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On logarithmic Sobolev inequalities for continuous time random walks on graphs

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Abstract. We establish modified logarithmic Sobolev inequalities for the path distributions of some continuous time random walks on graphs, including the simple examples of the discrete cube and the lattice $\mathbb{Z}^d$. Our approach is based on the Malliavin calculus on Poisson spaces developed by J. Picard and stochastic calculus. The inequalities we prove are well adapted to describe the tail behaviour of various functionals such as the graph distance in this setting.

1. Introduction

The classical logarithmic Sobolev inequality for Brownian motion $B = (B_t)_{t \geq 0}$ in $\mathbb{R}^d$ [Gr] indicates that for all functionals F in the domain of the Malliavin gradient operator $D : \mathrm{L}^2(\Omega, \mathbb{P}) \to \mathrm{L}^2(\Omega \times [0, T], \mathbb{P} \otimes dt)$,

$$\mathbb{E}(F^2 \log F^2) - \mathbb{E}(F^2) \log \mathbb{E}(F^2) \leq 2\, \mathbb{E}\left(\int_0^T |D_t F|^2 dt \right) . \tag{1.1}$$

In particular, if $F = f(B_{t_1}, \ldots, B_{t_n})$, $0 = t_0 \leq t_1 \leq \cdots \leq t_n$ for some smooth function $f : (\mathbb{R}^d)^n \to \mathbb{R}$,

$$D_t F = \sum_{i=1}^n \nabla_i F\, \mathrm{I}_{\{t \leq t_i\}} = \sum_{i=1}^n \mathrm{I}_{\{t_{i-1} < t \leq t_i\}} \left(\sum_{k=i}^n \nabla_k F \right)$$

where, with some abuse, $\nabla_i F = \nabla_i f(B_{t_1}, \ldots, B_{t_n})$ and $\nabla_i f$ is the gradient of f along the i-th direction, so that

$$\mathbb{E}(F^2 \log F^2) - \mathbb{E}(F^2) \log \mathbb{E}(F^2) \leq 2 \sum_{i=1}^n (t_i - t_{i-1}) \mathbb{E}\left(\left| \sum_{k=i}^n \nabla_k F \right|^2 \right) . \tag{1.2}$$

If $F = f(B_t)$, $t \geq 0$, for some smooth $f : \mathbb{R}^d \to \mathbb{R}$,

$$\mathbb{E}\big(f^2(B_t) \log f^2(B_t)\big) - \mathbb{E}\big(f^2(B_t)\big) \log \mathbb{E}\big(f^2(B_t)\big) \leq 2t\, \mathbb{E}\big(\big|\nabla f(B_t)\big|^2\big) . \tag{1.3}$$

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In the recent years, lots of efforts have been put in trying to extend the logarithmic Sobolev inequality (1.1) to Brownian motions on Riemannian manifolds M with geometric bounds on the Ricci curvature. This problem has been solved in the papers [Hs], [A-E], and is now pretty much well understood with the contribution [C-H-L] where it is shown how one can easily deduce a logarithmic Sobolev inequality in this setting from a Clark-Ocone formula. The latter is in turn a consequence of appropriate integration by parts formulae going back to the work by J.-M. Bismut [Bi]. In the flat case, such a representation formula indicates that for any smooth functional F,

$$F = \mathbb{E}(F) + \int_0^T \mathbb{E}\big(D_t F \mid \mathscr{F}_t\big) dB_t \ . \tag{1.4}$$

As a generic by-product of these logarithmic Sobolev inequalities, one can obtain tail estimates for Lipschitz functionals of large deviation type. For example (cf. [Le]), if M has non-negative Ricci curvature, for every $T > 0$,

$$\lim_{R\to\infty} \frac{1}{R^2} \log \mathbb{P}\Big\{ \sup_{0\le t\le T} d(B_t, x_0) \ge R \Big\} = -\frac{1}{2T} \tag{1.5}$$

where $d(B_t, x_0)$ is the Riemannian distance of Brownian motion B_t at time t from its starting point x_0.

The aim of this work is to investigate logarithmic Sobolev inequalities for Brownian motions with values in graphs, with some view to tail estimates of the type (1.5). Our study is very preliminary, and at this stage we only cover examples that would correspond to constant curvature spaces in a Riemannian setting. In order to introduce our purpose, and to understand better what kind of results can be expected, let us first discuss two simple examples.

Let $\chi = \{-1, +1\}^d$ be the discrete cube in $\mathbb{R}^d$, and let $B = (B_t)_{t\ge0}$ be the continuous time simple random walk on χ. In other words, B is the process that jumps, after an exponential waiting time, from one of the vertices of the cube to one of its neighbour with equal probability. The transition densities (with respect to the uniform probability measure on χ) of the process B are given by

$$p_t(x, y) = \prod_{i=1}^{d} (1 + x_i y_i \, \mathrm{e}^{-t}) \ ,$$

$x = (x_1, \dots, x_d)$, $y = (y_1, \dots, y_d) \in \chi$, $t \ge 0$. Let us assume $d = 1$ so as to make the notation more simple. If f is a function on $\{-1, +1\}$, and $F = f(B_t)$, $t \ge 0$, it is known that

$$\begin{aligned}\mathbb{E}\big(f^2(B_t) \log f^2(B_t)\big) &- \mathbb{E}\big(f^2(B_t)\big) \log \mathbb{E}\big(f^2(B_t)\big) \\ &\le \frac{1}{4}\,(1-\mathrm{e}^{-2t})\,\mathrm{e}^t \log\left(\frac{1+\mathrm{e}^{-t}}{1-\mathrm{e}^{-t}}\right) \mathbb{E}\big(\big|Df(B_t)\big|^2\big)\end{aligned} \tag{1.6}$$

where $Df(x) = f(-x) - f(x)$, $x \in \{-1, +1\}$. Since the law of B_t is a Bernoulli measure on $\{-1, +1\}$ with weigths $\frac{1}{2}(1 \pm \mathrm{e}^{-t})$, the inequality (1.6) is just the logarithmic Sobolev inequality for an asymmetric Bernoulli measure (cf. [SC]). One important feature of the constant in (1.6) is that it significantly differs, as $t \to 0$, from the corresponding one in the Poincaré inequality

$$\mathbb{E}\big(f^2(B_t)\big) - \big(\mathbb{E}\big(f(B_t)\big)\big)^2 \leq \frac{1}{4}\,(1-\mathrm{e}^{-2t})\,\mathbb{E}\big(\big|Df(B_t)\big|^2\big)\ . \tag{1.7}$$

As explained carefully in Section 2, these Poincaré and logarithmic Sobolev inequalities may be tensorized to arbitrary cylindrical functions. However, while the Poincaré inequalities may be extended to functions of all the path, this is no more the case for the logarithmic Sobolev inequalities, owing to the distortion of the constant by the factor $\mathrm{e}^t \log((1+\mathrm{e}^{-t})/(1-\mathrm{e}^{-t}))$ which tends to infinity as $t \to 0$.

A second example is provided by the graph $\chi = \mathbb{Z}^d$. Brownian motion B on $\mathbb{Z}^d$ is defined as before as the continuous time simple random walk on the integers (with generator one half of the discrete Laplacian). However, it turns out that we cannot expect here any kind of logarithmic Sobolev inequality contrary to the compact case $\{-1,+1\}$,. Assume again that $d = 1$ for simplicity. If $t > 0$, the law of B_t (starting from the origin) is easily seen as the convolution of the Poisson measure with parameter $t/2$ on $\mathbb{Z}_+$ with the Poisson measure with the same parameter on $\mathbb{Z}_-$. (This example may actually be analysed almost equivalently on the standard Poisson process.) Now, it is known that Poisson measures do not satisfy the standard logarithmic Sobolev inequality with respect to the discrete gradient on $\mathbb{Z}_+$ (see [B-L], [G-R]). Therefore, the same negative comment applies to B_t, and thus to the whole process B. In a sense, this observation is reasonable. Indeed, one cannot expect, for Brownian motions on $\mathbb{Z}$ or $\mathbb{Z}^d$, tail estimates that would be similar to the Gaussian large deviation result (1.5). For example, using Fourier series as well as a representation formula for the modified Bessel function, the heat kernel $p_t(x, y)$ of the discrete Laplacian on $\mathbb{Z}$ is given explicitely by

$$p_t(x,y) = \pi^{-1/2}\Gamma\big(\delta+\tfrac{1}{2}\big)\mathrm{e}^{-2t}t^\delta \int_{-1}^{+1} (1-u^2)^{\delta-1/2}\mathrm{e}^{2tu}du$$

for all $t > 0$ and $x, y \in \mathbb{Z}$, where δ is the distance $|x-y|$ from x to y. Now, as is shown in [Pa], for fixed $t > 0$, $p_t(x, y)$ behaves (at a logarithmic scale) as $\mathrm{e}^{-\alpha\delta^2/t}$ when the ratio δ/t is small, and as $\mathrm{e}^{-\beta\delta\log(\delta/t)}$ when δ/t is large ($\alpha, \beta > 0$). Thus the distribution of the distance of Brownian motion from its starting point entails a mixed Gaussian and Poisson behavior. Such a behavior cannot be reflected by a standard logarithmic Sobolev inequality that would only yield Gaussian tails (cf. [Le]).

In order to clarify these early observations, we will make use of a modified form of logarithmic Sobolev inequalities in discrete spaces recently put forward in the work [B-L]. To recall the main result of [B-L], let μ be the Poisson measure on $\mathbb{Z}_+$ with parameter $\theta > 0$. Then, for any f on $\mathbb{Z}_+$ with strictly positive values,

$$\int f\log f d\mu - \int f d\mu \log\int f d\mu \leq \theta \int \frac{1}{f}\,|Df|^2 d\mu \tag{1.8}$$

where $Df(x) = f(x+1) - f(x)$, $x \in \mathbb{Z}_+$. One main aspect of inequality (1.8) is that, due to the lack of chain rule for the discrete gradient D, the change of functions $f \mapsto f^2$ does not yield the standard logarithmic Sobolev inequality (which in case of Poisson measure is just not true). A similar inequality holds for the Bernoulli measure with this time a constant of the same order than the spectral gap.

Moreover, this form of logarithmic Sobolev inequality is well adapted to describe tail behaviors of Lipschitz functions. As was shown indeed in [B-L], if μ is any measure on $\mathbb{Z}_+$, satisfying an inequality such as (1.8) for some $C > 0$ and all f with strictly positive values, then, if f on $\mathbb{Z}_+$ is such that $\sup_{x\in\mathbb{Z}_+} |Df(x)| \leq 1$, we have that $\int |f| d\mu < \infty$ and, for every $r \geq 0$,

$$\mu\big(f \geq \textstyle\int f d\mu + r\big) \leq \exp\bigg(-\frac{r}{4}\log\Big(1+\frac{r}{2C}\Big)\bigg) .$$

In particular, the tail of the Lipschitz function f is Gaussian for the small values of r and Poissonian for the large values (with respect to C). This is of course the typical behaviour of $f(x) = x$ for example for Poisson measure, as well as the one put forward above for the heat kernel of the random walk on $\mathbb{Z}$.

As a consequence of these observations, we concentrate our investigation on modified logarithmic Sobolev inequalities of the (1.8) type. As a result, we will establish such an inequality for the continuous time process associated to the simple random walk on a locally uniformly finite graph jumping to the neighbours with equal probability. We will actually allow varying probabilities, but regardless of the position (which thus corresponds to some constant curvature setting). While for the simple examples of the cube and the lattice, the one-dimensional logarithmic Sobolev inequalities may be iterated to the family of all cylindrical functions, we follow a different route in the general case. Namely, our framework will be somewhat more general and enters the setting developed by J. Picard [Pi] in his investigation of Malliavin calculus on Poisson spaces. We actually take advantage of his integration by parts formulae to derive the appropriate representation formula. Provided with such a representation, the proof of the logarithmic Sobolev inequality simply relies on the stochastic calculus argument of [C-H-L]. Thus, we establish that for every positive functional F on the paths of the graph χ up to time $T > 0$,

$$\mathbb{E}(F\log F) - \mathbb{E}(F)\log\mathbb{E}(F) \leq \mathbb{E}\bigg(\frac{1}{F}\int_{[0,T]\times J} |D_{(t,j)}F|^2\, dt\otimes dn\bigg) , \quad (1.9)$$

where dn is the counting measure on the set J of the directions of the graph (see Section 5 for details) and $D_{(t,j)}$ is the gradient in Poisson spaces. For example, for the standard Poisson process $N = (N_t)_{t\geq 0}$ on $\mathbb{Z}_+$, the set of directions is reduced to the direction "+" (to the right), and if $F = f(N_{t_1}, \ldots, N_{t_n})$, $0 = t_0 \leq t_1 \leq \cdots \leq t_n$,

$$D_{(t,+)}F = \sum_{i=1}^{n} \mathrm{I}_{\{t_{i-1}<t<t_i\}} \sum_{k=i}^{n} D_k F \circ \tau_{i,k-1}$$

where, for $i < k$,

$$D_k F \circ \tau_{i,k-1} = D_k f\big(N_{t_1}, \ldots, N_{t_{i-1}}, N_{t_i}+1, \ldots, N_{t_{k-1}}+1, N_{t_k}, \ldots, N_{t_n}\big)$$

with $D_k f$ the discrete derivative of f along the k-th coordinate. Therefore, in this case,

$$
\begin{aligned}
\int_{[0,T]\times J} & |D_{(t,j)}F|^2 \, dt \otimes dn \\
&= \sum_{i=1}^{n} (t_i - t_{i-1}) \Big(\sum_{k=i}^{n} D_k F \circ \tau_{i,k-1} \Big)^2 \\
&= \sum_{i=1}^{n} (t_i - t_{i-1}) \big[f(N_{t_1}, \dots, N_{t_{i-1}}, N_{t_i} + 1, \dots, N_{t_n} + 1) - f(N_{t_1}, \dots, N_{t_n}) \big]^2
\end{aligned}
$$

and thus (1.9) provides the appropriate extension of (1.1) (as soon as the discrete derivative is infinitesimal). As a general consequence of (1.9), if F is 1-Lipschitz with respect to this gradient D, we show that

$$
\mathbb{P}\big(F \geq \mathbb{E}(F) + R\big) \leq \exp\bigg(-\frac{R}{4} \log\Big(1 + \frac{R}{2\alpha^2}\Big)\bigg)
$$

for every $R \geq 0$ where $\alpha^2 = \operatorname*{ess\,sup}_{\Omega} \int_{[0,T]\times J} |D_{(t,j)}F|^2 dt \otimes dn$. These conclusions are presented in Sections 4 and 5, which form the core of this paper, where the main results are further illustrated by various examples. As announced, we discuss more or less in depth the random walks on the cube in Section 2 and on the lattice $\mathbb{Z}^d$ in Section 3. In the last part, we apply the logarithmic Sobolev inequalities to prove the preceding tail estimate for Lipschitz functions following [B-L], [Le].

