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Chaotic time dependence in a disordered spin system

L. R. G. Fontes[1], M. Isopi[2], C. M. Newman[3]

[1] Instituto de Matemática e Estatística, Universidade de São Paulo, Cx. Postal 66281, 05315-970 São Paulo, SP, Brasil. e-mail: lrenato@ime.usp.br
[2] Dipartimento di Matematica, Università di Roma "La Sapienza", Piazzale Aldo Moro 2, I-00185 Roma, Italia. e-mail: isopi@mat.uniroma1.it
[3] Courant Institute of Mathematical Sciences, New York University, 251 Mercer Street, New York, NY 10012, USA. e-mail: newman@cims.nyu.edu

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Abstract. Stochastic Ising and voter models on $\mathbb{Z}^d$ are natural examples of Markov processes with compact state spaces. When the initial state is chosen uniformly at random, can it happen that the distribution at time t has multiple (subsequence) limits as $t \to \infty$? *Yes* for the $d = 1$ Voter Model with Random Rates (VMRR) – which is the same as a $d = 1$ rate-disordered stochastic Ising model at zero temperature – if the disorder distribution is heavy-tailed. *No* (at least in a weak sense) for the VMRR when the tail is light or $d \geq 2$. These results are based on an analysis of the "localization" properties of Random Walks with Random Rates.

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1. Introduction

The purpose of this paper is to initiate the study of a Markov process phenomenon we call *Chaotic Time Dependence* (CTD), where the distribution at time t has multiple subsequence limits as $t \to \infty$. For many interacting particle systems, such as stochastic Ising or voter models, it is not to hard to produce CTD by careful choice of the initial state. But when the initial state is chosen uniformly at random, it can be difficult to see whether or

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not CTD happens. We will demonstrate that CTD does occur in a certain class of such interacting particle systems with disorder (i.e., with a random environment).

These systems, (linear) Voter Models with Random Rates (VMRR's) on $\mathbb{Z}^d$, are somewhat special and CTD is shown to occur only for $d = 1$ (and heavy-tailed environments). Nevertheless, the results are of interest for at least two distinct reasons. First, they raise the issue of whether CTD occurs in other more physical spin systems with $d \geq 2$ – both with and without disorder – such as the Edwards–Anderson spin glass [1] and the homogeneous Ising ferromagnet (see, e.g., [2]). Indeed, our original motivation for considering CTD was as a dynamical analogue for the chaotic *size* dependence [3, 4, 5] of the finite-volume Gibbs measures that can occur in such models. Second, the mechanism that leads to CTD in VMRR's is a dimension-dependent localization property of Random Walks with Random Rates (RWRR's) that is of independent interest. The reader should be aware however that this mechanism is specific to VMRR's and any possible CTD in the more physical spin sytems mentioned above (and discussed immediately below) should arise differently. Thus our results about CTD for VMRR's shed little light on whether or not CTD occurs for these other spin systems. We also note (in the context of nonergodic interacting particle systems) that CTD is different than "local nonequilibration" [6] or "recurrence" [7]. The latter phenomena, which concern the subsequence limits of the *state* at time t rather than the *distribution* at time t, can and do occur even in the absence of CTD.

The interacting particle systems for which we will discuss the occurrence or non-occurrence of CTD may be called stochastic (nearest-neighbor) spin systems. These are continuous time Markov processes σ_t with state space $\mathcal{S} = \{\sigma(i) : i \in \mathbb{Z}^d\} = \{-1, +1\}^{\mathbb{Z}^d}$ and only transitions in which a single spin $\sigma(i)$ flips at a time, from θ to $-\theta$, with time independent rates r_i^θ that depend on the state only through the nearest neighbor spins $\{\sigma(j) : |j - i| = 1\}$. (Here $|\cdot|$ denotes Euclidean distance.) In stochastic Ising models, the rates are chosen depending on a temperature parameter $T \geq 0$ so that the infinite volume Gibbs measures for that T are invariant distributions for the process. Formally, these measures have densities (with respect to the symmetric product Bernoulli measure) proportional to $\exp(-H(\sigma)/T)$, where the *energy* $H(\sigma)$ is

$$H(\sigma) = -\frac{1}{2} \sum_{\{i,j\}:|j-i|=1} J_{\{i,j\}} \sigma(i)\sigma(j) \ . \tag{1.1}$$

The transition rates are usually taken as explicit functions of the *change in energy*,

$$\Delta H_i^\theta(\sigma) = \sum_{j:|j-i|=1} J_{\{i,j\}}\theta\sigma(j) \ . \tag{1.2}$$

The *coupling parameters* $J_{\{i,j\}}$ depend on the particular Ising model under consideration.

