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Probability Theory and Related Fields


# Freezing transition in the Ising model without internal contours

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**Abstract.** We consider the low temperature Ising model in a uniform magnetic field $h > 0$ with minus boundary conditions and conditioned on having no internal contours. This simple contour model defines a non-Gibbsian spin state. For large enough magnetic fields ($h > h_c$) this state is concentrated on the single spin configuration of all spins up. For smaller values ($h \leq h_c$), the spin state is non-trivial. At the critical point $h_c \neq 0$ the magnetization jumps discontinuously. Freezing provides also an example of a translation invariant weakly Gibbsian state which is not almost Gibbsian.

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## 1. Introduction

Contour models play an important role in the study of phase transitions for lattice systems. Since Peierls' original argument showing a phase transition in the standard Ising model, the method of contours has been generalized, leading to the modern Pirogov-Sinai theory, which enables one to draw the

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phase diagram of great many classical (and some quantum) equilibrium systems.

As often happens with powerful methods, contour models have gained an interest of their own. Quite independent from specific applications to statistical-mechanical lattice spin systems, contour models also appear as probability measures over geometric objects (paths, compatible contours, polymers,...). Looked upon in this way, the model we study here is conceptually rather simple. We consider Ising-like contours on the $d$-dimensional lattice but we only allow for external contours. The weight $W(\gamma)$ of a contour $\gamma$ is then determined by its length $|\gamma|$ and by the number $v(\gamma)$ of sites inside it: $W(\gamma) \sim \exp[-\beta|\gamma| + hv(\gamma)]$, $\beta, h > 0$. Previously, the model was used to study certain problems in a simplified setting. E.g. in [10], it was used in the study of the Ising model in a random magnetic field. Apart from its use as simple starting point for a more complicated analysis, the model also has an independent interest as stochastic geometric model in the context of image restoration and pattern recognition where the weight of a domain (that cannot contain other domains) not only depends on its surface but also on other morphological characteristics (such as its volume), see [2]. Our main motivation is described below.

Looked upon as a lattice spin system, the state we consider is non-Gibbsian and it is the simplest variant of the non-Gibbsian examples considered in [8]. Not only is our system non-Gibbsian because the measure is not nonnull (i.e. there are forbidden configurations) but more importantly, the question whether a set is closed in by an external contour may depend on behavior far away, and hence the state fails to be quasilocal. Here however we will concentrate on yet another non-Gibbsian property: the existence of a finite non-zero value of the magnetic field where a freezing transition occurs.

One of the important questions in the discussion Gibbs vs. non-Gibbs is to understand what can be saved of the Gibbs formalism in physically interesting (weakly) non-Gibbsian models, see e.g. [1], [15], [5], [4], [16]. In [8] one finds a warning and it is argued that it is perhaps too early to describe the non-Gibbsian pathologies (as in the title of [7]) as merely zero measure events not altering significantly the Gibbsian game. We think it is indeed not clear yet what are the essential ingredients needed to call a Gibbsian restoration of non-Gibbsian states satisfactory. We need examples and our model serves that purpose.

One of the things that cannot happen in the Gibbs formalism is the occurrence of frozen spin states. In fact, as first noticed in [14], one way to recognize non-Gibbsianness is finding such a state when turning on a finite magnetic field. In the present paper we concentrate on proving a phase transition towards a frozen state for our contour model. As we vary the parameters $\beta, h$ in the weight of a contour, the system goes from a nontrivial state into a frozen state (where each spin is up). In contrast with the usual picture for phase transitions where this is excluded (see e.g. [12]), here the free energy is linear in the external field if that field is above the critical value (which depends on the temperature). As a result, the magnetization has a flat piece (and equals one) as a function of the field. This is unheard of in ordinary (Gibbsian) phase transitions where the pressure can develop a flat piece as a function of the density but not vice versa. This phenomenon was studied before in one-dimensional models ([9], [11]) and it is sometimes called an anti-phase transition. As the applied field is lowered beyond the critical value, the magnetization jumps discontinuously to a smaller value.

In the following Section we explain the model and our results in a more precise way. In particular, we explain there the non-Gibbsian features of our field. The rest of the paper is devoted to proofs with a very short introduction to the theory of contour models in Section 3.

## 2. Main results

### *2.1. The model*

To each site $x$ of the $d$-dimensional lattice $\mathbb{Z}^d$ we assign a spin variable $\sigma_x \in \{+1, -1\}$. The set of all such configurations is $K = \{+1, -1\}^{\mathbb{Z}^d}$. Fixing a finite box $V \subset \mathbb{Z}^d$, the Hamiltonian (or energy function) corresponding to the standard Ising model is

$$H_V(\sigma) = -J \sum_{\langle xy\rangle \cap V \neq \emptyset} (\sigma_x\sigma_y - 1) - B \sum_{x\in V} (\sigma_x + 1) \ , \tag{1}$$

where the sum is over nearest neighbor pairs $\langle xy\rangle$, $J \geq 0$ is a coupling constant and $B \geq 0$ plays the role of an external field. As usual, (1) defines the finite volume Gibbs measures. The one we are interested in, $\nu_V = \nu_V^{-,\beta,J,B}$, corresponds to minus boundary conditions, and is defined on $K$ via

$$\nu_V(\sigma) = \exp[-\beta H_V(\sigma)] I[\sigma_y = -1, y \in V^c]/\mathscr{Z}_V \ . \tag{2}$$

The parameter $\beta \geq 0$ is called the inverse temperature, but since it multiplies in (2) the coupling $J$ and the external field $B$, we will simply set $J = 1/2$ and $B = h/(2\beta)$, $h \geq 0$. We now treat $\beta$ and $h$ as independent parameters. The partition function $\mathscr{Z}_V$ in (2) is then given by

$$\mathscr{Z}_V = \sum_{\sigma\in K:\sigma_y=-1, y\in V^c} \exp[\beta/2 \sum_{\langle xy\rangle\cap V\neq\emptyset} (\sigma_x\sigma_y - 1) + h/2 \sum_V (\sigma_x + 1)] \ . \tag{3}$$

The formulae (2) and (3) can be expressed in another way via the so called Ising contours. For convenience think of the square lattice $\mathbb{Z}^2$, and let us agree to draw in $V$ horizontal lines and vertical lines of unit length between neighboring sites with opposite spin values. The collection of all these lines is a disjoint union of contours, each contour being a closed non-self-intersecting polygonal curve. Since we have specified the configuration $\sigma$ to be $-1$ outside $V$, there is a one-to-one relation between specifying the Ising contours and giving the spin configuration. When the reader is actually trying this out, he or she will have to make a decision concerning the rounding of the corners of contours at places where four lines meet. But this is only a matter of convention.

The model we wish to study here is obtained by forbidding in (3) all spin configurations for which some contour $\gamma$ is inside another. We denote by $\mathrm{Int}(\gamma)$ the sites of the lattice which are inside the contour $\gamma$. A configuration $\theta = \{\gamma_1, ..., \gamma_n\}$ of contours will be called an allowed configuration in $V$, iff for all $i = 1, ..., n$ we have $\mathrm{Int}(\gamma_i) \subset V$, and also $\mathrm{Int}(\gamma_i) \cap \mathrm{Int}(\gamma_j) = \emptyset$. We denote it by $\theta \subset V$. In words, the contours of $\theta \subset V$ should be mutually external and having $+1$-spins inside. Let us denote by "$AE$" the corresponding event (All contours are External). The measure we are interested in is then

$$\mu_V(\sigma) \equiv \mu_V^{\beta,h}(\sigma) = \nu_V(\sigma|AE) \ . \tag{4}$$

Writing $K_V \subset K$ for the set of configurations $\sigma_V$ for which $\sigma_V \equiv -1$ on $V^c$ and for which all Ising contours are external, we clearly have $\mu_V(K_V) = 1$ and $\mu_V(\sigma_V) > 0$ for all $\sigma_V \in K_V$. If we denote by $\theta(\sigma_V)$ the (allowed) contour configuration determined by $\sigma_V$, then to each $\theta \subset V$ there is exactly one $\sigma_V \in K_V$ so that $\theta(\sigma_V) = \theta$ and

$$\mu_V^{\beta,h}(\sigma_V) = \prod_{\gamma\in\theta} W(\gamma)/Z(V,\beta,h) \ , \tag{5}$$

where $W(\gamma) = \exp\{-\beta|\gamma| + hv(\gamma)\}$ and

$$Z(V,\beta,h) = \sum_{\theta\subset V}\prod_{\gamma\in\theta} \exp\{-\beta\,|\gamma| + hv\,(\gamma)\} \ , \tag{6}$$

with notations $|\gamma|$ = length of contour, $v(\gamma)$ = volume (number of sites) inside $\gamma$.

By compactness, the family of probability measures $\mu_V^{\beta,h}$ (fixed $\beta, h \geq 0$) on $K$ has at least one limiting point $\mu^{\beta,h}$. We also introduce the state $\mu_V^{frozen}$. We obtain it by considering for the boundary condition a configuration which has a unique external contour, surrounding $V$. Evidently, the measures $\mu_V^{frozen}$ are supported by a single configuration. Their thermodynamic limit is denoted by $\mu^{frozen}$; its support is the configuration $\sigma^+ \equiv +1$.

Our main result is summarized as follows.