Since this work has been submitted, various authors developed results related to the present contribution. L. Miclo (personal communication) and L. Wu [Wu] observed in particular that for the Poisson measure with parameter θ and for every non-negative function f on $\mathbb{Z}_+$,

$$
\int f \log f d\mu - \int f d\mu \log \int f d\mu \leq \theta \int Df D(\log f) d\mu \quad , \tag{1.10}
$$

an inequality that actually follows from the corresponding one on the two-point space. While (1.8) and (1.10) (for Bernoulli or Poisson measures) are not comparable, the proof of (1.10) is actually more simple than the one of (1.8). Moreover, with respect to the latter, inequalities (1.10) do imply exponential decay of entropy. They also entail concentration properties similar to (1.8). The Poisson process version of (1.10) is investigated in [Wu] with arguments similar to the ones developed here. Most of our results may actually be expressed similarly with the form $Df D(\log f)$.

2. The discrete cube

We consider here the continuous time simple random walk $B = (B_t)_{t\geq 0}$ on the discrete cube, in dimension one for simplicity. Thus let $\chi = \{-1, +1\}$. We are looking for a logarithmic Sobolev inequality for the law of B. As discussed in the introduction, we already know that for a one-dimensional cylindrical function $F = f(B_t)$, $t \geq 0$,

$$\begin{aligned}&\mathbb{E}\big(f^2(B_t)\log f^2(B_t)\big) - \mathbb{E}\big(f^2(B_t)\big)\log\mathbb{E}\big(f^2(B_t)\big)\\&\quad\le \frac{1}{4}\,(1-\mathrm{e}^{-2t})\,\mathrm{e}^t\log\left(\frac{1+\mathrm{e}^{-t}}{1-\mathrm{e}^{-t}}\right)\mathbb{E}\big(|Df(B_t)|^2\big)\ .\end{aligned}\tag{2.1}$$

Recall that $Df(x) = f(-x) - f(x)$, $x \in \chi$. Our first aim is to properly tensorize this inequality to arbitrary cylindrical functions. At the level of Poincaré inequalities, a Markovian tensorization of (1.7) yields, for $F = f(B_{t_1}, \ldots, B_{t_n})$, $0 \le t_1 \le \cdots \le t_n$,

$$\mathbb{E}(F^2) - \big(\mathbb{E}(F)\big)^2 \le \frac{1}{2}\sum_{i=1}^{n}\big(1-\mathrm{e}^{-2(t_i-t_{i-1})}\big)\,\mathbb{E}\big(\Gamma^{i\ldots n}F\big)\tag{2.2}$$

where, for $i < k$,

$$D_kF\circ\tau_{i,k-1} = D_kf\big(B_{t_1},\ldots,B_{t_{i-1}},-B_{t_i},\ldots,-B_{t_{k-1}},B_{t_k},\ldots,B_{t_n}\big)$$

with D_kf the discrete derivative of f with respect to the k-th coordinate,

$$\begin{aligned}2\Gamma^{i\ldots n}F &= \left(\sum_{k=i}^{n} D_kF\circ\tau_{i,k-1}\right)^2\\&= \big[f(B_{t_1},\ldots,B_{t_{i-1}},-B_{t_i},\ldots,-B_{t_n}) - f(B_{t_1},\ldots,B_{t_n})\big]^2\ .\end{aligned}$$

To establish (2.2), note that the law of $(B_{t_1},\ldots,B_{t_n})$ is the measure on $\{-1,+1\}^n$ given by

$$dP(x_1,\ldots,x_n) = p_{t_1}(x_0,x_1)p_{t_2-t_1}(x_1,x_2)\cdots p_{t_n-t_{n-1}}(x_{n-1},x_n)dx_1dx_2\cdots dx_n$$

where we denote by dx the uniform probability measure on $\{-1,+1\}$. By induction on (1.7) we get that

$$\begin{aligned}&\mathbb{E}\big(f^2(B_{t_1},\ldots,B_{t_n})\big)\\&\quad=\int f^2dP \le \left(\int f dP\right)^2 + \frac{1}{4}\sum_{i=1}^{n}(1-\mathrm{e}^{-2(t_i-t_{i-1})})\int |D_if_i|^2dP\end{aligned}$$

where

$$\begin{aligned}f_i(x_1,\ldots,x_i) &= \int f_{i+1}(x_1,\ldots,x_{i+1})p_{t_{i+1}-t_i}(x_i,x_{i+1})dx_{i+1}\\&= \mathbb{E}\big(f(x_1,\ldots,x_i,B_{t_{i+1}},\ldots,B_{t_n})\mid B_{t_i}=x_i\big)\end{aligned}$$

for every $i = 1,\ldots,n-1$, and $f_n = f$. The result will be established as soon as we can show that

$$\begin{aligned}|D_if_i|^2 \le \int &\big[f(x_1,\ldots,x_{i-1},-x_i,\ldots,-x_n) - f(x_1,\ldots,x_n)\big]^2\\&\times p_{t_{i+1}-t_i}(x_i,x_{i+1})\cdots p_{t_n-t_{n-1}}(x_{n-1},x_n)dx_{i+1}\cdots dx_n\ .\end{aligned}\tag{2.3}$$

To achieve this task, write

$$\begin{aligned} & f_i(x_1, \ldots, x_{i-1}, -x_i) \\ & \quad = \int f(x_1, \ldots, x_{i-1}, -x_i, x_{i+1}, \ldots, x_n) \\ & \qquad \times p_{t_{i+1}-t_i}(-x_i, x_{i+1}) p_{t_{i+2}-t_{i+1}}(x_{i+1}, x_{i+2}) \cdots p_{t_n-t_{n-1}}(x_{n-1}, x_n) dx_{i+1} \cdots dx_n \\ & \quad = \int f(x_1, \ldots, x_{i-1}, -x_i, -x_{i+1}, x_{i+2}, \ldots, x_n) \\ & \qquad \times p_{t_{i+1}-t_i}(x_i, x_{i+1}) p_{t_{i+2}-t_{i+1}}(-x_{i+1}, x_{i+2}) \cdots p_{t_n-t_{n-1}}(x_{n-1}, x_n) dx_{i+1} \cdots dx_n \\ & \quad = \cdots \\ & \quad = \int f(x_1, \ldots, x_{i-1}, -x_i, \ldots, -x_n) \\ & \qquad \times p_{t_{i+1}-t_i}(x_i, x_{i+1}) \cdots p_{t_n-t_{n-1}}(x_{n-1}, x_n) dx_{i+1} \cdots dx_n \ . \end{aligned}$$

We have used here a basic commutation property expressed by the fact that

$$\int \varphi(-y) p_t(x, y) dy = \int \varphi(y) p_t(-x, y) dy \ . \tag{2.4}$$

As a consequence,

$$\begin{aligned} D_i f_i &= f_i(x_1, \ldots, x_{i-1}, -x_i) - f_i(x_1, \ldots, x_{i-1}, x_i) \\ &= \int \big[f(x_1, \ldots, x_{i-1}, -x_i, \ldots, -x_n) - f(x_1, \ldots, x_n) \big] \\ &\quad \times p_{t_{i+1}-t_i}(x_i, x_{i+1}) \cdots p_{t_n-t_{n-1}}(x_{n-1}, x_n) dx_{i+1} \cdots dx_n \end{aligned} \tag{2.5}$$

from which (2.3) follows by Jensen's inequality.

To develop the same argument at the level of the logarithmic Sobolev inequality, we first get, by induction on (2.1),

$$\begin{aligned} & \int f^2 \log f^2 dP - \int f^2 dP \log \int f^2 dP \\ & \qquad \le \frac{1}{4} \sum_{i=1}^n \big(1 - \mathrm{e}^{-2(t_i - t_{i-1})}\big) c(t_i - t_{i-1}) \int |D_i f_i|^2 dP \end{aligned}$$

where

$$c(t_i - t_{i-1}) = \mathrm{e}^{-(t_i - t_{i-1})} \log \left(\frac{1 + \mathrm{e}^{-(t_i - t_{i-1})}}{1 - \mathrm{e}^{-(t_i - t_{i-1})}} \right)$$

and now, the successive functions f_i are defined by

$$f_i^2(x_1, \ldots, x_i) = \int f_{i+1}^2(x_1, \ldots, x_{i+1}) p_{t_{i+1}-t_i}(x_i, x_{i+1}) dx_{i+1} \ .$$

In this case we get

$$\begin{aligned} |D_i f_i|^2 \le \int & \big[f(x_1, \ldots, x_{i-1}, -x_i, \ldots, -x_n) - f(x_1, \ldots, x_n) \big]^2 \\ & \times p_{t_{i+1}-t_i}(x_i, x_{i+1}) \cdots p_{t_n-t_{n-1}}(x_{n-1}, x_n) dx_{i+1} \cdots dx_n \end{aligned}$$

from the Minkowski inequality in L^2. Therefore,

$$\mathbb{E}(F^2 \log F^2) - \mathbb{E}(F^2) \log \mathbb{E}(F^2) \\ \leq \frac{1}{2} \sum_{i=1}^{n} \big(1 - \mathrm{e}^{-2(t_i - t_{i-1})}\big) c(t_i - t_{i-1})\, \mathbb{E}\big(\Gamma^{i\dots n} F\big) \ . \tag{2.6}$$

However, contrary to what occurs in (2.2), the factors $c(t_i - t_{i-1})$ which tend to infinity as $t_i - t_{i-1} \to 0$ would not allow us to extend this inequality to functions of all the path properly.

Together with the example of the lattice in the next section (for which the classical logarithmic Sobolev inequality just does not hold), this is why we will turn to some modified logarithmic Sobolev inequalities whose constants behave better as $t \to 0$. For example, as was shown in [B-L], if f is positive on $\{-1, +1\}$,

$$\mathbb{E}\big(f(B_t) \log f(B_t)\big) - \mathbb{E}\big(f(B_t)\big) \log \mathbb{E}\big(f(B_t)\big) \\ \leq \frac{1}{4} (1 - \mathrm{e}^{-2t}) \mathbb{E}\left(\frac{|Df(B_t)|^2}{f(B_t)}\right) \ . \tag{2.7}$$

The constant in (2.7) is now comparable to the one in the Poincaré inequality (1.7). This inequality can be tensorized to cylindrical functions $F = f(B_{t_1}, \dots, B_{t_n})$, $f > 0$, following the argument leading to (2.2) thus allowing extensions to the whole path. Indeed, as a consequence of (2.5) and the Cauchy-Schwarz inequality,

$$\frac{|D_i f_i|^2}{f_i} \leq \int \frac{1}{f} \big[f(x_1, \dots, x_{i-1}, -x_i, \dots, -x_n) - f(x_1, \dots, x_n)\big]^2 \\ \times p_{t_{i+1}-t_i}(x_i, x_{i+1}) \cdots p_{t_n - t_{n-1}}(x_{n-1}, x_n) dx_{i+1} \cdots dx_n$$

from which we get similarly that

$$\mathbb{E}(F \log F) - \mathbb{E}(F) \log \mathbb{E}(F) \leq \frac{1}{2} \sum_{i=1}^{n} \big(1 - \mathrm{e}^{-2(t_i - t_{i-1})}\big)\, \mathbb{E}\Big(\frac{\Gamma^{i\dots n} F}{F}\Big) \ . \tag{2.8}$$

Since (2.7) (and (2.8)) may be treated, to some extent, as the example of $\mathbb{Z}^d$ below, we simply turn to this case now. Actually, we will establish (2.7) in the next section in a setting that includes both examples.