The *homogeneous ferromagnet* corresponds to $J_{\{i,j\}} \equiv +1$, while Edwards–Anderson (EA) spin glasses correspond to $J_{\{i,j\}}$'s chosen as an i.i.d. random sample from some symmetric distribution on $\mathbb{R}$, such as standard normal. EA models are examples of disordered Ising models (i.e., Ising models in a random environment) and ideally one wishes to obtain results that are valid for almost every realization of the $J_{\{i,j\}}$'s. In the case of spin glasses, it is natural (and standard in the physics literature) to study the corresponding Markov process with an initial state $\sigma_0 = \xi$, where ξ is itself a random sample from a symmetric Bernoulli distribution. In the case of the homogeneous ferromagnet, an alternative, but less interesting choice for σ_0 is to take it as identically $+1$ or identically -1.

The homogeneous ferromagnet, for $d \geq 2$ and T sufficiently small, is known to have multiple Gibbs measures, and thus the corresponding Markov processes have multiple invariant distributions. Two of these are the measures μ^+ and μ^- which are the infinite volume limits of the finite volume Gibbs measures with respectively plus or minus boundary conditions. The distribution of σ_t when $\sigma_0 \equiv +1$ or $\sigma_0 \equiv -1$, denoted μ_t^+ or μ_t^-, is known to converge (in the natural sense) to respectively μ^+ or μ^- (see, e.g., Chapters III and IV of [8]). Using that result, it is not difficult to construct a σ_0 so that $\mu_t^{\sigma_0}$ has both μ^+ and μ^- among its subsequence limits. This can be done by choosing radii R_i with $R_{i+1} - R_i \to \infty$ rapidly and taking σ_0 to be constant on the annuli centered at the origin betwen R_i and R_{i+1}, and alternating in sign as i increases. What is an open problem however is the large t behavior of μ_t^ξ for the uniformly random ξ: Does $\mu_t^\xi \to \frac{1}{2}\mu^+ + \frac{1}{2}\mu^-$ (for almost every ξ) or does CTD occur with the set of subsequence limits being $\{\alpha\mu^+ + (1-\alpha)\mu^- : \alpha \in [0,1]\}$ or does ... ? In a somewhat heuristic sense, we would say in the first case that μ_t^ξ locally loses all memory of the initial state ξ, while in the second case some memory (or predictability) persists in the sense that $\mu_t^\xi \approx \alpha(t,\xi)\mu^+ + [1-\alpha(t,\xi)]\mu^-$, with $\alpha(t,\xi)$ depending nontrivially on ξ for arbitrarily large t's.

An even more wide open set of problems concerns EA spin glasses. Here, it is generally thought, but not proved, that for d sufficiently large and T sufficiently small, there are again multiple Gibbs measures, but perhaps "many more" than for the homogeneous ferromagnet (see [9] for a general mathematical discussion of these and related issues). Thus, if CTD were to occur for μ_t^ξ, the set of subsequence limits could require many more parameters for its description than a single α in $[0,1]$. One motivation for this idea is based on an analogy to Chaotic *Size* Dependence (CSD) [3, 4, 5, 9]. CSD is the existence of many subsequence limits for finite volume Gibbs measures ρ_L (on, say, the cube $\{-L, -L+1, \ldots, L\}^d$ with periodic boundary conditions) as $L \to \infty$. For the homogeneous ferromagnet, there is no CSD with periodic boundary conditions but there should be with random boundary conditions [4]; there, the boundary conditions play the role of the initial state ξ for CTD.

