of $\sigma_i(z)$ for which (3.5) holds. $\tau_i(z)$ is well defined since $\sigma_i(z)$ is bounded above by the diameter of the compact set B^*. It follows from the continuity of the inner product that $\tau_i(z)$ is a continuous function of z. Now define $\tau_i \doteq \inf\{\tau_i(z) : z \in \partial B \text{ and } \langle z, n_i \rangle = 0\}$. Since $\partial B \cap \{z : \langle z, n_i \rangle = 0\}$ is a compact set, the infimum is achieved on this set and therefore $\tau_i > 0$. This establishes the claim with $\tau \doteq \min_i \tau_i > 0$ and completes the proof that Assumption 3.1 implies Assumption 2.1.

We next show that if Assumption 2.1 is satisfied by a polytope B, then Assumption 3.1 holds. In analogy with the previous argument, we will show that the dual B^* of B satisfies Assumption 3.1.

Define

$$B^* \doteq \left\{ x : \sup_{y \in B} \langle y, x \rangle \leq 1 \right\}.$$

Since $B \in \mathscr{S}$, it follows that $B^* \in \mathscr{S}$. Let S be the set of extreme points of B^* so that $B^* = \text{conv}[S]$. Since B^* is a symmetric polytope, S is symmetric and finite. Define $H_i \doteq \{x : \langle x, d_i \rangle = 0\}$.

Fix any value of $i \in \{1, \ldots, N\}$. It will suffice to prove (3.3) for this given value of i. If $v \in S \cap H_i$, then $L_i^* v = v$ and (3.3) is satisfied. Next assume that $v \in S \setminus H_i$. According to Theorem 2.4, B is an invariant set for $\mathscr{L}$. Lemma 3.1 then implies that B^* is invariant for $\mathscr{L}^*$, and therefore $L_i^* v \in B^*$. If $L_i^* v \in (B^*)^\circ$ we are done, and thus we need only consider the case where $L_i^* v \in \partial B^*$.

We will use the facts that $\langle v, d_i \rangle \neq 0$ and $L_i^* v \in \partial B^*$ to arrive at a contradiction. Let $z \in \partial B$ be such that $\max_{y \in B} \langle y, L_i^* v \rangle = \langle z, L_i^* v \rangle$. Since $L_i^* v \in \partial B^*$, it follows from the definition of B^* as the dual of B that

$\langle z, L_i^* v\rangle = 1$. Define $\tilde{z} \doteq L_i z = z - \langle z, n_i\rangle d_i$. Since B is invariant for $\mathscr{L}$ and $z \in B$, $\tilde{z} \doteq L_i z \in B$. Note that

$$\sup_{y\in B}\langle y, v\rangle = 1 = \langle z, L_i^* v\rangle = \langle L_i z, v\rangle = \langle \tilde{z}, v\rangle.$$

Since $\langle \tilde{z}, v\rangle = 1$ it follows that $\tilde{z} \in \partial B$, and since $\langle \tilde{z}, v\rangle = \sup_{y\in B}\langle y, v\rangle$ it follows that $-v \in \nu(\tilde{z})$. Finally we make use of the normalization $\langle d_i, n_i\rangle = 1$ to obtain $\langle \tilde{z}, n_i\rangle = 0$. Then $\langle \tilde{z}, n_i\rangle = 0$, $\tilde{z} \in \partial B$, $-v \in \nu(\tilde{z})$, and $\langle -v, d_i\rangle \neq 0$ contradict the fact that the set B satisfies (2.2), and we conclude that $L_i^* v \in \partial B^*$ is impossible. This completes the proof that if B is a polytope then Assumption 2.1 implies Assumption 3.1. □

3.3. Structure of the dual set

A number of key qualitative properties of the sets B and B^* are developed in this section. The most important result is Theorem 3.4, which identifies a collection of vectors which *must* be used in the construction of B^*, in the sense that a scalar multiple of each of these vectors must be one of the extreme points of B^*.

In general one would like to derive algebraic conditions which guarantee the existence of a set B that satisfies (2.2). This is most easily accomplished by characterizing the conditions under which its dual B^* satisfies (3.3). By Theorem 3.2 it then follows that $B = (B^*)^*$ satisfies (2.2). Given any polytope C we use the term *minimal internal representation* to refer to the characterization of C as a convex hull of its vertices, and the term *minimal external representation* to refer to the characterization of the polytope as the intersection of a minimal number of halfspaces. The following lemma states a well known connection between internal and external representations of sets and their duals. A proof is given in [18].

Lemma 3.3. *A polytope B with $0 \in B^\circ$ has $B = \cap_{j=1,\ldots,J}\{x : \langle x, v_j\rangle \leq k_j\}$ as its minimal external representation for a set of unit vectors v_j and scalars $k_j > 0$, $j = 1, \ldots, J$ if and only if its dual B^* has the minimal internal representation* $\mathrm{conv}[v_j/k_j, j = 1, \ldots, J]$.

Given any polytope B, as illustrated by Figure 2, each vertex of its dual is equal to a scalar multiple of the normal to some face of B, and conversely. We exploit this to infer properties of the structure of the dual B^* from the fact that B satisfies (2.2). Note that (2.2) imposes restrictions on the inward normals of the set B. This naturally translates into a property of the vertices of B^*. Suppose there exists a set B^* that satisfies (3.3) for a SP

$\{(d_i, n_i, c_i), i = 1, \ldots, N\}$ whose directions of constraint span $\mathbb{R}^n$. Then we can associate a set of unit directions $\mathscr{V}$, such that B^* must have a vertex along every direction in $\mathscr{V}$.

We now describe how to construct $\mathscr{V}$ for a given SP and then demonstrate the significance of the set in Theorem 3.4. Observe that the vertex directions are chosen so that each vertex is left invariant by as many operators as possible in order to minimize the number of conditions in (3.3) that have to be verified. We use $\dim[M]$ to denote the dimension of a space M and $|S|$ to denote the cardinality of a set S.

Construction of the Set $\mathscr{V}$ of fundamental vertex directions

Consider a SP $\{(d_i, n_i, c_i), i = 1, \ldots, N\}$ with $\text{span}(\{d_i, i = 1, \ldots, N\}) = \mathbb{R}^n$.

1. Define $\mathscr{D} \doteq \{d_i, i = 1, \ldots, N\}$ to be the set of directions of constraint.
2. Define $\{\mathscr{D}_j, j = 1, \ldots, J\}$ to be the collection of maximal subsets of $\mathscr{D}$ that span distinct $(n-1)$-dimensional subspaces of $\mathbb{R}^n$. By a maximal subset, we mean that if $d \in \mathscr{D} \cap \text{span}(\mathscr{D}_j)$, then $d \in \mathscr{D}_j$.
3. Let d_j^* be a unit normal to the hyperplane $\text{span}(\mathscr{D}_j)$, so that $\langle d_j^*, d\rangle = 0$ for every $d \in \mathscr{D}_j$.
4. Then $\mathscr{V} \doteq \{\pm d_j^*, j = 1, \ldots, J\}$ is the set of fundamental vertex directions.

Note that the construction above assumes that $\text{span}(\{d_i, i = 1, \ldots, N\}) = \mathbb{R}^n$. This condition will be satisfied by all the classes of SPs that we consider. Although we do not give the details here, it is worth observing that this assumption is really without loss of generality, in that the regularity properties of a SP which does not satisfy this condition can be deduced from those of an associated SP which does satisfy the condition.

Theorem 3.4. *Let $\mathscr{V}$ be the set of fundamental vertex directions associated with a given SP $\{(d_i, n_i, c_i), i = 1, \ldots, N\}$. Then any polytope B^* that satisfies* (3.3) *for the SP must have a vertex along every direction in $\mathscr{V}$.*

Proof. For the proof we will use the same notation as was just used in the construction of the fundamental set of vertices.