Before turning to the next section, let us however briefly digress on a somewhat different tensorization procedure on the cube. As is usual, the cube is often seen as a discrete version of Gaussian spaces, and the process $(B_t)_{t \geq 0}$ ought to be compared to the (real-valued) Ornstein-Uhlenbeck process $X = (X_t)_{t \geq 0}$ (starting from the origin for example). Since X is Gaussian, it is a simple matter to see that if $F = f(X_{t_1}, \cdots, X_{t_n})$, $0 \leq t_1 \leq \dots t_n$, and f is smooth enough on $\mathbb{R}^n$,

$$\mathbb{E}(F^2) - \big(\mathbb{E}(F^2)\big) \leq \sum_{i=1}^{n} \big(1 - \mathrm{e}^{-2(t_i - t_{i-1})}\big) \mathbb{E}\left(\left(\sum_{k=i}^{n} \mathrm{e}^{-(t_k - t_i)} \partial_k F\right)^2\right) \tag{2.9}$$

where $\partial_k F = \partial_k f(X_{t_1}, \dots, X_{t_n})$. Similarly, we have a logarithmic Sobolev inequality replacing the variance of F on the left-hand side of (2.9) by one-half of the entropy of F^2. In (2.9), the coefficients $e^{-(t_k - t_i)}$ reflects a (constant) strictly positive curvature property of Gauss spaces (cf. e.g. [Ba1]). Now, the commutation property (2.4) on the cube does not reflect any kind of non-zero curvature. Using a different commutation argument, one can however show that for $F = f(B_{t_1}, \dots, B_{t_n})$ on the cube,

$$\begin{aligned} &\mathbb{E}(F^2) - \big(\mathbb{E}(F^2)\big) \\ &\quad \le \frac{1}{4} \sum_{i=1}^{n} \big(1 - e^{-2(t_i - t_{i-1})}\big) \mathbb{E}\Bigg(\Bigg(\sum_{k=i}^{n} e^{-(t_k - t_i)} \widetilde{D}_k F \circ \tau_{i,k-1} \Bigg)^2 \Bigg) \end{aligned} \tag{2.10}$$

where

$$\widetilde{D}_k F \circ \tau_{i,k-1} = B_{t_k} D_k f\big(B_{t_1}, \dots, B_{t_{i-1}}, -B_{t_i}, \dots, -B_{t_{k-1}}, B_{t_k}, \dots, B_{t_n}\big) \ .$$

The proof of (2.10) is similar to the one of (2.2). It is enough to show that, for every $i = 1, \dots, n$,

$$\begin{aligned} |D_i f_i|^2 \le \int &\Bigg(\sum_{k=i}^{n} e^{-(t_k - t_i)} x_k D_k f \circ \tau_{i,k-1} \Bigg)^2 \\ &\times p_{t_{i+1} - t_i}(x_i, x_{i+1}) \cdots p_{t_n - t_{n-1}}(x_{n-1}, x_n) dx_{i+1} \cdots dx_n \ . \end{aligned} \tag{2.11}$$

Let us sketch the argument when $n = 2$, $i = 1$. We can write

$$\begin{aligned} D_1 f_1(x_1) &= f_1(-x_1) - f_1(x_1) \\ &= \int \big[f(-x_1, x_2) p_{t_2 - t_1}(-x_1, x_2) - f(x_1, x_2) p_{t_2 - t_1}(x_1, x_2) \big] dx_2 \\ &= \int \big[f(-x_1, x_2) D_1 p_{t_2 - t_1}(x_1, x_2) + D_1 f(x_1, x_2) p_{t_2 - t_1}(x_1, x_2) \big] dx_2 \ . \end{aligned}$$

Observe now that

$$\begin{aligned} &\textstyle\int f(-x_1, x_2) D_1 p_{t_2 - t_1}(x_1, x_2) dx_2 \\ &\quad = e^{-(t_2 - t_1)} x_1 \int x_2 D_2 f(-x_1, x_2) p_{t_2 - t_1}(x_1, x_2) dx_2 \ . \end{aligned}$$

Therefore

$$\begin{aligned} x_1 D_1 f_1(x_1) = \int &\big[x_1 D_1 f(x_1, x_2) p_{t_2 - t_1}(x_1, x_2) \\ &+ e^{-(t_2 - t_1)} x_2 D_2 f(-x_1, x_2) p_{t_2 - t_1}(x_1, x_2) \big] dx_2 \end{aligned}$$

so that (2.11) follows from Jensen's inequality in this particular example. The general case is similar.

To develop the same argument at the level of the logarithmic Sobolev inequality, we would need the analogue of (2.11) when the f_i's are defined by

$$f_i^2(x_1, \dots, x_i) = \int f_{i+1}^2(x_1, \dots, x_{i+1}) p_{t_{i+1} - t_i}(x_i, x_{i+1}) dx_{i+1} \ .$$

It is however much less convenient to deal now with the discrete derivatives $D_i f_i$, and we actually do not know whether (2.11) still holds in this case. To be more precise, we have been able to check this inequality in dimension 2 directly (writing f as determined by 4 values). With the help of the computer, we also checked the result for a function f on $\{-1,+1\}^3$. However what is missing to us, is a generic argument that would yield the result for any cylindrical function. Thus we do not know at this point, although we strongly conjecture it, whether, for $F = f(B_{t_1},\dots,B_{t_n})$ and any n,

$$\begin{aligned}&\mathbb{E}(F^2\log F^2) - \mathbb{E}(F^2)\log\mathbb{E}(F^2)\\ &\quad\le \frac{1}{4}\sum_{i=1}^{n}\big(1-\mathrm{e}^{-2(t_i-t_{i-1})}\big)c(t_i-t_{i-1})\,\mathbb{E}\Bigg(\Bigg(\sum_{k=i}^{n} e^{-(t_k-t_i)}\widetilde{D}_k F\circ\tau_{i,k-1}\Bigg)^2\Bigg)\ .\end{aligned}$$

3. The lattice $\mathbb{Z}^d$

We consider here the continuous time random walk $B = (B_t)_{t\ge 0}$ on the lattice $\mathbb{Z}^d$ and start with the case $d = 1$ for simplicity. Thus B is the process on $\mathbb{Z}$ that jumps to one of its neighbour with equal probability and whose Markov generator is (half of) the discrete Laplacian. As we have seen it in the introduction, the standard logarithmic Sobolev inequality cannot hold and we have to consider a modified form of it. Our starting point is the following result for one-dimensional cylindrical functions. If f is a function on $\mathbb{Z}$, we set

$$2\Gamma f(x) = \big[f(x+1)-f(x)\big]^2 + \big[f(x)-f(x-1)\big]^2, \quad x\in\mathbb{Z}\ .$$

Proposition 3.1. *For any $t\ge 0$ and any function f on $\mathbb{Z}$,*

$$\mathbb{E}\big(f(B_t)^2\big) - \big(\mathbb{E}f(B_t)\big)^2 \le t\,\mathbb{E}\big(\Gamma f(B_t)\big)\ . \tag{3.1}$$

Moreover, if f takes stricly positive values,

$$\mathbb{E}\big(f(B_t)\log f(B_t)\big) - \mathbb{E}\big(f(B_t)\big)\log\mathbb{E}\big(f(B_t)\big) \le t\,\mathbb{E}\left(\frac{\Gamma f(B_t)}{f(B_t)}\right)\ . \tag{3.2}$$

Proof. We first prove the corresponding inequality for the Poisson process. This inequality is actually known [B-L] but we would like to provide here a new more simple proof based on the Γ_2 calculus of [Ba1], [Ba2]. In what follows, calculus are made on $\mathbb{Z}_+$ for simplicity, but it could be made on $\mathbb{Z}$ or $\mathbb{Z}^d$, or on the cube. The method is actually more general and includes further examples of discrete Markov generators.

Let L be the generator of the Poisson process on $\mathbb{Z}_+$, defined by

$$\mathrm{L}f(x) = f(x+1) - f(x), \quad x\in\mathbb{Z}_+\ .$$

Let $(P_t)_{t\ge 0}$ be the Markov semigroup with generator $\frac{1}{2}\mathrm{L}$, so that $P_t f(x) = \int f\,d\mu_t^x$ where μ_t^x is the Poisson distribution on $x+\mathbb{Z}_+$ with parameter $t/2$. The "carré du champ" operator Γ associated to L is defined by $2\Gamma(f,g) = \mathrm{L}(fg) - f\mathrm{L}g - g\mathrm{L}f$ and we have here

$$2\Gamma f(x) = 2\Gamma(f,f)(x) = \big[f(x+1) - f(x)\big]^2, \quad x \in \mathbb{Z}_+ \ .$$

The Γ_2 operator is defined similarly by the formula $2\Gamma_2 f = \mathrm{L}(\Gamma f) - 2\Gamma(f, \mathrm{L}f)$. We wish to compare $\Gamma_2 f$ and $\Gamma(\sqrt{\Gamma f})$. For this, note that

$$\begin{aligned} 0 \le 4\Gamma\Big(\sqrt{\Gamma f}\Big)(x) &= \big[|f(x+2) - f(x+1)| - |f(x+1) - f(x)|\big]^2 \\ &\le \big[\big(f(x+2) - f(x+1)\big) - \big(f(x+1) - f(x)\big)\big]^2 = 4\Gamma_2 f \ . \end{aligned}$$

The positivity of $\Gamma_2 f$ will imply a Poincaré inequality, while the more precise bound $\Gamma_2 f \ge \Gamma(\sqrt{\Gamma f})$ will lead to a modified logarithmic Sobolev inequality. What follows is classical in the case of diffusions (see [Ba2] for instance). Let f be any function on $\mathbb{Z}_+$. Set $\psi_1(s) = P_s \Gamma P_{t-s} f$, $\psi_2(s) = P_s(\sqrt{\Gamma P_{t-s} f})$, $0 \le s \le t$, and write $g = g_s = P_{t-s} f$ to make the notation more simple. Then

$$\psi_1'(s) = \frac{1}{2}\, P_s\big(\mathrm{L}\Gamma g - 2\Gamma(g, \mathrm{L}g)\big) = P_s(\Gamma_2 g) \ge 0$$

so that, for all $t \ge 0$, $\Gamma P_t f \le P_t \Gamma f$. In the same way,

$$\begin{aligned} \psi_2'(s) &= \frac{1}{2}\, P_s\bigg(\mathrm{L}\sqrt{\Gamma g} - \frac{1}{\sqrt{\Gamma g}}\, \Gamma(g, \mathrm{L}g)\bigg) \\ &= \frac{1}{2}\, P_s\bigg(\frac{1}{2\sqrt{\Gamma g}}\Big(2\sqrt{\Gamma g}\mathrm{L}\big(\sqrt{\Gamma g}\big) - 2\Gamma(g, \mathrm{L}g)\Big)\bigg) \\ &= \frac{1}{2}\, P_s\bigg(\frac{1}{\sqrt{\Gamma g}}\Big(\Gamma_2 g - \Gamma\big(\sqrt{\Gamma g}\big)\Big)\bigg) \ge 0 \ . \end{aligned}$$

Hence, $\psi_2(t) \ge \psi_2(0)$, that is to say $\sqrt{\Gamma(P_t f)} \le P_t(\sqrt{\Gamma f})$, for all $t \ge 0$.