**Theorem 1.** *Let the dimension $d > 1$. For all $\beta$ large enough, there is $b = b(\beta), 0 < b < \infty$ such that*:

1) *If $h \leq b$, there are at least two different states in our model*: *the frozen state $\mu^{frozen}$ and the state $\mu^{\beta,h}$. The measures $\mu^{\beta,h}$ are nontrivial. That means that the probability $\mu^{\beta,h}(\sigma_\Lambda = \eta$ on $\Lambda)$ of the appearance of any allowed configuration $\eta$ in a finite region $\Lambda$ is positive for any $\Lambda$;*

2) *For all $h > b$ the measures $\mu^{\beta,h}$ are concentrated on a single configuration $\sigma^+ \equiv +1$; in other words, $\mu^{\beta,h} = \mu^{frozen}$.*

Added to this are more details about the nature of the measure in finite volume:

**Theorem 2.** $1')$ *if $h \leq b$, for all contours $\gamma$,*

$$\mu_V^{\beta,h}[\sigma_V : \gamma \in \theta(\sigma_V)] \leq \exp[-\lambda v(\gamma) - m|\gamma|] \tag{7}$$

*uniformly in $V$, with $\lambda > 0$ whenever $h \leq b$, and $m > 0$ for $h < b$.*

$2')$ *if $h > b$, and $V$ is a cube with boundary $\partial V$, then, for all $x \in V$*

$$\mu_V^{\beta,h}[\sigma_x = -1] \leq e^{-c\,\mathrm{dist}(x,\partial V)} \tag{8}$$

*for $c = c(\delta, \beta) > 0, c \uparrow \infty$ as $\beta \uparrow \infty$.*

**Remark 1.** *As a function of $\beta \geq 0$, $b(\beta)$ is non-increasing. It is easy to see that there is a constant $c(d) < \infty$ for which $0 \leq b(\beta) \leq b(0) < c(d)$. On the other hand, we expect that $b(\beta) \sim e^{-c\beta}$ for $\beta$ large. That this behavior is correct as a lower bound for $b(\beta)$ follows from* (26) *below.*

**Remark 2.** *It is an immediate corollary of the previous theorem that the magnetization jumps discontinuously as a function of $h$ from negative values for $h < b$ to the value $+1$ for $h > b$.*

**Remark 3.** *The restriction to large $\beta$ in Theorem* 1 *is needed to have a first-order phase transition. It remains of course true that the frozen phase exists for any $\beta$, provided $h$ is large enough. However, when $h$ is lowered at high temperatures ($\beta$ small) there will be no jump in the magnetization and there is uniqueness for all $h$. On the contrary, for $\beta$ large and for a certain value of $h$ there is a coexistence of phases*: *one frozen and one "liquid".*

### *2.2. Non-Gibbsianness*

We start by recalling the definitions of Gibbs, almost Gibbs and weakly Gibbs states. We then argue that for $h \leq b$ our random field $\mu^{\beta,h}$ provides an example of a weakly Gibbs field, which is not a Gibbs field, nor even an almost Gibbs field. However, some probabilities vanish for the field $\mu^{\beta,h}$, which makes this example somewhat unsatisfactory. Because of that we present a family of random fields $\mu^{\beta,h,L}$, with the field $\mu^{\beta,h}$ being the limit point of the family $\mu^{\beta,h,L}$ as $L \to \infty$. These fields again are weakly Gibbs fields, which are not Gibbs, nor almost Gibbs fields, provided $L$ is big enough. The fields $\mu^{\beta,h,L}$ already have all probabilities positive. These fields and a related study already appeared in [8].

Let $X$ be a finite set. A Gibbsian potential $\mathcal{U} = (U_A(\sigma_A), A \subset \mathbb{Z}^d, 0 < |A| < \infty)$ is a system of functions $U_A(\sigma_A) \in \mathbb{R}^1 \cup (+\infty)$ of $\sigma_A \in X^A$, labelled by the system of all finite nonempty subsets $A \subset \mathbb{Z}^d$. We will use the following conventions: $a + \infty = +\infty$, $e^{-\infty} = 0$. For any finite $V \subset \mathbb{Z}^d$, any configuration $\bar\sigma_{\mathbb{Z}^d\setminus V} \in X^{\mathbb{Z}^d\setminus V}$, called a boundary condition, and any $\sigma_V \in X^V$ consider the relative energy

$$E_V^{\mathcal{U}}(\sigma_V|\bar\sigma_{\mathbb{Z}^d\setminus V}) = \sum_{A\subseteq V, A\neq\emptyset} U_A(\sigma_A) + \sum_{A\subset\mathbb{Z}^d: A\cap V\neq\emptyset, A\cap(\mathbb{Z}^d\setminus V)\neq\emptyset, |A|<\infty} U_A(\sigma_{A\cap V} \cup \bar\sigma_{A\cap(\mathbb{Z}^d\setminus V)}) \ . \tag{9}$$

Of course, the series in (9) might not converge for all boundary conditions $\bar\sigma_{\mathbb{Z}^d\setminus V}$ and configurations $\sigma_V$. So, for any finite $V \subset \mathbb{Z}^d$ we introduce the set $\bar X_V^{\mathcal{U}}$ of all boundary conditions $\bar\sigma_{\mathbb{Z}^d\setminus V}$ such that the series (9) either absolutely converges for all $\sigma_V \in X^V$, or goes to $+\infty$. For $\bar\sigma_{\mathbb{Z}^d\setminus V} \in \bar X_V^{\mathcal{U}}$ let

$$p_V^{\mathcal{U}}(\sigma_V|\bar\sigma_{\mathbb{Z}^d\setminus V}) = \frac{\exp\{-E_V^{\mathcal{U}}(\sigma_V|\bar\sigma_{\mathbb{Z}^d\setminus V})\}}{Z_V^{\mathcal{U}}(\bar\sigma_{\mathbb{Z}^d\setminus V})} \ , \tag{10}$$

where the partition function

$$Z_V^{\mathcal{U}}(\bar\sigma_{\mathbb{Z}^d\setminus V}) = \sum_{\sigma_V\in X^V} \exp\{-E_V^{\mathcal{U}}(\sigma_V|\bar\sigma_{\mathbb{Z}^d\setminus V})\} \ . \tag{11}$$

The system $p^{\mathcal{U}} = \{p_V^{\mathcal{U}}(\sigma_V|\bar\sigma_{\mathbb{Z}^d\setminus V}), \sigma_V \in X^V, \bar\sigma_{\mathbb{Z}^d\setminus V} \in \bar X_V^{\mathcal{U}}, |V| < \infty\}$ of probability distributions is called the Gibbs specification for the potential $\mathcal{U}$.

A probability measure $\mathbb{P}$ on the measurable space $(X^{\mathbb{Z}^d}, \mathfrak{B}_{\mathbb{Z}^d})$ is called a *Gibbs random field* with potential $\mathcal{U}$, if
i) $\bar X_V^{\mathcal{U}} = X^{\mathbb{Z}^d\setminus V}$,
ii) the functions $p_V^{\mathcal{U}}(\sigma_V|\bar\sigma_{\mathbb{Z}^d\setminus V})$ are continuous,
iii) for any finite $V \subset \mathbb{Z}^d$, any function $\phi(\sigma_V)$ of $\sigma_V \in X^V$ and any subset $B \in \mathfrak{B}_{\mathbb{Z}^d\setminus V}$ the DLR equation holds:

$$\int_B \phi(\sigma_V)\mathbb{P}(d\sigma) = \int_B \left( \sum_{\sigma_V \in X^V} \phi(\sigma_V) p_V^{\mathscr{U}}(\sigma_V|\bar{\sigma}_{\mathbb{Z}^d\setminus V}) \right) \mathbb{P}_{\mathbb{Z}^d\setminus V}(d\bar{\sigma}_{\mathbb{Z}^d\setminus V}) \ . \tag{12}$$

Here $\mathbb{P}_{\mathbb{Z}^d\setminus V}$ is the restriction of the measure $\mathbb{P}$ to the $\sigma$-algebra $\mathfrak{B}_{\mathbb{Z}^d\setminus V}$.

A probability measure $\mathbb{P}$ on $(X^{\mathbb{Z}^d}, \mathfrak{B}_{\mathbb{Z}^d})$ is called an *Almost Gibbs random field* with potential $\mathscr{U}$, if
i) $\mathbb{P}\left(\bar{X}_V^{\mathscr{U}}\right) = 1$ for all $V$,
ii) there exist functions $\tilde{p}_V^{\mathscr{U}}(\sigma_V|\bar{\sigma}_{\mathbb{Z}^d\setminus V})$ on $X^{\mathbb{Z}^d}$, which coincide $\mathbb{P}$-a.e. with the functions $p_V^{\mathscr{U}}(\sigma_V|\bar{\sigma}_{\mathbb{Z}^d\setminus V})$ (defined only for $\bar{\sigma}_{\mathbb{Z}^d\setminus V} \in \bar{X}_V^{\mathscr{U}}$) and are $\mathbb{P}$-a.e. continuous on $X^{\mathbb{Z}^d}$.
iii) the DLR equation (12) holds (for either $p$ or $\tilde{p}$).

A probability measure $\mathbb{P}$ on $(X^{\mathbb{Z}^d}, \mathfrak{B}_{\mathbb{Z}^d})$ is called a *Weakly Gibbs random field* with potential $\mathscr{U}$, if
i) $\mathbb{P}\left(\bar{X}_V^{\mathscr{U}}\right) = 1$ for all $V$,
ii) the functions $p_V^{\mathscr{U}}(\sigma_V|\bar{\sigma}_{\mathbb{Z}^d\setminus V})$ are $\mathbb{P}$-measurable (on $\bar{X}_V^{\mathscr{U}}$-s),
iii) the DLR equation (12) holds.

