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Transversal fluctuations for increasing subsequences on the plane

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Abstract. Consider a realization of a Poisson process in $\mathbb{R}^2$ with intensity 1 and take a maximal up/right path from the origin to (N, N) consisting of line segments between the points, where maximal means that it contains as many points as possible. The number of points in such a path has fluctuations of order N^χ, where $\chi = 1/3$, [BDJ]. Here we show that typical deviations of a maximal path from the diagonal $x = y$ is of order N^ξ with $\xi = 2/3$. This is consistent with the scaling identity $\chi = 2\xi - 1$ which is believed to hold in many random growth models.

1. Introduction and results

The fluctuations in many random growth models, for example in first-passage percolation, are described by two exponents, χ and ξ, see e.g. [KS] and [LNP]. The exponent χ describes the longitudinal whereas ξ describes the transversal fluctuations. In first-passage percolation the length of a minimizing path from the origin to (N, N) has fluctuations of order N^χ, and the minimizing path has typical deviations from the diagonal $x = y$ of order N^ξ. General heuristic arguments (see [KS]) suggest that the scaling identity $\chi = 2\xi - 1$ is valid in any dimension, compare the heuristic argument below. In two dimensions it is predicted that $\chi = 1/3$ and hence we should have $\xi = 2/3$. Since $\xi > 1/2$ one says that the minimizing path is *superdiffusive*.

We will consider a related model where it is known that $\chi = 1/3$ and prove that in this model we actually have $\xi = 2/3$. The model is a Poissonized version of the problem of the longest increasing subsequence in a random permutation introduced in [Ha], see also [AD]. In this model one considers a Poisson process with intensity 1 in $\mathbb{R}^2_+$ and looks at a maximal up/right path from the origin to (N, N) consisting of line segments between the Poisson points, where maximal means that it contains as many points as possible. The length of a path is the number of Poisson points in the path, and the length of a maximal path has fluctuations of order $N^{1/3}$, see [BDJ]. In this paper we will prove that the typical deviations of the maximal paths from $x = y$ are of order $N^{2/3}$.

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The proof uses the line of argument, for first-passage percolation models, initiated in [NP], to prove $\chi' \geq 2\xi - 1$ (where χ' is closely related to χ), and [LNP] to prove lower (superdiffusive) bounds on a suitably defined ξ. A related argument was used to analyze the corresponding problem for crossing Brownian motion in a Poissonian potential in [Wü], and the present paper follows the arguments in [Wü]. A heuristic argument goes as follows. The length of a typical maximal path from the origin to (x, y) is $\sim 2\sqrt{xy}$, see [AD]. Hence, a maximal path from the origin to (N, N) that passes through $(N(t-\delta), N(t+\delta))$, $0 < t < 1$, δ small, is shorter by the amount

$$2\sqrt{N(t-\delta)N(t+\delta)} + 2\sqrt{N(1-t+\delta)N(1-t-\delta)} - 2\sqrt{N^2} \ .$$

This should be of the same order as the length fluctuations, i.e. $O(N^\chi)$, which gives $\delta^2 = O(N^{\chi-1})$. Thus, $N^\xi \sim N\delta \sim N^{\chi/2+1/2}$, that is $2\xi - 1 = \chi$ and hence $\xi = 2/3$ since $\chi = 1/3$. The argument used below essentially makes this rigorous.

We will now give the precise definitions. Let $\mathbb{P}$ denote the Poissonian law with fixed intensity 1 on the space Ω of locally finite, simple, pure point measures on $\mathbb{R}^2$; $\omega = \sum_i \delta_{\zeta_i} \in \Omega$, $\zeta_i = (x_i, y_i)$ are the points in ω. Write $(x, y) \prec (x', y')$ if $x < x'$ it and $y < y'$. Given ω and two points $w \prec w'$ in $\mathbb{R}^2$ an *up/right path* π from w to w' is a subsequence $\{\zeta_{i_k}\}_{k=1}^M$ of points in ω such that

$$w \prec \zeta_{i_1} \prec \cdots \prec \zeta_{i_M} \prec w' \ .$$

The length, $|\pi|$, of π is M, the number of Poisson points in the path. Let $\Pi(w, w'; \omega)$ denote the set of all up/right paths from w to w' in ω. If K is a convex subset of $\mathbb{R}^2$ we let $\Pi^K(w, w'; \omega)$ denote all up/right paths π from w to w' inside K, i.e. $\pi \subseteq K$ and $w, w' \in K$. Let

$$d(w, w'; \omega) = \max\{|\pi|\,;\ \pi \in \Pi(w, w'; \omega)\} \ ,$$

and

$$d^K(w, w'; \omega) = \max\{|\pi|\,;\ \pi \in \Pi^K(w, w'; \omega)\} \ .$$

Let $\ell_N(\sigma)$ denote the length of a longest increasing subsequence in a random permutation $\sigma \in S_N$ (uniform distribution). If $i_1 < \cdots < i_n$ and $\sigma(i_1) < \cdots < \sigma(i_n)$ we have an increasing subsequence of length n and $\ell_N(\sigma)$ is the length of the longest such sequence. We define the Poissonized distribution function by

$$\phi_n(\lambda) = e^{-\lambda} \sum_{N=0}^{\infty} \frac{\lambda^N}{N!} P[\ell_N(\sigma) \leq n] \ ,$$