Although there are no rigorous results about the presence or absence of CTD for these models when $T > 0$ (and there are multiple invariant distributions), there are some results when $T = 0$ and also some open problems. Moreover, the $T = 0$ version of the stochastic Ising ferromagnet is closely related to the linear voter models treated in this paper. The $T = 0$ stochastic Ising models we refer to are the ones where r_i^θ is 1, 1/2 or 0 according to whether $\Delta H_i^\theta(\sigma)$ is $< 0, = 0$ or > 0. The absence of CTD has been proved for a large class of disordered Ising models when $T = 0$ as a consequence of the much stronger result that the spin configuration σ_t itself converges a.s. as $t \to \infty$ [10]; the class of models this applies to include those where the common distribution of the $J_{\{i,j\}}$'s is continuous with finite first moment for arbitrary d. For the homogeneous ferromagnet in $d = 1$, it is not hard to prove absence of CTD, but it is still an open problem for $d \geq 2$. (It is known however that σ_t does not converge a.s. as $t \to \infty$ for $d = 2$ [10].) There is however no proof of the presence of CTD for any of the stochastic Ising models so far described.

A main contribution of this paper is to prove CTD for certain $d = 1$ models that are modified versions of the $T = 0$ homogeneous ferromagnet (albeit by a mechanism that is rather special to these modified models). The modification is to introduce disorder, not via the $J_{\{i,j\}}$'s, but rather by multiplying the rate 1, 1/2 or 0 for a flip at location i by $1/\tau_i$, where the τ_i's are a random sample from some distribution on $(0, \infty)$. (In fact we take distributions on $[1, \infty)$ to avoid uninteresting complications.) For $d = 1$, this interacting particle system is exactly the same as the VMRR (which we will define soon and then more precisely in the next section). We will also consider the VMRR for $d > 1$, but that is *not* the same as modifying the stochastic homogeneous ferromagnet by a local factor of $1/\tau_i$. For the VMRR on $\mathbb{Z}^d$, the rate for making a transition at i from θ to $-\theta$ is $1/\tau_i$ times the fraction of the $2d$ nearest neighbors j of i with $\sigma(j) \neq \theta$.

There are other instances in the literature where particle systems with random rates are treated: In [11], survival in a biased voter model with a random environment is the issue. In [12, 13, 14, 15, 16, 17], contact (and percolation) processes in random environments are studied. References [18, 19] consider the exclusion and zero range processes with random rates.

Our results about CTD for VMRR's concern both its presence and absence for a.e. ξ and a.e. τ ($\tau = \{\tau_i : i \in \mathbb{Z}^d\}$) and they require some assumptions on the common distribution of the τ_i's. Although some of the results have weaker hypotheses, let us assume for simplicity in the rest of this Introduction that the common distribution of the τ_i's is such that for some $\alpha > 0$ and $c, c' \in (0, \infty)$, $ct^{-\alpha} \leq \mathbb{P}(\tau_0 > t) \leq c't^{-\alpha}$ for all large t. The main result of the paper is that CTD occurs (for a.e. ξ and τ) when $d = 1$ and $\alpha < 1$ (see Theorem 1). Some heavy-tail assumption (like $\alpha < 1$) is essential since we also show that for all d, CTD does not occur (for a.e. ξ and τ) when $\alpha > 1$ (see Theorem 2); i.e., $\mu_t^{\xi,\tau}$ converges as $t \to \infty$ for a.e. ξ and τ (we make explicit here the dependence of μ_t^{ξ} on τ). Furthermore, even when $\alpha < 1$, CTD seems restricted to low d. Indeed, when $d \geq 3$, CTD does not occur (for a.e. ξ and τ) for any $\alpha > 0$ (see Theorem 5). We conjecture that the same conclusion holds for $d = 2$, but have only proved a weaker result (see Theorem 3) that $\mu_t^{\xi,\tau}$ converges *in probability* as $t \to \infty$; i.e., every correlation $E\,[\sigma_t(i_1)\ldots\sigma_t(i_n)|\xi, \tau]$, regarded as a random variable through its dependence on ξ and τ, converges in probability as $t \to \infty$.