We now define the potential $\mathscr{U} = \mathscr{U}^\beta$ for our field $\mu^{\beta,h}$. Let $\Gamma \subset \mathbb{Z}^d$ be a finite Ising contour. We define first the set $\Delta(\Gamma) \subset \mathbb{Z}^d$ of points adjacent to $\Gamma$, as follows. For $\Delta \subset \mathbb{Z}^d$ let $\Delta^+ = \Delta \cap \mathrm{Int}\Gamma$, $\Delta^- = \Delta \cap \mathrm{Ext}\Gamma$. Then $\Delta(\Gamma)$ is the maximal set among the sets $\Delta$ with $\Delta^\pm \neq \emptyset$ satisfying the property: for every Ising configuration $\sigma$ having the contour $\Gamma$ among its contours and for any pair of points $t^+ \in \Delta^+$, $t^- \in \Delta^-$ we have $\sigma\left(t^+\right) = -\sigma\left(t^-\right)$. We now define

$$U_A^1(\sigma_A) = \begin{cases} \beta\,|\Gamma|\,, & \text{if } A = \Delta(\Gamma)\,, \sigma_A|_{\Delta^\pm(\Gamma)} = \pm 1, \\ \infty, & \text{if } A = \Delta(\Gamma)\,, \sigma_A|_{\Delta^\pm(\Gamma)} = \mp 1, \\ 0, & \text{otherwise}\ , \end{cases} \tag{13}$$

$$U_A^2(\sigma_A) = \begin{cases} -hv(\Gamma)\,, & \text{if } A = \mathrm{Int}\Gamma \cup \Delta^-(\Gamma) \text{ and} \\ & \quad \sigma_A|_{\mathrm{Int}\Gamma} = +1, \sigma_A|_{\Delta^-(\Gamma)} = -1, \\ 0, & \text{otherwise,} \end{cases} \tag{14}$$

$$U_A(\sigma_A) = U_A^1(\sigma_A) + U_A^2(\sigma_A)\ . \tag{15}$$

For any finite $V \subset \mathbb{Z}^d$ we define the set $\bar{X}_V^{\mathscr{U}}$ of boundary conditions $\bar{\sigma}_{\mathbb{Z}^d\setminus V}$ as follows: $\bar{\sigma}_{\mathbb{Z}^d\setminus V} \in \bar{X}_V^{\mathscr{U}}$ iff there exists a configuration $\sigma_{\mathbb{Z}^d}$ having only external contours and such that $\sigma_{\mathbb{Z}^d}|_{\mathbb{Z}^d\setminus V} = \bar{\sigma}_{\mathbb{Z}^d\setminus V}$.

*Checking of the DLR equation:* By definition, we have

$$\int_B \phi(\sigma_W)\mu^{\beta,h}(d\sigma) = \lim_{V\to\infty} \int_B \phi(\sigma_W)\mu_V^{\beta,h}(d\sigma)\ .$$

Evidently, for every finite $V$

$$\int_B \phi(\sigma_W)\mu_V^{\beta,h}(d\sigma) = \int_B \phi(\sigma_W) p_W^{\mathscr{U}}(\sigma_W|\bar{\sigma}_{\mathbb{Z}^d\setminus W})\mu_V^{\beta,h}(d\bar{\sigma}_{\mathbb{Z}^d\setminus W}) \ .$$

For our finite box $W$ let us introduce the set of configurations $X(W,n)$, $n \in \mathbb{N}$ as follows:

$$\sigma \in X(W,n) \iff \forall\gamma \in \theta(\sigma) \text{ with Int } \gamma \cap W \neq \emptyset \neq \text{ Int } \gamma \cap W^c$$
$$\text{we have } |\gamma| < n \ .$$

Then, according to the estimate (7) we have

$$\int_B \phi(\sigma_W) p_W^{\mathscr{U}}(\sigma_W|\bar{\sigma}_{\mathbb{Z}^d\setminus W})\mu_V^{\beta,h}(d\bar{\sigma}_{\mathbb{Z}^d\setminus W})$$
$$= \lim_{n\to\infty}\int_{B\cap X(W,n)} \phi(\sigma_W) p_W^{\mathscr{U}}(\sigma_W|\bar{\sigma}_{\mathbb{Z}^d\setminus W})\mu_V^{\beta,h}(d\bar{\sigma}_{\mathbb{Z}^d\setminus W});$$

moreover, this convergence is uniform in $V$. Therefore,

$$\lim_{V\to\infty}\int_B \phi(\sigma_W) p_W^{\mathscr{U}}(\sigma_W|\bar{\sigma}_{\mathbb{Z}^d\setminus W})\mu_V^{\beta,h}(d\bar{\sigma}_{\mathbb{Z}^d\setminus W})$$
$$= \lim_{n\to\infty}\lim_{V\to\infty}\int_{B\cap X(W,n)} \phi(\sigma_W) p_W^{\mathscr{U}}(\sigma_W|\bar{\sigma}_{\mathbb{Z}^d\setminus W})\mu_V^{\beta,h}(d\bar{\sigma}_{\mathbb{Z}^d\setminus W}) \ .$$

But

$$\lim_{V\to\infty}\int_{B\cap X(W,n)} \phi(\sigma_W) p_W^{\mathscr{U}}(\sigma_W|\bar{\sigma}_{\mathbb{Z}^d\setminus W})\mu_V^{\beta,h}(d\bar{\sigma}_{\mathbb{Z}^d\setminus W})$$
$$= \int_{B\cap X(W,n)} \phi(\sigma_W) p_W^{\mathscr{U}}(\sigma_W|\bar{\sigma}_{\mathbb{Z}^d\setminus W})\mu^{\beta,h}(d\bar{\sigma}_{\mathbb{Z}^d\setminus W}) \ ,$$

since the integrand is a local function, when restricted to $X(W,n)$. Finally,

$$\lim_{n\to\infty}\int_{B\cap X(W,n)} \phi(\sigma_W) p_W^{\mathscr{U}}(\sigma_W|\bar{\sigma}_{\mathbb{Z}^d\setminus W})\mu^{\beta,h}(d\bar{\sigma}_{\mathbb{Z}^d\setminus W})$$
$$= \int_B \phi(\sigma_W) p_W^{\mathscr{U}}(\sigma_W|\bar{\sigma}_{\mathbb{Z}^d\setminus W})\mu^{\beta,h}(d\bar{\sigma}_{\mathbb{Z}^d\setminus W}) \ ,$$

again because of (7).

Let us finally show that the conditional distributions $q(\cdot|\cdot)$ of the field $\mu^{\beta,h}$ are discontinuous almost everywhere. First we will write down the explicit expression for these distributions. Let $\sigma_{\mathbb{Z}^d}$ be any configuration with all contours finite and external, and which equals $-1$ outside its contours. Let $V \subset W \subset \mathbb{Z}^d$ be two finite simply-connected boxes. Let $\gamma_1, ..., \gamma_k$ be all the contours of $\sigma_{\mathbb{Z}^d}$ which are inside $V$, while $\Gamma_1, ..., \Gamma_K$ are all the contours of $\sigma_{\mathbb{Z}^d}$ which intersect both $V$ and $V^c$. Denote by $\bar{V}$ the box $V \setminus \bigcup_{i=1}^{K} \left(\mathrm{Int}\Gamma_i \cup \Delta^- (\Gamma_i)\right)$ . Then,

$$q \left(\sigma_V | \sigma_{W \setminus V}\right) = \prod_{i=1}^{k} \exp\left\{-\beta|\gamma_i| + hv(\gamma_i)\right\} / Z(\bar{V}, \beta, h) > 0 \ . \tag{16}$$

Let now $C \subset W \setminus V$ be a closed circuit surrounding $V$. Denote by $\sigma^C_{\mathbb{Z}^d}$ the configuration which equals $+1$ on $C$ and which coincide with $\sigma_{\mathbb{Z}^d}$ on its complement. Then $q \left(\sigma_V | \sigma^C_C\right) = 0$ unless $\sigma_V \equiv +1$, and so also $q \left(\sigma_V | \sigma^C_{W \setminus V}\right) = 0$. Since the circuit $C$ can be arbitrarily far away from $V$, that establishes the discontinuity, provided the reader is prepared to buy the argument that indeed $q \left(\sigma_V | \sigma^C_{W \setminus V}\right) = 0$. In fact, in most cases the probability $\mu^{\beta,h} \left(\sigma^C_{W \setminus V}\right)$ itself vanishes, so the statement that $q \left(\sigma_V | \sigma^C_{W \setminus V}\right) = 0$ might look rather as a convention. We cannot say more in its support, so in what follows we present yet another example of a random field, which exhibits all the qualitative features of the field $\mu^{\beta,h}$, while all the probabilities are positive, removing any ambiguity. *These fields are also only Weakly Gibbs, and not Almost Gibbs.*

The fields $\mu^{\beta,h,L}$ are defined by the $(-)$-boundary conditions and the potential $\mathscr{U}^L$ in the same way as the field $\mu^{\beta,h}$ is defined by the potential $\mathscr{U}$. The potential $\mathscr{U}^L$ is given by the following modification of the relations (13–15). Namely, the relation (13) is replaced by

$$U^{1,L}_A(\sigma_A) = \begin{cases} \beta\,|\Gamma|\,, & \text{if } A = \Delta\,(\Gamma)\,,\, \sigma_A|_{\Delta^{\pm}(\Gamma)} = \pm 1, \\ LF\,(\sigma_A)\,, & \text{if } A = \mathrm{Int}\,\Gamma \cup \Delta^-\,(\Gamma)\,,\, \sigma_A|_{\Delta^{\pm}(\Gamma)} = \mp 1, \\ 0, & \text{otherwise}\ , \end{cases} \tag{17}$$

where $F_\Gamma\,(\sigma_A)$ is a translation-invariant positive function, which grows to infinity with the number of $(-)$-spins inside $\Gamma$ extremely fast. For example,

$$F_\Gamma\,(\sigma_A) = \left[\sum_{x \in \mathrm{Int}\Gamma} (1 - \sigma_x)\right]^2$$

would go. (Notice that $F_\Gamma\,(\sigma_A) \geq |\Gamma|^2$ .) As $L \to \infty$, we recover the potential $U^1_A$. We also introduce the states $\mu^{frozen,L}_V$. They are defined by the following boundary condition $\bar{\sigma}_V$: the configuration $\bar{\sigma}_V$ has a unique contour $\Gamma$ (with $-1$ spins outside it), and $V = \mathrm{Int}\Gamma \setminus \Delta^+\,(\Gamma)$ . The notation $\mu^{frozen,L}$

refers to some limit point of the sequence $\mu_V^{frozen,L}$. Clearly, $\mu^{frozen,L} \to \mu^{frozen}$, as $L \to \infty$.