Fix $d_j^* \in \mathscr{V}$ such that d_j^* is orthogonal to $\text{span}(\mathscr{D}_j)$ for some subset $\mathscr{D}_j$ of $\mathscr{D}$. Let $E_j \subset \mathscr{D}_j$ be a basis for $\text{span}(\mathscr{D}_j)$, so that $|E_j| = \dim[\text{span}(D_j)] = n-1$, and let $F_j = \{n_i : d_i \in E_j\}$. Then clearly $\dim[\text{span}(F_j)] \leq n-1$ since $|F_j| = n-1$. Define $\bar{m}_j$ to be a non-zero vector orthogonal to $\text{span}(F_j)$. Since B^* satisfies (3.3), $B = (B^*)^*$ satisfies (2.2) by Theorem 3.2. Let $a_j \doteq \max\{s \geq 0 : s\bar{m}_j \in B\}$, so that $z_j \doteq a_j\bar{m}_j$ is the point on ∂B

along the direction $\bar{m}_j$. Since $\bar{m}_j$ is orthogonal to span(F_j), it follows that $\langle z_j, n_i \rangle = 0$ for all i such that $n_i \in F_j$. The continuity of the inner product implies that given $\delta > 0$, there exists a neighbourhood of z_j such that for all z in that neighbourhood $|\langle z, n_i \rangle| < \delta$ for all i such that $n_i \in F_j$. Let N_j be the intersection of the neighbourhood with ∂B. Then property (2.2) demands that for every $z \in N_j$, each inward normal $\nu \in \nu(z)$ to B at z must satisfy $\langle \nu, d_i \rangle = 0$ for all i such that $d_i \in E_j$. In other words, any $\nu \in \nu(z)$ must be orthogonal to span$(E_j) =$ span$(\mathscr{D}_j)$. Since $[\text{span}(\mathscr{D}_j)]^{\perp}$ has dimension 1 and contains d_j^*, we conclude that for every $z \in N_j$, $\nu(z) \subset \{\pm d_j^*\}$. The symmetry of B dictates that in fact both d_j^* and $-d_j^*$ must be inward normals of $(n-1)$-dimensional extreme faces of B, and consequently Lemma 3.3 implies that the dual set B^* must have vertices along the directions $\pm d_j^*$. $\square$

The symmetry of the set B^* dictates that the vertex directions of B^* always come in pairs, corresponding to parallel sides of the set B. Thus when trying to satisfy the projection property in (3.3), one always looks for polytopes $B^* = \text{conv}[v_j, j = 1, \ldots, J]$ of the form $\text{conv}[\pm a_k w_k, k = 1, \ldots, K]$ with $J = 2K$, the w_k fixed unit vectors and the a_k unknown constants. The goal then is to determine algebraic conditions on the problem data under which values a_k exist for which the projection property holds. In many examples (3.3) is satisfied by a polytope B^* whose set of vertex directions coincides with $\mathscr{V}$. This is true of most of the applications considered in [17]. In general, however, one may need to consider polytopes having additional vertex directions. This is illustrated by the example considered in [17, Section 2.4].

When considering directions in addition to those in $\mathscr{V}$, the following monotonicity property is very useful. Note that the requirement that the dual set B be bounded requires that B^* contain 0 in its interior. In this case if a set B^* with vertices of the form $\{\pm a_k w_k, k = 1, \ldots, K\}$ exists, then a set B^* will also exist of the form $\{\pm a_k w_k, k = 1, \ldots, K+1\}$ for any vector w_{K+1}. The total number of vertices of any polytope that satisfies (3.3) is an interesting quantity since it serves as an upper bound on the minimum number of faces that any set $B \in \mathscr{S}$ satisfying (2.2) must possess, and thus provides a measure of the complexity of B. It turns out that different SPs require sets of different complexity, and examples of this phenomenon are presented in [17]. The monotonicity property shows that if (3.3) is satisfied for a SP by a polytope that has a simple structure, then (3.3) will also be satisfied by a more complex polytope (i.e. one that has additional vertices.) However, although (3.3) will be satisfied even if one chooses a set that is more complicated than necessary, the verification of the property may become more difficult. In order to facilitate algebraic characterizations of the geometric property (3.3), it is always desirable to work with the simplest set that can possibly satisfy (3.3) for any given SP. Further discussion of this issue is provided in [17, Section 5.2].

Theorem 3.5 (Monotonicity Property). *Consider the SP $\{(d_i, n_i, c_i), i = 1, \ldots, N\}$. Suppose there exists a polytope whose set of vertex directions is given by $W = \{\pm w_i, i = 1, \ldots, M\}$ and which satisfies* (3.3) *for the SP. Then given any set $\tilde{W} \supset W$, there exists a polytope with $\tilde{W}$ as its vertex directions that also satisfies* (3.3) *for the SP.*

Proof. Let B^* be the polytope with vertex directions W that satisfies (3.3). Choose $v \in \tilde{W} \setminus W$. It is clearly sufficient to establish that (3.3) is satisfied by a polytope whose vertex directions are $W \cup \{v\}$. One can then appeal to an inductive argument in order to establish the theorem as stated. Let $\alpha = \sup\{\kappa > 0 : \kappa v \in B^*\}$. Then since B^* is compact and $0 \in (B^*)^\circ$, $\alpha \in (0, \infty)$ and $\alpha v \in \partial B^*$.

We show that there exists an $\varepsilon > 0$ such that $\tilde{B}^* \doteq \text{conv}[B^*, (\alpha + \varepsilon)v]$ satisfies (3.3). Since the vertices of B^* satisfy (3.3) and $B^* \subset \text{conv}[B^*, (\alpha + \varepsilon)v]$ for any $\varepsilon > 0$, it only remains to show that there exists $\varepsilon > 0$ such that for every $i = 1, \ldots, N$, either

$$L_i^*((\alpha + \varepsilon)v) \in (\tilde{B}^*)^\circ \text{ or } L_i^*((\alpha + \varepsilon)v) = (\alpha + \varepsilon)v.$$

We know that the last display is true when αv is substituted for $(\alpha+\varepsilon)v$ since $\alpha v \in B^*$ and B^* satisfies (3.3). For those operators for which $L_i^* \alpha v = \alpha v$, linearity dictates that $L_i^*(\alpha + \varepsilon)v = (\alpha + \varepsilon)v$. The remaining operators must satisfy $L_i^* \alpha v \in (B^*)^\circ$. Define $\rho \doteq \max_i ||L_i^* v||$, where the max is taken over i such that L_i^* does not leave v invariant, and let $\varepsilon > 0$ be small enough so that the $\rho\varepsilon$-neighbourhoods for all such $L_i^*(\alpha v)$ lie in the interior of B^*. Such an ε exists since $N < \infty$ and $(B^*)^\circ$ is open. Then

$$L_i^*((\alpha + \varepsilon)v) = L_i^*(\alpha v) + \varepsilon L_i^* v \subset N_{\rho\epsilon}(L_i^*(\alpha v)) \subset (B^*)^\circ \subset (\tilde{B}^*)^\circ,$$

and the theorem is established. □

Remark 3.6. Another obvious but very useful monotonicity property is the following. Suppose there exists a set B^* that satisfies (3.3) for the SP $\{(d_i, n_i, c_i), i = 1, \ldots, N\}$. Then given any subset $J \subset \{1, \ldots, N\}$, the set B^* also satisfies (3.3) for the SP defined by $\{(d_i, n_i, c_i), i \in J\}$.