Our main tool in obtaining these results is to use the relation between time reversed linear voter models and coalescing random walks (see, e.g., [8]). Thus, for example, $\mathbb{E}[\sigma_t(0)|\xi, \tau] = \sum_i \xi_i\, \mathbb{P}(X_t = i|\tau)$, where X_t is a (simple symmetric) continuous time random walk on $\mathbb{Z}^d$ starting at the origin with the rate for making steps from i (to one of its $2d$ nearest neighbors) given by $1/\tau_i$. The absence or presence of CTD for σ_t is then closely related to whether or not $\sup_i \mathbb{P}(X_t = i|\tau) \to 0$ as $t \to \infty$ (and if so, how fast): thus the mechanism for CTD in the VMRR is the occurrence of a kind of localization in the RWRR.

Localization and related issues such as subdiffusive behavior for RWRR's and other such models are of independent interest. There is, in particular, an extensive literature concerning both diffusion (see, e.g., [20]) and anomalous diffusion (see [21] for a review) in random media. Typically, the anomalous diffusion exponent ζ is defined so that the distance from the origin after time t scales like t^ζ. Remark 3.1 below implies that when $d = 1$ and $\alpha < 1$, at a time of order n^γ ($\gamma = \frac{1}{2\alpha} + \frac{1}{2}$), the distance is of order $n^{1/2} = (n^\gamma)^{1/(2\gamma)}$; thus $\zeta = 1/(2\gamma) = \frac{\alpha}{1+\alpha} < 1/2$ corresponding to subdiffusive behavior. But the localization phenomenon means that the distribution of X_t/t^ζ is also unusually singular so that there is a kind of localization or spatial intermittency accompanying the subdiffusion. This issue will be pursued in a future paper, but meanwhile we briefly discuss the dimension dependence of anomalous subdiffusion and of localization/intermittency. (We remark that certain random walk models without spatial disorder but with a heavy-tailed waiting time distribution also exhibit subdiffusion (see, e.g., [22]) – but presumably not localization. Since both our model and that of ref. [22] appear in the literature as toy models for transport in the same kind of disordered media (see, e.g., [23]), but they give rise to different behavior, this issue deserves further investigation. We also remark that in [24], a totally asymmetric RWRR is studied and a localization result similar to Lemma 3.2 below is presented. The rates there are taken to have a common marginal gamma distribution with a parameter corresponding to $\alpha < 1$.)

When $\alpha > 1$, for any d, there is ordinary diffusion with $\zeta = 1/2$ (see, e.g., [20]). For $\alpha < 1$ and $d \geq 2$, one expects subdiffusion, but with a different formula for ζ than $\frac{\alpha}{1+\alpha}$. The fact that individual sites are not visited so many times by an n-step simple symmetric random walk leads to the replacement of the formula $\gamma = \frac{1}{2\alpha} + \frac{1}{2}$ of Remark 3.1 with $\gamma = \frac{1}{\alpha}$ so that one expects $\zeta = \alpha/2$ for $\alpha < 1$ and $d \geq 2$. Another difference between $d = 1$ and $d \geq 2$ is in the nature of localization. Our results suggest for $d = 2$ and prove for $d \geq 3$ that there is no localization for the RWRR X_t starting at the origin and moving in all of $\mathbb{Z}^d$; this type of localization seems to be a strictly one-dimensional phenomenon. However, there is a modified type of localization for the RWRR, which has appeared in the physics literature [25], that occurs for any d when $\alpha < 1$.

To explain it, consider an RWRR X_t^L in the box $\Lambda_L = \{-L, -L + 1, \ldots, L\}^d$ (with, say, reflecting boundary conditions). If we look at the fraction of time up to t spent at site i, take the limit as $L \to \infty$ then as $t \to \infty$, we are in the situation studied in this paper. The modified localization studied in [25] corresponds to taking the limit in the opposite order. Then the first limit results in the invariant distribution for X_t^L, whose discrete density is $\tau_i / \sum_{j \in \Lambda_L} \tau_j$, and the $L \to \infty$ limit yields localization for $\alpha < 1$ and any d.

In the next section we describe in detail the Voter Model with Random Rates and our results concerning chaotic time dependence. Proofs are given in subsequent sections.

2. The Voter Model with Random Rates

We call the following model a spin system, although it is a disordered version of the well known and much studied (linear) voter model [8, 26], which is usually described as a particle system.