The statements of the Theorems 1, 2 hold true for the measures $\mu^{\beta,h,L}$, when $L$ is large, with obvious small modifications:

**Theorem 3.** *Let the dimension $d > 1$. For all $\beta$ and $L$ large enough, there is $b = b(\beta, L), 0 < b < \infty$ such that:*
1) *If $h \leq b$, the measures $\mu^{frozen,L}$ and $\mu^{\beta,h,L}$ are manifestly different.*
2) *For all $h > b$ the measures $\mu^{\beta,h,L} \to \mu^{frozen}$, as $L \to \infty$.*

In Theorem 2 the statement (7) holds for the states $\mu_V^{\beta,h,L}$, while (8) is true provided the site $x$ can be connected by a $(-)$-path to the outside of the box $V$.

The set $\bar{X}_V^{\mathscr{U}^L}$ is now the collection of all configurations such that every site $x$ is surrounded by only finitely many contours. The checking of the DLR condition is done by repeating the argument above. That implies weak Gibbsianity.

The conditional distributions are now discontinuous everywhere on $\bar{X}_V^{\mathscr{U}^L}$, so we do not have Almost Gibbsianity for the field $\mu^{\beta,h,L}$. To see it, consider the conditional distribution $q\left(\sigma_V|\sigma_{W\setminus V}\right)$, and let us take the case when all the contours of $\sigma_V \cup \sigma_{W\setminus V}$ are external with respect to $W$, to make the argument simpler. Then

$$q\left(\sigma_V|\sigma_{W\setminus V}\right) = q\left(\sigma_V|\sigma_{W\setminus V}, \mathscr{C}\right)\mathbb{P}\left(\mathscr{C}\right) + q\left(\sigma_V|\sigma_{W\setminus V}, \mathscr{C}^c\right)\mathbb{P}\left(\mathscr{C}^c\right) \quad ,$$

where the event $\mathscr{C}$ means that there is a $+$-circuit, surrounding $V$. (Necessarily, it has to be outside $W$.) Let us further restrict the situation by considering the case of $\sigma_V \neq (+1)_V$ . Now, by choosing the functions $F_\Gamma$ growing fast enough, and $L$ large, one can make the probability $q\left(\sigma_V|\sigma_{W\setminus V}, \mathscr{C}\right)$ to be arbitrarily small, so

$$q\left(\sigma_V|\sigma_{W\setminus V}\right) - q\left(\sigma_V|\sigma_{W\setminus V}, \mathscr{C}^c\right)\mathbb{P}\left(\mathscr{C}^c\right) \to 0, \quad \mathbb{P}\left(\mathscr{C}^c\right) \to 1$$

as $L \to \infty$. Incidentally, the probability $q\left(\sigma_V|\sigma_{W\setminus V}, \mathscr{C}^c\right)$ converges in this limit to the expression given by (16) above. However, when we modify the condition $\sigma_{W\setminus V}$ to $\sigma_{W\setminus V}^C$ by introducing a $+$-circuit $C$ in $W\setminus V$ surrounding $V$, then $q\left(\sigma_V|\sigma_{W\setminus V}^C\right)$ goes to zero as $L \to \infty$, provided again that $\sigma_V \neq (+1)_V$.

Finally we note that in our situation the free energy depends on boundary conditions, and is different for the pair of states $\mu^{\beta,h,L}$, $\mu^{frozen,L}$, as well as for $\mu^{\beta,h}$, $\mu^{frozen}$. This is yet another feature that weakly Gibbsian fields share with Gibbs random fields on a noncompact spin space (see, e.g. [18]).

## 3. Preliminaries about contour functionals and cluster expansions

In what follows we will use extensively the theory of contour functionals and cluster expansions. We mostly combine the set-up's of [3], [13] and [19].

The contours we study here are the regular Ising contours as introduced in the previous Section. Two contours are compatible if they do not share a bond and if at every common site, they do not make illegal turns. Contour functionals are translation invariant real-valued functions on the set of these contours. We will consider the Banach space $B$ of translation invariant contour functionals $\phi$ for which the norm

$$\|\phi\| = \sup_{\gamma} \frac{|\phi(\gamma)|}{v(\gamma)}$$

is finite.

Given such a contour functional $\phi$ and a finite volume $V \subset \mathbb{Z}^d$ we consider the sets (called, boundaries) $\partial \subset V$ of pairwise compatible contours $\gamma \in \partial$ in $V$. The partition function $Z(V|\phi)$ is then defined as usual by

$$Z(V|\phi) = \sum_{\partial \subset V} \prod_{\gamma \in \partial} e^{-\phi(\gamma)} \quad . \tag{18}$$

We adopt the convention that $Z(\emptyset|\phi) = 1$.

It is a matter of simple algebra to see that

$$Z(V|\phi) = \sum \prod_{\gamma \in \partial} e^{-\phi(\gamma)} Z(\mathrm{Int}(\gamma)|\phi) \tag{19}$$

where the sum is over boundaries $\partial \subset V$ consisting only of (their own) external contours.

Most important for us are $\tau$-functionals, which are functionals $\phi$, satisfying the estimate

$$\phi(\gamma) \geq \tau |\gamma| \quad , \tag{20}$$

with $\tau$ large enough. The subset $B_\tau \subset B$ of all $\tau$-functionals from $B$ is a closed convex subset for every $\tau$.

The machinery of the cluster expansions, applied to $\tau$-functionals, results in the following decomposition of the logarithm of the partition function (18) in any finite box $V$ of size $|V|$:

$$\ln Z(V|\phi) = f(\phi)|V| + \sum_{t \in \partial V} g^{\phi}(t, V) \quad , \tag{21}$$

where $f(\phi)$ is the free energy of the contour model, and the function $g^{\phi}(t, V)$, where $t \in \partial V$, is regular:

for any $x$, we have $g^{\phi}(t, V) = g^{\phi}(t + x, V + x)$;
$\left|g^{\phi}(t, V)\right| \leq \exp\{-\tau\}$,
if $t \in \partial V_1 \cap \partial V_2$, then

$$|g^{\phi}(t, V_1) - g^{\phi}(t, V_2)| \leq exp\{-cdist(t, V_1 \triangle V_2)\} \ ,$$

where $V_1 \triangle V_2$ is the symmetric difference, and $c = c(\phi) > 0$, $c(\phi) \to \infty$ as $\tau \to \infty$, uniformly in $\phi$. Below we will give more explicit information on the function $g^{\phi}(t, V)$.

We now follow [3], where the results we need are given in a form convenient for our applications. As usual, we call two contours $\gamma_1, \gamma_2$ incompatible, $\gamma_1 \not\sim \gamma_2$, if they either share a bond or if at a common site they make an illegal turn – i.e. one which is not compatible with the rule of rounding the corners. The relation $\not\sim$ defines the structure of a graph on any set of contours in a natural way. A gang of contours $\rho = (\bar{\rho}, \alpha)$ in the volume $V$ is a non-empty connected (in the graph sense) family $\bar{\rho}$ of distinct contours in $V$, $\bar{\rho} = \{\gamma, \gamma \subset V\}$, together with an integer-valued function $\alpha(\cdot)$ on $\bar{\rho}$. The set of all such gangs will be denoted by $G(V)$. Introduce the function $b(\gamma)$ - the might of the contour - to be $\exp\{|\gamma|\}$, and consider the weight $w_0(\gamma) = \exp\left\{-\frac{\tau}{2}|\gamma|\right\}$. It is easy to see that for any $\gamma$ the following relation holds:

$$\begin{aligned} &\exp\left\{\sum_{\widetilde{\gamma}:\widetilde{\gamma}\not\sim\gamma} w_0(\widetilde{\gamma})\, b(\widetilde{\gamma}) + w_0(\gamma)\, b(\gamma)\right\} \\ &\quad\equiv \exp\left\{\sum_{\widetilde{\gamma}:\widetilde{\gamma}\not\sim\gamma} \exp\left\{(-\frac{\tau}{2}+1)|\widetilde{\gamma}|\right\} + \exp\left\{(-\frac{\tau}{2}+1)|\gamma|\right\}\right\} \leq b(\gamma) \\ &\quad\equiv \exp\{|\gamma|\} \ , \end{aligned}$$

provided $\tau$ is large enough. Hence we can apply Theorem 2.2 of [3] to any $\tau$-functional $\phi$ and obtain the following decomposition:

$$\ln Z(V|\phi) = \sum_{\rho\in G(V)} q_{\phi}(\rho) \equiv \sum_{\rho\in G(V)} r(\rho) \prod_{\gamma\in\bar{\rho}} \exp\{-\alpha(\gamma)\,\phi(\gamma)\} \ . \quad (22)$$

Here the numbers $r(\rho)$ depend only on the graph structure on the set $\bar{\rho}$. For every gang $\rho$ the following estimate holds:

$$\left|q_{\phi}(\rho)\right| \leq \left(\sum_{\gamma\in\bar{\rho}} \exp\left\{(-\frac{\tau}{2}+1)|\gamma|\right\}\right)\left(\prod_{\gamma\in\bar{\rho}} \exp\left\{-\frac{\tau}{2}\alpha(\gamma)|\gamma|\right\}\right) \ . \quad (23)$$

It follows from (22) that the free energy of the contour functional $\phi$ is given by the relation

$$f(\phi) = \sum_{\substack{\rho:\rho\in G(\mathbb{Z}^d),\\ 0\in\mathrm{Int}(\rho)}} \frac{1}{|\mathrm{Int}(\rho)|} q_\phi(\rho) \quad , \tag{24}$$

where $\mathrm{Int}\,(\rho) = \cup_{\gamma\in\bar{\rho}} \mathrm{Int}\,(\gamma)$ . The boundary term is then equal to

$$\sum_{t\in\partial V} g^\phi(t, V) = - \sum_{\substack{\rho:\mathrm{Int}(\rho)\cap V\neq\emptyset,\\ \mathrm{Int}(\rho)\cap V^c\neq\emptyset}} \frac{|Int\,(\rho)\cap V|}{|\;Int\,(\rho)|} q_\phi\,(\rho) \quad .$$

The function $g^\phi(\cdot,\cdot)$ can be defined, for example, by

$$g^\phi(t, V) = - \sum_{\substack{\rho:t\in\mathrm{Int}(\rho),\\ \mathrm{Int}(\rho)\cap V^c\neq\emptyset}} \frac{1}{|Int(\rho)\cap\partial V||Int(\rho)\cap V^c|} q_\phi(\rho) \quad . \tag{25}$$

The regularity of $g^\phi(\cdot,\cdot)$ for $\tau$ large is standard combinatorics.