$[\ell_0(\sigma) \equiv 0]$. Let $a(w, w')$ denote the area of the rectangle $[w, w']$ with corners at w and w'. Now,

$$\mathbb{P}[d(w, w') \leq n] = \sum_{N=0}^{\infty} \mathbb{P}[d(w, w') \leq n \,|\, \omega([w, w']) = N]\mathbb{P}[\omega([w, w']) = N] \ ,$$

and, see [Ha] or [AD], $\mathbb{P}[d(w, w') \leq n \,|\, \omega([w, w']) = N] = P[\ell_N(\sigma) \leq n]$. Hence

$$\mathbb{P}[d(w,w') \le n] = \phi_n(a(w,w')) \ . \tag{1.1}$$

By Lemma 7.1 in [BDJ] we have a very good control of the function $\phi_n(\lambda)$. Let

$$t = 2^{1/3}(n+1)^{-1/3}(n+1-2\sqrt{\lambda}) \ . \tag{1.2}$$

Then for any fixed t in $\mathbb{R}$,

$$\lim_{\lambda\to\infty} \phi_n(\lambda) = F(t) \ , \tag{1.3}$$

where $F(t)$ is the Tracy-Widom largest eigenvalue distribution for GUE, see [TW] and [BDJ]. The distribution function $F(t)$ is given by

$$F(t) = \exp\big(-\int_t^\infty (x-t)u(x)^2 dx\big) \ ,$$

where $u(x)$ is the solution of the Painlevé II equation

$$u''(x) = 2u(x)^3 + xu(x), \quad \text{and} \quad u(x) \sim \mathrm{Ai}\,(x) \text{ as } x \to \infty \ ,$$

where Ai (x) is the Airy function. From this formula and the asymptotics of $u(x)$, see [BDJ], it follows that $0 < F(0) < 1$, which will be used below. Furthermore we have the following estimates. There are positive constants δ, T_0, c_1, c_2 so that if $T_0 \le t \le 2^{-2/3}(n+1)^{2/3}$, then

$$|\log \phi_n(\lambda)| \le c_1 \exp(-c_2 t^{3/2}) \ , \tag{1.4}$$

and if $-\delta(n+1)^{2/3} \le t \le -T_0$, then

$$\phi_n(\lambda) \le c_1 \exp(c_2 t^3) \ , \tag{1.5}$$

for all sufficiently large n. The estimate (1.4) also follows from the results in [Se]. These estimates will be important in the proof of our theorem.

Let $C(\gamma, N)$ be the cylinder of width N^γ from 0 to $w_N = (N, N)$:

$$C(\gamma, N) = \{(x,y)\,;\ 0 \le x+y \le 2N\,,\ -\sqrt{2}N^\gamma \le -x+y \le \sqrt{2}N^\gamma\} \ .$$

Denote by

$$\Pi_{\max}(w,w';\omega) = \{\pi \in \Pi(w,w';\omega)\,;\ |\pi| = d(w,w';\omega)\} \ ,$$

the set of maximal paths from w to w'. We are interested in the size of the fluctuations of maximal paths around the diagonal $x = y$, *the transversal fluctuations*. Let A_N^γ be the event that all maximal paths from 0 to w_N are contained in the cylinder $C(\gamma, N)$,

$$A_N^\gamma = \{\omega \in \Omega\,;\ \text{for all } \pi \in \Pi_{\max}(0, w_N;\omega) \text{ we have } \pi \subseteq C(\gamma, N)\} \ .$$

The *exponent of transversal fluctuations*, ξ, is then defined by

$$\xi = \inf\{\gamma > 0\,;\ \liminf_{N\to\infty} \mathbb{P}[A_N^\gamma] = 1\} \ . \tag{1.6}$$

We can now state the main result of the paper.

Theorem 1.1. *For the model defined above the exponent of transversal fluctuations* $\xi = 2/3$.

The proof of the theorem occupies the next sections.