Now, let $\phi_1(s) = P_s((P_{t-s} f)^2)$, and $\phi_2(s) = P_s(P_{t-s} f \log P_{t-s} f)$ if f is non-negative. With the preceding, we can bound the derivatives of ϕ_1 and ϕ_2. Namely,

$$\phi_1'(s) = P_s(\Gamma P_{t-s} f) \le P_s(P_{t-s}\Gamma f) = P_t(\Gamma f) \ .$$

This implies the Poincaré inequality for P_t in the form of

$$P_t(f^2) - (P_t f)^2 \le t P_t(\Gamma f) \ . \tag{3.3}$$

In order to bound the derivative

$$\phi_2'(s) = \frac{1}{2}\, P_s\Big(\mathrm{L}(P_{t-s} f \log P_{t-s} f) - (1 + \log P_{t-s} f)\mathrm{L}(P_{t-s} f)\Big) \ ,$$

notice that for all $g \ge 0$,

$$\mathrm{L}(g \log g) - (1 + \log g)\, \mathrm{L}g \le \frac{2\Gamma g}{g} \ .$$

This last estimate is a consequence of the inequality $\log b - \log a - (b-a)/a \leq 0$, $a, b > 0$. Hence,

$$\phi_2'(s) \leq P_s\left(\frac{1}{P_{t-s}f}\,\Gamma(P_{t-s}f)\right) \leq P_s\left(\frac{1}{P_{t-s}f}\left(P_{t-s}\sqrt{\Gamma f}\right)^2\right) \ ,$$

where we used that $\sqrt{\Gamma(P_{t-s}f)} \leq P_{t-s}(\sqrt{\Gamma f})$. By the Cauchy-Schwarz inequality

$$\left(P_{t-s}(X)\right)^2 \leq P_{t-s}(X^2/Y)P_{t-s}(Y)$$

we get that

$$\phi_2'(s) \leq P_s\left(P_{t-s}\left(\frac{\Gamma f}{f}\right)\right) = P_t\left(\frac{\Gamma f}{f}\right) \ .$$

Finally, as $\phi_2(t) - \phi_2(0) = P_t(f\log f) - P_t f \log P_t f$, we have shown that for all $f > 0$ on $\mathbb{Z}_+$,

$$P_t(f\log f) - P_t f\log(P_t f) \leq tP_t\left(\frac{\Gamma f}{f}\right) \ . \tag{3.4}$$

We now relate (3.3) and (3.4) to the inequalities of the proposition. Recall thus the process B on $\mathbb{Z}$. Assume it starts at $x \in \mathbb{Z}$. It is known and easy to see [G-R] that the law $p_t(x,\cdot) = p_t^x$ of B_t is the convolution product

$$p_t^x = \mu_t^x * \tilde{\mu}_t^0$$

where μ_t^x is the Poisson measure of parameter $t/2$ on $x+\mathbb{Z}_+$, and $\tilde{\mu}_t^0$ is the reversed Poisson measure of parameter $t/2$ on $\mathbb{Z}_-$. (To prove this equality, just verify that these measures coincide on the characters $\mathrm{e}^{i\theta\cdot}$.) In the preceding language, the Markov semigroup of the process $(B)_{t\geq 0}$ has generator $\frac{1}{2}\mathrm{L}$ where L is the discrete Laplacian $\mathrm{L}f(x) = f(x+1) + f(x-1) - 2f(x)$ on $\mathbb{Z}$. Since (3.3) and (3.4) apply to both μ_t^x and $\tilde{\mu}_t^0$, it is an easy task to deduce (3.1) and (3.2) by a classical tensorization argument. Let us deal with the logarithmic Sobolev inequality (3.2). Let $f > 0$ on $\mathbb{Z}$. We can write

$$\begin{aligned}\int f\log f dp_t^x &= \iint f(y+z)\log f(y+z)d\mu_t^x(z)d\tilde{\mu}_t^0(y)\\ &= \iint f\log f d\mu_t^{x+y}d\tilde{\mu}_t^0(y) \ .\end{aligned}$$

From (3.4) applied to μ_t^{x+y}, we get

$$\int f\log f d\mu_t^{x+y} \leq \int f d\mu_t^{x+y}\log\int f d\mu_t^{x+y} + t\int\frac{1}{2f}\left[f(\cdot+1)-f\right]^2 d\mu_t^{x+y} \ .$$

If we let $h(y) = \int f d\mu_t^{x+y}$ and apply (3.4) to $\tilde{\mu}_t^0$, it follows that

$$\int h\log h d\tilde{\mu}_t^0 \leq \int h d\tilde{\mu}_t^0 \log\int h d\tilde{\mu}_t^0 + t\int\frac{1}{2h}\left[h(\cdot-1)-h\right]^2 d\tilde{\mu}_t^0 \ .$$

But

$$\big[h(\cdot-1)-h\big](y)=\int\big[f(\cdot-1)-f\big]d\mu_t^{x+y}\ ,$$

and, by the Cauchy-Schwarz inequality,

$$\bigg(\int\big[f(\cdot-1)-f\big]d\mu_t^{x+y}\bigg)^2\le\int\frac{1}{2f}\big[f(\cdot-1)-f)\big]^2d\mu_t^{x+y}\int f d\mu_t^{x+y}\ .$$

It follows from these bounds that

$$\int f\log f dp_t^x\le\int f dp_t^x\log\int f dp_t^x+t\int\frac{1}{2f}\big[f(\cdot+1)-f]\big)^2dp_t^x$$
$$+t\int\frac{1}{2f}\big[f(\cdot-1)-f)\big]^2dp_t^x\ .$$

Since B_t has law p_t^x, inequality (3.2) follows. The Poincaré inequality (3.1) is established in the same way from (3.3). This completes the proof of Proposition 3.1. □

It should be mentioned that the preceding inequalities are sharp. (3.3) is sharp on the function $f(x)=x$ while (3.4) applied to the functions $f_\varepsilon(x)=\varepsilon^x$, $x\in\mathbb{Z}_+$, with $\varepsilon>0$ yields

$$\frac{\big(P_t(f_\varepsilon\log f_\varepsilon)-P_tf_\varepsilon\log(P_tf_\varepsilon)\big)(0)}{P_t\big(\frac{\Gamma f}{f}\big)(0)}=t\ \frac{\varepsilon\log\varepsilon+(1-\varepsilon)}{(1-\varepsilon)^2}$$

which tends to t as ε tends to 0. Note that applying (3.4) to $1+\varepsilon f$ and letting ε tend to 0 only yields (3.3) up to a factor 2.

The tensorization argument used in the preceding proof may be used similarly to tensorize Proposition 3.1 to the d-dimensional continuous time random walk $B=(B_t)_{t\ge0}$ on the lattice $\mathbb{Z}^d$. Indeed, the law of $B_t=(B_t^1,\dots,B_t^d)$ is the product measure of the laws of the marginals. We get in this way (3.1) and (3.2) with Γ defined in this case by

$$2\Gamma f=\sum_{j=1}^d\Big(\big[f(\cdot+e_j)-f\big]^2+\big[f(\cdot-e_j)-f\big]^2\Big)$$

where $(e_1,\dots,e_d)$ is the canonical basis of $\mathbb{Z}^d$.

The proof developed in Proposition 3.1 applies similarly to the cube, by means of the generator $\mathrm{L}f(x)=f(-x)-f(x)$ for which $2\Gamma f=|Df|^2$. It should be mentioned however that we do not recover exactly (1.7) and (2.7), but only their analogues in finite time using that $1-\mathrm{e}^{-2t}\le 2t$, $t\ge0$.

Although we will not follow this route in the sequel, it is tempting to tensorize Proposition 3.1 to cylindrical functions $F=f(B_{t_1},\dots,B_{t_n})$, $0\le t_1<\cdots<t_n$, as we described it on the cube in Section 2. By induction on (3.2), we get that

$$\mathbb{E}(F\log F)-\mathbb{E}(F)\log\mathbb{E}(F)\le\sum_{i=1}^n(t_i-t_{i-1})\mathbb{E}\left(\frac{\Gamma^iF_i}{F_i}\right)$$

where $F_i = f_i(B_{t_1}, \dots, B_{t_i})$ and

$$f_i(x_1, \dots, x_i) = \mathbb{E}\big(f(x_1, \dots, x_i, B_{t_{i+1}}, \dots B_{t_n}) \mid B_{t_i} = x_i\big)$$

and where Γ^i is the Γ operator acting on the i-th coordinate. Using the commutation property

$$\int \varphi(y+1) dp_t^x(y) = \int \varphi(y) dp_t^{x+1}(y)$$

analogous to (2.4), we bound as in Section 2 (cf. the proof of (2.2) and (2.9))

$$\mathbb{E}\left(\frac{\Gamma^i F_i}{F_i}\right)$$

by

$$\mathbb{E}\left(\frac{\Gamma^{i\dots n} F}{F}\right)$$

using the Cauchy-Schwarz inequality, where

$$\begin{aligned} 2\Gamma^{i\dots n} F &= \big[f(B_{t_1}, \dots, B_{t_{i-1}}, B_{t_i}+1, \dots, B_{t_n}+1) - f(B_{t_1}, \dots, B_{t_n})\big]^2 \\ &\quad + \big[f(B_{t_1}, \dots, B_{t_{i-1}}, B_{t_i}-1, \dots, B_{t_n}-1) - f(B_{t_1}, \dots, B_{t_n})\big]^2 \ . \end{aligned}$$

Thus we get

$$\mathbb{E}(F \log F) - \mathbb{E}(F) \log \mathbb{E}(F) \leq \sum_{i=1}^n (t_i - t_{i-1})\, \mathbb{E}\left(\frac{\Gamma^{i\dots n} F}{F}\right) \ . \tag{3.5}$$

The corresponding Poincaré inequality

$$\mathbb{E}(F^2) - \big(\mathbb{E}(F)\big)^2 \leq \sum_{i=1}^n (t_i - t_{i-1})\, \mathbb{E}\big(\Gamma^{i\dots n} F\big) \tag{3.6}$$

is obtained in the same way. On the cube,

$$2\Gamma^{i\dots n} F = \big[f(B_{t_1}, \cdots, B_{t_{i-1}}, -B_{t_i}, \dots, -B_{t_n}) - f(B_{t_1}, \dots, B_{t_n})\big]^2$$

so that, at the expense of the bounds $1 - \mathrm{e}^{-2(t_i - t_{i-1})} \leq 2(t_i - t_{i-1})$, (3.5) is directly comparable to (2.8), and (3.6) to (2.2).

As we will realize it later on, this tensorization procedure heavily relies on commutativity in $\mathbb{Z}$ or $\mathbb{Z}^d$. In order to reach some more general statements, we will rather consider a path space approach based on the stochastic calculus of variation in Poisson spaces developed by J. Picard [Pi] to which we turn now.

4. Modified logarithmic Sobolev inequality on Poisson spaces

In this part, we consider random Poisson measures, and establish a modified logarithmic Sobolev inequality in this context. The correspondance between Poisson measures and continous time Markov processes will be developed in Section 5 in order to deduce logarithmic Sobolev inequalities for various kinds of continuous random walks on graphs.

We follow the notation of [Pi]. Let J be a finite set, which will be later on the set of directions taken by the process, and n a positive measure $n(\{j\}) = \frac{1}{2}\lambda_j$ on J. The space $U = \mathbb{R}^+ \times J$ is endowed with the measure $d\lambda^-(u) = dt \otimes dn(j)$. We call Ω the set of measures ω on U such that $\omega(\{u\}) = 0$ or 1 for all $u \in U$ and $\omega(A) < \infty$ whenever $\lambda^-(A) < \infty$. Let also λ^+ be the random measure on U defined by $\lambda^+(\omega, A) = \omega(A)$ for $\omega \in \Omega$. We will denote by $\mathbb{P}$ the probability measure on Ω under which λ^+ is a random Poisson measure of intensity λ^-, and by λ the compound Poisson measure $\lambda = \lambda^+ - \lambda^-$.

It is clear that, almost surely, the random atomic measure λ^+ has at most one atom at time t for all $t \geq 0$. Thus, we can restrict Ω to such measures (that have at most one atom at each time t), and we can order the random atoms $(T_k, j_k)_{k\geq 1}$, $0 < T_1 < T_2 < \cdots < T_k < \cdots$ of the measure λ^+. Recall that $(T_{k+1} - T_k)_{k\geq 1}$ is a sequence of i.i.d. random variables of exponential law with parameter $\Lambda = n(J) = \frac{1}{2}\sum_{j\in J}\lambda_j$, and that $(j_k)_{k\geq 1}$ is also a sequence of i.i.d. random variables with law $n/n(J)$ on J, and that these two sequences of random variables are independent.

The filtration which will be used is the right-continuous filtration $\mathscr{F}_t = \sigma(\lambda^+(A), A \in \mathscr{B}([0,t] \times J), t \geq 0$. More generally, for any interval T of $\mathbb{R}_+$, $\mathscr{F}_T$ will denote the σ-algebra $\sigma\left(\lambda^+(A), A \in \mathscr{B}(T \times J)\right)$, and $\mathscr{F}$ will be $\mathscr{F}_{[0,\infty[}$. It is clear that $\mathscr{F}_{t^-} = \bigcup_{s<t}\mathscr{F}_s$ is $\mathscr{F}_{[0,t[}$ with our notation. Notice also that an $\mathscr{F}$-measurable function F is $\mathscr{F}_T$-measurable if and only if $F(\omega) = F(\omega_{|T\times J})$, where $\omega_{|T\times J}$ is the measure ω restricted to $T \times J$. Indeed, the condition is necessary because it is true for functions $\lambda^+(A)$, $A \in \mathscr{B}(T \times J)$, and sufficient because the map

$$\begin{aligned}(\Omega, \mathscr{F}_T) &\longrightarrow (\Omega, \mathscr{F})\\ \omega &\longmapsto \omega_{|T\times J}\end{aligned}$$

is measurable. In a similar way, a process Y is predictable if and only if $Y_t(\omega) = Y_t(\omega_{|[0,t[\times J})$. This condition is necessary because it holds for processes of the type

$$\mathrm{I}_{]s_1,s_2]}(t)\,\mathrm{I}_{\Omega_1}(\omega), \quad \Omega_1 \in \mathscr{F}_{s_1},\ s_1 < s_2\ .$$

It is sufficient since the map

$$\begin{aligned}(\mathbb{R}_+ \times \Omega, \mathscr{P}) &\longrightarrow (\mathbb{R}_+ \times \Omega, \mathscr{B}(\mathbb{R}_+) \otimes \mathscr{F})\\ (t,\omega) &\longmapsto (t, \omega_{[0,t[\times J})\end{aligned}$$

is measurable. Here, $\mathscr{P}$ denotes the predictable σ-algebra on $\mathbb{R}_+ \times \Omega$.

