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Chaos decomposition of multiple fractional integrals and applications*

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Abstract. Chaos decomposition of multiple integrals with respect to fractional Brownian motion (with $H > 1/2$) is given. Conversely the chaos components are expressed in terms of the multiple fractional integrals. Tensor product integrals are introduced and series expansions in those are considered. Strong laws for fractional Brownian motion are proved as an application of multiple fractional integrals.

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1. Introduction

In Dasgupta and Kallianpur [2] multiple integrals w.r.t fractional Brownian motion (multiple fractional integrals from now on) are constructed in the mean square sense when the index H of fractional Brownian motion lies between 1/2 and 1. In this paper we consider the chaos decomposition of the multiple fractional integrals which gives information about their structure.

It is known that L^2 functionals of certain Gaussian processes admit a chaos expansion in terms of Hermite polynomials. For a detailed discussion on chaos expansion of an L^2-functional of a Gaussian process we refer the reader to chapter 6 of Kallianpur [8]. The discussion there is of great

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generality. We shall state the main result in a form suitable for our purpose. For $X(t), t \in [0, T]$ a real Gaussian process, one starts with a fixed but arbitrary complete orthonormal set $\{e_i\}_1^\infty$ in the reproducing kernel Hilbert space (RKHS) of the process. Using the isometric isomorphism between the RKHS and the linear space of the process one chooses i.i.d. normal random variables $\{\xi_i\}_1^\infty$ from the linear space of the process corresponding to $\{e_i\}_1^\infty$. Then theorem 6.6.1 of Kallianpur [8] states that

$$L^2(\Omega, \mathscr{F}^X, P) = \sum_{p \geq 0} \oplus \bar{G}_p \tag{1}$$

where $\bar{G}_p$ is the closure of the linear subspace spanned by products of Hermite polynomials of total degree p in $\{\xi_i\}_1^\infty$, and is called the p-th homogeneous chaos. It is also proved that the p-th homogeneous chaos is isometrically isomorphic to the p-fold symmetric tensor product of the linear space of the process.

For us the process $X(t)$ is fractional Brownian motion $B_H(t), t \in [0, T]$, $H > 1/2$. The multiple fractional integrals in Dasgupta and Kallianpur [2] are L^2 functionals of fractional Brownian motion. In this paper we consider their chaos decomposition, that is we identify the elements in $\bar{G}_m, m \leq p$ which correspond to a multiple fractional integral of order p. These elements are called the chaos components of the multiple fractional integral. This is done in sections 2 and 3. Conversely the chaos components are expressed in terms of multiple fractional integrals of successively lower orders. This is done in sections 4 and 5. In section 6 we consider infinite series in tensor product integrals. Finally in section 7 we extend a strong law due to Gladyshev for fractional Brownian motion as an application of multiple fractional integrals.

2. Chaos components of multiple fractional integrals of elementary functions

For the sake of notational simplicity we shall first consider the double integral in detail. Let $a_{ij}, i, j = 1, 2, \ldots, n$, be real numbers and $\Delta_i = [t_i, t_{i+1}), i = 1, 2, \ldots, n$, be a partition of a finite interval $[0, T]$. A function of the form

$$\sum_{i,j} a_{ij} 1_{\Delta_i}(x) 1_{\Delta_j}(y)$$

will be called an elementary function. We define the double integral of $\sum_{i,j} a_{ij} 1_{\Delta_i}(x) \times 1_{\Delta_j}(y)$ with respect to $\{B_H(t), t \in [0, T]\}$ as

$$\sum_{i,j} a_{ij} \Delta B_H(t_i) \Delta B_H(t_j) \tag{2}$$

where $\Delta B_H(t_i) = B_H(t_{i+1}) - B_H(t_i)$. Without loss of generality we assume $a_{i,j}$ to be symmetric. The main tool in considering the structure of (2) is the following expansion in the notation of Dasgupta and Kallianpur [2]

$$\Delta B_H(t_k) = \sum_{i=1}^{\infty} \xi_i(\omega) c_i(t_k) \ a.s.$$

where the $\xi_i(\omega)$'s are i.i.d. normal random variables corresponding to a complete orthonormal set e_i from the RKHS of fractional Brownian motion and $c_i(t_k) = e_i(t_{k+1}) - e_i(t_k)$. Consider a finite sum from the above expansion and call it

$$\Delta B_H^{n'}(t_k) = \sum_{i=1}^{n'} \xi_i(\omega) c_i(t_k) \ . \tag{3}$$

Do the same for all $\Delta B_H(t_k), k = 1, 2, \ldots, n$ and denote the image of $\Delta B_H^{n'}(t_k)$ in the RKHS by

$$f_k^{n'} \ . \tag{4}$$

Then

$$\begin{aligned} \sum_{l,m} a_{lm} \Delta B_H^{n'}(t_l) \Delta B_H^{n'}(t_m) &= \sum_{i=1}^{n'} \sum_{j=1}^{n'} \{\sum_{l,m} a_{lm} c_i(t_l) c_j(t_m)\} \xi_i \xi_j \\ &= \sum_{i=1}^{n'} \sum_{j=1}^{n'} b_{ij} \xi_i \xi_j \end{aligned} \tag{5}$$

where $b_{ij} = \sum_{l,m} a_{lm} c_i(t_l) c_j(t_m)$.

In Johnson and Kallianpur [7] the authors represent a p-form

$$\sum_{i_1,\ldots,i_p=1}^{n} b_{i_1,\ldots,i_p} \xi_{i_1} \ldots, \xi_{i_p}$$

on the Wiener space as a sum of multiple Wiener integrals. These multiple Wiener integrals can be represented in terms of Hermite polynomials in $\xi_{i_1} \ldots, \xi_{i_p}$, for example a multiple Wiener integral of order m there will consist of sums of products of Hermite polynomials of total degree m. We will make explicit use of these ideas and so restate their lemma 4.2 here. Let (ϕ_i) be a CONS for $L^2(R_+)$ so that $(\phi_{i_1} \otimes \cdots \otimes \phi_{i_p})$ is a CONS for $L^2(R_+^p)$.

Let $a_{i_1,\ldots,i_p}$ be symmetric and (ξ_i) be the element in the linear space of the Wiener process corresponding to (ϕ_i). Let

$$f_p = \sum_{i_1,\ldots,i_p=1}^{n} a_{i_1,\ldots,i_p}(\phi_{i_1} \otimes \cdots \otimes \phi_{i_p}) \ . \tag{6}$$

Then

$$\sum_{i_1,\ldots,i_p=1}^{n} a_{i_1,\ldots,i_p}\xi_{i_1,\ldots,i_p} = \sum_{k=0}^{[p/2]} C_{p,k} I_{p-2k}(Tr^k f_p) \ , \tag{7}$$

where

$$C_{p,k} = \frac{p!}{(p-2k)!2^k k!} \tag{8}$$

and for f_p defined in (6) one defines

$$Tr^k f_p = \sum_{i_{2k+1},\ldots,i_p=1}^{n} \Big(\sum_{j_1,\ldots,j_k=1}^{n} a_{j_1,j_1,\ldots,j_k,j_k,i_{2k+1},\ldots,i_p} \Big)(\phi_{i_{2k+1}} \otimes \cdots \otimes \phi_{i_p}) \ . \tag{9}$$

and I_{p-2k} is a multiple Wiener integral of order $p-2k$. For simplicity we discuss the case $p=2$ in detail. We have

$$\sum_{i,j=1}^{n} b_{i,j}\xi_i\xi_j = \sum_{i=1}^{n} b_{i,i} + I_2(\sum_{i,j=1}^{n} b_{i,j}\phi_i \otimes \phi_j) \ . \tag{10}$$