Remark 1.2. We can consider the analogous problem for the growth model introduced in [Jo]. Let $w(i, j)$, $(i, j) \in \mathbb{Z}_+^2$, be independent geometrically (or exponentially) distributed random variables and consider

$$G(N) = \max\{ \sum_{(i,j)\in\pi} w(i, j); \pi \text{ an up/right path from } (1, 1) \text{ to } (N, N)\} \ .$$

In [Jo] it is proved that there are positive constants a and b so that $(G(N) - aN)/bN^{1/3}$ converges in distribution to a random variable with distribution function $F(t)$. In analogy with above we can consider the transversal deviations of a maximal path and define the exponent ξ. If we had large deviation estimates for $\mathbb{P}[G(N) \leq n]$ analogous to (1.4) and (1.5) we could copy the proof given in the next section and show that $\xi = 2/3$ in this case also. In [Jo] an estimate like (1.4) is proved, but (1.5) is open. It follows from [BR] that $\mathbb{P}[G(N) \leq n]$ is given by a certain $n \times n$ Toeplitz determinant just as $\phi_n(\lambda)$, and it might be possible to prove the analogue of (1.5) using Riemann-Hilbert techniques as in [BDJ].

2. Proof of $\xi \geq 2/3$

We will first prove that $\xi \geq 2/3$. Pick $\gamma \in (\xi, 1)$ and $\epsilon > 0$ (small). That $\xi < 1$ follows from the proof in sect. 3 that $\xi \leq 2/3$, which is independent of the present section. By the definition of ξ there is an N_0 such that

$$\mathbb{P}[A_N^\gamma] \geq 1 - \epsilon \tag{2.1}$$

for all $N \geq N_0$. If $\omega \in A_N^\gamma$, then every maximal path from 0 to w_N is contained in the cylinder $C(\gamma, N)$, so writing $C_1 = C(\gamma, N)$, we see that $d^{C_1}(0, w_N; \omega) = d(0, w_N; \omega)$. Hence, by (2.1),

$$\mathbb{P}[d^{C_1}(0, w_N) = d(0, w_N)] \geq 1 - \epsilon \ , \tag{2.2}$$

if $N \geq N_0$.

Set $\mathbf{v}_1 = (1/\sqrt{2}, 1/\sqrt{2})$ and $\mathbf{v}_2 = (-1/\sqrt{2}, 1/\sqrt{2})$. Let $m_N = 3N^\gamma \mathbf{v}_2$ and let C_2 be the cylinder $C_2 = C_1 + m_N$. Pick a b such that $\gamma < b < 1$, and assume that N is so large that $N^b - 4N^\gamma > 0$. Define the points A, B, C on the sides of C_2 by

$$\begin{aligned} \overline{OA} &= (N^b + 2N^\gamma)\mathbf{v}_1 + 2N^\gamma \mathbf{v}_2, \\ \overline{OB} &= (N^b + 4N^\gamma)\mathbf{v}_1 + 4N^\gamma \mathbf{v}_2, \\ \overline{OC} &= N^b \mathbf{v}_1 + 4N^\gamma \mathbf{v}_2 \ . \end{aligned}$$

ABC is a right angle triangle with the right angle at A, the side AB is vertical with A on the lower side of C_2 and B on the upper side. Divide the vertical side AB into $K = K(N)$ segments $z_{i-1}z_i$, $i = 1, \ldots, K$, where $z_0 = A$ and $z_K = B$. Let

L_i be the part of the straight line through z_i, parallel to the x-axis, lying in C_2. The parallelogram between L_{i-1} and L_i in C_2 is denoted by F_i, $i = 1, \ldots, K$. We also define the analogous geometrical objects at the other end of the cylinder, close to $m_N + w_N$, by translating the whole picure by $t_N = \sqrt{2}N - 6N^\gamma - 2N^b$, $z_i' = z_i + t_N \mathbf{v}_1$, $F_i' = F_i + t_N \mathbf{v}_1$, $\overline{OA}' = \overline{OA} + t_N \mathbf{v}_1$ and $\overline{OB}' = \overline{OB} + t_N \mathbf{v}_1$.

Given a Borel set F, $\omega(F)$ is the number of Poisson points in F. Let $\pi = \{\zeta_1, \ldots, \zeta_M\}$, $\zeta_1 \prec \cdots \prec \zeta_M$, be a maximal path in $\Pi^{C_2}(m_N, m_N + w_N; \omega)$ and let π^* be the curve obtained by joining ζ_i and ζ_{i+1}, $i = 0, \ldots, K$, by straight line segments, $\zeta_0 = m_N$ and $\zeta_{K+1} = m_N + w_N$. The curve π^* intersects AB at some point P and $A'B'$ at some point Q. The point P belongs to $\bar{F}_i$ and Q to $\bar{F}_j'$ for some i, j. We will write $z(\omega) = z_i$ and $z'(\omega) = z_j'$. (If $P = z_i$ for some i we let $z(\omega) = z_i$ and analogously for Q.) If we set $D_N(\omega) = \max_i \omega(\bar{F}_i) + \max_j \omega(\bar{F}_j')$, then

$$\begin{aligned} d^{C_2}(m_N, m_N + w_N) &\le d^{C_2}(m_N, z(\omega)) + d^{C_2}(z(\omega), z'(\omega)) \\ &\quad + d^{C_2}(z'(\omega), m_N + w_N) + D_N(\omega) \ . \end{aligned} \tag{2.3}$$