3.4. Algebraic conditions and transformation techniques

Once one assumes that B^* is of the form $\text{conv}[\pm a_k w_k, k = 1, \ldots, K]$ with w_k fixed unit vectors and the $a_k > 0$ unknown, it is desirable to derive *finite* algebraic conditions under which such a polytope satisfies (3.3) for the SP. In general, one needs to derive a finite external representation from the given internal representation for the set B^*. Then clearly (3.3) can be expressed algebraically in the form of a finite number of linear inequalities. We first use the fact that $C = (C^*)^*$ for any closed convex set C to obtain an external representation (though not necessarily finite) from a given internal representation. It will be convenient to express $C = (C^*)^*$ in the following form:

$$C = \left\{ x \in I\!R^n : \langle x, \nu \rangle \le 1 \text{ for all } \nu \text{ satisfying } \max_{u \in C} \langle u, \nu \rangle \le 1 \right\}. \quad (3.6)$$

Given the internal representation of any polytope, (3.6) automatically gives us an external representation. However, in order to derive a useful algebraic condition for (3.3) from the external representation, we require the minimal or at least a finite external representation of the polytope. This requires the identification of a finite set of vectors $\mathscr{K}$ such that any ν satisfying $\max_{u \in C} \langle u, \nu \rangle \le 1$ can be written as the convex combination of vectors in $\mathscr{K}$. The external representation then takes the form

$$C = \cap_{\nu \in \mathscr{K}} \{x : \langle x, \nu \rangle \le 1\}. \quad (3.7)$$

Such a representation provides a finite algebraic characterization of the set of normals associated with those fixed direction of constraints for which (3.3) is satisfied. In general, this characterization is expressed in terms of a finite number of algebraic conditions on the matrix

$$V = [v_{ij}] \doteq [\langle d_i, n_j \rangle].$$

The precise form of the algebraic condition is dictated by the class of the SP, which is in turn determined by the number and structure of the directions of constraint in the representation of the SP. This is illustrated by the concrete examples studied in [17]. The purpose of this section is to describe how transformation techniques can be used to extend the algebraic condition available for a particular set of constraint directions to other directions of constraint within the same class. Given a SP that satisfies Assumption 3.1, Theorem 3.6 identifies a corresponding class of SPs which also satisfy Assumption 3.1. Recall that A^* denotes the adjoint of the operator A.

Theorem 3.6. *Suppose there exists a polytope B^* that satisfies Assumption* 3.1 *for the SP* $\{(z_i, w_i, k_i), i = 1, \ldots, N\}$. *Then given any invertible transformation $A : I\!R^n \to I\!R^n$, $A^* B^*$ satisfies Assumption* 3.1 *for the SP* $\{(A^{-1} z_i, A^* w_i, c_i), i = 1, \ldots, N\}$.

Proof. Note that the particular values of k_i and c_i are of no consequence in proving the theorem. Let L_i^* and $\tilde{L}_i^*$, $i = 1, \ldots, N$ be the collections of projection operators associated with the SPs $\{(z_i, w_i, k_i), i = 1, \ldots, N\}$ and $\{(A^{-1}z_i, A^* w_i, c_i), i = 1, \ldots, N\}$, respectively. Since

$$\begin{aligned} A^* L_i^* (A^*)^{-1} x &= A^*[(A^*)^{-1}x - \langle (A^*)^{-1}x, z_i\rangle w_i] \\ &= x - \langle x, A^{-1}z_i\rangle A^* w_i, \end{aligned}$$

the two collections of projection operators are related by

$$\tilde{L}_i^* = A^* L_i^* (A^*)^{-1}.$$

If v_j, $j = 1, \ldots, J$ are the vertices of B^*, then since A is an invertible transformation $A^* v_j$, $j = 1, \ldots, J$ are the vertices of $A^* B^*$. Moreover, $L_i^* v_j$ lies in the interior of B^* or $L_i^* v_j = v_j$ if and only if $\tilde{L}_i^*(A^* v_j)$ lies in the interior of $A^* B^*$ or $\tilde{L}_i^*(A^* v_j) = A^* v_j$, respectively. Thus $A^* B^*$ satisfies (3.3) for the SP $\{(A^{-1}z_i, A^* w_i, c_i), i = 1, \ldots, N\}$. □

We conclude from Theorem 3.6 that if Assumption 3.1 is satisfied for a collection $\mathscr{C}$ of SPs of the form $\{(z_i, w_i, k_i), i = 1, \ldots, N\}$, then Assumption 3.1 is also satisfied for all SPs in the class $\mathscr{F}$ defined by

$$\mathscr{F} \doteq \bigcup_{\{(z_i, w_i, k_i), i=1,\ldots,N\} \in \mathscr{C}} \; \bigcup_{\mathbf{c} \in \mathbb{R}^N} \; \bigcup_{A : \det A \neq 0} \{(A^{-1}z_i, A^* w_i, c_i), i = 1, \ldots, N\}, \tag{3.8}$$

where $\mathbf{c} = (c_1, \ldots, c_N)$.

In [17] we apply the methods of this section to examine concrete classes of SPs. Sections 2 and 3 of [17] show how the transformation technique described above can be used to extend results within two classes of SPs – the generalized Harrison–Reiman class and the generalized processor sharing class of SPs. In the following section, we develop new methods for the construction of solutions to the SP.

4. Existence of solutions to the Skorokhod Problem

4.1. Introduction

In this section we develop a new set of methods for the construction of solutions to the SP. Under Assumption 2.1 the existence of solutions to the SP for paths ψ of bounded variation is equivalent to the existence of a certain non-linear projection on $\mathbb{R}^n$. In Section 4.2 we introduce the definitions of local and global projections for a SP. Theorem 4.2 summarizes the extent to which the existence of a projection guarantees existence of solutions to the SP. In Section 4.3 we develop a decomposition approach to constructing solutions to SPs that satisfy Assumption 2.1. In Section 4.3.2 we introduce a broad class of SPs, which we refer to as *simple SPs*, that can be represented as the union of SPs with a more elementary structure, which we term *standard SPs*. We present criteria for the existence of solutions to standard SPs and indicate how they can be used to obtain existence of solutions to simple SPs. In Section 4.4 we show that any SP on a polyhedral domain can in a sense be approximated by a sequence of simple SPs. Theorem 4.12 proves that if Assumption 2.1 is satisfied by the limit SP, then the existence of projections for the elements of the approximating sequence of SPs guarantees that for the limit. These results are applied in [17, Sections 2.3, 2.5 and 3.4] to establish the existence of solutions to SPs from two classes of SPs. In the case of the generalized processor sharing SP considered in [17, Section 3], existence is (necessarily) obtained only for a strict subset of $D([0,\infty) : \mathbb{R}^n)$ that includes all functions of bounded variation.

4.2. *The projection*

The existence of solutions to the SP has been shown to be predicated on the existence of a projection π of $\mathbb{R}^n$ onto the domain G [7, 13, 29, 40]. For $\delta > 0$, we define $N_\delta \doteq \{x \in \mathbb{R}^n : \inf_{y \in G} \|x - y\| < \delta\}$.

Definition 4.1 (Projection). *A mapping $\pi : \mathbb{R}^n \to G$ is said to be a projection for the SP with domain G and directions of constraint $d(x)$ if*

1. $\pi(y) = y$ *for* $y \in G$.
2. $\pi(y) \in \partial G$ *for* $y \notin G$, *and* $\pi(y) - y = \alpha\gamma$ *for some* $\alpha \geq 0$ *and* $\gamma \in d(\pi(y))$.

When a mapping with the above properties can only be defined on some neighbourhood $N_\delta(G)$ with $\delta > 0$, then it is said to be a local projection.