Now, we want to introduce the derivative of a function F defined on Ω. For $u \in U$, let $D_u F = F \circ \varepsilon_u^+ - F \circ \varepsilon_u^-$, where the transformation $\varepsilon_{(t,j)}^+$ (resp. $\varepsilon_{(t,j)}^-$) of Ω adds an atom at time t in direction j if there was none and removes all other atoms at time t (resp. removes the probably existing atom at time t). This definition is not the one of [Pi], where the path correspondance we will make is not considered, and where the existence of several simultaneous atoms is not annoying. In [Pi], transformations $\tilde{\varepsilon}_u^+$ and $\tilde{\varepsilon}_u^-$ on the measure space are considered. The first one just adds an atom at u while the second one removes it if there was one. But actually, it is clear that $\tilde{\varepsilon}_{(s,j)}^{\pm}(\omega) \neq \varepsilon_{(s,j)}^{\pm}(\omega)$ if and only if ω admits an atom at (s, j') with $j \neq j'$. Almost surely, λ^+ has at most one atom at time s, and as soon as (s, j) is charged by λ^+, then (s, j') is not. Consequently, $\lambda^+\{\tilde{\varepsilon}^{\pm} \neq \varepsilon^{\pm}\} = 0$. Moreover, as λ^- is non-atomic, $\lambda^-\{\tilde{\varepsilon}^{\pm} \neq \varepsilon^{\pm}\} = 0$. Thus, $\tilde{\varepsilon}_u^{\pm}(\omega) = \varepsilon_u^{\pm}(\omega)$ holds for $(\lambda^+ + \lambda^-) \otimes \mathbb{P}$ almost all (u, ω). Therefore, all the results of [Pi] are still valid in our context.

Our task in this section will be to prove the following modified logarithmic Sobolev inequality for the law $\mathbb{P}$ of the random Poisson measure.

Theorem 4.1. *Let $0 < T \leq \infty$ and let F be an $\mathscr{F}_T$-measurable and integrable function on Ω. Then*

$$\mathbb{E}(F^2) - \big(\mathbb{E}(F)\big)^2 \leq \mathbb{E}\left(\int_{[0,T]\times J} |D_u F|^2 d\lambda^-(u)\right) . \tag{4.1}$$

Moreover, if F takes strictly positive values, then

$$\mathbb{E}(F \log F) - \mathbb{E}(F) \log \mathbb{E}(F) \leq \mathbb{E}\left(\frac{1}{F} \int_{[0,T]\times J} |D_u F|^2 d\lambda^-(u)\right) . \tag{4.2}$$

These inequalities are sharp for every choice of λ_j.

Before we turn to the proof of Theorem 4.1, let us comment on optimality. Let $N_t = \lambda^+([0, t] \times J)$ the number of atoms before time t. N_t has a Poisson law of parameter Λt. Let $F_\varepsilon(\omega) = \varepsilon^{N_t}$ with $0 < \varepsilon < 1$. Then for $\lambda^- \otimes \mathbb{P}$ almost all $((s, j), \omega)$, $s \leq t$,

$$\left|D_{(s,j)} N_t\right| = |N_t + 1 - N_t| = 1 \quad \text{and}$$
$$\left|D_{(s,j)} F_\varepsilon(\omega)\right| = \left|\varepsilon^{N_t+1} - \varepsilon^{N_t}\right| = (1 - \varepsilon) F_\varepsilon(\omega) \ ,$$

so that

$$\mathbb{E}(N_t^2) - \big(\mathbb{E}(N_t)\big)^2 = \Lambda t = \mathbb{E}\left(\int_{[0,T]\times J} 1 \, d\lambda^-(u)\right) ,$$

and (4.1) is sharp. Concerning (4.2), let

$$r_\varepsilon = \frac{\mathbb{E}(F_\varepsilon \log F_\varepsilon) - \mathbb{E} F_\varepsilon \log \mathbb{E}(F_\varepsilon)}{\mathbb{E}\Big(\int_{[0,t]\times J} \frac{(D_u F_\varepsilon)^2}{F_\varepsilon} d\lambda^-(u)\Big)} = \frac{\mathbb{E}(F_\varepsilon \log F_\varepsilon) - \mathbb{E} F_\varepsilon \log \mathbb{E}(F_\varepsilon)}{\Lambda t (1 - \varepsilon)^2 \mathbb{E}(F_\varepsilon)} .$$

Using the corresponding comment in Section 3, it is easily seen that

$$r_\varepsilon = \frac{\varepsilon \log \varepsilon - (1 - \varepsilon)}{(1 - \varepsilon)^2} \to 1$$

as ε tends to 0 so that (4.2) is sharp also.

Let us recall that applying (4.2) to $F = 1 + \varepsilon G$ and letting $\varepsilon \to 0$ only yields (4.1) up to a constant 2 so that (4.1) and (4.2) will be proved separately.

As described in the introduction, the idea of the proof of Theorem 4.1 is the following. We first describe the Clark-Ocone representation formula following Corollary 6 of [Pi]. Once this representation has been established, we simply adapt the stochastic calculus proof [C-H-L] to our setting. The following is the announced representation formula. It will play the role of (1.4) in this discrete framework.

Theorem 4.2. *Denote by $\omega_t^s = \omega_{|[s,t[\times J}$ the measure ω restricted to $[s,t[\times J$. Let F be integrable on Ω. Then,*

$$Z_{(s,j)}(\omega) = \mathbb{E}\big(D_{(s,j)}F \mid \mathscr{F}_{s-}\big) = \int (D_{(s,j)}F)(\omega_s^0 + \bar{\omega}_\infty^s)\, d\mathbb{P}(\bar{\omega})$$

is almost surely defined, and

$$\mathbb{E}(F \mid \mathscr{F}_t) = \mathbb{E}(F) + \int_{[0,t]\times J} Z_{(s,j)} d\lambda(s,j) \ .$$

Proof. It is based on the isometry formula of [Pi], Theorem 1.

Proposition 4.3. *Let Z_u be $\lambda^- \otimes \mathbb{P}$ or $\lambda^+ \otimes \mathbb{P}$ integrable and such that $D_u Z_u = 0$. Then*

$$\mathbb{E}\left(\int Z_u d\lambda^+(u)\right) = \mathbb{E}\left(\int Z_u d\lambda^-(u)\right) \ .$$

Actually this proposition is equivalent to the following integration by parts formula (Theorem 2 of [Pi]).

Proposition 4.4. *Let Z_u^1 and Z_u^2 be two processes on $U \times \Omega$. Define $D_u Z_u^i = Z_u^i \circ \varepsilon_u^+ - Z_u^i \circ \varepsilon_u^-$ $i = 1, 2$, and assume that $Z_u^1 D_u Z_u^2$ and $Z_u^2 D_u Z_u^1$ are $(\lambda^+ + \lambda^-) \otimes \mathbb{P}$ integrable. Then*

$$\begin{aligned}\mathbb{E}\left(\int Z_u^1\,(D_u Z_u^2)\, d\lambda(u)\right) &= \mathbb{E}\left(\int (D_u Z_u^1)\, Z_u^2\, d\lambda(u)\right)\\ &= \mathbb{E}\left(\int D_u Z_u^1\, D_u Z_u^2\, d\lambda^-(u)\right) \ .\end{aligned}$$

Provided with these isometry and duality formulae, the proof of Theorem 4.2 will consist in using the following existence theorem of a martingale representation (cf. [Br], Chap. III, T9 and [Pi]).

Proposition 4.5. *Let $M_t = \mathbb{E}(M_\infty \mid \mathscr{F}_t)$ be a uniformly integrable martingale. Then there exists a previsible process Z_u such that $\int_{[0,t]\times J} |Z_u| d\lambda^-(u) < \infty$ $\mathbb{P}$-almost surely for all $t \geq 0$, satisfying*

$$M_t = M_0 + \int_{[0,t]\times J} Z_u d\lambda(u) \quad \mathbb{P}\text{-almost surely} \ .$$

The process Z is unique in the sense that if $\widetilde{Z}$ satisfies the same conditions, then $Z = \widetilde{Z}$ $(\lambda^- + \lambda^+) \otimes \mathbb{P}$-almost everywhere.

To prove Theorem 4.2, we first show that the processes (when $j \in J$ varies)

$$W_{(t,j)}(\omega) = \int D_{(t,j)} F(\omega_t^0 + \bar{\omega}_\infty^t)\, d\mathbb{P}(\bar{\omega})$$

are well defined and integrable, and then that $W_u(\omega) = Z_u(\omega)$ for $(\lambda^- + \lambda^+) \otimes \mathbb{P}$-almost every (u, ω). What follows is just an adaptation of [Pi]. Let us prove first that

$$\widetilde{W}_{(t,j)}(\omega) = \int \big|D_{(t,j)} F(\omega_t^0 + \bar{\omega}_\infty^t)\big| d\mathbb{P}(\bar{\omega}) < \infty \tag{4.3}$$

for $\lambda^- \otimes \mathbb{P}$- almost all $((t, j), \omega)$. Let $Y_{(t,j)}$ be previsible non-negative processes, $j \in J$. Then

$$Y_{(t,j)}(\omega) = Y_{(t,j)}(\omega_t^0) = Y_{(t,j)}(\omega_t^0 + \bar{\omega}_\infty^t) \ .$$

Therefore,

$$\begin{aligned}
&\mathbb{E}\bigg(\int \widetilde{W}_u Y_u \, d\lambda^-(u)\bigg) \\
&\quad = \iiint \big|D_{(t,j)} F(\omega_t^0 + \bar{\omega}_\infty^t)\big| Y_{(t,j)}(\omega_t^0 + \bar{\omega}_\infty^t) d\mathbb{P}(\omega) d\mathbb{P}(\bar{\omega}) d\lambda^-(t, j) \\
&\quad = \mathbb{E}\bigg(\int |D_u F| Y_u \, d\lambda^-(u)\bigg) \\
&\quad \le \mathbb{E}\bigg(|F| \int Y_u \, d\lambda^-(u) + \int |F \circ \varepsilon_u^+| Y_u \, d\lambda^-(u)\bigg) \\
&\quad = \mathbb{E}\bigg(|F| \int Y_u \, d\lambda^-(u) + \int |F| Y_u \, d\lambda^+(u)\bigg) \\
&\quad = \mathbb{E}\bigg(|F| \int Y_u \, d(\lambda^- + \lambda^+)(u)\bigg) \ .
\end{aligned}$$

In the last but one line, we made use of the isometry formula of Proposition 4.3 since $|F \circ \varepsilon_u^+| Y_u$ does not depend on $\lambda^+(\{u\})$. Taking $Y_t = \mathrm{I}_{[0,\zeta_k]}(t)$ with

$$\zeta_k = \inf\big\{t;\ (\lambda^- + \lambda^+)\big([0, t] \times J\big) \ge k - 1\big\} \ ,$$

we get that

$$\mathbb{E}\bigg(\int_{[0,\zeta_k]\times J} \widetilde{W}_u \, d\lambda^-(u)\bigg) \le k\, \mathbb{E}\big(|F|\big) \ ,$$

which proves the integrability property (4.3). The process W_u is thus well defined.