Note that $z(\omega) \in \mathscr{A} \doteq \{z_0, \ldots, z_K\}$ and $z'(\omega) \in \mathscr{A}' \doteq \{z_0', \ldots, z_K'\}$.

Lemma 2.1. *Let $K = [8N^{2\gamma}] + 1$. Then*

$$\mathbb{P}[D_N(\omega) \ge d] \le C(8N^{2\gamma} + 1)e^{-d/2} \ , \tag{2.4}$$

for all $d \ge 1$, where C is a numerical constant.

Proof. Since

$$\{D_N(\omega) \ge d\} \subseteq \{\max_i \omega(\bar{F}_i) \ge \frac{d}{2}\} \cup \{\max_j \omega(\bar{F}_j') \ge \frac{d}{2}\}$$

we have

$$\mathbb{P}[D_N(\omega) \ge d] \le 2K\mathbb{P}[\omega(\bar{F}_1) \ge d/2] \ . \tag{2.5}$$

Here we have used the fact that all the random variables $\omega(\bar{F}_i), \omega(\bar{F}_j')$ are identically distributed. The area of $\bar{F}_1$ is $8N^{2\gamma}/K = \lambda$, and thus

$$\mathbb{P}[\omega(\bar{F}_1) \ge d/2] \le \sum_{j=[d/2]}^{\infty} e^{-\lambda} \frac{\lambda^j}{j!} \le C \sum_{j=[d/2]}^{\infty} e^{-\lambda f(j/\lambda)} \ , \tag{2.6}$$

where C is a numerical constant and $f(x) = x \log x + 1 - x$. Here we have used Stirling's formula. Note that $f(x) \ge x$ if $x \ge 9$ say. Choose $K = [8N^{2\gamma}] + 1$, so that $\lambda \le 1$, and assume that $d \ge 18$. Then, by (2.6),

$$\mathbb{P}[\omega(\bar{F}_1) \ge d/2] \le C \sum_{j=[d/2]}^{\infty} e^{-j} \le Ce^{-d/2}$$

and introducing this estimate into (2.5) yields

$$\mathbb{P}[\omega(\bar{F}_1) \ge d] \le C(1 + 8N^{2\gamma})e^{-d/2}$$

for all $N \ge 1$, $d \ge 1$. Q.E.D.

It follows from the estimate (2.4) that

$$\mathbb{P}[D_N(\omega) \le 5\log N] \ge 1-\epsilon \ , \tag{2.7}$$

for all sufficiently large N.

Next, choose κ_1 and κ_2 so that $0 < \kappa_1 < 1/3 < \kappa_2 < 1$.

Lemma 2.2. *Assume that (2.1) holds. There is a numerical constant $\eta \in (0,1)$, such that if $\epsilon \le \eta$ and N is sufficiently large, then*

$$\mathbb{P}[d^{C_1}(0,w_N) - d^{C_2}(m_N, m_N + w_N) \le -N^{\kappa_1}] \ge \eta \ . \tag{2.8}$$

Furthermore, for N sufficiently large,

$$\mathbb{P}[|d(0,z(\omega)) - 2\sqrt{a(0,z(\omega))}| \le N^{b\kappa_2}] \ge 1-\epsilon \ , \tag{2.9}$$

$$\mathbb{P}[|d(m_N,z(\omega)) - 2\sqrt{a(m_N,z(\omega))}| \le N^{b\kappa_2}] \ge 1-\epsilon \ , \tag{2.10}$$

$$\mathbb{P}[|d(z'(\omega),w_N) - 2\sqrt{a(z'(\omega),w_N)}| \le N^{b\kappa_2}] \ge 1-\epsilon \ , \tag{2.11}$$

$$\mathbb{P}[|d(z'(\omega),w_N+m_N) - 2\sqrt{a(z'(\omega),w_N+m_N)}| \le N^{b\kappa_2}] \ge 1-\epsilon \ , \tag{2.12}$$