Let $\mathscr{S} = \{-1, +1\}^{\mathbb{Z}^d}$ be the set of spin configurations in $\mathbb{Z}^d$. A point of $\mathscr{S}$, a configuration, will be denoted by $\sigma = \{\sigma(i),\ i \in \mathbb{Z}^d\}$. The system evolves stochastically from an initial configuration as follows. Each site $i \in \mathbb{Z}^d$ has a Poisson clock that rings at times whose increments are i.i.d exponential random variables with positive mean τ_i. The clocks for different sites are independent of each other. We denote by $\tau = \{\tau_i,\ i \in \mathbb{Z}^d\}$

the family of means and by $\tau^{-1} = \{\tau_i^{-1},\ i \in \mathbb{Z}^d\}$, the family of rates of the process. At each ring of the clock at i, its spin is updated, taking the value of the spin at a site chosen uniformly at random among its $2d$ nearest neighbors.

Since almost surely no two clocks ring simultaneously, the dynamics is almost surely well defined locally. For the process to be almost surely well defined globally for all times, one has to insure that the event that infinitely many jumps occur in finite time (in the dual random walk process, as discussed below) has probability zero. It suffices, for example, to take τ uniformly positive. In the random version to be discussed below, the uniform positivity is not necessary. Actually, in that case, it suffices to assume that the (common) distribution of the i.i.d. τ_i's has no atom at 0.

The formal generator of this continuous time Markov process, denoted by $\{\sigma_t,\ t \geq 0\}$ or simply σ_t, can be given by

$$\mathscr{G} f(\sigma) = \sum_{i \in \mathbb{Z}^d} \frac{1}{\tau_i} \sum_{j:|i-j|=1} \frac{1}{2d} \left[f(\sigma^{ij}) - f(\sigma) \right] \ , \tag{2.1}$$

where f is an arbitrary real function on $\mathscr{S}$ depending on finitely many coordinates only, $|\cdot|$ is the L_2 norm in $\mathbb{Z}^d$ and σ^{ij} denotes the modification of σ at i, where it takes the spin $\sigma(j)$, namely

$$\sigma^{ij}(k) = \begin{cases} \sigma(k), & \text{if } k \neq i, \\ \sigma(j), & \text{if } k = i \ . \end{cases} \tag{2.2}$$

This is a nonhomogeous linear voter model.

We want to put random disorder in the rates, so τ will be taken as a family of i.i.d. strictly positive random variables, independent of the dynamics. The resulting process is almost surely well defined, since the probability for an infinite number of jumps in finite time (in the dual random walk process) is zero even without assuming uniform positivity of τ. Nevertheless, to simplify the construction (and the analysis of other issues), we take

$$\mathbb{P}(\tau_0 \geq 1) = 1 \ . \tag{2.3}$$

This will be called a Voter Model with Random Rates (VMRR).

We take uniformly random initial conditions for σ_t and thus introduce another family ξ of i.i.d. random variables, which will be taken independent of the rates and the dynamics. Let $\xi = \{\xi_i,\ i \in \mathbb{Z}^d\}$, with common symmetric Bernoulli distribution, that is, we set $\mathbb{P}(\xi_0 = +1) = \mathbb{P}(\xi_0 = -1) = 1/2$. ξ will be taken as the initial state of σ_t, that is , we set $\sigma_0 = \xi$. We want to investigate in this model the phenomenon of *chaotic time dependence*.

Definition 2.1. *For given realizations of the initial configuration of spins ξ and configuration of rates τ, we say that σ_t exhibits* chaotic time dependence (*CTD*) *if it does* not *converge weakly as* $t \to \infty$ (*i.e., if some finite dimensional distribution – or spatial correlation* $E[\sigma_t(i_1)\dots\sigma_t(i_n)]$ *– of* σ_t *does not converge to a single limit as* $t \to \infty$).

Remark 2.1. *A similar notion of chaotic time dependence can be defined for any stochastic dynamical system: presence of multiple weak limits along different subsequences of time (for given choices of parameters and initial distribution).*

We now state our result on the presence of CTD. For this and our other results no attempt has been made to obtain optimal hypotheses.