From theorem 6.6.4 of Kallianpur [8] we know that the double Wiener integral in the above equation can be written in terms of symmetric tensor products of ξ_i. This gives

$$\sum_{i,j=1}^{n} b_{i,j}\xi_i\xi_j = \sum_{i=1}^{n} b_{i,i} + \sum_{i,j=1}^{n} b_{i,j}(\xi_i \tilde{\otimes} \xi_j) \ , \tag{11}$$

where $\tilde{\otimes}$ denotes a symmetric tensor product of two elements from the linear space of fractional Brownian motion. We are going to use the expression (11) for (5). It does not make any difference that our ξ_i's are from the linear space of fractional Brownian motion, because by definition the $(\xi_i \tilde{\otimes} \xi_j)$'s are just Hermite polynomials in i.i.d. normal random variables. Our main problem is in making n' go to ∞. For this we put the expressions for $b_{i,j}$'s back and get from (5) and (11) after interchange of summation

$$\begin{aligned}
&\sum_{l,m} a_{lm} \Delta B_H^{n'}(t_l) \Delta B_H^{n'}(t_m) \\
&\quad = \sum_{i=1}^{n} b_{i,i} + \sum_{i,j=1}^{n} b_{i,j} (\xi_i \tilde{\otimes} \xi_j) \\
&\quad = \sum_{l,m} a_{lm} \sum_{i=1}^{n'} c_i(t_l) c_i(t_m) + \sum_{l,m} a_{lm} \sum_{i,j=1}^{n'} (c_i(t_l)\xi_i \tilde{\otimes} c_j(t_m)\xi_j) \\
&\quad = \sum_{l,m} a_{lm} \langle f_l^{n'}, f_m^{n'} \rangle + \sum_{l,m} a_{lm} (\sum_{i=1}^{n'} c_i(t_l)\xi_i \tilde{\otimes} \sum_{j=1}^{n'} c_j(t_m)\xi_j) \\
&\quad = \sum_{l,m} a_{lm} \langle f_l^{n'}, f_m^{n'} \rangle + \sum_{l,m} a_{lm} \Delta B_H^{n'}(t_l) \tilde{\otimes} \Delta B_H^{n'}(t_m) \ , \qquad (12)
\end{aligned}$$

where one has to remember that the components with different orders of symmetric tensor products are orthogonal. Making n' go to ∞ we get

Theorem 2.1.

$$\begin{aligned}
&\sum a_{lm} \Delta B_H(t_l) \Delta B_H(t_m) \\
&\quad = \sum a_{lm} \langle f_l, f_m \rangle + \sum a_{lm} \Delta B_H(t_l) \tilde{\otimes} \Delta B_H(t_m) \ , \qquad (13)
\end{aligned}$$

where f_l is the image of $\Delta B_H(t_l)$ in the RKHS. □

The moments of theorem 3.1 of Dasgupta and Kallianpur [2] follow from this theorem.

The argument for multiple integrals is similar and gives us

Theorem 2.2.

$$\begin{aligned}
&\sum_{i_1,\ldots,i_p} a_{i_1 \ldots i_p} \Delta B_H(t_{i_1}) \cdots \Delta B_H(t_{i_p}) \\
&\quad = \sum_{k=0}^{[p/2]} C_{p,k} \sum_{i_{2k+1},\ldots,i_p} (\sum_{i_1,\ldots,i_{2k}} a_{i_1,i_2,\ldots,i_{2k},i_{2k+1},\ldots,i_p} \\
&\quad \times \langle f_{i_1}, f_{i_2} \rangle \cdots \langle f_{i_{2k-1}}, f_{i_{2k}} \rangle) \Delta B_H(t_{i_{2k+1}}) \tilde{\otimes} \cdots \tilde{\otimes} \Delta B_H(t_{i_p}) \ . \qquad (14)
\end{aligned}$$

□

The formulas for the moments in theorem 5.1 of Dasgupta and Kallianpur [2] follow from the above theorem.

From theorem 2.2 it is seen that multiple integrals of even or odd orders consist of only even or odd chaos components respectively. This shows multiple integrals of even order are orthogonal to multiple integrals of odd order, but even (resp. odd) order multiple integrals are not in general orthogonal to one another.

Theorem 2.2 can be stated for a finite number of elements from the linear space of fractional Brownian motion instead of the increments of fractional Brownian motion. Suppose $X_1, \ldots, X_n$ are n elements from the linear space of $B_H(t), 0 \le t \le T$ and let g_i be the RKHS image of $X_i, i = 1, \ldots, n$. For symmetric $a_{i_1,\ldots,i_p}$ we have the following decomposition

$$\begin{aligned}
&\sum_{i_1,\ldots,i_p} a_{i_1\ldots i_p} X_{i_1} \cdots X_{i_p} \\
&= \sum_{k=0}^{[p/2]} C_{p,k} \sum_{i_{2k+1},\ldots,i_p} \Big(\sum_{i_1,\ldots,i_{2k}} a_{i_1,i_2,\ldots,i_{2k},i_{2k+1},\ldots,i_p} \langle g_{i_1}, g_{i_2}\rangle \cdots \langle g_{i_{2k-1}}, g_{i_{2k}}\rangle\Big) \\
&\quad \times X_{i_{2k+1}} \tilde{\otimes} \cdots \tilde{\otimes} X_{i_p} \ . \qquad (15)
\end{aligned}$$

If we consider a complete orthonormal system of g_i's from the RKHS then this would lead to an analogue of theorem 5.1 of Johnson and Kallianpur [7] if we replace $L^2(R_+^p)$ of their consideration by a p-fold symmetric tensor product of the RKHS of fractional Brownian motion.

3. Chaos components of general multiple fractional integrals

We now generalise (14) to multiple fractional integrals of more general functions which are defined in section 5 of Dasgupta and Kallianpur [2]. Consider the elementary function

$$F_n(x_1, \ldots, x_p) = \sum_{i_1,\ldots,i_p} a_{i_1\ldots i_p} 1_{\Delta_{i_1}}(x_1) \cdots 1_{\Delta_{i_p}}(x_p)$$

The chaos components of the MFI of F are identified in theorem 2.2. For notational convenience we introduce

$$\psi(ds, dt) = \frac{c\, ds\, dt}{|s-t|^{1-2\alpha}} \ , \qquad (16)$$

where

$$c = H(2H-1), \quad \alpha = H - 1/2 \ . \qquad (17)$$

Noticing that (from formulas (14) and (17) of Dasgupta and Kallianpur [2])

$$\langle f_i, f_j\rangle = \int_0^T \int_0^T 1_{\Delta_i}(s) 1_{\Delta_j}(t) \psi(dsdt) \ ,$$

the chaos component

$$\begin{aligned}
&C_{p,k} \sum_{i_{2k+1},\ldots,i_p} \Big(\sum_{i_1,\ldots,i_{2k}} a_{i_1,i_2,\ldots,i_{2k},i_{2k+1},\ldots,i_p} \langle f_{i_1}, f_{i_2}\rangle \cdots \langle f_{i_{2k-1}}, f_{i_{2k}}\rangle\Big) \\
&\quad \times \Delta B_H(t_{i_{2k+1}}) \tilde{\otimes} \cdots \tilde{\otimes} \Delta B_H(t_{i_p})
\end{aligned}$$