We sometimes refer to the projection as global when it is defined on all of $\mathbb{R}^n$. Consider the space $D([0,\infty) : \mathbb{R}^n)$ equipped with the topology of uniform convergence on compact sets and the metric

$$\rho(\psi_1, \psi_2) \doteq \sum_{i=1}^{\infty} \frac{1}{2^{-i}} \left[\sup_{0 \leq t \leq i} \|\psi_1(t) - \psi_2(t)\| \wedge 1 \right].$$

For any given set G, let $D_G([0,\infty) : \mathbb{R}^n)$ be the set of trajectories in $D([0,\infty) : \mathbb{R}^n)$ that start in G. Define

$$F_G([0,\infty):\mathbb{R}^n) \doteq \left\{ \begin{array}{l} \psi \in D_G([0,\infty):\mathbb{R}^n) : \psi \text{ is piecewise} \\ \text{constant with a finite number of} \\ \text{jumps on each interval } [0,T],\, T<\infty \end{array} \right\}. \quad (4.1)$$

Given any SP, if the projection associated to that SP exists, then it follows immediately that a solution to the SP exists for all $\psi \in F_G([0,\infty):\mathbb{R}^n)$. Note that this set is dense in $D_G([0,\infty):\mathbb{R}^n)$ with respect to the sup norm. If the SP also satisfies Assumption 2.1 then the Skorokhod Map is Lipschitz continuous on $F_G([0,\infty):\mathbb{R}^n)$. Hence, by a standard theorem from analysis [37, p.149], there exists a unique Lipschitz continuous extension of the Skorokhod Map to $D_G([0,\infty):\mathbb{R}^n)$. The extension may not define a solution to the SP for all paths in $D_G([0,\infty):\mathbb{R}^n)$ since the boundedness of the total variation of the constraining term η, though true for every element of the sequence, is not guaranteed in the limit. Such "generalized" solutions to the SP are still useful. In particular, one often considers a functional of ϕ or η that is well-behaved even when the total variation of η is not bounded. An example of such a situation arises in the generalized processor sharing SP that is discussed in [17, Section 3.4]. Solutions to the SP that satisfy Definition 1.1 can be obtained as limits of sequences of functions in $F_G([0,\infty):\mathbb{R}^n)$ when there exists an *a priori* bound on $|\eta|(T)$ for every $T<\infty$. In other words, solutions exist for functions in the class

$$\mathscr{F}_G = \cap_{T<\infty} \cup_{s<\infty} \overline{\{\psi \in F_G([0,\infty):\mathbb{R}^n) : |\eta|(T) \leq s\}}, \quad (4.2)$$

where the closure is taken with respect to the sup norm [13]. When Assumption 2.1 is satisfied, the class $\mathscr{F}_G$ contains all functions of bounded variation in $D_G([0,\infty):\mathbb{R}^n)$. Under an additional condition that is stated below as Assumption 4.1, the class $\mathscr{F}_G$ can be shown to coincide with $D_G([0,\infty):\mathbb{R}^n)$.

Assumption 4.1. *For every $x \in \partial G$, there is $n \in n(x)$ such that $\langle d, n\rangle > 0$ for all $d \in d(x)$.*

Assumption 4.1 states that for every $x \in \partial G$, there exists a hyperplane of support to G through x that separates $x - d(x)$ and G. For SPs with $G = \mathbb{R}^n_+$ and n directions of constraint this geometric assumption is equivalent to the algebraic one used in Reiman and Williams [35] and Bernard and El Kharroubi [3], where it is referred to as the condition that the matrix formed by the directions of constraint is completely-$\mathscr{S}$. For such SPs, it serves as a necessary and sufficient condition for the existence of solutions to the SP on $D_G([0,\infty):\mathbb{R}^n)$ [3]. The existence results discussed above are summarized in the following theorem [13].

Theorem 4.2. *Given a SP with domain G, the existence of a projection $\pi : \mathbb{R}^n \to G$ implies the existence of solutions for all $\psi \in \mathscr{F}_G$. If the SP satisfies Assumption* 2.1*, then solutions are unique for all such ψ and there exists a unique Lipschitz continuous extension of the Skorokhod Map to $D_G([0,\infty) : \mathbb{R}^n)$. If in addition Assumption* 4.1 *holds, then $\mathscr{F}_G = D_G([0,\infty) : \mathbb{R}^n)$.*

Note that if for some $\delta > 0$ a local projection is defined for a SP on the neighbourhood $N_\delta(G)$, then one obtains existence of solutions for the subset $H_G([0,\infty) : \mathbb{R}^n)$ of functions in $F_G([0,\infty) : \mathbb{R}^n)$ whose maximum jump size is less than δ. As in (4.2), we can then obtain solutions to the SP on the closure (with respect to the sup norm) of the set of functions in $H_G([0,\infty) : \mathbb{R}^n)$ for which there exists an a priori bound on $|\eta|(T)$ for every $T < \infty$. When Assumption 2.1 holds for the SP this includes all continuous functions of bounded variation.

4.3. The existence of a projection for simple Skorokhod Problems

From Theorem 4.2 it is apparent that a crucial step in proving existence of solutions to the SP is establishing the existence of a global projection for the SP. Section 4.3.1 shows how local projections can be pieced together in order to obtain global projections for a SP. Consider a SP that satisfies Assumption 2.1. For such SPs, Theorem 4.4 shows that the existence of a local projection is equivalent to that of a global one. SPs with a complex structure can often be represented as a combination of smaller sub-SPs which are easier to analyze. Theorem 4.5 derives conditions under which a global projection for the SP can be synthesized from local projections for the sub-SPs. In Section 4.3.2 we introduce a class of SPs, which we call simple SPs, for which a natural decomposition exists. Simple SPs can be represented as the union of SPs with a more elementary structure, which we term standard SPs.

4.3.1. Construction of global projections

Suppose there exists a local projection π for a SP that is defined on a neighbourhood $N_\delta(G)$ of the domain for $\delta > 0$. Then for $\alpha \in [0,\infty)$ we define the map F_α on $N_\delta(G) \cap G^c$ by

$$F_\alpha(x) \doteq \pi(x) + \frac{x - \pi(x)}{||x - \pi(x)||}(\alpha + ||x - \pi(x)||). \tag{4.3}$$

When $\alpha = 0$ F_α is simply the identity map. In general $F_\alpha(x)$ locates a point α units away from x on the ray joining $\pi(x)$ to x. It is evident that the point

$\pi(x)$ also serves as a projection for all points on this ray. In other words, we can define $\pi(F_\alpha(x)) \doteq \pi(x)$. To see that $\pi(x)$ defines a valid projection for $F_\alpha(x)$ note that $\pi(x) \in \partial G$, and from (4.3) we get

$$\pi(F_\alpha(x)) - F_\alpha(x) = \pi(x) - F_\alpha(x) = (\pi(x) - x)\left(\frac{\alpha + ||x - \pi(x)||}{||x - \pi(x)||}\right),$$

which is contained in $d(\pi(x)) = d(\pi(F_\alpha(x)))$ since $\pi(x)$ is a valid projection for x. Thus we can use F_α to extend the domain of definition of the local projection to $G \cup \mathscr{C}$, where

$$\mathscr{C} \doteq \cup_{\alpha \geq 0} F_\alpha(N_\delta(G) \cap G^c), \tag{4.4}$$

by defining $\pi(x) \doteq x$ for $x \in G$ and $\pi(x) \doteq \pi(y)$ for any $x \in \mathscr{C}$ such that $x = F_\alpha(y)$ for some $y \in N_\delta(G) \cap G^c$ and $\alpha \geq 0$. Suppose $x = F_\alpha(y)$ and $x = F_{\alpha'}(y')$ for some $(y, y') \in (N_\delta(G) \cap G^c)^2$ and $(\alpha, \alpha') \in [0, \infty)^2$. Then, as just shown, both $\pi(y)$ and $\pi(y')$ are valid projections for x. Let x_0 be any point in G, and let x_1 and x_2 be points in G^c. Define $\psi_i(t)$ to be x_0 if $t \in [0, 1)$ and x_i if $t \in [1, \infty)$. If a projection π is defined at both x_1 and x_2, then a solution to the SP for ψ_i, $i = 1, 2$ is given by

$$\phi_i(t) = \begin{cases} x_0 & \text{for } t \in [0, 1) \\ \pi(x_i) & \text{for } t \in [1, \infty). \end{cases}$$