Theorem 1. *For* $d = 1$, *if the distribution of* τ_0 *is such that there exist constants* $0 < c, c' < \infty$ *and* $0 < \alpha < 1$ *for which*

$$ct^{-\alpha} \leq \mathbb{P}(\tau_0 > t) \leq c' t^{-\alpha} \tag{2.4}$$

for all large t, *then the VMRR exhibits chaotic time dependence for almost every* ξ *and* τ.

The following results are convergence theorems showing or suggesting absence of CTD.

Theorem 2. *For all* d, *if* $\mathbb{E}(\tau_0^\alpha) < \infty$ *for some* $\alpha > 1$, *then the VMRR converges weakly for almost every* ξ *and* τ.

Theorem 3. *For* $d = 2$, *if the distribution of* τ_0 *satisfies* (2.4) *for some* $\alpha > 0$ *and* $c, c' \in (0, \infty)$, *then the VMRR converges weakly in probability with respect to* ξ *and* τ (*i.e., every spatial correlation of* σ_t *converges in probability with respect to* ξ *and* τ).

Theorem 4. *For* $d \geq 3$, *for every* τ (*with* $\tau_i \geq 1$ *for all* i), *the voter model with inhomogeneous rates* τ *converges weakly in probability with respect to* ξ.

We can strengthen the type of convergence in Theorem 4 but at the expense of restricting the allowed τ's:

Theorem 5. *For* $d \geq 3$, *if* τ_0 *satisfies* $\mathbb{E}(\tau_0^\alpha) < \infty$ *for some* $\alpha > 0$, *then the VMRR converges weakly for almost every* ξ *and* τ.

The arguments for these results are based on the (well known) dual representation of linear voter models as coalescing random walks [8, 26], defined as follows. Consider a system of particles initially placed one on each site of $\mathbb{Z}^d$ evolving according to continuous-time simple symmetric coalescing random walks with inhomogeneous rates τ^{-1}. This means that there is an exponential clock sitting at every site i ringing at rate τ_i^{-1}. When it does ring, if there is a particle at that site, it chooses a site uniformly at random among its nearest neighbors and jumps there. Two particles at the same site coalesce immediately into a single particle.

Let us denote this particle system by η_t and call it a Coalescing Random Walk with Random Rates (CRWRR). It has state space $\tilde{\mathcal{S}} = \{0,1\}^{\mathbb{Z}^d}$ and generator

$$\tilde{\mathcal{G}}h(\eta) = \sum_{i\in\mathbb{Z}^d} \frac{\eta(i)}{\tau_i} \sum_{j:|i-j|=1} \frac{1}{2d}\,[h(\eta^{ij}) - h(\eta)] \ , \tag{2.5}$$

where h is an arbitrary real function on $\tilde{\mathcal{S}}$ depending on finitely many coordinates only and η^{ij} denotes the possible modification of η at i and j, where it takes the values 0 and 1 respectively, namely

$$\eta^{ij}(k) = \begin{cases} \eta(k), & \text{if } k \neq i \text{ and } j, \\ 0, & \text{if } k = i, \\ 1, & \text{if } k = j \ . \end{cases} \tag{2.6}$$

Let $X_t(i)$ denote the position at time t of the tagged particle of η initially at the site i.

Remark 2.2. *We note, for use in the proofs below, that for a single* i $\{X_t(i),\ t \geq 0\}$ *is an ordinary simple symmetric random walk in* $\mathbb{Z}^d$ *(with inhomogeneous rates). We will mostly denote* $X_t(0)$ *more compactly by* X_t*. We will also consider the discrete-time random walk embedded in* X_t*, denoted by* $\tilde{X}_n$*.*

There is then the following weak (or dual) representation of the VMRR in terms of the CRWRR (see [8, 26] for the case with uniform rates). For every fixed $t \geq 0$, we have that

$$\{\sigma_t(i),\ i \in \mathbb{Z}^d\} = \{\sigma_0(X_t(i)),\ i \in \mathbb{Z}^d\} \tag{2.7}$$

in distribution.