## 4. The free energy

We start by introducing (minus) the (specific) free energy

$$f\,(\beta, h) = \lim_{V\to\mathbb{Z}^d} \frac{1}{|V|} \ln Z\,(V, \beta, h) \quad .$$

**Lemma 4.** *The above limit exists with values in* $[0, h + \ln 2]$. *The function* $f\,(\beta, h)\,,\ \beta, h \geq 0$ *is convex, continuous and increasing in* $h$.

*Proof.* This follows from standard subadditivity arguments. □

**Lemma 5.** *There exists a value* $b = b(\beta), 0 < b < \infty$ *of the magnetic field such that*

$$f(\beta, h) \begin{cases} > h & if\ h < b, \\ = h & if\ h \geq b \ . \end{cases}$$

*Proof.* Because of the convexity of the function $f\,(\beta, h)$ the uniqueness of the value $b$ follows immediately. The only statement that requires a proof is the finiteness (≡existence) and positivity of $b$. That can be seen as follows. To see that $b > 0$, consider the limit of $f\,(\beta, h)$ as $h \to 0$. It is easy to see that this limit is positive. Indeed, by restricting the range of summation in the partition function (6) to the configurations which are allowed to contain only contours of unit volume surrounding the sites of the sublattice $3\mathbb{Z}^d$, we obtain

$$f(\beta, h) \geq \frac{1}{3^d} \ln\left(1 + \exp\{-2\beta d + h\}\right) \quad , \tag{26}$$

which, of course, is larger than $h$ if $h > 0$ is small. On the other hand, it is straightforward to see by a standard Peierls argument that for $h$ large the probability that a given site $x \in V$ is outside all contours $\gamma$ decays as $\exp\{-c(h)\,\mathrm{dist}(x, \partial V)\}$, with $c(h) \to \infty$ as $h \to \infty$, uniformly in $x$, $V$ and $\beta$. It implies that $b$ is finite. □

## 5. A representation via a contour model

In order to prove our Theorems we will express the partition function (6) by means of a certain contour model, corresponding to a certain contour functional. Note however that the contour models which will appear below, are for some values of $h$ not the "usual" contour models of the Pirogov Sinai theory. Namely, for $h < b$ the contour functionals we will obtain, are growing with the contour as its volume, rather than its boundary. For $h = b$ we obtain the usual contour model, while for $h > b$ we also get the usual contour model but with a positive parameter.

The first half of the Theorem 2 for the case of $h$ small together with the outline of the proof appeared in [13]. The idea of the proof in [13] (Chapter 7, Section 6) is similar to ours.

**Lemma 6.** *There is a uniquely defined contour functional $\phi = \phi^{\beta,h}$ as a solution of the system of equations*, (*see* (18))

$$\exp\left\{[h-b]^+ v(\gamma)\right\} \exp\{-\phi(\gamma)\}\, Z(\mathrm{Int}(\gamma)\,|\phi) = \exp\{-\beta|\gamma| + hv(\gamma)\} \quad , \tag{27}$$

*where $[\cdot]^+ = \max\{\cdot, 0\}$. For all $h \leq b$, $f(\phi^{\beta,h}) = f(\beta, h)$ while for $h \geq b$, $\phi^{\beta,h} = \phi^{\beta,b}$ and $f(\phi^{\beta,h}) = b$. For all $h \leq b$ we have the estimate*

$$\infty > \phi^{\beta,h}(\gamma) \geq (f(\beta, h) - h)v(\gamma) + \tau\,|\gamma| \quad , \tag{28}$$

*and for $h \geq b$ we have*

$$\tau'\,|\gamma| \geq \phi^{\beta,h}(\gamma) \geq \tau\,|\gamma| \quad , \tag{29}$$

*with $\tau, \tau' \to \infty$ as $\beta \to \infty$.*

*Proof.* The equation (27) clearly has a unique solution $\phi^{\beta,h}$, which can be found by induction in the volume of Int $(\gamma)$. What we need is the statement that it is a $\tau$-functional. Suppose for a moment that $\phi^{\beta,h}$ is indeed a $\tau$-functional. Then we can write down the cluster expansion for the partition function $Z\left(\text{Int}\,(\gamma)\,|\phi^{\beta,h}\right)$ :

$$\ln Z\left(\text{Int}\,(\gamma)\,|\phi^{\beta,h}\right) = f(\phi^{\beta,h})|\text{Int}\,(\gamma)\,| + \sum_{t\in\partial(\text{Int}(\gamma))} g^{\beta,h}(t, \text{Int}\,(\gamma)) \;\;,$$

where $g^{\beta,h}(\cdot,\cdot) \equiv g^{\phi_{\beta,h}}(\cdot,\cdot)$ is a regular function, defined by the contour functional $\phi^{\beta,h}$. Inserting the last decomposition into (27), we get:

$$\phi^{\beta,h}(\gamma) = \beta|\gamma| + \sum_{t\in\partial(\text{Int}(\gamma))} g^{\beta,h}(t, Int(\gamma)) + \left[f(\phi^{\beta,h}) - h\right]^{+} v(\gamma) \;\; . \quad (30)$$

On the other hand, if (30) has a solution, which is a $\tau$-functional, then this solution clearly satisfies (27). To show that such a solution of (30) does exist, we follow the strategy of [19], and we will prove that the transformation $\phi \to T\,(\phi)$ , defined by

$$T\,(\phi)\,(\gamma) = \sum_{t\in\partial(\text{Int}(\gamma))} g^{\phi}(t, \text{Int}\,(\gamma)) + [f\,(\phi) - h]^{+}\, v\,(\gamma) \;\;, \qquad (31)$$

is a contraction on a convex subset $B_\tau$ (see after (20)) of the corresponding Banach space of contour functionals. As before, $[\cdot]^{+} = \max\{\cdot, 0\}$ .

Since the function $[f\,(\phi) - h]^{+}\, v\,(\gamma)$ is nonnegative, the transformation $\phi\,(\cdot) \to \beta\,|\cdot| + T\,(\phi)\,(\cdot)$ preserves the set $B_\tau$ as soon as $\tau$ is large enough and $\beta > 2\tau$. We will show that $T$ is a contraction on each $B_\tau$, again for $\tau$ large enough.

To do this we will use the two lemmas which follow. The first lemma estimates the difference $f\,(\phi_1) - f\,(\phi_2)$ , while the second one estimates the difference $g^{\phi_1} - g^{\phi_2}$. These lemmas are the analogues of Propositions 4 and 5 in Chapter 2 of [19].

**Lemma 7.** *Let $\phi_1, \phi_2 \in B_\tau$ be two $\tau$-functionals. Then*

$$|f\,(\phi_1) - f\,(\phi_2)| \le c\,(\tau)\,\|\phi_1 - \phi_2\|\,,$$

*with $c\,(\tau) \to 0$ as $\tau \to \infty$.*

*Proof.* We use the formulas (22) and (24) to write:

$$f(\phi_1) - f(\phi_2) = \sum_{\rho:0\in\text{Int}(\rho)} \frac{1}{|\text{Int}(\rho)|}(q_{\phi_1}(\rho) - q_{\phi_2}(\rho))$$

$$= \sum_{\rho:0\in \mathrm{Int}(\rho)} \frac{r(\rho)}{|\mathrm{Int}(\rho)|} \left( \prod_{\gamma\in\bar\rho} \exp\{-\alpha(\gamma)\phi_1(\gamma)\} - \prod_{\gamma\in\bar\rho} \exp\{-\alpha(\gamma)\phi_2(\gamma)\} \right) .$$

Each difference can be rewritten as a sum of $\sum_{\gamma\in\bar\rho}\alpha(\gamma)$ terms in the following way. Let us enumerate the elements of $\bar\rho$ by numbers 1, ..., $l$ in an arbitrary way, and for any $k$ between 1 and $l$ denote by $\bar\rho_{<k}$ (resp. $\bar\rho_{>k}$) the family of contours $\gamma$ in $\bar\rho$ with indices less than $k$ (resp. greater than $k$). Then the generic term will be of the form

$$\prod_{\gamma\in\bar\rho_{<k}} \exp\{-\alpha(\gamma)\phi_1(\gamma)\}\exp\{-s\phi_1(\gamma_k)\} \times (\exp\{-\phi_1(\gamma_k)\} - \exp\{-\phi_2(\gamma_k)\})\exp\{-(\alpha(\gamma_k)-s-1)\phi_2(\gamma_k)\} \times \prod_{\gamma\in\bar\rho_{>k}} \exp\{-\alpha(\gamma)\phi_2(\gamma)\} ,$$

with the index $s$ running from 0 to $\alpha(\gamma_k)-1$. Note now, that there exist a value $t = t(\phi_1,\phi_2,\gamma_k)$, $0 \le t \le 1$, such that for $\bar\phi(\gamma_k) = t\phi_1(\gamma_k) + (1-t)\phi_2(\gamma_k)$ we have $\exp\{-\phi_1(\gamma_k)\} - \exp\{-\phi_2(\gamma_k)\} = \exp\left\{-\bar\phi(\gamma_k)\right\}(\phi_2(\gamma_k)-\phi_1(\gamma_k))$, and hence $\bar\phi$ is also a $\tau$-functional. As a result, we can bound the absolute value of the last expression by

$$\|\phi_2-\phi_1\| \prod_{\gamma\in\bar\rho_{<k}} \exp\{-\alpha(\gamma)\phi_1(\gamma)\}\exp\{-s\phi_1(\gamma_k)\}\exp\left\{-\bar\phi(\gamma_k)\right\} v(\gamma_k) \times \exp\{-(\alpha(\gamma_k)-s-1)\phi_2(\gamma_k)\} \prod_{\gamma\in\bar\rho_{>k}} \exp\{-\alpha(\gamma)\phi_2(\gamma)\} .$$