can be written as

$$C_{p,k}\int_{u_{2k+1}=0}^{T}\cdots\int_{u_p=0}^{T}\Big\{\int_{s_1=0}^{T}\int_{t_1=0}^{T}\cdots\int_{s_k=0}^{T}\int_{t_k=0}^{T}F_n(s_1,t_1,\ldots,s_k,t_k,u_{2k+1},\ldots,u_p)\times\psi(ds_1dt_1)\cdots\psi(ds_kdt_k)\Big\}d\Big(B_H(u_{2k+1})\tilde{\otimes}\cdots\tilde{\otimes}B_H(u_p)\Big)\ ,$$

so that theorem 2.2 reads as

$$\int_0^T\cdots\int_0^T F_n(x_1,\ldots,x_p)dB_H(x_1)\cdots dB_H(x_p) = \sum_{k=0}^{[p/2]}C_{p,k}\int_{u_{2k+1}=0}^{T}\cdots\int_{u_p=0}^{T}\times\Big\{\int_{s_1=0}^{T}\int_{t_1=0}^{T}\cdots\int_{s_k=0}^{T}\int_{t_k=0}^{T}F_n(s_1,t_1,\ldots,s_k,t_k,u_{2k+1},\ldots,u_p)\times\psi(ds_1dt_1)\cdots\psi(ds_kdt_k)\Big\}d\Big(B_H(u_{2k+1})\tilde{\otimes}\cdots\tilde{\otimes}B_H(u_p)\Big)\ . \quad (18)$$

We introduce some notation to bring out the formal similarity of the above result to the expansion of a multiple Stratonovich integral in terms of multiple Wiener integrals. Let us denote left hand side of (18) by $\delta_p(F)$ and the tensor product integrals on the RHS are denoted by $\mathscr{I}_{p-2k}(.)$ respectively. For an elementary F_n define the k-th trace of F_n by

$$Tr_{\psi}^{k}F_n(u_{2k+1},\ldots,u_p) = \int_{s_1=0}^{T}\int_{t_1=0}^{T}\cdots\int_{s_k=0}^{T}\int_{t_k=0}^{T}F_n(s_1,t_1,\ldots,s_k,t_k,u_{2k+1},\ldots,u_p)\times\psi(ds_1dt_1)\cdots\psi(ds_kdt_k)\ . \quad (19)$$

Then (18) reads as

$$\delta_p(F_n)=\sum_{k=0}^{[p/2]}C_{p,k}\mathscr{I}_{p-2k}(Tr_{\psi}^{k}F_n)\ , \quad (20)$$

Our object is to extend (20) to functions in $L^2(\mu_p)$. From section 5 of Dasgupta and Kallianpur [2] we know that if F_n converges to $F\in L^2(\mu_p)$, with μ_p defined as in (24), in the $L^2(\mu_p)$ norm then $\delta_p(F_n)$ converges to $\delta_p(F)$ in mean square. So we now have to show that if F_n converges to $F\in L^2(\mu_p)$ in the $L^2(\mu_p)$ norm then the chaos components on the right of (20) individually converge in mean square. We proceed to show this.

Repeated application of Cauchy-Schwarz inequality shows that (see formulas (42) to (45) of Dasgupta and Kallianpur [2])

$$
\begin{aligned}
&E\{C_{p,k}\mathscr{I}_{p-2k}(Tr_{\psi}^{k}F_n)\}^2 \\
&\quad= C_{p,k}^2(p-2k)!\int_{u_{2k+1}=0}^{T}\int_{v_{2k+1}=0}^{T}\cdots\int_{u_p=0}^{T}\int_{v_p=0}^{T} \\
&\qquad\times\Big\{\int_{s_1=0}^{T}\int_{t_1=0}^{T}\cdots\int_{s_k=0}^{T}\int_{t_k=0}^{T}F_n(s_1,t_1,\dots,s_k,t_k,u_{2k+1},\dots,u_p) \\
&\qquad\quad\times\psi(ds_1dt_1)\cdots\psi(ds_kdt_k)\Big\} \\
&\qquad\times\Big\{\int_{s_1=0}^{T}\int_{t_1=0}^{T}\cdots\int_{s_k=0}^{T}\int_{t_k=0}^{T}F_n(s_1,t_1,\dots,s_k,t_k,v_{2k+1},\dots,v_p) \\
&\qquad\quad\times\psi(ds_1dt_1)\cdots\psi(ds_kdt_k)\Big\}\psi(du_{2k+1}dv_{2k+1})\cdots\psi(du_pdv_p) \\
&\quad\le C_{p,k}^2(p-2k)!c^{p-2k}(1/(2\alpha)^{p-2k})\int_{u_{2k+1}=0}^{T}\cdots\int_{u_p=0}^{T} \\
&\qquad\Big\{\int_{s_1=0}^{T}\int_{t_1=0}^{T}\cdots\int_{s_k=0}^{T}\int_{t_k=0}^{T}F_n(s_1,t_1,\dots,s_k,t_k,v_{2k+1},\dots,v_p) \\
&\qquad\times\psi(ds_1dt_1)\cdots\psi(ds_kdt_k)\Big\}^2\times[u_{2k+1}^{2\alpha}+(T-u_{2k+1})^{2\alpha}]\cdots \\
&\qquad\times[u_p^{2\alpha}+(T-u_p)^{2\alpha}]du_{2k+1}\cdots du_p \\
&\quad= C_{p,k}^2(p-2k)!c^{p-2k}(1/(2\alpha)^{p-2k})\int_{u_{2k+1}=0}^{T}\cdots\int_{u_p=0}^{T}(Tr_{\psi}^{k}F_n)^2 \\
&\qquad\times[u_{2k+1}^{2\alpha}+(T-u_{2k+1})^{2\alpha}]\cdots[u_p^{2\alpha}+(T-u_p)^{2\alpha}]du_{2k+1}\cdots du_p \qquad (21) \\
&\quad\le\int_0^T\cdots\int_0^T F_n(x_1,\dots,x_p)^2\nu_{p-2k}(dx_1,\dots,dx_p)\ , \qquad (22)
\end{aligned}
$$

where

$$
\begin{aligned}
&\nu_{p-2k}(dx_1,dx_2,\dots,dx_{2k-1},dx_{2k},dx_{2k+1},\dots,dx_p) \\
&\quad= c^p\{2T^{2\alpha+1}/(2\alpha)(2\alpha+1)\}^k(C_{p,k})^2(p-2k)!(1/(2\alpha)^{p-2k}) \\
&\qquad\times|x_1-x_2|^{2\alpha-1}\cdots|x_{2k-1}-x_{2k}|^{2\alpha-1}[x_{2k+1}^{2\alpha}+(T-x_{2k+1})^{2\alpha}]\cdots \\
&\qquad\times[x_p^{2\alpha}+(T-x_p)^{2\alpha}]dx_1dx_2\cdots dx_{2k-1}dx_{2k}dx_{2k+1}\cdots dx_p
\end{aligned}
$$

$$= c^{p-k}\{2T^{2\alpha+1}/(2\alpha)(2\alpha+1)\}^k (C_{p,k})^2 (p-2k)!(1/(2\alpha)^{p-2k})$$
$$\times \psi(dx_1 dx_2)\cdots\psi(dx_{2k-1}dx_{2k}) \times [x_{2k+1}^{2\alpha} + (T-x_{2k+1})^{2\alpha}]\cdots$$
$$\times [x_p^{2\alpha} + (T-x_p)^{2\alpha}] dx_{2k+1}\cdots dx_p \quad . \tag{23}$$

Now if F_n converges to F in $L^2(\mu_p)$ where μ_p is the measure on $[0, T]^p$ defined by

$$\mu_p((dx_1,\ldots,dx_p) = \sum_{k=0}^{[p/2]} \nu_{p-2k}(dx_1,\ldots,dx_p) \tag{24}$$

then from (22) and (24) it is seen that $\mathscr{I}_{p-2k}(Tr_\psi^k F_n)$ is Cauchy in mean square for each k. This mean square limit is the $p-2k$-th chaos component of $\delta_p(F)$. We shall briefly discuss what happens to $Tr_\psi^k F_n$ as $n \to \infty$.