Now we want to show the integrability of W_u. It follows from $|W_u| \le \widetilde{W}_u$ that

$$\mathbb{E}\bigg(\int_{[0,\zeta_k]\times J} |W_u| \, d\lambda^-(u)\bigg) \le \mathbb{E}\bigg(\int_{[0,\zeta_k]\times J} \widetilde{W}_u \, d\lambda^-(u)\bigg) \le k\, \mathbb{E}\big(|F|\big)$$

which proves that for all $t > 0$,

$$\int_{[0,t]\times J} |W_u| \, d\lambda^-(u)$$

is almost surely finite. Notice now that $W_{(s,j)}$ is also equal to $\mathbb{E}(D_{(s,j)}F \mid \mathscr{F}_s)$ almost surely. Indeed, $D_{(s,j)}F(\omega)$ does not depend on $\omega_{|\{(s,j)\}}$, so that

$$\begin{aligned} W_{(s,j)}(\omega) &= \int D_{(s,j)}F(\omega_{|[0,t[\times J} + \bar{\omega}_{|[t,\infty[\times J})d\mathbb{P}(\bar{\omega}) \\ &= \int D_{(s,j)}F(\omega_{|[0,t]\times J} + \bar{\omega}_{|]t,\infty[\times J})d\mathbb{P}(\bar{\omega}) \\ &= \mathbb{E}\big(D_{(s,j)}F \mid \mathscr{F}_s\big) \ . \end{aligned}$$

It still remains to identify Z_u and W_u. To this end, we claim that for all previsible bounded processes $Y_{(t,j)}$ whose supports are in $[0, \zeta_k \wedge \zeta'_k]$, where $\zeta'_k = \inf\big\{t;\ \int_{[0,t]\times J} Z_u\, d\lambda^-(u) \geq k\big\}$,

$$\mathbb{E}\left(\int_{[0,t]\times J} W_u Y_u d\lambda^-(u)\right) = \mathbb{E}\left(\int_{[0,t]\times J} Z_u Y_u d\lambda^-(u)\right) \ . \tag{4.4}$$

As before, using Proposition 4.3, we get that

$$\begin{aligned} &\mathbb{E}\left(\int_{[0,t]\times J} W_u Y_u\, d\lambda^-(u)\right) \\ &\quad = \int_{[0,t]\times J} \mathbb{E}(W_u Y_u) d\lambda^-(u) = \mathbb{E}\left(\int_{[0,t]\times J} D_u F\ Y_u\, d\lambda^-(u)\right) \\ &\quad = \mathbb{E}\left(\int_{[0,t]\times J} F \circ \varepsilon_u^+\ Y_u\, d\lambda^-(u)\right) - \mathbb{E}\left(\int_{[0,t]\times J} F \circ \varepsilon_u^-\ Y_u\, d\lambda^-(u)\right) \\ &\quad = \mathbb{E}\left(\int_{[0,t]\times J} F \circ \varepsilon_u^+\ Y_u\, d\lambda^+(u)\right) - \mathbb{E}\left(\int_{[0,t]\times J} F \circ \varepsilon_u^-\ Y_u\, d\lambda^-(u)\right) \ . \end{aligned}$$

But $\varepsilon_u^+(\omega) = \omega$ for λ^+-almost all u and $\varepsilon_u^-(\omega) = \omega$ for λ^--almost all u. Thus,

$$\begin{aligned} \mathbb{E}\left(\int_{[0,t]\times J} W_u Y_u\, d\lambda^-(u)\right) &= \mathbb{E}\left(\int_{[0,t]\times J} F\ Y_u\, d(\lambda^+ - \lambda^-)(u)\right) \\ &= \mathbb{E}\left(F \int_{[0,t]\times J} Y_u\, d\lambda(u)\right) = \mathbb{E}(M_t N_t) \end{aligned}$$

where M_t is the uniformly integrable martingale $\mathbb{E}(F \mid \mathscr{F}_t)$ and N_t is the bounded martingale $\int_{[0,t]\times J} Y_u\, d\lambda(u)$. Let $\lambda(\cdot, j)$ be the measure λ restricted to $\mathbb{R}_+ \times \{j\}$. We know that

$$dM_t = \sum_{j\in J} Z_{(t,j)} \lambda(dt, j)$$

and that

$$dN_t = \sum_{j\in J} Y_{(t,j)} \lambda(dt, j) \ .$$

It follows that the previsible bracket $\langle M, N\rangle$ is given by

$$d\langle M, N\rangle_t = \sum_{j\in J} Z_{(t,j)} Y_{(t,j)}\ \lambda^-(dt, j) \ .$$

Indeed, if we call $\Lambda_j(t)$ the martingale $\int_0^t \lambda(ds, j)$, then $d\langle \Lambda_j, \Lambda_{j'}\rangle_t$ is equal to 0 for $j \neq j'$ and equal to $\lambda^-(dt, j)$ if $j = j'$. As

$$\langle M, N\rangle_t = \int_{[0,t]\times J} Z_u Y_u d\lambda^-(u)$$

is bounded (by k) by the assumption on the support of Y_u, and similarly N_t, the process $M_t N_t - \langle M, N\rangle_t$ is a true martingale since M_t is uniformly integrable. Thus

$$\begin{aligned}\mathbb{E}\left(\int_{[0,t]\times J} W_u Y_u \, d\lambda^-(u)\right) &= \mathbb{E}(M_t N_t)\\ &= \mathbb{E}\big(\langle M, N\rangle_t\big)\\ &= \mathbb{E}\left(\int_0^t \sum_{j\in J} Z_{(s,j)} Y_{(s,j)} \; \lambda^-(ds, j)\right)\\ &= \mathbb{E}\left(\int_{[0,t]\times J} Z_u Y_u \, d\lambda^-(u)\right) .\end{aligned}$$

Thus (4.4) is established. It follows that $Z = W$, $\lambda^- \otimes \mathbb{P}$-almost everywhere. Moreover, as Z_u and W_u do not depend on $\lambda^+(\{u\})$, it follows from Proposition 4.3 that Z and W also coincide $\lambda^+ \otimes \mathbb{P}$-almost everywhere. The proof of Theorem 4.2 is complete. □

Now we turn to the proof of Theorem 4.1.

Proof of Theorem 4.1. We start with the Poincaré inequality (4.1). Suppose first that F is bounded, so that the martingale $M_t = \mathbb{E}(F \mid \mathscr{F}_t)$ is uniformly bounded. By Itô's formula,

$$d(M_t^2) = 2M_{t^-} dM_t + (\Delta M_t)^2 \ .$$

Recall here that $\Delta N_t = N_t - N_{t^-}$ denotes the jump of the process N_t at time t. As the process $\int M_{t-} dM_t$ is a martingale, we get, by taking expectation,

$$\mathbb{E}(F^2) - \big(\mathbb{E}(F)\big)^2 = \mathbb{E}\left(\sum_{0\le t\le T} (\Delta M_t)^2\right) .$$

But the Clark-Ocone formula of Theorem 4.2 implies that $dM_t = \sum_{j\in J} Z_j(t) d\lambda(t, j)$. Therefore the jumping part is given by $\Delta M_t = \sum_{j\in J} Z_j(t) d\lambda^+(t, j)$, and

$$(\Delta M_t)^2 = \sum_{j\in J} (Z_j(t))^2 d\lambda^+(t, j) \ ,$$

as atoms in different directions occur at distinct times $\mathbb{P}$-almost surely. It follows that

$$\begin{aligned}\mathbb{E}(F^2) - \big(\mathbb{E}(F)\big)^2 &= \mathbb{E}\left(\int_{[0,T]\times J} Z^2_{(t,j)} d\lambda^+(t, j)\right)\\ &= \mathbb{E}\left(\int_{[0,T]\times J} Z^2_{(t,j)} d\lambda^-(t, j)\right)\end{aligned}$$

where we used Proposition 4.3. Then, by the Cauchy-Schwarz inequality

$$(Z_{(t,j)})^2 = \mathbb{E}\big(D_{(t,j)}F \mid \mathscr{F}_t\big)^2 \le \mathbb{E}\big((D_{(t,j)}F)^2 \mid \mathscr{F}_t\big)$$

so that

$$\begin{aligned}\mathbb{E}(F^2) - \big(\mathbb{E}(F)\big)^2 &\le \mathbb{E}\bigg(\int_{[0,T]\times J} \mathbb{E}\big((D_{(t,j)}F)^2 \ \mid \mathscr{F}_t\big) d\lambda^-(t,j)\bigg)\\ &= \mathbb{E}\bigg(\int_{[0,T]\times J} (D_{(t,j)}F)^2 d\lambda^-(t,j)\bigg)\end{aligned}$$

which is (4.1). To handle arbitrary functionals F, consider $F_A = \max(\min(F, A), -A)$. Then $|F_A|$ is increasing and tends to $|F|$ as A tends to infinity, and $|D_u F_A|$ is also increasing to $|D_u F|$. The conclusion easily follows.

The proof of the modified logarithmic inequality (4.2) is similar. Again assume first F is bounded from above, and bounded from below by $\varepsilon > 0$. Itô's formula shows here that

$$d(M_t \log M_t) = (\log M_{t-} + 1)(dM_t - \Delta M_t) + \Delta(M_t \log M_t) \ .$$

The process $\int (\log M_{t-} + 1) dM_t$ is a martingale, and taking expectation, we get

$$\begin{aligned}&\mathbb{E}(F \log F) - \mathbb{E}(F) \log \mathbb{E}(F)\\ &\quad = \mathbb{E}(M_T \log M_T - M_0 \log M_0)\\ &\quad = \mathbb{E}\bigg(\sum_{0<t\le T} (M_t \log M_t - M_{t-} \log M_{t-}) - (M_t - M_{t-})(\log M_{t-} + 1)\bigg) \ .\end{aligned}$$

Using that $b \log b - a \log a - (b-a)(\log a + 1) \le (b-a)^2/a$, $a, b > 0$, we have

$$\begin{aligned}\mathbb{E}(F \log F) - \mathbb{E}(F) \log \mathbb{E}(F) &\le \mathbb{E}\bigg(\sum_{0<t\le T} \frac{1}{M_{t-}} (\Delta M_t)^2\bigg)\\ &= \mathbb{E}\bigg(\int_{[0,T]\times J} \frac{1}{M_{t-}} Z^2_{(t,j)} d\lambda^+(t,j)\bigg)\\ &= \mathbb{E}\bigg(\int_{[0,T]\times J} \frac{1}{M_{t-}} Z^2_{(t,j)} d\lambda^-(t,j)\bigg) \ .\end{aligned}$$

In the first step, we used the Clark-Ocone formula while in the last one, we used the isometry formula of Proposition 4.3. Next, notice that $M_{t-} = M_t$ for λ^--almost every t, and that, by the Cauchy-Schwarz inequality,

$$(Z_{(t,j)})^2 = \mathbb{E}\big(D_{(t,j)}F \mid \mathscr{F}_t\big)^2 \le \mathbb{E}\Big(\frac{1}{F} (D_{(t,j)}F)^2 \mid \mathscr{F}_t\Big)\mathbb{E}\big(F \mid \mathscr{F}_t\big) \ .$$

Therefore,

$$\begin{aligned}\mathbb{E}(F \log F) - \mathbb{E}(F) \log \mathbb{E}(F) &\le \mathbb{E}\bigg(\mathbb{E}\Big(\frac{1}{F} (D_{(t,j)}F)^2 \mid \mathscr{F}_t\Big) d\lambda^-(t,j)\bigg)\\ &= \mathbb{E}\bigg(\frac{1}{F} \int_{[0,T]\times J} (D_{(t,j)}F)^2 d\lambda^-(t,j)\bigg)\end{aligned}$$

and (4.2) is established in this case. When $0 < F \leq M$, we may work with the functions $F_\varepsilon = \max(F, \varepsilon)$ and a standard approximation leads to the result. The case when F is not bounded above is somewhat more subtle. We may assume that the right-hand side of (4.2) is finite. Let $F_A = \min(F, A)$. The entropy $\mathbb{E}(F_A \log F_A) - \mathbb{E}(F_A)\log \mathbb{E}(F_A)$ still converges towards $\mathbb{E}(F \log F) - \mathbb{E}(F)\log \mathbb{E}(F)$ as $A \to \infty$. To get the convergence of the energy, notice that $|D_u F_A|$ is bounded by $|D_u F|$ and by A. We then consider two parts

$$\mathbb{E}\left(\frac{1}{F_A}\int_{[0,T]\times J}(D_u F_A)^2 d\lambda^-(u)\right) = \int_{\{F\leq A\}}\frac{1}{F}\int_{[0,T]\times J}(D_u F_A)^2 d\lambda^-(u)d\mathbb{P} + \int_{\{F>A\}}\frac{1}{A}\int_{[0,T]\times J}(D_u F_A)^2 d\lambda^-(u)d\mathbb{P} \ .$$

The first term converges towards $\mathbb{E}\big(\int_{[0,T]\times J}\frac{1}{F}(D_u F)^2 d\lambda^-(u)\big)$ by monotone convergence. To prove that the second term converges to zero, we use the dominated convergence theorem together with the fact that, on $\{F > A\}$,

$$\frac{(D_u F_A)^2}{A} \leq \frac{(D_u F)^2}{F} \ . \tag{4.5}$$

Indeed, on $\{F > A\}$, $F_A = A$. Hence, $|D_u F_A| = \big|F_A \circ \varepsilon_u - F_A\big| = |F_A \circ \varepsilon_u - A|$. Therefore,

$$\frac{(D_u F_A)^2}{A} = \Big(\frac{F_A \circ \varepsilon_u}{A} - 1\Big)\big(F_A \circ \varepsilon_u - A\big) \ .$$

Similarly,

$$\frac{(D_u F)^2}{F} = \Big(\frac{F \circ \varepsilon_u}{F} - 1\Big)\big(F \circ \varepsilon_u - F\big) \ ,$$

and, according as $F_A \circ \varepsilon_u \leq A$ or $\geq A$, (4.5) follows. This completes the proof of the main Theorem 4.1. □

5. Modified logarithmic Sobolev inequalities on discrete path spaces

We now apply the results of the preceding section to some classes of continuous time Markov processes $B = (B_t)_{t\geq 0}$ on a graph χ. The basic assumption we make is that the generator $\frac{1}{2}\mathrm{L}$ of the process B may be written as

$$\mathrm{L}f = \sum_{j\in J}\lambda_j (f \circ \tau_j - f) \ , \tag{5.1}$$

where J is a finite set, τ_j are transformations of the set of vertices of the graph χ, and λ_j are positive constants. The oriented edges of the graph are the couples $(x, \tau_j(x))$. This means that the transformations τ_j give the directions taken by the process B as a random walk on the graph χ.