Thus Lipschitz continuity of the SM implies Lipschitz continuity of π on its domain of definition. Therefore if Assumption 2.1 is satisfied for the SP, then $\pi(y) = \pi(y')$ since the projection must be unique and hence the extension is well defined. In fact, as established in Lemma 4.3, when Assumption 2.1 holds $F_\alpha : N_\delta(G) \cap G^c \to F_\alpha(N_\delta(G) \cap G^c)$ is a homeomorphism for every $\alpha \geq 0$.

Lemma 4.3. *Suppose Assumption* 2.1 *is satisfied for a SP for which a local projection π exists on some neighbourhood $N_\delta(G)$ of the domain G. Then for any $\alpha \geq 0$ the mapping $F_\alpha : N_\delta(G) \cap G^c \to F_\alpha(N_\delta(G) \cap G^c)$ defined in* (4.3) *is a homeomorphism.*

Proof. For $\alpha = 0$ F_α is just the identity map which is clearly a homeomorphism. Fix $\alpha > 0$. Since the SP satisfies Assumption 2.1, the extension of π defined on (4.4) is continuous. Moreover, since $||x - \pi(x)|| > 0$ on $N_\delta(G) \cap G^c$, F_α is continuous on $N_\delta(G) \cap G^c$. Now suppose there exists $(y, y') \in (N_\delta(G) \cap G^c)^2$ such that $x \doteq F_\alpha(y) = F_\alpha(y')$. Then by the definition of the extension, $\pi(F_\alpha(y)) = \pi(y)$ and $\pi(F_\alpha(y')) = \pi(y')$ and thus

$$
\begin{aligned}
x &= \pi(y) + \frac{y - \pi(y)}{||y - \pi(y)||} (\alpha + ||y - \pi(y)||) \\
&= \pi(y') + \frac{y' - \pi(y')}{||y' - \pi(y')||} \left(\alpha + ||y' - \pi(y')||\right).
\end{aligned}
\tag{4.5}
$$

However, $\pi(y) = \pi(y')$ since both define valid projections for x and Assumption 2.1 implies that the projection is unique. Substituting this in (4.5) we obtain $\|y - \pi(y)\| = \|y' - \pi(y')\|$, and thus $y = y'$. This implies that F_α is one-to-one. The continuity of the inverse mapping is then apparent from the explicit expression

$$
F_\alpha^{-1}(x) = \pi(x) + \frac{x - \pi(x)}{\|x - \pi(x)\|} (||x - \pi(x)|| - \alpha)
$$

for $x \in F_\alpha(N_\delta(G) \cap G^c)$. This is well defined because the extension defines $\pi(x)$ uniquely for all $x \in \mathscr{C}$. Thus for every $\alpha \geq 0$, F_α is a homeomorphism from $N_\delta(G) \cap G^c$ to $F_\alpha(N_\delta(G) \cap G^c)$. □

The homeomorphism F_α is used in Theorem 4.4 to demonstrate that when Assumption 2.1 holds, $\mathscr{C} = G^c$ and so the extension of π to $\mathscr{C}$ given in (4.4) yields a projection for the SP that is defined on the whole of $I\!R^n$.

Theorem 4.4. *Suppose Assumption* 2.1 *is satisfied for the* $SP\{d_i, n_i, c_i),$ $= 1, \ldots, N\}$ *with domain* G. *Moreover suppose that there exists a local projection* π *for the SP defined on* $N_\delta(G)$ *for some* $\delta > 0$. *Then* π *can be uniquely extended to define a projection on* $I\!R^n$.

Proof. On G, $\pi(x) = x$. Given a local projection π for the SP, for $\alpha \geq 0$ consider the map F_α defined in (4.3) and the extension of π to $\mathscr{C}$ described in (4.4). We prove the theorem by showing that $\mathscr{C}$ is both open and closed relative to G^c. Since F_α is a homeomorphism by Lemma 4.3 and $N_\delta(G) \cap G^c$ is open, $F_\alpha(N_\delta(G) \cap G^c)$ is also open for every $\alpha \geq 0$. Consequently $\mathscr{C}$ is a union of open sets and is therefore also open.

Now let $\{x_n\}$ be a sequence in $\mathscr{C}$ that converges to $x \in G^c$. If $x \in N_\delta(G) \cap G^c$, then it lies in the domain of definition of the local projection and so $x \in \mathscr{C}$. Suppose $x \notin N_\delta(G)$. Since $\lim_{n\to\infty} x_n = x$, given any $\gamma \in (0, \delta)$, there exists $N < \infty$ such that $x_n \in [N_\gamma(G)]^c$ for $n \geq N$. Therefore $||\pi(x_n) - x_n|| \geq \gamma$ since $\inf_{x\in\mathscr{C}\setminus N_\gamma(G)} ||x - \pi(x)|| \geq \inf_{x\in\mathscr{C}\setminus N_\gamma(G)} d(x, G) \geq \gamma$. Now set $\gamma = \delta/2$ and choose $N < \infty$ such that $||x_n - \pi(x_n)|| \geq \gamma$ for $n \geq N$. Then since $x_n \in \mathscr{C}$ there exist $y_n \in N_\delta(G) \cap G^c$ and $\alpha_n \geq 0$ such that $x_n = F_{\alpha_n}(y_n)$, and thus

$$
x_n = \pi(y_n) + \frac{y_n - \pi(y_n)}{||y_n - \pi(y_n)||} (\alpha_n + ||y_n - \pi(y_n)||).
$$

With no loss it can be assumed that $y_n \in \{y : ||y - \pi(y)|| = \gamma\} \subset N_\delta(G)$. This is because if $||y_n - \pi(y_n)|| \neq \gamma$ we can always choose y'_n and α'_n to be

$$y'_n \doteq \pi(y_n) + \frac{y_n - \pi(y_n)}{||y_n - \pi(y_n)||}\gamma,$$

and

$$\alpha'_n \doteq \alpha_n + ||y_n - \pi(y_n)|| - \gamma,$$

so that $\pi(y'_n) = \pi(y_n) = \pi(x_n)$. It is easy to verify that $||y'_n - \pi(y'_n)|| = \gamma$, $y'_n \in N_\delta(G) \setminus G$ and $x_n = F_{\alpha'_n}(y'_n)$. Moreover, $\alpha'_n \geq 0$ since $||x_n - \pi(x_n)|| = \alpha_n + ||y_n - \pi(y_n)|| \geq \gamma$.