Proof. The random variables $d^{C_1}(0,w_N)$ and $d^{C_2}(m_N, m_N+w_N)$ are independent. Thus

$$\begin{aligned}
&\mathbb{P}[d^{C_1}(0,w_N) - d^{C_2}(m_N,m_N+w_N) \le -N^{\kappa_1}] \\
&\quad \ge \mathbb{P}[d^{C_1}(0,w_N) - 2N \le 0 \text{ and } d^{C_2}(m_N,m_N+w_N) - 2N \ge N^{\kappa_1}] \\
&\quad = \mathbb{P}[d^{C_1}(0,w_N) - 2N \le 0]\cdot \mathbb{P}[d^{C_1}(0,w_N) - 2N \ge N^{\kappa_1}] \ .
\end{aligned} \tag{2.13}$$

If $\omega \in A_N^\gamma$, then $d^{C_1}(0,w_N) = d(0,w_N)$, and consequently the last expression in (2.13) is greater than or equal to

$$\begin{aligned}
&\mathbb{P}[\{d(0,w_N) - 2N \le 0\} \cap A_N^\gamma]\cdot \mathbb{P}[\{d(0,w_N) - 2N \ge N^{\kappa_1}\} \cap A_N^\gamma] \\
&\quad \ge (\mathbb{P}[d(0,w_N) - 2N \le 0] + \mathbb{P}[A_N^\gamma] - 1) \\
&\qquad \times (\mathbb{P}[d(0,w_N) - 2N \ge N^{\kappa_1}] + \mathbb{P}[A_N^\gamma] - 1) \ .
\end{aligned} \tag{2.14}$$

By (1.1),

$$\mathbb{P}[d(0,w_N) - 2N \le 0] = \phi_{2N}(N^2) \ .$$

It follows from (1.3) that $\phi_{2N}(N^2) \to F(0)$ as $N \to \infty$. Furthermore, since $\kappa_1 < 1/3$,

$$\mathbb{P}[d(0,w_N) - 2N \ge N^{\kappa_1}] = 1 - \phi_{[2N+N^{\kappa_1}]}(N^2) \to 1 - F(0) \ ,$$

as $N \to \infty$, again by (1.3) and the fact that $\phi_n(\lambda)$ is increasing in n. Let $\eta = \frac{1}{3}F(0)(1 - F(0)) > 0$. If N is sufficiently large then $\mathbb{P}[d(0, w_N) - 2N \le 0] \ge F(0) - \eta$ and $\mathbb{P}[d(0, w_N) - 2N \ge N^{\kappa_1}] \ge 1 - F(0) - \eta$. Since $\mathbb{P}[A_N^\gamma] \ge 1 - \epsilon$ by (2.1) we see that the right hand side of (2.14) is $\ge (F(0) - 2\eta)(1 - F(0) - 2\eta) \ge \eta$. This proves (2.8).

Next, we will prove (2.9). The proofs of (2.10), (2.11) and (2.12) are completely analogous. Note that

$$\mathbb{P}[|d(0, z(\omega)) - 2\sqrt{a(0, z(\omega))}| \le N^{b\kappa_2}] \ge \mathbb{P}[\bigcap_{j=0}^{K}\{|d(0, z_j) - 2\sqrt{a(0, z_j)}| \le N^{b\kappa_2}\}]$$

so it suffices to show that

$$\sum_{j=1}^{K} \mathbb{P}[|d(0, z_j) - 2\sqrt{a(0, z_j)}| \ge N^{b\kappa_2}] \le \epsilon \tag{2.15}$$

for all sufficiently large N. We have $z_j = \frac{1}{\sqrt{2}}(N^b, N^b + r_j)$, where $4N^\gamma \le r_j \le 8N^\gamma$, so $a(0, z_j) = \frac{1}{2}(N^{2b} + N^b r_j) \doteq a_j$. Now,

$$\mathbb{P}[d(0, z_j) - 2\sqrt{a_j} \le -N^{b\kappa_2}] = \phi_{[2\sqrt{a_j} - N^{b\kappa_2}]}(a_j) \quad .$$

In this case t defined by (1.2) is $\sim -2^{1/6} N^{b\kappa_2 - b/3}$ and since $1/3 < \kappa_2 < 1$, the condition for (1.5) is fulfilled if N is sufficiently large and we get

$$\mathbb{P}[d(0, z_j) - 2\sqrt{a_j} \le -N^{b\kappa_2}] \le c_3 \exp(-c_4 N^{3b(\kappa_2 - 1/3)}) \tag{2.16}$$

for some positive constants c_3, c_4 and all j. Similarly we can use (1.4) to prove that

$$\mathbb{P}[d(0, z_j) - 2\sqrt{a_j} \ge N^{b\kappa_2}] \le c_5 \exp(-c_6 N^{\frac{3}{2}b(\kappa_2 - 1/3)}) \tag{2.17}$$

for some positive constants c_5, c_6 if N is sufficiently large. Using (2.16) and (2.17) we see that (2.15) holds if N is sufficiently large since $K = [8N^{2\gamma}] + 1$. This completes the proof of the lemma. Q.E.D.