For the existence of the limiting traces we need to show that $||F_n - F||_{\mu_p}^2 \to 0$ implies $Tr_\psi^k F_n(u_{2k+1},\ldots,u_p)$ is Cauchy in the L^2 norm of the measure (see (21))

$$\Gamma(du_{2k+1}\cdots du_p) = c^{p-2k}(C_{p,k})^2(p-2k)!(1/(2\alpha)^{p-2k})$$
$$\times [u_{2k+1}^{2\alpha} + (T-u_{2k+1})^{2\alpha}]\cdots[u_p^{2\alpha} + (T-u_p)^{2\alpha}]$$
$$\times du_{2k+1}\cdots du_p \quad .$$

An application of Cauchy-Schwarz inequality shows that

$$\int_0^T \cdots \int_0^T \Big(Tr_\psi^k F_m(u_{2k+1},\ldots,u_p) - Tr_\psi^k F_n(u_{2k+1},\ldots,u_p)\Big)^2$$
$$\times \Gamma(du_{2k+1}\cdots du_p) = \int_0^T\cdots\int_0^T \Big\{\int_{s_1=0}^T \int_{t_1=0}^T \cdots \int_{s_k=0}^T \int_{t_k=0}^T (F_m - F_n)$$
$$\times (s_1, t_1, \cdots, s_k, t_k, u_{2k+1}, \cdots, u_p)$$
$$\times \psi(ds_1 dt_1)\cdots\psi(ds_k dt_k)\Big\}^2 \Gamma(du_{2k+1}\cdots du_p)$$
$$\le c^k\{2T^{2\alpha+1}/(2\alpha)(2\alpha+1)\}^k$$
$$\times \int_{u_{2k+1}=0}^T \cdots \int_{u_p=0}^T \int_{s_1=0}^T \int_{t_1=0}^T \cdots \int_{s_k=0}^T \int_{t_k=0}^T$$
$$\times (F_m - F_n)(s_1, t_1, \cdots, s_k, t_k, u_{2k+1}, \cdots, u_p)^2$$
$$\times \psi(ds_1 dt_1)\ldots\psi(ds_k dt_k)\Gamma(du_{2k+1}\ldots du_p)$$
$$= \int_0^T \cdots \int_0^T (F_m - F_n)(x_1,\ldots,x_p)^2$$
$$\times \nu_{p-2k}(dx_1,\ldots,dx_p)$$

$$\leq \int_0^T \cdots \int_0^T (F_m - F_n)(x_1, \ldots, x_p)^2 \mu_p(dx_1, \ldots, dx_p)$$
$$\to 0 \ . \tag{25}$$

Thus we see that $Tr_\psi^k F_n$ is Cauchy in $L^2(\Gamma)$ and this limit is defined to be $Tr_\psi^k F$. We have seen that $||F_n - F||_{\mu_p}^2 \to 0$ implies $\delta_p(F_n) \to \delta_p(F)$ and $\mathscr{I}_{p-2k}(Tr_\psi^k F_n) \to \mathscr{I}_{p-2k}(Tr_\psi^k F)$ for each k in mean square. Hence generalising (20) we have the following generalization of theorem 2.2.

Theorem 3.1. *For symmetric $F \in L^2(\mu_p)$ we have*

$$\delta_p(F) = \sum_{k=0}^{[p/2]} C_{p,k} \mathscr{I}_{p-2k}(Tr_\psi^k F) \ . \tag{26}$$

□

The formulas for the moments in theorem 5.2 of Dasgupta and Kallianpur [2] follow from the above theorem. Further the remarks following theorem 2.2 about orthogonality also hold in the same way.

4. Representation of chaos components of multiple fractional integrals of elementary functions

We want to write the $p - 2k$th chaos component on the right hand side of (26) of theorem 3.1 in terms of multiple fractional integrals of orders less than or equal to $p - 2k$. In this section we do this for the chaos components on the right hand side of (14) of theorem 2.2. The general case needs a limiting argument dealt with in the next section.

The main tool is formula (4.9) of lemma 4.2 of Johnson and Kallianpur [7]. There the authors have represented a multiple Wiener integral in terms of p-forms in i.i.d. normal random variables. In our context we started with a p-form (as in (5))

$$\sum_{i_1,\ldots,i_p=1}^{n'} b_{i_1,\ldots,i_p} \xi_{i_1} \cdots \xi_{i_p} \tag{27}$$

and obtained its $p - 2k$th chaos component consisting of sums of products of Hermite polynomials of total degree $p - 2k$, which is same as

$$I_{p-2k}(Tr^k \sum_{i_1,\ldots,i_p=1}^{n'} b_{i_1,\ldots,i_p} \phi_{i_1} \otimes \cdots \otimes \phi_{i_p})$$
$$= I_{p-2k}(\sum_{i_1,\ldots,i_k=1}^{n'} \sum_{i_{2k+1},\ldots,i_p=1}^{n'} b_{i_1,i_1,\ldots i_k,i_k,i_{2k+1},\ldots,i_p} \phi_{i_{2k+1}} \otimes \cdots \otimes \phi_{i_p}) \tag{28}$$

of Johnson and Kallianpur [7] where one has to remember that in our context I_{p-2k} has meaning only as a sum of products of Hermite polynomials of total degree $p-2k$. For us it is convenient to write (28) in terms of tensor products following theorem 6.6.4 of Kallianpur [8]. This gives

$$\begin{aligned} & I_{p-2k}(\sum_{i_1,\ldots,i_k=1}^{n'} \sum_{i_{2k+1},\ldots,i_p=1}^{n'} b_{i_1,i_1,\ldots i_k,i_k,i_{2k+1},\ldots,i_p} \phi_{i_{2k+1}} \otimes \cdots \otimes \phi_{i_p}) \\ & = \sum_{i_1,\ldots,i_k=1}^{n'} \sum_{i_{2k+1},\ldots,i_p=1}^{n'} b_{i_1,i_1,\ldots i_k,i_k,i_{2k+1},\ldots,i_p} \xi_{i_{2k+1}} \tilde{\otimes} \cdots \tilde{\otimes} \xi_{i_p} \quad . \end{aligned} \tag{29}$$

Johnson and Kallianpur [7] also represent a multiple Wiener integral of the above form as a sum of p-forms in their formula (4.7) of lemma 4.2. It is this formula that will give us multiple fractional integrals for the chaos components of (27). We do this now keeping n' fixed and then we will make $n' \to \infty$.