In the proof of Theorem 1 in the next section, we use the representation (2.7) to show that the expected value of the spin at the origin at time t, given τ and σ_0, does not converge as $t \to \infty$. As noted in Remark 2.2, the object that comes into play is the random walk X_t. The quantity we will study, already using the representation (2.7), is thus

$$\mathbb{E}[\sigma_t(0)|\tau, \sigma_0 = \xi] = \mathbb{E}[\xi_{X_t}|\tau, \xi] = \sum_{i\in\mathbb{Z}^d} \xi_i\, \mathbb{P}(X_t = i|\tau) \ , \tag{2.8}$$

where τ refers to the rate configuration and ξ is the initial spin configuration of the VMRR.

The almost sure CTD of Theorem 1 follows from the facts, to be argued below, that, for almost every τ,

$$\lim_{t\to\infty} \mathbb{P}(X_t = i|\tau) = 0 \tag{2.9}$$

for all i but $\limsup_{t\to\infty} \sup_i \mathbb{P}(X_t = i|\tau) > 0$. The mechanism behind these facts, as we will show below, is the localization or concentration at large times of $\mathbb{P}(X_t = \cdot|\tau)$ in sites where τ is large. The confinement of $\mathbb{Z}^1$ makes it predictable (with probability bounded away from 0) which large τ sites X_t is located in.

This concentration does not happen when $E(\tau_0^\alpha) < \infty$ for some $\alpha > 1$. Then, one has almost surely that $\sup_i \mathbb{P}(X_t = i|\tau) \to 0$ as $t \to \infty$ at least as fast as a fractional inverse power and this rapid dilution of the distribution of X_t is the key to the convergence in Theorem 2, proved in Section 4. The proofs of Theorems 3–5 are similar; these are given in Section 5.

Remark 2.3. *A relevant point with respect to the weak limits of Theorems* 2–5 (*to be discussed below*) *and concerning the CRWRR is that it can be shown without much difficulty that the behavior of the coalescing random walk with uniform rates is inherited by the CRWRR. That is, in one and two dimensions all random walks coalesce eventually* (*almost surely with respect to the dynamics*) *for every* τ. *In three and higher dimensions, some walks will remain forever distinct* (*almost surely with respect to the dynamics*) *for every* τ.

3. Chaotic time dependence for $d = 1$

In what follows we will express the time spent by X_t at a given site i between jumps by $\tau_i T$, where T is an exponential random variable with mean one. In this way, we have three independent families of random variables involved in X_t. The embedded walk $\tilde{X}$, τ and a sequence of i.i.d. exponentials of mean one, $T_1, T_2, \ldots$

Next we state a result about the scaling of the time spent by X_t in the first n jumps, denoted S_n. We can write

$$S_n = \sum_{i=1}^{n} \tau_{\tilde{X}_i} T_i \quad . \tag{3.1}$$

Let $\gamma = \dfrac{1}{2\alpha} + \dfrac{1}{2}$.

Lemma 3.1. *For* $d = 1$ *and* $\alpha < 1$, *suppose that the distribution of* τ_0 *satisfies the right hand side inequality of* (2.4) *for some finite constant* c'

and all large enough t. Then S_n/n^γ is bounded in probability as $n \to \infty$; i.e., $\lim_{B\to\infty} \sup_n \mathbb{P}(S_n > Bn^\gamma) = 0$.

The proof of Lemma 3.1 is given later in this section. We next state another lemma that will be used together with Lemma 3.1 to prove Theorem 1; its proof is given at the end of the section.

Lemma 3.2. *To prove Theorem* 1, *it is sufficient to show that*

$$\mathbb{P}\left(\limsup_{t\to\infty} \sup_i \mathbb{P}(X_t = i|\tau) > 0\right) = 1 \ . \tag{3.2}$$

3.1. Proof of Theorem 1

Let $I = I(n, \tau)$ be the leftmost integer i in $[-\sqrt{n}, \sqrt{n}]$ where τ_i achieves its maximum. That is,

$$I = \min\{i : -\sqrt{n} \le i \le \sqrt{n} \text{ and } \tau_i = \max_{-\sqrt{n}\le j\le\sqrt{n}} \tau_j\} \ . \tag{3.3}$$

Let B and ϵ be positive numbers. We will first show that

$$\lim_{B\to\infty} \liminf_{\epsilon\to 0} \liminf_{n\to\infty} \mathbb{P}\left(\frac{1}{Bn^\gamma}\int_0^{Bn^\gamma} 1\{X_s = I\}\, ds > \epsilon\right) = 1 \tag{3.4}$$

and from this conclude that (3.2) holds.