Thus there always exists a sequence $y_n \in \{y : ||y - \pi(y)|| = \gamma\}$ and $\alpha_n \geq 0$ such that $x_n = F_{\alpha_n}(y_n)$. Then x_n can be expressed as

$$x_n = \pi(y_n) + \frac{\alpha_n + \gamma}{\gamma}(y_n - \pi(y_n)).$$

Since x_n converges to x, $\{x_n, n \in I\!\!N\}$ is bounded and so lies inside some neighbourhood $N_{K_1}(0)$ for $K_1 < \infty$. Since $x_n \in \mathscr{C}$ $\pi(x_n)$ exists, and thus the boundedness of $\{x_n, n \in I\!\!N\}$ along with the Lipschitz continuity of π implies that there exists $K_2 < \infty$ such that $\{\pi(x_n), n \in I\!\!N\} \subset N_{K_2}(0)$. Noting that $\pi(y_n) = \pi(x_n)$ and $\alpha_n \geq 0$, from the last display it follows that $\|y_n - \pi(x_n)\| \leq \|x_n - \pi(x_n)\|$. This shows that $\{y_n, n \in I\!\!N\}$ is a bounded sequence. Moreover, $\{y : ||y - \pi(y)|| = \gamma\}$ is closed because π is continuous. Thus since $\|y_n - \pi(y_n)\| = \gamma$, $\{y_n, n \in I\!\!N\}$ lies in a compact set and therefore there exists a subsequence, which we denote again by $\{y_n\}$, that converges to some $y \in N_\delta(G) \cap G^c$ with $||y - \pi(y)|| = \gamma$. The fact that π is Lipschitz continuous then implies that $\lim_{n\to\infty} \pi(y_n) = \pi(y)$. Since $x_n \to x$, α_n must also converge to some limit $\alpha \in [0, \infty)$. So we obtain

$$\begin{aligned} x = \lim_{n\to\infty} x_n &= \pi(y) + \frac{y - \pi(y)}{\gamma}(\alpha + \gamma) \\ &= \pi(y) + \frac{y - \pi(y)}{||y - \pi(y)||}(\alpha + ||y - \pi(y)||). \end{aligned}$$

In other words $x = F_\alpha(y)$ where $y \in N_\delta(G) \cap G^c$ and $\alpha \geq 0$, which shows that $x \in \mathscr{C}$. Thus we conclude that $x_n \in \mathscr{C} \to x \in G^c$ implies $x \in \mathscr{C}$. Thus $\mathscr{C}$ is closed relative to G^c and since it is also open, $\mathscr{C} = G^c$. Since Assumption 2.1 holds for the SP, the extension of the projection to $I\!\!R^n$ is unique. □

Consider the SP $P_j \doteq \{(d_i^j, n_i^j, c_i^j), i = 1, \ldots, I^j\}$ with domain G_j for $j = 1, \ldots, J$. Let $G \doteq \cap_{j=1}^J G_j$ and suppose that each of the domains G

and G_j, $j = 1, \ldots, J$ has nonempty interior relative to $\cup_{j=1}^J G_j$. Define the composite SP associated with the collection $\{P_j, j = 1, \ldots, J\}$ to be

$$Q \doteq \{(d, n, c) : (d, n, c) = (d_i^j, n_i^j, c_i^j) \text{ for some } i \in \{1, \ldots, I^j\} \text{ and } j \in \{1, \ldots, J\} \text{ such that } \{x : \langle x, n_i^j \rangle = c_i^j\} \cap \partial G \neq \emptyset\}. \tag{4.6}$$

The collection $\{P_j, j = 1, \ldots, J\}$ of SPs is said to be *consistent* if $(d, n, c) \in Q$, $(d', n', c') \in Q$, and $(n, c) = (n', c')$, imply $d = d'$. This simply ensures that every face of G has a single direction of constraint associated with it and consequently that Q represents a well defined SP on the polyhedral domain G. Suppose that the SP Q satisfies Assumption 2.1 and that each P_j has a local projection π_j. Theorem 4.5 shows that if the domains G_j overlap sufficiently, as prescribed by condition (4.7), then the projections π_j can be patched together to obtain a local projection for Q. The local projection can then be extended to all of $\mathbb{R}^n$ using Theorem 4.4. Let rel int A and rel ∂A refer to the interior and boundary respectively of a set A relative to some set $C \supset A$, which is specified explicitly according to the use. More precisely, rel int A with respect to $C \supset A$ is defined to be

$$\{x \in A : \exists \varepsilon > 0 \text{ such that } y \in C \text{ and } d(x, y) < \varepsilon \Rightarrow y \in A\},$$

and rel ∂A is defined to be $\mathrm{cl}(A) \backslash$rel int A. If no subset is explicitly specified, then the set C is taken to be the affine hull of A.

Theorem 4.5. *Suppose P_j is a Skorokhod Problem with local projection π_j and domain G_j for $j = 1, \ldots, J$. Let $G \doteq \cap_{j=1}^J G_j$. Suppose that the collection $\{P_j, j = 1, \ldots, J\}$ is consistent and the domains satisfy*

$$\bigcap_{j=1}^{J} \overline{\left[\partial G \setminus \left(\partial G \cap \partial G_j\right)\right]} = \emptyset. \tag{4.7}$$

If the composite Skorokhod Problem Q defined in (4.6) *satisfies Assumption* 2.1*, then a unique global projection exists for Q.*

Proof. As stated just before the theorem, Q is a well-defined SP with domain $G = \cap_{j=1}^J G_j$. For $j = 1, \ldots, J$ and $\varepsilon > 0$ define

$$O_j(\varepsilon) \doteq \{x \in \partial G \cap \partial G_j : d(x, \text{rel } \partial(\partial G \cap \partial G_j)) > \varepsilon\},$$

where the relative boundary is considered with respect to ∂G_j. By (4.7) and the fact that the G_j are polyhedral, there exists $\varepsilon_0 > 0$ such that for $\varepsilon < \varepsilon_0$

$$\bigcap_{j=1}^{J} N_{2\varepsilon}\left(\partial G \setminus \left(\partial G \cap \partial G_j\right)\right) = \emptyset.$$

Since by the definition of $O_j(\varepsilon)$, $\partial G \setminus O_j(\varepsilon) \subset N_{2\varepsilon}(\partial G \setminus (\partial G \cap \partial G_j))$ for $j = 1, \ldots, J$, the last display implies $\partial G \setminus \left[\cup_{j=1}^{J} O_j(\varepsilon)\right] = \emptyset$ for $\varepsilon < \varepsilon_0$. Since $\cup_{j=1}^{J} O_j(\varepsilon) \subset \partial G$, this implies that $\partial G = \cup_{j=1}^{J} O_j(\varepsilon)$. Fix $\varepsilon < \varepsilon_0$ and to simplify notation we no longer explicitly specify the dependence of O_j on ε.

Suppose π_j is defined on the neighbourhood $N_{\delta_j}(G_j)$ for $\delta_j > 0$, $j = 1, \ldots, J$. Let $\delta_0 \doteq \min_{i \in \{1,\ldots,J\}} \delta_i$ so that $N_{\delta_0}(G) \subset \cup_{j=1}^{J} N_{\delta_j}(G_j)$. Then π_j is defined on $N_{\delta_0}(G) \cap N_{\delta_0}(G_j)$ for every $j \in \{1, \ldots, J\}$. Since Assumption 2.1 holds for Q, it also holds for every P_j. Thus the projections π_j are unique and Lipschitz continuous. By definition of the projection, $\pi_j(y) = y$ for each $y \in O_j$, $j = 1, \ldots, J$. From the uniform Lipschitz continuity of the projections, given $\varepsilon > 0$ there exists $\delta \in (0, \delta_0)$ such that for $x \in I\!R^n$ and any $j = 1, \ldots, J$,

$$y \in O_j \text{ and } d(x, y) < \delta \Rightarrow d(\pi_j(x), \pi_j(y)) = d(\pi_j(x), y) < \varepsilon. \quad (4.8)$$

For $\delta \in (0, \delta_0)$ chosen so that (4.8) holds, for each $j \in \{1, \ldots, J\}$ let

$$O_j^{\delta} \doteq \{x \in G_j^c : d(x, y) < \delta \text{ for some } y \in O_j\}.$$