Formula (4.7) from Johnson and Kallianpur [7] reads as

$$\begin{aligned} & I_p(\sum_{i_1,\ldots,i_p=1}^{n'} a_{i_1,\ldots,i_p} \phi_{i_1} \otimes \cdots \otimes \phi_{i_p}) \\ & = \sum_{k=0}^{[p/2]} (-1)^k C_{p,k} \sum_{i_{2k+1},\ldots,i_p=1}^{n'} \sum_{i_1,\ldots,i_k=1}^{n'} a_{i_1,i_1,\ldots,i_k,i_k,i_{2k+1},\ldots,i_p} \xi_{i_{2k+1}} \cdots \xi_{i_p} \quad . \end{aligned}$$

Following this we can write from (29)

$$\begin{aligned} & \sum_{i_1,\ldots,i_k=1}^{n'} \sum_{i_{2k+1},\ldots,i_p=1}^{n'} b_{i_1,i_1,\ldots i_k,i_k,i_{2k+1},\ldots,i_p} \xi_{i_{2k+1}} \tilde{\otimes} \cdots \tilde{\otimes} \xi_{i_p} \\ & = \sum_{m=0}^{[p-2k/2]} (-1)^m C_{p-2k,m} \sum_{i_{2m+1},\ldots,i_p=1}^{n'} \sum_{i_{k+1},\ldots,i_m=1}^{n'} \\ & \quad \times (\sum_{i_1,\ldots,i_k=1}^{n'} b_{i_1,i_1,\ldots,i_k,i_k,i_{k+1},i_{k+1},\ldots,i_{k+m},i_{k+m},i_{2(k+m)+1},\ldots,i_p}) \xi_{i_{2m+1}} \cdots \xi_{i_p} \\ & = \sum_{m=k}^{k+[p-2k/2]} (-1)^{m-k} C_{p-2k,m-k} \\ & \quad \times \sum_{i_{2m+1},\cdots,i_p=1}^{n'} \sum_{i_1,\cdots,i_m=1}^{n'} b_{i_1,i_1,\cdots,i_m,i_m,i_{2m+1},\cdots,i_p} \xi_{i_{2m+1}} \cdots \xi_{i_p} \quad . \end{aligned} \tag{30}$$

Recall that the $b_{i_1,\ldots,i_p}$'s were obtained from

$$\sum_{l_1,\ldots,i_p} a_{l_1,\ldots,l_p} \Delta B_H(t_{l_1}) \cdots \Delta B_H(t_{l_1})$$
$$= \sum_{i_1=1}^{n'} \cdots \sum_{i_p=1}^{n'} \{ \sum_{l_1,\ldots,l_p} a_{l_1,\ldots,l_p} c_{i_1}(t_{l_1}) \cdots c_{i_p}(t_{l_p}) \} \xi_{i_1} \cdots \xi_{i_p}$$
$$= \sum_{i_1,\ldots,i_p=1}^{n'} b_{i_1,\ldots,i_p} \xi_{i_1} \cdots \xi_{i_p} \quad .$$

Now putting the expression for b_i's and interchanging order of summation in (30) we get

$$\sum_{i_1,\ldots,i_k=1}^{n'} \sum_{i_{2k+1},\ldots,i_p=1}^{n'} b_{i_1,i_1,\ldots i_k,i_k,i_{2k+1},\ldots,i_p} \xi_{i_{2k+1}} \tilde{\otimes} \cdots \tilde{\otimes} \xi_{i_p}$$
$$= \sum_{m=k}^{k+[p-2k/2]} (-1)^{m-k} C_{p-2k,m-k} \sum_{l_1,\ldots,l_p} a_{l_1,\ldots,l_p} \langle f^{n'}_{l_1}, f^{n'}_{l_2} \rangle \cdots \langle f^{n'}_{l_{2m-1}}, f^{n'}_{l_{2m}} \rangle$$
$$\times \{ \sum_{j_{l_{2m+1}}=1}^{n'} c_{j_{l_{2m+1}}}(t_{l_{2m+1}}) \xi_{j_{l_{2m+1}}} \} \cdots \{ \sum_{j_{l_p}=1}^{n'} c_{j_{l_p}}(t_{l_p}) \xi_{j_{l_p}} \}$$
$$= \sum_{m=k}^{k+[p-2k/2]} (-1)^{m-k} C_{p-2k,m-k}$$
$$\times \sum_{l_1,\ldots,l_p} a_{l_1,\ldots,l_p} \langle f^{n'}_{l_1}, f^{n'}_{l_2} \rangle \cdots \langle f^{n'}_{l_{2m-1}}, f^{n'}_{l_{2m}} \rangle \Delta B^{n'}_H(t_{l_{2m+1}}) \cdots \Delta B^{n'}_H(t_{l_p}) \quad . \tag{31}$$

The right hand side of (31) $a.s.$ goes to

$$\sum_{m=k}^{k+[p-2k/2]} (-1)^{m-k} C_{p-2k,m-k}$$
$$\times \sum_{l_1,\ldots,l_p} a_{l_1,\ldots,l_p} \langle f_{l_1}, f_{l_2} \rangle \cdots \langle f_{l_{2m-1}}, f_{l_{2m}} \rangle \Delta B_H(t_{l_{2m+1}}) \cdots \Delta B_H(t_{l_p}) \quad .$$

But we know from (14) of theorem 2.2 (as in the passage from (12) to (13)) that the left hand side of (31) goes to

$$\sum_{i_{2k+1},\ldots,i_p} (\sum_{i_1,\ldots,i_{2k}} a_{i_1,i_2,\ldots,i_{2k},i_{2k+1},\ldots,i_p} \langle f_{i_1}, f_{i_2} \rangle \cdots \langle f_{i_{2k-1}}, f_{i_{2k}} \rangle)$$
$$\times \Delta B_H(t_{i_{2k+1}}) \tilde{\otimes} \cdots \tilde{\otimes} \Delta B_H(t_{i_p}) \tag{32}$$

in mean square. So we have the following representation of the chaos components of theorem 2.2

Theorem 4.1.

$$\sum_{i_{2k+1},\dots,i_p} \Big(\sum_{i_1,\dots,i_{2k}} a_{i_1,i_2,\dots,i_{2k},i_{2k+1},\dots,i_p} \langle f_{i_1}, f_{i_2}\rangle \cdots \langle f_{i_{2k-1}}, f_{i_{2k}}\rangle \Big)$$
$$\times \Delta B_H(t_{i_{2k+1}}) \tilde{\otimes} \cdots \tilde{\otimes} \Delta B_H(t_{i_p})$$
$$= \sum_{m=k}^{k+[p-2k/2]} (-1)^{m-k} C_{p-2k,m-k} \sum_{l_1,\dots,l_p} a_{l_1,\dots,l_p} \langle f_{l_1}, f_{l_2}\rangle \cdots$$
$$\times \langle f_{l_{2m-1}}, f_{l_{2m}}\rangle \Delta B_H(t_{l_{2m+1}}) \cdots \Delta B_H(t_{l_p}) \tag{33}$$

□

5. Representation of chaos components of general multiple fractional integrals

For the elementary function

$$F_n(x_1,\dots,x_p) = \sum_{i_1,\dots,i_p} a_{i_1\dots i_p} 1_{\Delta_{i_1}}(x_1) \cdots 1_{\Delta_{i_p}}(x_p)$$

theorem 4.1 reads as

$$\int_{u_{2k+1}=0}^{T} \cdots \int_{u_p=0}^{T} \Big\{ c^k \int_{s_1=0}^{T} \int_{t_1=0}^{T} \cdots$$
$$\times \int_{s_k=0}^{T} \int_{t_k=0}^{T} F_n(s_1,t_1,\dots,s_k,t_k,u_{2k+1},\dots,u_p)$$
$$\times \frac{ds_1 dt_1}{|s_1-t_1|^{1-2\alpha}} \cdots \frac{ds_k dt_k}{|s_k-t_k|^{1-2\alpha}} \Big\} d\Big(B_H(u_{2k+1}) \tilde{\otimes} \cdots \tilde{\otimes} B_H(u_p) \Big)$$
$$= \sum_{m=k}^{k+[p-2k/2]} (-1)^{m-k} C_{p-2k,m-k} \times \int_{u_{2m+1}=0}^{T} \cdots \int_{u_p=0}^{T}$$
$$\times \Big(c^m \int_{s_1=0}^{T} \int_{t_1=0}^{T} \cdots \int_{s_m=0}^{T} \int_{t_m=0}^{T} F_n(s_1,t_1,\dots,s_m,t_m,u_{2m+1},\dots,u_p)$$
$$\times \frac{ds_1 dt_1}{|s_1-t_1|^{1-2\alpha}} \cdots \frac{ds_m dt_m}{|s_m-t_m|^{1-2\alpha}} \Big) dB_H(u_{2m+1}) \cdots dB_H(u_p) \ . \tag{34}$$