Let Ω and $\mathbb{P}$ as in Section 4. It is then possible to construct the process B with the sequence $(T_k)_{k\geq 1}$ as jumping times and the sequence $(j_k)_{k\geq 1}$ as successive directions. More precisely, denote by $N_t = \sup\{k \geq 1;\ T_k \leq t\} = \lambda^+([0,t] \times J)$ the number of jumps before time t. Then the process

$$B_t = (\tau_{j_{N_t}} \circ \cdots \circ \tau_{j_1})(B_0)$$

has the expected distribution. This procedure defines a map from Ω onto the space of càd-làg paths on χ. Thus, Theorem 4.1 applies to this setting. Inequalities (4.1) and (4.2) of Theorem 4.1 may not be sharp for all graphs, since there are more functions on Ω than on the space of paths on the graph. At least, they are sharp on $\mathbb{Z}^d$.

Recall now that the process B has the following probabilistic interpretation. Starting from some $x_0 \in \chi$, it jumps at time T_1 to a neighbour $x_1 = \tau_{j_1}(x_0)$ of x_0. The law of T_1 is exponential with parameter $\Lambda = \frac{1}{2}\sum_{j=1}^d \lambda_j$, and $j_1 = j$ with probability $\lambda_j/2\Lambda$. Then, the process waits an exponential time $T_2 - T_1$ with parameter Λ before jumping to a neighbour $x_2 = \tau_{j_2}(x_1)$ of x_1, and so on. If it happens that, for example, $\tau_{j_1}(x_0) = x_0$, then the process does not perform a true jump at time T_1, and x_1 is not a true neighbour of x_0. We thus consider the sets of true jump times and true jump directions $(T'_k, j'_k)_{k\geq 1}$. The direction j'_1 belongs to the set of true directions $J(x_0) = \{j;\ \tau_j(x_0) \neq x_0\}$, and $j'_1 = j$ with probability proportional to λ_j. Time T'_1 is exponential with parameter $\Lambda(x_0) = \frac{1}{2}\sum_{j\in J(x_0)} \lambda_j$. Denote by Y'_k the (true) successive positions $Y'_k = \tau_{j'_k}(Y'_{k-1})$. Y'_1 is chosen among Y'_0's neighbours, and $Y'_1 = y$ with probability proportional to $\sum_{j;\tau_j(Y'_0)=y} \lambda_j$. The next steps are similar. Conditionally to Y'_k, the waiting time $T'_{k+1} - T'_k$ is exponential with parameter $\Lambda(Y'_k)$, and j_{k+1} is chosen in $J(Y'_k)$ with probability proportional to λ_j. Notice also that $\mathscr{F}_t = \sigma(B_s, s \leq t)$ is the usual filtration.

Now we discuss somewhat in depth a few examples entering this setting. In particular, we need to interpret, if possible, the gradient that comes into (4.1) and (4.2).

Our first examples connect with Section 2 and 3. “Brownian motion" on the lattice $\mathbb{Z}$ may be described in the preceding terminology with the translations $\tau_1(x) = x+1$ and $\tau_{-1}(x) = x-1$, and the constants $\lambda_1 = \lambda_{-1} = 1$, the generator L being thus the discrete Laplacian on $\mathbb{Z}$. The choice $\lambda_1 = 1$, $\lambda_{-1} = 0$ leads to the standard Poisson process, and the case $\lambda_1 \neq \lambda_{-1}$ corresponds to an asymmetric continuous Markov chain. The example of the two-point space $\{-1,+1\}$ is described similarly, and these examples are easily extended in dimension d. More generally, a process B on a group χ generated by a finite number of elements $e_1, \ldots, e_d \in \chi$ may be defined by (5.1) with $\tau_1, \ldots, \tau_d$ the translations $\tau_j(x) = x \cdot e_j$. The process B then corresponds to a continuous random walk on χ. One may for example consider the symmetric group $\mathscr{S}_n$ generated by the set of transpositions $\{\tau_j, j \in J\}$, with all $\lambda_j = 1$ for instance.

This framework allows us to consider continuous time random walks on locally uniformly finite graphs. Let χ be an oriented graph, such that the number $d(x)$ of edges starting from any vertex x is uniformly bounded. Let L be the generator defined by $\mathrm{L}(x,y) = 1$ if (x,y) is an edge, 0 otherwise. Then

$$(\mathrm{L}f)(x) = \sum_{y \leftarrow x} \big(f(y) - f(x)\big)\ ,$$

so that it can be written in the form (5.1). Indeed, let d be the maximum degree $d = \max_{x \in \chi} d(x)$. Define $\lambda_1 = \cdots = \lambda_d = 1$. Now, fix a vertex x, and let $y_1, \ldots, y_{d(x)}$ be its neighbours. Define $\tau_j(x) = y_j$ for $j \leq d(x)$ and $\tau_j(x) = x$ for $d(x) < j \leq d$. Then we have

$$\mathrm{L}f = \sum_{j=1}^{d} (f \circ \tau_j - f) \ .$$

We may also define L by $\mathrm{L}(x, y) = 1/d(x)$ if (x, y) is an edge, 0 otherwise, so that

$$(\mathrm{L}f)(x) = \frac{1}{d(x)} \sum_{y \leftarrow x} \big(f(y) - f(x)\big) \ .$$

Such a choice still enters the setting of (5.1). Choose d to be the least common multiple of the set $d(x); \ s \in \chi$, so that for every vertex x, $d/d(x) \in \mathbb{N}$. Take $\lambda_1 = \cdots = \lambda_d = 1/d$. Fix a vertex x, and let $y_1, \ldots, y_{d(x)}$ be its neighbours. For $1 \leq j \leq d/d(x)$, define $\tau_j(x) = y_1$, and $\tau_j(x) = y_k$ for $1 + (k-1)d/d(x) \leq j \leq k\, d/d(x)$, $k \leq d(x)$. (If $d(x) = 0$, i.e. if x does not have any neighbour, then define $\tau_j(x) = x$ for every j.) Then we have

$$\mathrm{L}f = \frac{1}{d} \sum_{j=1}^{d} (f \circ \tau_j - f) \ .$$

The preceding two choices correspond to two different extensions of the continuous random walks on the commutative graphs $\{-1, +1\}^d$ or $\mathbb{Z}^d$. In the second case, the process B jumps with equal probability to one of its neighbour point after an exponential waiting time of parameter $1/2$, while in the first one, the waiting time is exponential with parameter (one half of) the number of neighbours of the position of B.

Finite graphs provide also a wide class of examples. Indeed, if L is any generator on a finite graph χ, define J as the set of edges, $J \subset \{(x, y), \ x \neq y, \ x, y \in \chi\}$, and for $j = (x, y) \in J$, $\lambda_j = \mathrm{L}(x, y)$ and $\tau_j(z) = y$ if $z = x$, z otherwise. It is easy to see again that this example may be treated as before.

In the last part of this section, we discuss the form of the energy functional that appear in the Poincaré and logarithmic Sobolev inequalities of Theorem 4.1 for some of these examples. For simplicity, let us deal with

$$\mathcal{E}(F) = \mathbb{E}\bigg(\int_{[0,T] \times J} |D_u F|^2 d\lambda^-(u)\bigg)$$

of (4.1), the study of the one in (4.2) being entirely similar. In the case when B is the continuous time random walk on $\mathbb{Z}$ and F is a cylindrical function $F(B) = f(B_{t_1}, \ldots, B_{t_n})$, it is easy to see that

$$\mathcal{E}(F) = \sum_{i=1}^{n} (t_i - t_{i-1})\, \mathbb{E}\big(\Gamma^{i \ldots n} F\big)$$

which we obtained in (3.6) by a Markov tensorization of Proposition 3.1. Indeed, recall here the two translations τ_1 and τ_{-1} by $+1$ and -1. Fix a time t between, say, t_{i-1} and t_i. Then, for $j = +1$ or $j = -1$, we have almost surely

$$\left(\varepsilon^+_{(t,j)}(B)\right)_s = \begin{cases} B_s & \text{for } s < t, \\ B_s + j & \text{for } s \geq t \end{cases}$$

and $\varepsilon^-_{(t,j)}(B) = B$. Hence

$$\begin{aligned} D_{(t,\pm 1)} F &= \sum_{k=i}^{n} D_k F \circ \tau_{i,k-1} \\ &= f(B_{t_1}, \ldots, B_{t_{i-1}}, B_{t_i} \pm 1, \ldots, B_{t_n} \pm 1) - f(B_{t_1}, \ldots, B_{t_n}) \end{aligned}$$

almost surely, and $\left(D_{(t,+1)}F\right)^2 + \left(D_{(t,-1)}F\right)^2 = 2\Gamma^{i\ldots n}F$ for $t_{i-1} < t < t_i$. The claim follows. The examples of the standard Poisson process (cf. the introduction) and of the cube (cf. Section 2) are similar.

In the general case, the description of $\mathscr{E}$ is not so simple and usually takes into account the whole path before a given time. For simplicity, let F be a one-dimensional cylindrical functional $F = f(B_t)$, $t \geq 0$. We claim that

$$\mathscr{E}(F) = t \sum_{j \in J} \lambda_j \mathbb{E}\left(\frac{1}{N_t + 1} \sum_{k=0}^{N_t} \left[f(\tau_{j_{N_t}} \circ \cdots \circ \tau_{j_{k+1}} \circ \tau_j Y_k) - f(B_t) \right]^2 \right) . \tag{5.2}$$

Recall that here N_t is the number of (hidden) jumps before time t, Y_k is the position and $j_1, \ldots, j_{N_t}$ are the random directions taken by the process. When the transformations τ_j, $j \in J$, of (5.1) commute, we have that

$$\tau_{j_{N_t}} \circ \cdots \circ \tau_{j_{k+1}} \circ \tau_j Y_k = \tau_j \left(\tau_{j_{N_t}} \circ \cdots \circ \tau_{j_{k+1}} Y_k \right) = \tau_j(B_t) \ ,$$

so that in this case (5.2) amounts to

$$\mathscr{E}(F) = t \sum_{j \in J} \lambda_j \, \mathbb{E}\left(\left[f\left(\tau_j(B_t)\right) - f(B_t) \right]^2 \right)$$

which corresponds, in the terminology of Section 3, to the Γ operator associated to the generator L. Even in cases where the τ_j's do not commute, there are instances in which (5.2) takes a more simple form. For example, in case of the symmetric group $\mathscr{S}_n$ generated by the transpositions, fix $t \geq 0$ and $k \leq N_t$, and set $\sigma = \tau_{j_{N_t}} \circ \cdots \circ \tau_{j_{k+1}}$. Then the sets $\{\sigma \circ \tau;\ \tau \text{ transposition}\}$ and $\{\tau \circ \sigma;\ \tau \text{ transposition}\}$ are equal since the transpositions form a conjugancy class. Hence

$$\begin{aligned} \sum_{j \in J} \lambda_j \left[f(\sigma \circ \tau_j Y_k) - f(B_t) \right]^2 &\leq \left(\max_{\ell \in J} \lambda_\ell \right) \sum_{j \in J} \left[f(\tau_j \circ \sigma Y_k) - f(B_t) \right]^2 \\ &= \left(\max_{\ell \in J} \lambda_\ell \right) \sum_{j \in J} \left[f(\tau_j B_t) - f(B_t) \right]^2 \end{aligned}$$

and

$$\mathcal{E}(F) \le t\big(\max_{\ell\in J} \lambda_\ell\big) \sum_{i\in J} \mathbb{E}\big([f \circ \tau_i - f)]^2\big) \ .$$

We now prove (5.2). We first write

$$\mathcal{E}(F) = \sum_{\substack{j\in J\\ n\ge 0}} \lambda_j \ \sum_{k=0}^{n} \int_0^t \mathbb{E}\Big(\mathrm{I}_{\substack{\{T_k<s\le T_{k+1},\\ T_n<t\le T_{n+1}\}}} \big[f(\tau_{j_n} \circ \cdots \circ \tau_{j_{k+1}} \circ \tau_j Y_k) \\ - f(\tau_{j_n} \circ \cdots \circ \tau_{j_{k+1}} Y_k)\big]^2\Big) ds \ .$$

By independence of the sequences $(T_k)_{k\ge 0}$ and $(j_k)_{k\ge 0}$, we get

$$\mathcal{E}(F) = \sum_{j\in J, n\ge 0} \lambda_j \sum_{k=0}^{n} \Big(\int_0^t \mathbb{P}\{T_k < s \le T_{k+1},\ T_n < t \le T_{n+1}\}\, ds\Big) \\ \times\ \mathbb{E}\Big(\big[f(\tau_{j_n} \circ \cdots \circ \tau_{j_{k+1}} \circ \tau_j Y_k) - f(\tau_{j_n} \circ \cdots \circ \tau_{j_{k+1}} Y_k)\big]^2\Big) \ .$$

Let

$$I_{n,k}(t) = \int_0^t \mathbb{P}\{T_k < s \le T_{k+1},\ T_n < t \le T_{n+1}\}\, ds$$

for $k \le n$. Then,

$$\begin{aligned} I_{n,k}(t) &= \int_0^t \mathbb{P}\big\{\omega([0,s[\times J) = k,\ \omega([s,t[\times J) = n-k\big\}\, ds \\ &= \int_0^t \frac{(\Lambda s)^k}{k!}\, \mathrm{e}^{-\Lambda s}\, \frac{(\Lambda(t-s))^{n-k}}{(n-k)!}\, \mathrm{e}^{-\Lambda(t-s)}\, ds \\ &= \frac{1}{\Lambda}\, \mathrm{e}^{-\Lambda t} \int_0^{\Lambda t} \frac{u^k}{k!} \cdot \frac{(\Lambda t - u)^{n-k}}{(n-k)!}\, du \\ &= \frac{1}{\Lambda}\, \mathrm{e}^{-\Lambda t}\, \frac{(\Lambda t)^{n+1}}{(n+1)!} = \frac{t}{n+1}\, \mathbb{P}\{T_n < t \le T_{n+1}\} \ . \end{aligned}$$

Therefore, coming back to $\mathcal{E}$,

$$\begin{aligned} \mathcal{E}(F) \ &= \sum_{j\in J, n\ge 0} \lambda_j \sum_{k=0}^{n} \frac{t}{n+1}\, \mathbb{P}\{T_n < t \le T_{n+1}\} \\ &\quad \times\ \mathbb{E}\Big(\big[f(\tau_{j_n} \circ \cdots \circ \tau_{j_{k+1}} \circ \tau_j Y_k) - f(\tau_{j_n} \circ \cdots \circ \tau_{j_{k+1}} Y_k)\big]^2\Big) \\ &= t \sum_{j\in J} \lambda_j \mathbb{E}\Big(\frac{1}{N_t+1} \sum_{k=0}^{N_t} \big[f(\tau_{j_{N_t}} \circ \cdots \circ \tau_{j_{k+1}} \circ \tau_j Y_k) - f(B_t)\big]^2\Big) \ , \end{aligned}$$

and thus the proof of (5.2) is complete.