Denote by B_N^γ the set of ω that satisfy all the inequalities inside $\mathbb{P}[\quad]$ in (2.7) – (2.12). Then, by (2.7) and Lemma 2.2,

$$\mathbb{P}[B_N^\gamma] \ge \eta - 5\epsilon \quad . \tag{2.18}$$

Note that for any ω,

$$d(0, w_N) \ge d(0, z(\omega)) + d(z(\omega), z'(\omega)) + d(z'(\omega), w_N) \quad . \tag{2.19}$$

The inequalities (2.3) and (2.19) give

$$\begin{aligned} d^{C_2}(m_N, m_N + w_N) - d(0, w_N) \le\ & d^{C_2}(m_N, z(\omega)) + d^{C_2}(z'(\omega), m_N + w_N) \\ & - d(0, z(\omega)) - d(z'(\omega), w_N) + D_N(\omega) \quad . \end{aligned} \tag{2.20}$$

Now, using (2.20), we see that for $\omega \in B_N^\gamma$,

$$\begin{aligned} d^{C_1}(0, w_N) - d(0, w_N) &= d^{C_1}(0, w_N) - d^{C_2}(m_N, m_N + w_N) \\ &\quad + d^{C_2}(m_N, m_N + w_N) - d(0, w_N) \\ &\leq - N^{\kappa_1} + 4N^{b\kappa_2} + 2\sqrt{a(m_N, z(\omega))} \\ &\quad + 2\sqrt{a(z'(\omega), m_N + w_N)} - 2\sqrt{a(0, z(\omega))} \\ &\quad - 2\sqrt{a(z'(\omega), w_N)} + 5\log N \ . \end{aligned} \tag{2.21}$$

To proceed we need the following purely geometric lemma.

Lemma 2.3. *For all sufficiently large N,*

$$\sqrt{a(m_N, z)} - \sqrt{a(0, z)} \leq 10N^{2\gamma - b} \tag{2.22}$$

for any $z \in \mathscr{A}$ and

$$\sqrt{a(z', w_N + m_N)} - \sqrt{a(z', w_N)} \leq 10N^{2\gamma - b} \tag{2.23}$$

for any $z' \in \mathscr{A}'$.

Proof. We will prove (2.22). The inequality (2.23) then follows by symmetry. Now, $a(m_N, z_j) = (N^b + 3N^\gamma)(N^b - 3N^\gamma + r_j)/2$, $a(0, z_j) = (N^{2b} + r_j N^b)/2$ and hence

$$\sqrt{a(m_N, z)} - \sqrt{a(0, z)} = \frac{a(m_N, z) - a(0, z)}{\sqrt{a(m_N, z)} + \sqrt{a(0, z)}} \leq \frac{3r_j N^\gamma}{2\sqrt{2}N^b} \leq 10N^{2\gamma - b} \ ,$$

since $r_j \leq 8N^\gamma$. Q.E.D.

Introducing the estimates (2.22) and (2.23) into (2.21) we obtain

$$d^{C_1}(0, w_N) - d(0, w_N) \leq -N^{\kappa_1} + 5N^{b\kappa_2} + 40N^{2\gamma - b}$$

for all $\omega \in B_N^\gamma$ if N is sufficiently large. Thus, by (2.18),

$$\mathbb{P}[d^{C_1}(0, w_N) - d(0, w_N) \leq -N^{\kappa_1} + 5N^{b\kappa_2} + 40N^{2\gamma - b}] \geq \eta - 5\epsilon \geq \frac{\eta}{2} \ , \tag{2.24}$$

if $\epsilon < \eta/10$ and N is sufficiently large. But we also have the estimate (2.2). These estimates are consistent for large N only if

$$\kappa_1 \leq \max\{b\kappa_2, 2\gamma - b\} \ . \tag{2.25}$$

In this inequality we can let $\kappa_1 \nearrow 1/3$ and $\kappa_2 \searrow 1/3$ to get $1/3 \leq \max\{b/3, 2\gamma - b\}$ and since $b < 1$, we must have $1/3 \leq 2\gamma - b$. Here we can let $\gamma \searrow \xi$ and $b \nearrow 1$ to get $1/3 \leq 2\xi - 1$, i.e. $\xi \geq 2/3$.