We want to extend (34) to more general functions F. We know from section 3 that if F_n converges to F in $L^2(\mu_p)$ then the left hand side of (34) converges to the corresponding tensor product integral of $Tr_{\psi}^{k} F$. By applying Cauchy Schwarz inequality in the manner of (25) it can be seen that if F_n converges to F in $L^2(\mu_p)$ then the functions

$$c^m \int_{s_1=0}^{T} \int_{t_1=0}^{T} \cdots \int_{s_m=0}^{T} \int_{t_m=0}^{T} F_n(s_1, t_1, \ldots, s_m, t_m, u_{2m+1}, \ldots, u_p) \times \frac{ds_1 dt_1}{|s_1 - t_1|^{1-2\alpha}} \cdots \frac{ds_m dt_m}{|s_m - t_m|^{1-2\alpha}}$$

converge to

$$c^m \int_{s_1=0}^{T} \int_{t_1=0}^{T} \cdots \int_{s_m=0}^{T} \int_{t_m=0}^{T} F(s_1, t_1, \cdots, s_m, t_m, u_{2m+1}, \cdots, u_p) \times \frac{ds_1 dt_1}{|s_1 - t_1|^{1-2\alpha}} \cdots \frac{ds_m dt_m}{|s_m - t_m|^{1-2\alpha}}$$

in the norm

$$\sum_{l=m}^{m+[p-2m/2]} c^{p-2m} (1/2\alpha)^{p-2l} (C_{p-2m,l-m})^2 \left\{ \frac{T^{2\alpha+1}}{\alpha(2\alpha+1)} \right\}^{l-m} \times \int_{u_{2m+1}=0}^{T} \int_{v_{2m+1}=0}^{T} \cdots \int_{u_l=0}^{T} \int_{v_l=0}^{T} \int_{x_{2l+1}=0}^{T} \cdots \int_{x_p=0}^{T} c^{2m} \left\{ \int_{s_1=0}^{T} \int_{t_1=0}^{T} \cdots \times \int_{s_m=0}^{T} \int_{t_m=0}^{T} F(s_1, t_1, \ldots, s_m, t_m, u_{2m+1}, v_{2m+1}, \ldots, u_{2l}, v_{2l}, x_{2l+1}, \ldots, x_p) \frac{ds_1 dt_1}{|s_1 - t_1|^{1-2\alpha}} \cdots \frac{ds_m dt_m}{|s_m - t_m|^{1-2\alpha}} \right\}^2 \times [x_{2l+1}^{2\alpha} + (T - x_{2l+1})^{2\alpha}] \cdots [x_p^{2\alpha} + (T - x_p)^{2\alpha}] dx_{2l+1} \cdots dx_p \times \frac{du_m dv_m}{|u_m - v_m|^{1-2\alpha}} \cdots \frac{du_l dv_l}{|u_l - v_l|^{1-2\alpha}} \ . \tag{35}$$

By theorem 5.2 of Dasgupta and Kallianpur [2] this shows that the terms on the right hand side of (34) converge to the corresponding multiple fractional integrals. Thus adopting the notation introduced in section 3 for F symmetric and in $L^2(\mu_p)$ we have the following relation between the chaos components and the multiple fractional integrals of lower orders

Theorem 5.1.

$$\mathscr{I}_{p-2k}(Tr_\psi^k F) = \sum_{m=k}^{k+[p-2k/2]} (-1)^{m-k} C_{p-2k,m-k} \delta_{p-2m}(Tr_\psi^m F) \ . \tag{36}$$

□

6. Series expansions

In sections 2 and 3 it was seen that multiple fractional integrals of different orders are in general not orthogonal. In this section we consider tensor product integrals which are orthogonal and then consider infinite series in those.

In theorem 3.1 we considered the relation between the two types of integrals $\delta_p(F)$ and $\mathscr{I}_{p-2k}(Tr_\psi^k F)$. To define the first one it was found necessary to restrict F to $L^2(\mu_p)$. However it can be seen from (22) that for symmetric $F \in L^2(\nu_p)$ the integral $\mathscr{I}_p(F)$ exists. These integrals have nice properties and we summarise them below:

(i)$E\mathscr{I}_p(F)\mathscr{I}_q(F) = 0, \quad p \neq q,$
(ii) $E\mathscr{I}_p(F) = 0,$

$$\begin{aligned}\text{(iii)}E\Big\{\mathscr{I}_p(F)\Big\}^2 &= \int_{s_1=0}^{T} \cdots \int_{s_p=0}^{T} \int_{t_1=0}^{T} \int_{t_p=0}^{T} F(s_1, \cdots, s_p)F(t_1, \ldots, t_p) \\ &\quad \times\psi(ds_1, dt_1) \cdots \psi(ds_p, dt_p) \\ &= \langle F, F\rangle_\psi \quad . \end{aligned} \tag{37}$$

Now if $F_p \in L^2(\nu_p)$, $p = 1, 2, \ldots$ are such that

$$\sum_{p=1}^{\infty} ||F_p||_{\nu_p}^2 < \infty \quad , \tag{38}$$

then the sum

$$\sum_{p=1}^{\infty} \mathscr{I}_p(F_p) \tag{39}$$

converges in mean square, has mean zero, and second moment

$$E\Big\{\sum_{p=1}^{\infty} \mathscr{I}_p(F_p)\Big\}^2 = \sum_{p=1}^{\infty} \langle F_p, F_p\rangle_\psi \quad . \tag{40}$$

Suppose $f \in L^2(\nu_1)$. Then $f(x_1)f(x_2)\cdots f(x_p)$ denoted simply by $f^{\otimes p}$ belongs to $L^2(\nu_p)$ and the following series

$$\sum_{p=0}^{\infty} \frac{\lambda^p}{p!} \mathscr{I}_p(f^{\otimes p}) \tag{41}$$

converges in mean square where λ is a scalar. We are considering this series to show that

$$e^{\mathscr{I}(f)-(1/2)\langle f,f\rangle_\psi} = \sum_{p=0}^{\infty} \frac{1}{p!} \mathscr{I}_p(f^{\otimes p}) \quad . \tag{42}$$

This is done with the help of the following two lemmas.