Note finally that $\widehat\phi(\gamma_k) = -\ln\left(\exp\left\{-\bar\phi(\gamma_k)\right\} v(\gamma_k)\right)$ is yet another $\tau$-functional (or, rather, $\frac{\tau}{2}$-functional, to be precise). So it is natural now to introduce the set $G^3\left(\mathbb{Z}^d\right)$ of all gangs of 3-colored contours in $\mathbb{Z}^d$, that is, we generalize the definition of $G\left(\mathbb{Z}^d\right)$ by allowing gangs $\rho = (\bar\rho,\alpha)$, with the connected families $\bar\rho$ to consist of colored contours, which are either geometrically distinct or differently colored. We then introduce the weights $q_{\phi_1,\phi_2,\widehat\phi}(\rho)$ as in (22); the difference is that we now use different functionals $\phi_1$, $\phi_2$ or $\widehat\phi$, according to the colors of the contours. Summarizing, we have the bound, which should be compared with the representation (24):

$$|f(\phi_1) - f(\phi_2)| \le \|\phi_1-\phi_2\| \sum_{\substack{\rho:\rho\in G^3\left(\mathbb{Z}^d\right),\\ 0\in\mathrm{Int}(\rho)}} \frac{1}{|\,\mathrm{Int}(\rho)|} \left|q_{\phi_1,\phi_2,\widehat\phi}(\rho)\right| . \quad (32)$$

(For reasons of clarity we note that the estimate (24) contains an overcounting: a very small portion of the set $G^3\left(\mathbb{Z}^d\right)$ should appear in this estimate.) An application of standard combinatorics completes the proof of the lemma. □

**Lemma 8.** *Let $\phi_1, \phi_2 \in B_\tau$ be two $\tau$-functionals. Then for all $V, t \in \partial V$,*

$$\left|g^{\phi_1}(t, V) - g^{\phi_2}(t, V)\right| \le c(\tau)\, \|\phi_1 - \phi_2\|,$$

*uniformly in $V, t$, with $c(\tau) \to 0$ as $\tau \to \infty$.*

*Proof.* Comparing the formulas (24) and (25), we see that the only difference between them is the range of summation in $\rho$, and the lemma would follow once we would be able to estimate the differences $q_{\phi_1}(\rho) - q_{\phi_2}(\rho)$. This, however, was done in the proof of the previous lemma. □

**Proposition 9.** *Suppose $\tau$ is large enough. Then the transformation $T$, defined by the formula* (31)*, is a contraction on $B_\tau$.*

*Proof.* We will estimate the difference between $T(\phi_1)(\gamma)$ and $T(\phi_2)(\gamma)$:

$$\begin{aligned}
&\frac{|T(\phi_1)(\gamma) - T(\phi_2)(\gamma)|}{v(\gamma)} \\
&\quad \le \frac{1}{v(\gamma)}\left[\sum_{t\in\partial(\mathrm{Int}(\gamma))}\left|g^{\phi_1}(t, \mathrm{Int}(\gamma)) - g^{\phi_2}(t, \mathrm{Int}(\gamma))\right|\right] \\
&\qquad + \left([f(\phi_1) - h]^+ - [f(\phi_2) - h]^+\right) \\
&\quad \le \sup_{V\subset\mathbb{Z}^d, t\in\partial V}\left|g^{\phi_1}(t, V) - g^{\phi_2}(t, V)\right| + |f(\phi_1) - f(\phi_2)| \\
&\quad \le 2c(\tau)\, \|\phi_1 - \phi_2\|.
\end{aligned}$$

□

$>$ From the previous Proposition it follows that there is a $\tau$-functional $\phi = \phi^{\beta,h}$ so that

$$\phi^{\beta,h}(\gamma) = \beta\,|\gamma| + \sum_{t\in\partial(\mathrm{Int}(\gamma))} g^{\beta,h}(t, \mathrm{Int}(\gamma)) + \left[f\left(\phi^{\beta,h}\right) - h\right]^+ v(\gamma) \quad , \tag{33}$$

where $g^{\beta,h}(\cdot,\cdot) \equiv g^{\phi_{\beta,h}}(\cdot,\cdot)$ is a regular function, defined via (21) by the contour functional $\phi^{\beta,h}$. On the other hand, because $\phi^{\beta,h} \in B_\tau$,

$$\ln Z\left(\mathrm{Int}(\gamma)\,|\phi^{\beta,h}\right) = f(\phi^{\beta,h})v(\gamma) + \sum_{t\in\partial(\mathrm{Int}(\gamma))} g^{\beta,h}(t, \mathrm{Int}(\gamma)),$$

Hence, upon combining these identities,

$$\exp\{a(h)v(\gamma)\}\exp\left\{-\phi^{\beta,h}(\gamma)\right\}Z\left(\mathrm{Int}(\gamma)\,|\phi^{\beta,h}\right)=\exp\{-\beta\,|\gamma|+hv(\gamma)\}, \tag{34}$$

where

$$a(h)=\left[h-f\left(\phi^{\beta,h}\right)\right]^{+}. \tag{35}$$

It immediately follows that $f(\phi^{\beta,h}) = f(\beta, h - [h - f(\phi^{\beta,h})]^{+})$. If $h \leq b$, then $h - a(h) \leq b$ and thus, from Lemma 4, $f(\phi^{\beta,h}) = f(\beta, h - a(h)) \geq h$. This implies that then $a(h) = 0$ and that $\phi^{\beta,h}$ solves (27), while $f(\phi^{\beta,h}) = f(\beta, h)$. In particular, $a(b) = 0$, hence $f(\phi^{\beta,b}) = f(\beta, b - a(b)) = f(\beta, b)$. But $f(\beta, b) = b$, and so $f(\phi^{\beta,b}) = b$. Together with (34) it implies immediately that the functional $\phi^{\beta,b}$ solves the equation (27) not only for $h = b$, but for all $h \geq b$ as well. Because the solution to (27) is unique, we have $\phi^{\beta,h} = \phi^{\beta,b}$ for $h \geq b$. The relation (29) follows from (33) and the regularity properties of the function $g^{\beta,h}(\cdot,\cdot)$. That completes the proof of Lemma 5. □

Proof of the first half of Theorems 1 and 2. >From Lemma 5 it follows that the contours of our model are described by the exterior contours of the contour model with $\tau$-functional $\phi^{\beta,h}$ for $h \leq b$. The inequality (7) follows immediately from (28).

The first half of Theorem 1 (about the positivity) follows from the cluster expansion (21). Indeed, it is easily seen to be equivalent to the statement that for every finite box $V$ the limit

$$\lim_{W\to\mathbb{Z}^d}\frac{Z(W\setminus V,\beta,h)}{Z(W,\beta,h)}>0\ .$$

But due to the regularity properties of the function $g^{\beta,h}$, for every $t \in \partial(W \setminus V)$ adjacent to $V$ the following limit exists:

$$g^{\beta,h}\left(t,\mathbb{Z}^d\setminus V\right)=\lim_{W\to\mathbb{Z}^d}g^{\beta,h}(t,W\setminus V)\ .$$

Hence the relation (21) implies that

$$\lim_{W\to\mathbb{Z}^d}\frac{Z(W\setminus V,\beta,h)}{Z(W,\beta,h)}=\exp\left\{-f\left(\phi^{\beta,h}\right)|V|+\sum_{t\in\partial\left(\mathbb{Z}^d\setminus V\right)}g^{\beta,h}\left(t,\mathbb{Z}^d\setminus V\right)\right\}\ .$$

Since it is manifestly positive, the positivity required follows. □

## 6. Contour models with positive parameter

Here we will prove the second part of the theorem. We fix the cube $V$ having side length $N$. If $\theta = \{\gamma_1, ..., \gamma_n\} \subset V$ is a configuration of external contours in $V$, we denote by $\mathrm{Ext}_V\theta$ their exterior, $\mathrm{Ext}_V\theta = V \setminus \cup_i \mathrm{Int}\,(\gamma_i)$ . We must show that this external volume is typically small and has no fingers reaching deep inside the volume $V$. It is the proof of the present statement, where the representation of our model of external contours with the help of the usual contour model (with parameter) is especially important technically. The reason is that the proof is based on an application of a version of the Peierls transformation, used to estimate the probability of appearance of protruding fragments (≡fingers) of the random set $\mathrm{Ext}_V\theta$. This estimate is obtained by performing a surgery on such a fragment, which cuts off the finger. However, in general such a surgery creates a configuration of contours which are not necessarily mutually external. This, happily, is not an issue for the usual contour model, to which, at this stage of the argument, we have already reduced our problem.