More generally, if F is now a cylindrical function $F = f(B_{t_1}, \dots, B_{t_n})$, a similar calculus yields

$$\mathscr{E}(F) = \sum_{i=1}^{n} (t_i - t_{i-1}) \sum_{j \in J} \lambda_j \mathbb{E}\bigg(\frac{1}{N_{t_i} - N_{t_{i-1}} + 1} \sum_{k=N_{t_{i-1}}}^{N_{t_i}} \Delta(f, B, i, j, k) \bigg)$$

where

$$\Delta(f, B, i, j, k) = \big[f(B_{t_i} \dots, B_{t_{i-1}}, \tau_{j_{N_{t_i}}} \circ \cdots \circ \tau_{j_{k+1}} \circ \tau_j Y_k, \dots, \tau_{j_{N_{t_n}}} \circ \cdots \circ \tau_{j_{k+1}} \circ \tau_j Y_k) - f(B_{t_1}, \dots, B_{t_n}) \big]^2 .$$

As before, this expression may be simplified in some cases (commutation of the τ_j's, on the symmetric group, etc). For example, if the τ_j's commute,

$$\mathscr{E}(F) = \sum_{i=1}^{n} (t_i - t_{i-1}) \sum_{j \in J} \lambda_j \mathbb{E}\Big(\big[f\big(B_{t_1}, \dots, B_{t_{i-1}}, \tau_j(B_{t_i}), \dots, \tau_j(B_{t_n})\big) - f(B_{t_1}, \dots, B_{t_n}) \big]^2 \Big) .$$

Let us note to conclude, that in the latter example of commuting transformations τ_j, the induction method detailled in Section 2 on the cube may be adapted to yield a usual logarithmic Sobolev inequality for cylindrical functions $F = f(B_{t_1}, \dots, B_{t_n})$ with energy

$$\mathscr{E}(F) = \sum_{i=1}^{n} \alpha(t_i - t_{i-1}) \sum_{j \in J} \lambda_j \mathbb{E}\Big(\big[f\big(B_{t_1}, \dots, B_{t_{i-1}}, \tau_j(B_{t_i}), \dots, \tau_j(B_{t_n})\big) - f(B_{t_1}, \dots, B_{t_n}) \big]^2 \Big)$$

provided there is one for the law of B_t with constant $\alpha(t)$.

6. Applications to tail estimates

In this last section, we show how the path space logarithmic Sobolev inequalities of Sections 4 and 5 may be used to deduce a tail estimate similar to (1.5) in the context of continuous random walks on graphs. As was discussed in the introduction on the basis of the random walk on $\mathbb{Z}^d$, we expect tail behaviours with mixed Gaussian and Poisson components. This is exactly what is provided by the preceding logarithmic Sobolev inequalities. We follow here [B-L] and [Le].

With the notation and hypotheses of Sections 4 and 5, let us agree that a cylindrical function $F = f(B_{t_1}, \dots, B_{t_n})$, $0 \le t_1 \le \cdots \le t_n$, is K-Lipschitz if $|D_u F(\omega)| \le K$, for $\lambda^- \otimes \mathbb{P}$-almost every (u, ω). An $\mathscr{F}_T$-measurable function F is said to be K-Lipschitz if it is the almost sure limit of a sequence (F_k) of cylindrical K-Lipschitz functions.

Proposition 6.1. *Let F be $\mathscr{F}_T$-measurable and K-Lipschitz function. Then F is integrable, and, for every $R \geq 0$,*

$$\mathbb{P}\big\{F \geq \mathbb{E}(F) + R\big\} \leq \exp\bigg(-\frac{R}{4K}\log\bigg(1 + \frac{KR}{2\alpha^2}\bigg)\bigg) \tag{6.1}$$

where $\alpha^2 = \operatorname*{ess\,sup}_{\Omega} \int |D_u F|^2 d\lambda^-(u)$. In particular,

$$\limsup_{R\to\infty} \frac{1}{R\log R}\, \log \mathbb{P}\{F \geq R\} \leq -\frac{1}{4K}\ . \tag{6.2}$$

As announced, this result describes a Gaussian tail when R is small with respect to α^2/K, and a Poisson tail for its large values. The constants in Proposition 6.1 are not sharp.

Before proving Proposition 6.1, let us illustrate the statement. Let B be the Brownian motion on the lattice $\mathbb{Z}^d$ or on the cube starting from x_0. Then (6.1) and (6.2) apply to $F = \sup_{0\leq t\leq T} d(\cdot, x_0)$ with $K = 1$, where d is the graph distance. One may also consider the L^p-distances

$$\bigg(\int_0^T d(\cdot, x_0)^p dt\bigg)^{1/p}\ .$$

Proof of Proposition 6.1. By a simple approximation procedure, it is enough to consider the case of a bounded cylindrical 1-Lipschitz function F (cf. [Le] for details in this respect.) We apply the logarithmic Sobolev inequality (4.2) to $\mathrm{e}^{\tau F}$ for every $\tau \geq 0$. Since for every u,

$$\big|D_u(\mathrm{e}^{\tau F})\big| \leq \mathrm{e}^{\tau F}\big|\mathrm{e}^{\tau D_u F} - 1\big| \leq \tau \mathrm{e}^{\tau}|D_u F|\,\mathrm{e}^{\tau F}\ ,$$

we get that

$$\mathbb{E}\big(\tau F\mathrm{e}^{\tau F}\big) - \mathbb{E}(e^{\tau F})\log \mathbb{E}(\mathrm{e}^{\tau F}) \leq \alpha^2\tau^2\mathrm{e}^{2\tau}\mathbb{E}(\mathrm{e}^{\tau F})\ .$$

If we let $H(\tau) = \log(\mathbb{E}\mathrm{e}^{\tau F})/\tau$, the preceding amounts to $H'(\tau) \leq \alpha^2\mathrm{e}^{2\tau}$ while $H(0) = \mathbb{E}(F)$. Therefore,

$$\mathbb{E}(\mathrm{e}^{\tau F}) \leq \exp\bigg(\frac{\alpha^2}{2}\,\tau(\mathrm{e}^{2\tau} - 1) + \tau\mathbb{E}(F)\bigg)\ .$$

By Chebychev's inequality, for all $R \geq 0$ and $\tau \geq 0$,

$$\mathbb{P}\big\{F \geq \mathbb{E}(F) + R\big\} \leq \exp\bigg(-\tau R + \frac{\alpha^2}{2}\,\tau(\mathrm{e}^{2\tau} - 1)\bigg)\ .$$

Choose then $\tau = \frac{1}{2}\log(\frac{R}{\alpha^2})$ for $R \geq 2\alpha^2$, and $\tau = \frac{R}{4\alpha^2}$ for $0 \leq R \leq 2\alpha^2$ which immediately yields (6.1). (6.2) is an easy consequence of (6.1). Proposition 6.1 is established. □

While the proof of the upper limit in (1.5) follows a very similar scheme, the lower limit in (1.5) relies on comparison theorems in manifolds with non-negative

Ricci curvature. To conclude this work, we present one instance for which the bound (6.2) is sharp.

We consider a graph χ such that each edge is oriented in both sense. Assume B starts at x_0 and set $D = \sup_{0 \le t \le T} d(\cdot, x_0)$ where d is the graph distance. Recall that $\Lambda = \frac{1}{2} \sum_{j \in J} \lambda_j$.

Proposition 6.2. *There exists R_0 such that for all $R \ge R_0$ and all $T \ge 0$,*

$$\mathbb{P}\{D \ge R\} \ge \exp\Big(-R \log\Big(\frac{R}{\lambda T}\Big) - \Lambda T \Big)$$

where $\lambda = \frac{1}{2} \min_{j \in J} \lambda_j$. Hence, in this case

$$\liminf_{R \to \infty} \frac{1}{R \log R} \log \mathbb{P}\{D \ge R\} \ge -1 \ .$$

Proof. For $R \ge 0$, denote by $[R] = \inf\{n \in \mathbb{N},\, n \ge R\}$ its upper integer part. Choose a sequence $j^{(n,R)} \in J$, $n \le [R]$, such that $\tau_{j^{([R],R)}} \circ \cdots \circ \tau_{j^{(1,R)}}(x_0)$ is at distance $[R]$ from x_0. Then

$$\begin{aligned} \mathbb{P}\{D \ge R\} &= \mathbb{P}\Big\{ \sup_{0 \le t \le T} d(B_t, x_0) \ge [R] \Big\} \ge \mathbb{P}\Big\{ d(B_T, x_0) \ge [R] \Big\} \\ &\ge \mathbb{P}\Big\{ N_T = [R] \Big\}\, \mathbb{P}\Big\{ \forall n \le [R],\, j_n = j^{(n,R)} \Big\} \end{aligned}$$

where we recall that N_T is the number of jumps before time T. As N_T is Poisson with parameter Λt and as $\lambda = \frac{1}{2} \min_{j \in J} \lambda_j$,

$$\mathbb{P}\{D \ge R\} \ge \Big(\mathrm{e}^{-\Lambda T} \frac{(\Lambda T)^{[u]}}{[u]!} \Big) \Big(\frac{\lambda}{\Lambda} \Big)^{[R]} \ .$$

Choose then R_0 such that $[R]! \le \exp(R \log R)$ for $R \ge R_0$, and the proof is easily completed. □

In case $\chi = \mathbb{Z}^d$, the distances $d(B_t, x_0)$ and $\sup_{0 \le s \le t} d(B_s, X_0)$ have a Poissonian tail for all t, in the sense that, (for all directions j)

$$\begin{aligned} \mathrm{e}^{-(\lambda_j + \lambda_{-j})t/2} \sum_{k \ge R} \frac{(\lambda_j t/2)^k}{k!} &\le \mathbb{P}\Big\{ d(B_t, x_0) \ge R \Big\} \\ &\le \mathbb{P}\Big\{ \sup_{0 \le s \le t} d(B_t, x_0) \ge R \Big\} \qquad (6.3) \\ &\le \mathrm{e}^{-\Lambda t} \sum_{k \ge R} \frac{(\Lambda t)^k}{k!} \end{aligned}$$

so that

$$\log \mathbb{P}\Big\{ d(B_t, x_0) \ge R \Big\} \sim \mathbb{P}\Big\{ \sup_{0 \le s \le t} d(B_t, x_0) \ge R \Big\} \sim -R \log R \ .$$

The right-hand side inequality in (6.3) follows from $\sup_{0\leq s\leq t} d(B_s, B_0) \leq N_t$ and from the fact that the number of jumps N_t is Poissonnian with parameter Λt. The left-hand side inequality is a consequence of the decomposition $p_t^x = \otimes_{j=1}^d \mu_{\lambda_j t} * \tilde{\mu}_{\lambda_{-j} t}$. It namely implies that for every $j \leq d$,

$$\begin{aligned}\mathbb{P}\big\{d(B_t, x_0) \geq R\big\} &\geq \mathbb{P}\big\{|B_t^j - B_0^j| \geq R\big\}\\ &= (\mu^0_{\lambda_j t} * \tilde{p}^0_{\lambda_{-j} t})\big([R, \infty[\big)\\ &\geq \mu^0_{\lambda_j t}\big([R, \infty[\big)\tilde{\mu}^0_{\lambda_{-j} t}(0) = \mathrm{e}^{-(\lambda_j + \lambda_{-j})t/2} \sum_{k\geq R} \frac{(\lambda_j t/2)^k}{k!} \;.\end{aligned}$$

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