Lemma 6.1.

$$\mathscr{I}_p(f^{\otimes p}) = \mathscr{I}(f)\tilde{\otimes}\mathscr{I}(f)\cdots\tilde{\otimes}\mathscr{I}(f) \ . \tag{43}$$

Proof. Suppose f_n is a sequence of elementary functions converging to f in the $L^2(\nu_1)$ norm. This gives

$$\mathscr{I}(f_n) \to \mathscr{I}(f) \ . \tag{44}$$

Let us write $f_n = \sum a_i 1_{\Delta_i}$. Then equating the terms corresponding to $k = 0$ of formulas (14) and (20) we have

$$\begin{aligned}\mathscr{I}_p(f_n^{\otimes p}) &= \sum_{i_1,\ldots,i_p} a_{i_1}\ldots a_{i_p}\Delta B_H(t_{i_1})\tilde{\otimes}\cdots\tilde{\otimes}\Delta B_H(t_{i_1}) \\ &= (\sum a_{i_1}\Delta B_H(t_{i_1}))\tilde{\otimes}\cdots\tilde{\otimes}(\sum a_{i_p}\Delta B_H(t_{i_p})) \\ &= \mathscr{I}(f_n)\tilde{\otimes}\mathscr{I}(f_n)\cdots\tilde{\otimes}\mathscr{I}(f_n) \ .\end{aligned} \tag{45}$$

Now $f_n^{\otimes p}$ converges to $f^{\otimes p}$ in the $L^2(\nu_p)$ norm showing that the LHS of (45) converges to the LHS of (43). On the other hand (44) shows that the RHS of (45) converges to the RHS of (43). □

Lemma 6.2.

$$\mathscr{I}_p(f^{\otimes p}) = \mathscr{I}_{p-1}(f^{\otimes p-1})\mathscr{I}(f) - \langle f, f\rangle_\psi (p-1)\mathscr{I}_{p-2}(f^{\otimes p-2}) \ . \tag{46}$$

Proof. $\mathscr{I}(f)$ being in the linear space of fractional Brownian motion can be expanded as a series of the form $\sum a_i\xi_i$. In the following argument the interchange of summation is first performed for a truncated part of the above series and then a limiting argument is applied as was done in section 2. We first restate the result of lemma 4.1 of Johnson and Kallianpur [7] in a form suitable for our purpose. For a complete orthonormal system $\{\phi_i\}$ from $L^2(R)$ the following consequence of the Ito decomposition formula holds:

$$\begin{aligned}&\sum_{i_1,\ldots,i_p=1}^{n'} a_{i_1,\ldots,i_p} I_p(\phi_{i_1}\otimes\cdots\otimes\phi_{i_p}) \\ &\quad = \sum_{i_1,\ldots,i_p=1}^{n'} a_{i_1,\ldots,i_p} I_{p-1}(\phi_{i_1}\otimes\cdots\otimes\phi_{i_{p-1}})\xi_{i_p} \\ &\qquad -(p-1)\sum_{i_1,\ldots,i_p=1}^{n'} a_{i_1,i_1,i_2\ldots,i_{p-1}} I_{p-2}(\phi_{i_2}\otimes\cdots\otimes\phi_{i_{p-1}}) \ ,\end{aligned} \tag{47}$$

where I_p denotes a multiple Wiener integral. Referring to the above formula in terms of tensor products of ξ_i's (as we did in (11)) consider the LHS of (46):

$$\begin{aligned}
&\mathscr{I}(f)\tilde{\otimes}\cdots\tilde{\otimes}\mathscr{I}(f)\\
&\quad= (\sum a_{i_1}\xi_{i_1})\tilde{\otimes}\cdots\tilde{\otimes}(\sum a_{i_p}\xi_{i_p})\\
&\quad= \sum a_{i_1}\cdots a_{i_p}\xi_{i_1}\tilde{\otimes}\cdots\tilde{\otimes}\xi_{i_p}\\
&\quad= \sum a_{i_1}\cdots a_{i_p}(\xi_{i_1}\tilde{\otimes}\cdots\tilde{\otimes}\xi_{i_{p-1}})\xi_{i_p} - (p-1)\\
&\qquad\times\sum a_{i_1}a_{i_1}a_{i_2}\cdots a_{i_{p-1}}(\xi_{i_2}\tilde{\otimes}\cdots\tilde{\otimes}\xi_{i_{p-2}})\\
&\quad= \Big\{(\sum a_{i_1}\xi_{i_1})\tilde{\otimes}\cdots\tilde{\otimes}(\sum a_{i_{p-1}}\xi_{i_{p-1}})\Big\}(\sum a_{i_p}\xi_{i_p})\\
&\qquad -(p-1)(\sum a_{i_1}a_{i_1})\Big\{(\sum a_{i_2}\xi_{i_2})\tilde{\otimes}\cdots\tilde{\otimes}(\sum a_{i_{p-1}}\xi_{i_{p-1}})\\
&\quad= \Big\{\underbrace{\mathscr{I}(f)\tilde{\otimes}\cdots\tilde{\otimes}\mathscr{I}(f)}_{p-1}\Big\}\mathscr{I}(f) - (p-1)\langle f, f\rangle_\psi\underbrace{\mathscr{I}(f)\tilde{\otimes}\cdots\tilde{\otimes}\mathscr{I}(f)}_{p-2}\\
&\quad= \mathscr{I}_{p-1}(f^{\otimes p-1})\mathscr{I}(f) - \langle f, f\rangle_\psi(p-1)\mathscr{I}_{p-2}(f^{\otimes p-2}) \;. \qquad (48)
\end{aligned}$$

□

We can now prove

Theorem 6.1. *For $f \in L^2(\nu_1)$ we have*

$$e^{\mathscr{I}(f)-(1/2)\langle f,f\rangle_\psi} = \sum_{p=0}^{\infty}\frac{1}{p!}\mathscr{I}_p(f^{\otimes p}) \;. \qquad (49)$$

Proof. Consider

$$F(\lambda) = 1 + \lambda\mathscr{I}(f) + \sum_{p=2}^{\infty}\frac{\lambda^p}{p!}\mathscr{I}_p(f^{\otimes p}) \;.$$

Differentiating the above and using (48) we get

$$\begin{aligned}
F'(\lambda) &= \mathscr{I}(f) + \sum_{p=2}^{\infty}\frac{\lambda^{p-1}}{(p-1)!}\mathscr{I}_p(f^{\otimes p})\\
&= \mathscr{I}(f) + \sum_{p=2}^{\infty}\frac{\lambda^{p-1}}{(p-1)!}\mathscr{I}_{p-1}(f^{\otimes p-1})\mathscr{I}(f)
\end{aligned}$$

$$-\langle f, f\rangle_\psi \sum_{p=2}^{\infty} \frac{(p-1)\lambda^{p-1}}{(p-1)!} \mathscr{I}_{p-2}(f^{\otimes p-2})$$
$$= F(\lambda)\mathscr{I}(f) - \lambda\langle f, f\rangle_\psi F(\lambda) \quad . \tag{50}$$

From (50) it now easily follows that

$$F(\lambda) = e^{\lambda \mathscr{I}(f) - (\lambda^2/2)\langle f, f\rangle_\psi}.$$

Putting $\lambda = 1$ gives the desired result. □

7. Strong laws

In this section we consider a strong law proved by Gladyshev [4]. Gladyshev's result is for a class of Gaussian processes that includes fractional Brownian motion as a special case. For simplicity we restrict ourselves to fractional Brownian motion only. In this case Gladyshev's result can be recast in our language of multiple fractional integrals and the second moments of multiple fractional integrals play an important role in the proof. We shall first discuss Gladyshev's result using our formulas of section 2 and then use the techniques to prove other strong laws. We should mention that the same techniques apply in the same way to prove similar other strong laws for the class of Gaussian processes considered by Gladyshev.