For $x \in N_\delta(G) \cap G^c$, let $\theta(x) \doteq \min\{j \in \{1, \ldots, J\} : x \in O_j^{\delta}\}$. Since $G^c = (\cap_{j=1}^{J} G_j)^c = \cup_{j=1}^{J} G_j^c$ and $\cup_{j=1}^{J} O_j = \partial G$, it is evident that $\cup_{j=1}^{J} O_j^{\delta} = N_\delta(G) \cap G^c$. Thus $\theta(x)$ is well defined on $N_\delta(G) \cap G^c$. Define

$$\pi(x) = \begin{cases} x & \text{for } x \in G, \\ \pi_{\theta(x)}(x) & \text{for } x \in N_\delta(G) \cap G^c. \end{cases} \quad (4.9)$$

We now show that π is a local projection for Q. Fix $x \in N_\delta(G) \cap G^c$ and suppose $j = \theta(x)$ so that $\pi(x) = \pi_j(x)$. The fact that $\pi_j(x)$ is a valid projection for P_j implies that $\pi_j(x) \in \partial G_j$. Then (4.8) and the fact that $x \in O_j^{\delta}$ imply that $\pi_j(x) \in N_\varepsilon(O_j) \cap \partial G_j$. From the definition of O_j it follows that $N_\varepsilon(O_j) \cap \partial G_j \subset \operatorname{rel\,int}(\partial G \cap \partial G_j)$ (where the interior is considered relative to ∂G_j) and thus $\pi_j(x) \in \partial G$. Consequently $\pi(x) = \pi_j(x) \in \partial G$. Moreover, $\pi_j(x) - x$ belongs to the allowed directions of constraint for P_j at $\pi_j(x)$. Since the SPs $\{P_j, j = 1, \ldots, J\}$ are consistent, for $y \in \operatorname{rel\,int}(\partial G \cap \partial G_j)$ (again relative to ∂G_j) the allowed directions of constraint $d(y)$ are the same for P_j and Q. Thus $\pi(x) - x = \pi_j(x) - x \in d_{P_j}(\pi_j(x)) = d_Q(\pi_j(x)) = d_Q(\pi(x))$, where $d_{P_j}(x)$ and $d_Q(x)$ represent the directions of constraint associated with the

SPs P_j and Q, respectively. This establishes that $\pi(x)$ given in (4.9) is a well-defined local projection for the SP Q on $N_\delta(G)$. Since we also assume Assumption 2.1 holds for the SP, the existence of a unique global projection follows from Theorem 4.4. $\square$

A complex SP can sometimes be decomposed into a collection of consistent SPs which have simpler structures. For example, see Figure 7 in [17]. Existence of a projection is usually more easily verified for the simpler SPs. Theorem 4.5 is very useful in such situations as it states conditions under which a projection for the complex SP can be inferred from the existence of projections for the collection of consistent SPs. In the next section, we introduce a class of SPs called simple SPs, which lends itself naturally to such a decomposition.

4.3.2. Existence for simple Skorokhod Problems

In this section we introduce two classes of SPs – standard SPs and simple SPs in Definitions 4.6 and 4.7, respectively. Conditions that guarantee existence of projections and solutions are best understood for the class of standard SPs. After discussing these conditions we indicate how they can be used along with Theorem 4.5 to verify existence of solutions to simple SPs, which locally resemble standard SPs. In [17, Section 3.4] we apply this technique to prove the existence of solutions to SPs that arise from a generalized processor sharing network, on a subset of $D_G([0,\infty) : \mathbb{R}^n)$.

Definition 4.6 (Standard SP). *The SP $\{(d_i, n_i, c_i), i = 1, \ldots, N\}$ is said to be standard if $N = n$ and* $\text{span}(\{n_1, n_2, \ldots, n_N\}) = \mathbb{R}^n$.

Definition 4.7 (Simple SP). *A polyhedron $G \subset \mathbb{R}^n$ is simple if each of its vertices lies in exactly n of its faces. The SP $\{(d_i, n_i, c_i), i = 1, \ldots, N\}$ is simple if its domain $G = \cap_{i=1}^N \{x : \langle x, n_i\rangle \geq c_i\}$ is simple.*

Let a simple domain have vertices v_j, $j = 1, \ldots, K$. Then we define $I_j \subset \{1, \ldots, N\}$ to be the set that identifies the n hyperplanes that contain the vertex v_j, so that

$$v_j = \cap_{i \in I_j} \{x : \langle x, n_i\rangle = c_i\}.$$

The standard SP associated with the vertex v_j is defined to be

$$P_j \doteq \{(d_i, n_i, c_i), i \in I_j\}, \tag{4.10}$$

and its domain is given by

$$G_j = \cap_{i \in I_j} \{x : \langle x, n_i \rangle \geq c_i\}.$$

We first consider two criteria for the existence of solutions to standard SPs. Let $D \doteq [d_1, d_2, \ldots, d_n]$ be the matrix of directions of constraint for a standard SP. As mentioned earlier, for standard SPs with domain $G = I\!R^n_+$, Assumption 4.1 is equivalent to the condition that the matrix D be completely-$\mathscr{S}$. This latter condition is a necessary and sufficient condition for the existence of solutions to the SP on $D([0, \infty) : I\!R^n)$ [3, 30, 35]. In particular, it is sufficient for the existence of a global projection for the associated SP. In [13, Theorem 3.1] another sufficient condition for the existence of a projection was introduced for SPs $\{(d_i, n_i, c_i), i = 1, \ldots, N\}$ whose domain G is a closed convex cone with vertex $w = \cap_{i=1}^N \{x : \langle x, n_i \rangle = c_i\}$. The condition requires the existence of an $n \times n$ matrix R that satisfies $d_i = Rn_i$ for $i = 1, \ldots, N$, and $\langle \nu, R\nu \rangle \geq a > 0$ for all $\nu \in \nu(w)$. (Recall that $\nu(w)$ is the set of unit inward normals to G at w.) The simplest situation in which this assumption is applicable is when G is the intersection of N half-spaces with $N \leq n$ and $d_i = Rn_i$, $i = 1, \ldots, N$ for some $n \times n$ matrix R. In particular, it is applicable to standard SPs.

Now suppose a projection exists for every standard SP associated with a simple SP. It is natural to ask whether a projection can then be constructed for the simple SP. Theorem 4.5 shows that this is possible if Assumption 2.1 holds for the simple SP and the domains of the constituent standard SPs overlap sufficiently to satisfy the condition in (4.7).

Theorem 4.8. *Consider the simple SP $Q = \{(d_i, n_i, c_i), i = 1, \ldots, N\}$ whose domain G has vertices v_j, $j = 1, \ldots, K$. Suppose that the associated standard SPs P_j have local projections π_j and domains G_j such that* (4.7) *is satisfied. If Assumpion* 2.1 *holds for Q, then there exists a global projection π for Q.*

Proof. The proof is a direct consequence of Theorem 4.5. It is easy to see that the simple SP Q can be represented as the composite SP for the collection P_j, $j = 1, \ldots, J$ of standard SPs. Suppose G_j is the domain of the standard SP P_j. Since the standard SPs arise from the simple SP Q, they automatically satisfy the consistency condition that the directions of constraint coincide on faces that are common to G, G_j and G_k for any $j, k \in \{1, \ldots, N\}$. By assumption, the domains satisfy (4.7) and Assumption 2.1 holds for Q. Thus by Theorem 4.5, Q has a unique global projection. □

4.4. Existence of solutions to general Skorokhod Problems

We now study existence of solutions to any SP on a polyhedral domain that has constant directions of constraint defined on each face. Theorem 4.11

asserts that any such SP can be approximated by a sequence of simple SPs. Theorem 4.12 shows that the existence of projections for SPs is in a certain sense closed under limits. More precisely, it proves that under Assumption 2.1 the existence of projections for the approximating simple SPs implies the existence of a projection for the limit. This suggests a method of obtaining existence of solutions for any SP by first constructing projections for an appropriate sequence of approximating simple SPs, which in turn can be obtained by building local projections for the corresponding standard SPs. The following lemma shows that any polytope can be approximated arbitrarily closely (using just radial perturbations) by a simplical polytope having the same set of vertex directions. This lemma is used in Theorem 4.11 below. A proof can be found in [18, Lemma A.4].