3. Proof of $\xi \leq 2/3$

We turn now to the proof of the opposite inequality $\xi \leq 2/3$. By the definition (1.6) of ξ we see that we have to show that if $\gamma > 2/3$, then

$$\lim_{N\to\infty} \mathbb{P}[\Omega \setminus A_N^\gamma] = 0 \ . \tag{3.1}$$

If $\omega \in \Omega \setminus A_N^\gamma$, then there is a path $\pi_0 \in \Pi_{\max}(0, w_N; \omega)$ such that π_0 is not contained in $C(\gamma, N)$. We take one such path. Fix $\gamma \in (2/3, 1)$. Let π_0^* be the curve associated to π_0. Then π_0^* intersects the upper and/or the lower sides of $C(\gamma, N)$. Assume that it intersects the upper side. Define a sequence of points on the upper side of $C(\gamma, N)$, $z_j = (jM/K, jM/K + \sqrt{2}N^\gamma)$, $0 \leq j \leq K$, where $M = N - \sqrt{2}N^\gamma$ and $K = [2\sqrt{2}N^{1+\gamma}] + 1$. Let D_j be the parallelogram with corners at z_{j-1}, z_j, $(jM/K, jM/K - \sqrt{2}N^\gamma)$ and $((j-1)M/K, (j-1)M/K - \sqrt{2}N^\gamma)$, $1 \leq j \leq K$.

The curve π_0^* intersects the upper side for the first time, going from 0 to w_N, in the line segment $z_{j-1}z_j$ for some j. We set $z(\omega) = z_{j-1}$. By the choice of $z(\omega)$ we have that

$$d(0, w_N) \leq d(0, z(\omega)) + d(z(\omega), w_N) + \max_{1\leq j\leq K} \omega(D_j) \ . \tag{3.2}$$

In the case when π_0^* does not intersect the upper side but only the lower side, there is a last time where it intersects the lower side and we can assign a point $z(\omega)$ on the lower side so that (3.2) holds. This case is the image under the map $T_N : (x, y) \to (N - x, N - y)$ of the first case. Let $\mathscr{C} = \{z_j\}_{j=0}^K$ and let $\mathscr{C}'$ be the image of $\mathscr{C}$ under T_N.

Lemma 3.1. *Set*

$$\Lambda_N = \{\omega\,;\ \max_{1\leq j\leq K} \omega(D_j) \leq 2\log N\} \ ,$$

and for each $z \in \mathscr{C} \cup \mathscr{C}'$, $\delta \in (1/3, 2\gamma - 1)$,

$$E_z = \{\omega\,;\ d(0, z) \leq 2\sqrt{a(0, z)} + a(0, z)^{\delta/2} + N^\delta \quad \textit{and} \quad d(z, w_N) \leq 2\sqrt{a(z, w_N)} + a(z, w_N)^{\delta/2} + N^\delta\} \ .$$

For any given $\epsilon > 0$, *there is an* N_0 *such that if* $N \geq N_0$, *then*

$$\mathbb{P}\Big[\bigcup_{z\in\mathscr{C}\cup\mathscr{C}'} (\Omega \setminus E_z) \cup (\Omega \setminus \Lambda_N)\Big] \leq \epsilon \ . \tag{3.3}$$

Proof. An argument analogous to the one used in the proof of Lemma 2.1 shows that there is a numerical constant C so that

$$\mathbb{P}[\Omega \setminus \Lambda_N] \leq CN^{\gamma-1} \ .$$

We consider $z \in \mathscr{C}$, the case $z \in \mathscr{C}'$ is analogous by symmetry. Recall that $[z, w]$ denotes the rectangle with corners at z and w. If $a(0, z) \leq N^{\delta/2}$, then $\mathbb{P}[\omega([0, z]) \geq$

$N^\delta] \le C\exp(-N^\delta/2)$ for some numerical constant C, by Chebyshev's inequality. Since we trivially have $d(0,z;\omega) \le \omega([0,z])$, we obtain

$$\mathbb{P}[d(0,z) > 2\sqrt{a(0,z)} + a(0,z)^{\delta/2} + N^\delta] \le C\exp(-N^\delta/2)\ , \tag{3.4}$$

provided $a(0,z) \le N^{\delta/2}$. Now, with $a = a(0,z)$,

$$\mathbb{P}[d(0,z) > 2\sqrt{a} + a^{\delta/2} + N^\delta] \le 1 - \phi_{[2\sqrt{a}+a^{\delta/2}]}(a)\ .$$

This last expression can be estimated using (1.4), which gives

$$1 - \phi_{[2\sqrt{a}+a^{\delta/2}]}(a) \le c_1' \exp(-c_2' a^{(\delta-1/3)/2})\ .$$

If $a \ge N^{\delta/2}$, the right hand side is $\le c_1'\exp(-c_2' N^{\delta(\delta-1/3)/4})$ and thus

$$\mathbb{P}[d(0,z) > 2\sqrt{a} + a^{\delta/2} + N^\delta] \le c_1' \exp(-c_2' N^{\delta(\delta-1/3)/4})\ . \tag{3.5}$$

We can prove estimates analogous to (3.4) and (3.5) with $d(0,z)$ replaced by $d(z,w_N)$ in the same way. Bringing everything together we see that (3.3) holds if N is sufficiently large. The lemma is proved. Q.E.D.