Let us consider a partition $0, 1/2^n, 2/2^n, \ldots, (2^n-1)/2^n, 1$ of $[0, 1]$. Then Gladyshev's result is that

$$(1/2^n)^{1-2H} \sum_{k=1}^{2^n} \left(B_H(k/2^n) - B_H((k-1)/2^n)\right)^2 \to 1 \; a.s. \tag{51}$$

In our notation the random variables in (51) are double integrals of the functions

$$g_n(x, y) = \sum_{k=1}^{2^n} (1/2^n)^{1-2H} 1_{\Delta_k}(x) 1_{\Delta_k}(y) \tag{52}$$

w.r.t. $B_H(t)$, i.e. $\delta_2(g_n)$. It can be checked that $E\delta_2(g_n) = 1$. The main idea is to find $Var(\delta_2(g_n))$ and apply Borel-Cantelli lemma. From our theorem 2.1 we know

$$Var(\delta_2(g_n)) = 2|| \sum (1/2^n)^{1-2H} f_k \otimes f_k||^2 \tag{53}$$

where f_k is the RKHS image of $B_H(k/2^n) - B_H((k-1)/2^n)$. Gladyshev provides the following bounds for the various elements of (53):

$$|d_{jk,}| = |(1/2^n)^{1-2H}\langle f_j, f_k\rangle| \le \frac{const.}{2^n(j-k-1)^{2-2H}}\ for\ j \ge k+2,$$
$$|d_{k,k}| \le const./2^n$$
$$|d_{k+1,k}| \le const./2^n \quad . \tag{54}$$

Notice that since

$$(1/2^n)^{2-4H}\langle f_j \otimes f_k, f_j \otimes f_k\rangle = (1/2^n)^{2-4H}\langle f_j, f_k\rangle^2 = d_{j,k}^2 \quad , \tag{55}$$

our second moment (53) can be expressed in terms of $d_{j,k}$'s and this gives

$$\begin{aligned} 2\|\Sigma(1/2^n)^{1-2H} f_k \otimes f_k\|^2 &\le const. \sum_{j,k=1,j\ge k}^{2^n} d_{j,k}^2 \\ &\le const. \left\{1/2^n + (1/2^n)^{4-4H}\right\} \\ &\quad \text{(proved in Gladyshev [4])} \quad . \end{aligned} \tag{56}$$

After this Borel-Cantelli lemma can be applied to obtain the result.

Using the previous discussion we now prove the following strong law

Theorem 7.1.

$$(1/2^n)^{1-4H}\sum_{k=1}^{2^n}\left(B_H(k/2^n) - B_H((k-1)/2^n)\right)^4 \to 3\ a.s. \tag{57}$$

Proof. Using our notation of multiple fractional integrals of sections 2 and 3 we see that the random variables in (57) are multiple fractional integrals of

$$g_n(x_1,x_2,x_3,x_4) = \sum_{k=1}^{2^n}(1/2^n)^{1-4H} 1_{\Delta_k}(x_1)1_{\Delta_k}(x_2)1_{\Delta_k}(x_3)1_{\Delta_k}(x_4) \tag{58}$$

i.e. $\delta_4(g_n)$. We note that $E\delta_4(g_n) = 3$. By theorem 2.2 we have for $p = 4$

$$\begin{aligned} Var(\delta_4(g_n)) &= (C_{4,0})^2 24||(1/2^n)^{1-4H}\sum f_k \otimes f_k \otimes f_k \otimes f_k||^2 \\ &\quad + (C_{4,1})^2||(1/2^n)^{1-4H}\sum\langle f_k, f_k\rangle f_k \otimes f_k||^2 \quad . \end{aligned} \tag{59}$$

We now work towards finding bounds for the two terms in (59).

Recall that $\langle f_k, f_k\rangle = (1/2^n)^{2H}$. This shows that the second term

$$
\begin{aligned}
(C_{4,1})^2 \quad & ||(1/2^n)^{1-4H} \sum \langle f_k, f_k\rangle f_k \otimes f_k||^2 \\
& \leq const.|| \sum (1/2^n)^{1-2H} f_k \otimes f_k||^2 \\
& \leq const. \left\{ 1/2^n + (1/2^n)^{4-4H} \right\} \\
& \text{by (56)} \quad . \qquad (60)
\end{aligned}
$$

Now the other term in (59) is

$$
\begin{aligned}
& (C_{4,0})^2 24||(1/2^n)^{1-4H} \sum f_k \otimes f_k \otimes f_k \otimes f_k||^2 \\
& \leq const.2^{2n}||(1/2^n)^{2-4H} \sum f_k \otimes f_k \otimes f_k \otimes f_k||^2 \\
& \leq const.2^{2n} \left\{ \sum_{j,k} (1/2^n)^{4-8H} \langle f_j, f_k\rangle^4 \right\} \\
& = const.2^{2n} \left\{ \sum_{j,k} d_{j,k}^4 \right\} \quad . \qquad (61)
\end{aligned}
$$

We need a precise estimate of (61). For this we use the bounds (54) and a technique due to Gladyshev [4]. First of all we have

$$
\sum d_{k,k}^4 \leq const.2^n/2^{4n} \leq const.(1/2^n)^3 \qquad (62)
$$

and

$$
\sum d_{k,k+1}^4 \leq const.(1/2^n)^3 \text{ similarly} \quad . \qquad (63)
$$

Now from the first inequality in (54) we get

$$
\begin{aligned}
& \sum_{j,k=1, j\geq k+2}^{2^n} d_{j,k}^4 \leq const.(1/2^n)^4 \sum_{j,k=1, j\geq k+2}^{2^n} \frac{1}{(j-k-1)^{8-8H}} \\
& = const.(1/2^n)^4 \sum_{t=1}^{2^n-2} \frac{2^n-2-t}{t^{8-8H}} \\
& < const.(1/2^n)^4 2^n \sum_{t=1}^{2^n-2} \frac{1}{t^{8-8H}} \\
& < const.(1/2^n)^3 \left(1 + \int_1^{2^n} \frac{dt}{t^{8-8H}}\right) \\
& \leq const.(1/2^n)^3 (1/2^n)^{7-8H} \\
& = const.(1/2^n)^{10-8H} \quad . \qquad (64)
\end{aligned}
$$

(62), (63) and (64) give from (61) that

$$
\begin{aligned}
&(C_{4,0})^2 24||(1/2^n)^{1-4H} \sum f_k \otimes f_k \otimes f_k \otimes f_k||^2 \\
&\quad \leq const.2^{2n} \left\{(1/2^n)^3 + (1/2^n)^{10-8H}\right\} \\
&\quad = const. \left\{(1/2^n) + (1/2^n)^{8-8H}\right\} \quad . \qquad (65)
\end{aligned}
$$

Combining (59) with (60) and (65) we get

$$
\begin{aligned}
Var(\delta_4(g_n)) &\leq const. \left\{1/2^n + (1/2^n)^{8-8H}\right\} \\
&\quad + const. \left\{1/2^n + (1/2^n)^{4-4H}\right\} \quad . \qquad (66)
\end{aligned}
$$

Now we are ready to apply the Borel-Cantelli lemma. Consider $0 < \beta < 1$ such that $\beta - 1 + (4 - 4H) > 0$. Then

$$
\begin{aligned}
&\sum_{n=1}^{\infty} P\left(|\delta_4(g_n) - E\delta_4(g_n)| > 2^{n(\beta-1)/2}\right) \\
&\quad \leq \sum_{n=1}^{\infty} 2^{n(1-\beta)} Var(\delta_4(g_n)) \\
&\quad \leq \sum_{n=1}^{\infty} 2^{n(1-\beta)} const. \left\{1/2^n + (1/2^n)^{8-8H}\right\} \\
&\qquad + \sum_{n=1}^{\infty} 2^{n(1-\beta)} const. \left\{1/2^n + (1/2^n)^{4-4H}\right\} \\
&\quad < \infty \quad . \qquad (67)
\end{aligned}
$$

By an application of Borel-Cantelli lemma the proof is complete. □

Remarks. From the proof it is clear that similar strong laws can be proved for higher powers of increments of fractional Brownian motion. The technique of the proof will be the same, however the formulas get more involved. So we merely state the form of the strong law:

$$
(1/2^n)^{1-2pH} \sum_{k=1}^{2^n} \left\{B_H(k/2^n) - B_H((k-1)/2^n)\right\}^{2p} \to \frac{(2p)!}{p!2^p} a.s. \qquad (68)
$$

where p is a positive integer.

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