Definition 4.9 (Simplical Polytope). *A simplical polytope is a polytope in $\mathbb{R}^n$ for which each $(n-1)$-dimensional face is the convex hull of n vertices.*

Lemma 4.10. *Consider a polytope $Z \doteq \operatorname{conv}[C]$ for some finite set of points $C = \{v_j, j = 1, \ldots, J\}$ and suppose that $0 \in Z^\circ$. Then given any $\varepsilon > 0$ there exists a set $S = \{(1+\varepsilon_j)v_j, j = 1, \ldots, J\}$ such that $\operatorname{conv}[S]$ is a simplical polytope and $|\varepsilon_j| < \varepsilon$ for all $j \in \{1, ..., J\}$.*

Theorem 4.11. *Given any SP $\{(d_i, n_i, c_i), i = 1, \ldots, N\}$ and $\delta > 0$, there exists a simple SP of the form $\{(d_i, n_i, c_i'), i = 1, \ldots, N\}$ such that $c_i' \in (c_i - \delta, c_i)$ for all $i \in \{1, \ldots, N\}$.*

Proof. This is simply the statement dual to Lemma 4.10, which states that given any polytope Z, a simplical polytope can be obtained using arbitrarily small radial perturbations of the vertices of Z. One can without loss of generality assume that 0 lies in the relative interior of G since properties of the SP are invariant to translation of the domain. Therefore there exists an external representation of the domain $G = \cap_{i=1}^N \{x : \langle x, n_i\rangle \geq c_i\}$ with all $c_i \leq 0$. Now let $b_i \doteq c_i - \delta/3$ and define $F \doteq \cap_{i=1}^N \{x : \langle x, n_i\rangle \geq b_i\}$. Then $b_i < 0$ for every $i \in \{1, \ldots, N\}$ and 0 lies in the interior of F. This implies that the dual domain F^* is bounded, and by Lemma 3.3, the internal representation of F^* is $F^* \doteq \operatorname{conv}[\{n_i/b_i, i = 1, \ldots, N\}]$. Given any $\delta > 0$, by Lemma 4.10 there exists a simplical polytope of the form $F_\delta^* \doteq \operatorname{conv}[\{n_i/\tilde{b}_i, i = 1, \ldots, N\}]$, where $\tilde{b}_i \in (b_i - \delta/3, b_i)$. The dual $F_\delta \doteq (F_\delta^*)^*$ is then a simple domain with representation $F_\delta = \cap_{i=1}^N \{x : \langle x, n_i\rangle \geq \tilde{b}_i\}$. Thus $\{(d_i, n_i, \tilde{b}_i), i = 1, \ldots, N\}$ is a simple SP and since $\tilde{b}_i \in (c_i - \delta, c_i)$, the theorem is established. □

Thus if we use a description of the SP for which 0 lies in the relative interior of the domain, then it is always possible to choose the approximating sequence of simple domains to be such that $c_i' < c_i$. We will assume without loss of generality for the rest of this section that 0 lies in the relative interior of the domain of the SP. However, note that the proof of Theorem 4.12 given below only uses the fact that $0 \in G$.

Theorem 4.12. *Consider a sequence* $P^k \doteq \{(d_i, n_i, c_i^k), i = 1, \ldots, N\}$, $k \in I\!N$, *of SPs such that for every* i, *the sequence* c_i^k *monotonically converges up to* c_i. *Suppose there exists a projection* π^k *for each* P^k. *If the SP* $P = \{(d_i, n_i, c_i), i = 1, \ldots, N\}$ *satisfies Assumption* 2.1, *then it has a unique global projection.*

Proof. Let G^k be the domain and $d^k(\cdot)$ the directions of constraint associated with the SP P^k. Since $0 \in G$ and c_i^k converges monotonically up to c_i, $G = \cap_k G^k$. Note that $0 \in \cap_k G^k$ implies $\pi^k(0) = 0$ for every $k \in I\!N$. Fix $x \notin G$. Since Assumption 2.1 holds for the limit SP, it holds for every P^k because Assumption 2.1 is independent of the values of c_i in the representation of the SP. Thus every π^k is Lipschitz continuous with a common Lipschitz constant M. Fix $x \notin G$. We now show that $\{\pi^k(x), k \in I\!N\}$ is contained in a compact set. The uniform Lipschitz continuity of the projections dictates that

$$\|\pi^k(x)\| = \|\pi^k(x) - \pi^k(0)\| \leq M\|x\|.$$

This shows that the sequence $\{\pi^k(x), k \in I\!N\}$ is bounded. Hence there exists a convergent subsequence, which we also label by $\pi^k(x)$, that has limit $\pi(x) \doteq \lim_{k\to\infty} \pi^k(x)$. Now because $x \notin G$ and $c_i^k \uparrow c_i$, there exists $K < \infty$ such that for $k \geq K$, $x \notin G^k$ and hence $\pi^k(x) \in \partial G^k$. Then since $\{\pi^k(x), k \in I\!N\}$ is a bounded sequence, $c_i^k \uparrow c_i$ and $\pi^k(x) \in \partial G^k$, $\pi(x) = \lim_{k\to\infty} \pi^k(x) \in \partial G$.

We next state an upper semicontinuity property of the set of directions of constraint. If $y^k \in \partial G^k$ and $y^k \to y \in \partial G$, then there exists $K < \infty$ such that for $k \geq K$,

$$d^k(y^k) \subset d(y). \tag{4.11}$$

We know that $\pi^k(x) - x \in d^k(\pi^k(x))$. Substituting $y^k = \pi^k(x)$ and $y = \pi(x)$ in (4.11), for large enough k we obtain $\pi^k(x) - x \in d(\pi(x))$. Taking the limit in k and noting that $d(\pi(x))$ is a closed cone we conclude that $\pi(x) - x \in d(\pi(x))$. Thus $\pi(x)$ is a projection for all $x \notin G$.

If $x \in G$, then $x \in G^k$ for all k and so $\pi(x) = \pi^k(x) = x$. Therefore $\pi(x)$ is a valid global projection for P. □

5. Conclusion

In summary, this paper develops methods for constructing solutions to the SP and for verifying Lipschitz continuity of the SM for SPs on polyhedral domains. The key step is the construction of a set B^* that satisfies Assumption 3.1. Once the existence of B^* is verified, the consequent Lipschitz properties are also useful in establishing existence of solutions to the SP. Assumption 3.1 is geometric in nature. However, the paper shows how convex analysis and transformation methods can be applied to obtain algebraic conditions on the problem data that guarantee that Assumption 3.1 is satisfied. It is this interesting interplay between convex geometry and algebraic characterizations that makes the general study of the SP challenging.

The duality methods developed in this paper are applied to concrete classes of SPs in the sequel [17] to this paper. All SPs outside the generalized Harrison–Reiman class turn out to be much harder to analyze, in part because techniques more sophisticated than contraction mapping techniques are required, and also because of the greater complexity of the associated set B.

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