*Proof.* Since, from Lemma 5, the distribution of the contours in our model of exterior contours coincides with the distribution of exterior contours in the contour model with a parameter, from now on we will consider the latter. We remind the reader its definition. We suppose that a contour functional $\phi$ and a parameter $a > 0$ are given. Then the partition function $Z\,(V|\phi, a)$ of the model in the box $V$ is given by

$$Z\,(V|\phi, a) = \sum_n \sum_{\theta=\{\gamma_1,\dots,\gamma_n\}} \prod_{i=1}^{n} e^{av(\gamma_i)} e^{-\phi(\gamma_i)} Z\,(\mathrm{Int}\,(\gamma_i)\,|\phi) \quad ,$$

where the summation is taken over all collections $\theta$ of exterior contours in $V$, including the empty one, whose contribution is 1. The corresponding probability distribution $p_V^{\phi,a}\,(\kappa)$ on the set $K\,(V)$ of all configurations $\kappa$ of compatible contours in $V$ is given by

$$p_V^{\phi,a}\,(\kappa) = Z\,(V|\phi, a)^{-1} \prod_{\lambda\in\kappa} e^{-\phi(\lambda)} \prod_{\gamma\in E(\kappa)} e^{av(\gamma)} \quad ,$$

where $E\,(\kappa)$ is the set of all external contours in $\kappa$. Comparing always with Lemma 5, taking $\phi = \phi^{\beta,h}$ for $h > b$ we have $a = h - b$ and we can assume (29):

$$\tau'\,|\gamma| \geq \phi\,(\gamma) \geq \tau\,|\gamma| \quad .$$

In addition we have the "almost locality" of the functional $\phi$, which we have thanks to relation (33):

$$\phi\,(\gamma) = \beta\,|\gamma| + \sum_{t\in\partial(\mathrm{Int}(\gamma))} g(t, \mathrm{Int}\,(\gamma)) \quad . \tag{36}$$

Here $g$ is a regular function, and so the value of the functional $\phi(\gamma)$ can be thought of as a result of integrating along $\gamma$ of a certain function on $\gamma$, which is almost local, i.e. which depends weakly on the distant fragments of $\gamma$.

Our treatment of the problem follows closely the strategy of the paper [6]. The first step is very simple.

**Lemma 10.** *Let $\lambda$ be a contour in $V$. Consider the event $I(\lambda) = \{\kappa \in K(V) : \lambda \in \kappa, \lambda \notin E(\kappa)\}$. Then*

$$p_V^{\phi,a}(I(\lambda)) \leq e^{-\phi(\lambda)} \quad .$$

*Proof.* The proof is a straightforward application of the classical Peierls' argument. The required estimate is obtained by comparing the weight of a configuration $\kappa \in I(\lambda)$ with that of configuration $\kappa \setminus \lambda$. □

The next statement is the analogue of Lemma 1 in [6].

**Lemma 11.** *Let $V$ be a finite box, such that for every contour $\lambda$ in $V$ we have* Int $(\lambda) \subset V$. *For every configuration $\kappa \in K(V)$ introduce the random variable $u_V(\kappa) = |V| - \sum_{\gamma \in E(\kappa)} v(\gamma)$. Then, for $s > 0$,*

$$p_V^{\phi,a}(\{\kappa : u_V(\kappa) \geq s\}) \leq \exp\{-as + C|\partial V|\} \quad ,$$

*where $C = C(\tau)$.*

*Proof.* The probability in question equals

$$\begin{aligned} &p_V^{\phi,a}(\{\kappa : u_V(\kappa) > s\}) \\ &\quad = Z(V|\phi,a)^{-1} \sum_n \sum_{\substack{\theta=\{\gamma_1,\dots,\gamma_n\}: \\ u_V(\theta)>s}} \prod_{i=1}^n e^{av(\gamma_i)} e^{-\phi(\gamma_i)} Z(\mathrm{Int}(\gamma_i)|\phi) \quad . \end{aligned}$$

The idea of the proof of the upper bound is to replace the partition function in the denominator of the last expression by a lower bound which has the form of one of the factors of the numerator. To do this we consider the configuration $\sigma_V$ on $\mathbb{Z}^d$, which is equal to $+1$ inside the volume $V$ and equals $-1$ outside. Let $\theta(V) = \{\Gamma_1, \dots, \Gamma_k, k = k(V)\}$ be the collection of all its contours. Clearly, all these contours are external contours of $\sigma_V$, and $u_V(\theta(V)) = 0$. Hence

$$p_V^{\phi,a}(\{\kappa : u_V(\kappa) > s\}) \leq \frac{\sum_{\theta: u_V(\theta)>s} \prod_{i=1}^{n(\theta)} e^{av(\gamma_i)} e^{-\phi(\gamma_i)} Z(\mathrm{Int}(\gamma_i)|\phi)}{\prod_{i=1}^{k(V)} e^{av(\Gamma_i)} e^{-\phi(\Gamma_i)} Z(\mathrm{Int}(\Gamma_i)|\phi)}$$

$$\leq e^{-as} \frac{1}{\prod_{i=1}^{k(V)} e^{-\phi(\Gamma_i)}} \frac{Z(V|\phi)}{\prod_{i=1}^{k(V)} Z(\text{Int}(\Gamma_i)|\phi)} \quad .$$

We now claim that each of the last two factors admits an upper bound of the order of $\exp\{C|\partial V|\}$ for some $C$. For the first one we use the upper bound in (29). For the last factor this follows from the expansion (21) and the fact that the complement $V \setminus \cup_{\Gamma\in\Theta(V)}\text{Int}(\Gamma)$ is contained in the neighborhood of $\partial V$ of radius 2 – hence the volume terms cancel out. □

To proceed, we have to define the fingers of the configuration $\kappa$ of contours. So let again $V = V_N$ be a cube of size $N$, $\theta = \{\gamma_1, ..., \gamma_n\} = E(\kappa) \subset V$ be a configuration of external contours in $V$, and $\text{Ext}_V\theta$ be their exterior. A natural candidate to be called an $l$-finger seems to be any connected component $\bar{F}$ of the intersection $\text{Ext}_V\theta \cap V_{N-l}$. However, such a component might have holes in it. Because of that we will call such a component an $l$-prefinger, and by an $l$-finger $F = F_l$ we will call a result of filling in all these holes: $F = \bar{F} \cup_{\gamma_i \subset \bar{F}} \text{Int}(\gamma_i)$. A base $B = B_l$ of a finger $F_l$ is by definition the set $F_l \setminus V_{N-l-1}$. A finger $F_l$ with $l < \frac{1}{2}\delta N$ will be called a long finger, if $F \cap V_{(1-\delta)N} \neq \emptyset$. A cutting of a finger $F_l$ of a configuration $\kappa$ results in a new configuration $\Sigma(\kappa)$; it corresponds to "flipping all the spins in the base $B_l$". Precisely it means the following: consider the family k of contours, such that $\text{Int}(\text{k}) = B_l$. The configuration $\Sigma(\kappa)$ as a subset of faces in $\mathbb{R}^d$ is just the symmetric difference; $\Sigma(\kappa) = \kappa \triangle \text{k}$. (What we are doing here is called in topology 'surgery' or 'attaching a handle'.) Clearly, the complement $\text{Ext}_V\theta\backslash B_l$ is disconnected, and has two connected components. One is $\bar{F}_l\backslash B_l$, another is $\text{Ext}_V\theta\backslash\bar{F}_l$. Let $\bar{\Gamma} = \bar{\Gamma}(F_l) \in \Sigma(\kappa)$ be the contour, which is the exterior boundary of $\bar{F}_l\backslash B_l$; in other words, $\bar{\Gamma} = \partial(F_l\backslash B_l) \subseteq \partial\left(\bar{F}_l\backslash B_l\right)$. Note, that $\bar{\Gamma}$ is an interior contour of the family $\Sigma(\kappa)$. The exterior contour of $\Sigma(\kappa)$, which contains $\bar{\Gamma}$ in its interior, will be denoted by $\Gamma$.

Let us estimate from above the probability $p_V^{\phi,a}(\kappa)$ in terms of $p_V^{\phi,a}(\Sigma(\kappa))$. By making the surgery we force the configuration to have longer contours; in fact, we are making $2|B_l|$ new bonds – at most. For that we have to compensate; on the other hand, the volume of the exterior contours only increases after the surgery, so the volume factor changes in the right way. We claim that the resulting estimate is:

$$p_V^{\phi,a}(\kappa) \leq \exp\{3\beta|B_l|\}\, p_V^{\phi,a}(\Sigma(\kappa)) \quad . \tag{37}$$

To see it we use the "almost locality" property (36) of the functional $\phi$. It ensures that the change in the total contribution from the contour functional to the weights – i.e. the difference $\left|\sum_{\lambda\in\kappa}\phi(\lambda) - \sum_{\lambda\in\Sigma(\kappa)}\phi(\lambda)\right|$ – which comes, of course, only from the contours affected by the surgery – is bounded from above by $c(\beta)|B_l|$, with $c(\beta) \to 0$ as $\beta \to \infty$.

The next step is now clear: we have to use the fact that the configuration $\Sigma(\kappa)$ will have quite a big interior contour $\bar{\Gamma}$ as soon as the initial configuration $\kappa$ has a long finger, and the surgery is made at the level $l < \frac{1}{2}\delta N$. However, that would work only in case when the finger responsible is 'thin' at the level $l$; otherwise the smallness of the factor $\exp\left\{-\phi\left(\bar{\Gamma}\right)\right\}$ we are counting on would be killed by the factor $\exp\{3\beta|B_l|\}$. This argument has to justify the need for the following definition:

**Definition 1.** *We will call a long finger $F_l$ to be $(N^s, \alpha)$-thin, for some $s \geq 1$ and $0 \leq \alpha < 1$, iff for some $c > 0$*

$$\left|\bar{\Gamma}\right| > cN^s, \text{ while } |B_l| < N^{s\alpha} \ .$$

(Here one should think about $N$ large and $c$ fixed.) As we know from the Lemma 10, the probability $p_V^{\phi,a}\left(\left\{\kappa : \bar{\Gamma} \in \kappa\right\}\right) \leq \exp\left\{-\phi\left(\bar{\Gamma}\right)\right\}$. The number of different contours inside $V_N$ with the length $j$ does not exceed $d(2N)^d 3^j$. Finally, the map $\kappa \to \Sigma(\kappa)$ is many-to-one; the reason is that different sets $B_l$ can lead to the same configuration $\Sigma(\kappa)$. If we are talking about the $(N^s, \alpha)$-thin fingers, then the number of such sets – or the number of the preimages – can be bounded from above by $\left[(2N)^d\right]^{N^{s\alpha}}$. Putting all this together with the estimate (37), we have the thin finger estimate:

$$\begin{aligned} & p_{V_N}^{\phi,a}\left(\left\{\kappa : \kappa \text{ has a } \left(N^s, \alpha\right)\text{-thin finger}\right\}\right) \\ & \quad \leq (2N)^{dN^{s\alpha}} \exp\left\{3\beta N^{s\alpha}\right\} \sum_{j > cN^s} d(2N)^d 3^j \exp\{-\tau j\} \\ & \quad \leq \exp\left\{-\frac{c\tau}{2} N^s\right\} \ , \end{aligned} \tag{38}$$

provided $N$ is large.