Set

$$B_N^\gamma = (\Omega \setminus A_N^\gamma) \cap (\bigcap_{z\in\mathscr{C}\cup\mathscr{C}'} E_z) \cap \Lambda_N\ .$$

By Lemma 3.1, for $N \ge N_0$,

$$\mathbb{P}[\Omega \setminus A_N^\gamma] \le \epsilon + \mathbb{P}[B_N^\gamma]\ . \tag{3.6}$$

Since $a(0,z) \le N^2$ and $a(z,w_N) \le N^2$ for any $z \in \mathscr{C}\cup\mathscr{C}'$, we see from (3.2) that for $\omega \in B_N^\gamma$,

$$d(0,w_N) \le 2\log N + 4N^\delta + 2\sqrt{a(0,z(\omega))} + \sqrt{a(z(\omega),w_N)}\ . \tag{3.7}$$

We need one more geometric lemma.

Lemma 3.2. *For any $z \in \mathscr{C}\cup\mathscr{C}'$,*

$$\sqrt{a(0,z)} + \sqrt{a(z,w_N)} - \sqrt{a(0,w_N)} \le -N^{2\gamma-1}\ , \tag{3.8}$$

if N is sufficiently large.

Proof. Again, by symmetry, it suffices to consider the case $z \in \mathscr{C}$. Now, $a(0,z_j) = j\frac{M}{K}(j\frac{M}{K} + \sqrt{2}N^\gamma)$ and $a(z_j,w_N) = (N - j\frac{M}{K})(N - j\frac{M}{K} - \sqrt{2}N^\gamma)$. where $1 \le j \le K = [2\sqrt{2}N^{1+\gamma}] + 1$ and $M = N - \sqrt{2}N^\gamma$. Write $x = jM/KN$ and $y = \sqrt{2}N^{\gamma-1}$, so that $0 \le x \le 1-y$. Then,

$$\sqrt{a(0,z)} + \sqrt{a(z,w_N)} - \sqrt{a(0,w_N)} = Nf(x,y) \ , \tag{3.9}$$

where

$$f(x,y) = \sqrt{x^2 + xy} + \sqrt{(1-x)^2 - (1-x)y} \ .$$

For a fixed $y \in (0,1)$ this function assumes its maximum in $[0, 1-y]$ at $x = (1-y)/2$, which gives $f(x,y) \le -y^2/2$. Inserting this estimate into (3.9) and taking $y = \sqrt{2}N^{\gamma-1} < 1$, which is true if N is large enough, proves the lemma.

Q.E.D.

Combining the estimates (3.7) and (3.8), we see that

$$\mathbb{P}[B_N^\gamma] \le \mathbb{P}[d(0,w_N) - 2\sqrt{a(0,w_N)} \le 2\log N + 4N^\delta - 2N^{2\gamma-1}] \ . \tag{3.10}$$

To finish the proof we need

Lemma 3.3. *If* $\delta \in (1/3, 2\gamma-1)$, $\gamma > 2/3$, *then*

$$\lim_{N\to\infty} \mathbb{P}[d(0,w_N) - 2\sqrt{a(0,w_N)} \le 2\log N + 4N^\delta - 2N^{2\gamma-1}] = 0 \ . \tag{3.11}$$

Proof. Since $\delta < 2\gamma - 1$, we have that $2\log N + 4N^\delta - 2N^{2\gamma-1} \le -N^{2\gamma-1}$ if N is sufficiently large. Thus, by (1.1),

$$\begin{aligned}\mathbb{P}[d(0,w_N) \le 2N + 2\log N + 4N^\delta - 2N^{2\gamma-1}] &\le \mathbb{P}[d(0,w_N) \le 2N - N^{2\gamma-1}] \\ &= \phi_{[2N-N^{2\gamma-1}]}(N^2) \ .\end{aligned}$$

The identity (1.2) with $n = [2N - N^{2\gamma-1}]$ and $\lambda = N^2$ gives $t \sim -N^{2\gamma-4/3}$, and hence (1.5) gives us the estimate

$$\phi_{[2N-N^{2\gamma-1}]}(N^2) \le c_1 \exp(-c_2' N^{6\gamma-4}) \ ,$$

where $c_2' > 0$. This proves the lemma. Q.E.D.

Combining (3.11) with (3.6) and (3.10) we have proved (3.1). Thus $\xi \le 2/3$ and we are done.

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