To deal with the rest of the fingers we introduce the fat fingers.

**Definition 2.** *We will call a long finger $F_l$ to be $(N^s, \alpha)$-fat, for some $s \geq 1$ and $0 \leq \alpha < 1$, iff for some $c > 0$*

$$\left|\bar{\Gamma}\right| < N^{s\alpha}, \quad \text{ while } \left|\bar{F}_l\right| > cN^s \ .$$

The fatness of the finger implies that in the box $F_l$ the following event happens: the number of sites outside all the exterior contours is anomalously large. In such a situation we can use the Lemma 11 for estimating its probability. (For the sake of clarity we stress here that we do not perform a surgery on a fat finger.) The number of different $\bar{\Gamma}$-s which might appear in such a procedure is bounded by $\left[d(2N)^d\right]^{N^{s\alpha}}$. Therefore we have:

$$p_{V_N}^{\phi,a}\left(\left\{\kappa : \kappa \text{ has a } \left(N^s,\alpha\right)\text{-fat finger}\right\}\right)$$
$$\leq \left[d\,(2N)^d\right]^{N^{s\alpha}} \exp\left\{-acN^s + CN^{s\alpha}\right\} \leq \exp\left\{-\frac{ac}{2}N^s\right\} \quad , \qquad (39)$$

for $N$ large.

With these ingredients the proof of (8) proceeds as follows. We first take $\alpha$ to be close enough to 1, so that

$$\frac{\alpha}{1-\alpha} > d \ .$$

We then perform the following steps:

Step 1. Consider these $\kappa$, which have a long finger $F_l$ with the base $B_l$, which is short:

$$|B_l| < N^{\alpha} \ . \qquad (40)$$

Then such a finger is $(N,\alpha)$-thin with $c = \delta$, so due to the (38) we have that the probability of these configurations is bounded from above by

$$\exp\left\{-\frac{\delta\tau}{2}N\right\} \ .$$

Step 2. For the remaining configurations the condition (40) is violated for all $l < \frac{\delta}{2}N$. Consider the following part of them: these configurations, which have a long finger $F_l$, such that its base $B_l$ satisfies a weaker restriction

$$|B_l| < N^{\alpha(\alpha+1)} \qquad (41)$$

for some $l < \frac{\delta}{2^2}N$. The step splits into two substeps:

Step 2.1. The size of the contour $\bar{\Gamma}$ is of larger order than that of the base $B_l$, namely

$$\left|\bar{\Gamma}\right| \geq N^{(\alpha+1)^2/2} \ .$$

Then our finger is $\left(N^{(\alpha+1)^2/2}, \frac{2\alpha}{\alpha+1}\right)$-thin, so the corresponding probability is bounded from above by

$$\exp\left\{-\frac{\tau}{2}N^{(\alpha+1)^2/2}\right\} \ .$$

Step 2.2. The size of the contour $\bar{\Gamma}$ satisfies the opposite estimate:

$$\left|\bar{\Gamma}\right| < N^{(\alpha+1)^2/2} \ .$$

We claim that in that case we are dealing with the $\left(N^{\alpha+1}, \frac{\alpha+1}{2}\right)$-fat finger, with $c = c_2 = \frac{\delta}{2^2}$. Indeed, since (40) is violated, we have $|B_l| \geq N^\alpha$ for all $l$ between $\frac{\delta}{2^2}N$ and $\frac{\delta}{2}N$. Hence the corresponding probability is bounded from above by

$$\exp\left\{-\frac{a\delta}{2^2}N^{\alpha+1}\right\} \quad .$$

Proceeding by induction, we arrive at

Step m. Introduce the quantity $r_m = \alpha + \alpha^2 + \cdots + \alpha^{m-1} = \frac{\alpha-\alpha^m}{1-\alpha}$. During that step we consider the following portion of the configurations not treated before – they have long fingers with bases $B_l$ which satisfy

$$|B_l| \geq N^{r_m} \tag{42}$$

for all $l < \frac{\delta}{2^{m-1}}N$, while for some $l < \frac{\delta}{2^m}N$ they satisfy

$$|B_l| < N^{\alpha(r_m+1)} \left(= N^{r_{m+1}}\right) \quad .$$

Again we have two substeps:

Step m.1. Here we consider the case of fingers $F_l$ with

$$\left|\bar{\Gamma}\right| \geq N^{(r_m+1)(\alpha+1)/2} \quad .$$

They are $\left(N^{(r_m+1)(\alpha+1)/2}, \frac{2\alpha}{\alpha+1}\right)$-thin, and the corresponding probability is bounded from above by

$$\exp\left\{-\frac{\tau}{2}N^{(r_m+1)(\alpha+1)/2}\right\} \quad .$$

Step m.2. Here we consider what is left after the first substep, namely the fingers with

$$\left|\bar{\Gamma}\right| < N^{(r_m+1)(\alpha+1)/2} \quad .$$

So, because of (42), we are dealing with the $\left(N^{r_m+1}, \frac{\alpha+1}{2}\right)$-fat fingers, with $c = c_m = \frac{\delta}{2^m}$. Such fingers can be observed with probability at most

$$\exp\left\{-\frac{a\delta}{2^m}N^{r_m+1}\right\} \quad .$$

The exponent in the last expression goes to 0 as $m \to \infty$. However, this is not the issue here, since as soon as $r_m$ becomes bigger than $d$, the process terminates since no configuration satisfies (42) for $N$ large enough. That proves part $2'$ of Theorem 2 with $c(\delta, \beta) = \frac{\delta\tau}{2}$; this value corresponds to the smallest exponent, which appeared at the first step of our procedure. □

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## References

[1] Bricmont, J., Kupiainen, A., Lefevere R.: Renormalization group pathologies and the definition of Gibbs states, Comm. Math. Phys. **194**, 359–388 (1998)

[2] Baddeley, A.J., van Lieshout, M.N.M.: Area-interaction point processes, Annals of the Institute of Statistical Mathematics, **47**, 601–619 (1995)

[3] Dobrushin, R.L.: Estimates of semi-invariants for the Ising model at low temperatures, in Beresin memorial volume, AMS Translations (2) vol. **177**, 59–81 (1996)

[4] Dobrushin, R.L.: A Gibbsian representation for non-Gibbsian fields. Lecture given at the workshop 'Probability and Physics', September 1995, Renkum (the Netherlands).

[5] Dobrushin, R.L., Shlosman, S.B.: Gibbsian description of 'non-Gibbsian' fields, Russian Math. Surveys **52**, 285–297 (1997)

[6] van Enter, A.C.D., Fernandez, R., Schonmann, R., Shlosman, S.: Complete analyticity of the 2D Potts model above the critical temperature, Commun. Math. Phys. **189**, 373–393 (1997)

[7] van Enter, A.C.D., Fernandez, R., Sokal, A.D.: Regularity properties and pathologies of position-space renormalization transformations: scope and limitations of Gibbsian theory, J. Stat. Phys. **72**, 879–1167 (1993)

[8] van Enter, A.C.D., Shlosman, S.B.: (Almost) Gibbsian description of the sign-fields of SOS-fields, J. Stat. Phys. **92**, 353–368 (1998)

[9] Fisher, M.E.: On discontinuity of the pressure, Comm. Math. Phys. **26**, 6–14 (1972)

[10] Fisher, D.S., Fröhlich, J., Spencer, T.: The Ising Model in a Random Magnetic Field J. Stat. Phys. **34**, 863–870 1984

[11] Fisher, M.E., Milton, G.W.: Classifying first-order phase transitions, Physica A **138**, 22–54 (1986)

[12] Griffiths, R.B., Ruelle, D.: Strict convexity ("continuity") of the pressure in lattice systems, Comm. Math. Phys. **23**, 169–175 (1973)

[13] Malyshev, V.A., Minlos, R.A.: Gibbs random fields. Cluster expansion, Kluwer Academic Publishers, (1991)

[14] Lebowitz, J.L., Maes, C.: The effect of an external field on an interface, entropic repulsion, J. Stat. Phys. **46**, 39–49 (1987)

[15] Lörinczi, J., Maes, C.: Weakly Gibbsian measures for lattice spin systems, J. Stat. Phys. **89**, 561–579 (1997)

[16] Maes, C., Vande Velde, K.: Relative energies for non-Gibbsian states, Commun. Math. Phys. **189**, 277–286 (1997)

[17] Maes, C., Redig, F., Van Moffaert, A.: Almost Gibbsian versus weakly Gibbsian measures, Stoch. Proc. Appl. **79**, 1–15 (1999)

[18] Shlosman, S.: Non-translation-invariant states in two dimension, Commun. Math. Phys. **87**, 497–504 (1983)

[19] Sinai, Ya.G.: Theory of Phase Transitions: Rigorous Results, Oxford Pergamon Press, (1982)