The above probability is bounded from below by

$$\mathbb{P}\left(\frac{1}{n^\gamma}\int_0^{S_n} 1\{X_s = I\}\, ds > B\epsilon\right) - \mathbb{P}(S_n > Bn^\gamma) \ . \tag{3.5}$$

By Lemma 3.1, the last probability in (3.5) can be made arbitrarily small uniformly in n by choosing B large. Thus

$$\limsup_{B\to\infty} \limsup_{\epsilon\to 0} \limsup_{n\to\infty} \mathbb{P}(S_n > Bn^\gamma) = 0 \ . \tag{3.6}$$

The left hand side of the expression inside the first probability in (3.5) (which we will later denote by U) equals

$$\frac{1}{n^\gamma}\tau_I G(L_{n,I}) \ , \tag{3.7}$$

in distribution, where $L_{n,k}$ denotes the local time of $\tilde{X}$ up to discrete time n at site k, that is

$$L_{n,k} = \sum_{i=0}^{n} 1\{\tilde{X}_i = k\} \ , \tag{3.8}$$

and $G(m)$ is a gamma random variable with mean and variance m. Now (3.7) is the product of

$$\frac{\tau_I}{(\sqrt{n})^{\frac{1}{\alpha}}} \quad \text{and} \quad \frac{G(L_{n,I})}{\sqrt{n}} \ , \tag{3.9}$$

which are asymptotically strictly positive in probability (this is easy to check for the former random variable from the left hand side inequality of (2.4); for the latter, it follows from Theorems 9.13 and 10.1 in [27] and from the right hand side inequality of (2.4)). Thus

$$\liminf_{\epsilon\to 0} \liminf_{n\to\infty} \mathbb{P}\left(\frac{1}{n^\gamma}\int_0^{S_n} 1\{X_s = I\}\, ds > B\epsilon\right) = 1 \tag{3.10}$$

for all B and (3.4) follows from (3.10) and (3.6) via (3.5).

Next it follows from (3.4) by fairly standard arguments that

$$\lim_{B\to\infty} \liminf_{\epsilon\to 0} \liminf_{n\to\infty} \mathbb{P}\left(\frac{1}{Bn^\gamma}\int_0^{Bn^\gamma} \mathbb{P}(X_s = I|\tau)\, ds > \epsilon\right) = 1 \ . \tag{3.11}$$

To see this, replace the ϵ in (3.4) by $\epsilon/(1-\sqrt{\delta})$ for some small δ and use the fact that

$$\mathbb{P}\left(U \geq \frac{\epsilon}{1-\sqrt{\delta}}\right) \geq 1-\delta \quad \text{implies} \quad \mathbb{P}\left(E[U|\tau] \geq \epsilon\right) \geq \sqrt{\delta} \ .$$

The estimate (3.11) now implies that

$$\lim_{\epsilon\to 0} \liminf_{n\to\infty} \mathbb{P}\left(\frac{1}{n}\int_0^{n} \sup_i \mathbb{P}(X_s = i|\tau)\, ds > \epsilon\right) = 1 \tag{3.12}$$

and thus that $\lim_{\epsilon\to 0} \lim_{t\to\infty} \mathbb{P}\left(\sup_{s>t} \sup_i \mathbb{P}(X_s = i|\tau) > \epsilon\right) = 1$, which implies the condition (3.2) of Lemma 3.2. □

It remains to prove Lemmas 3.1 and 3.2.

3.2. Proof of Lemma 3.1

We will argue that

$$\lim_{\lambda\to 0^+} \liminf_{n\to\infty} \mathbb{E}\left(\exp\left\{-\frac{\lambda}{n^\gamma} S_n\right\}\right) = 1 \ . \tag{3.13}$$

This immediately implies the desired result.

We rewrite the expectation in (3.13) as $\mathbb{E}[\mathbb{E}(\exp\{-\lambda S_n/n^\gamma\}|\tau, \tilde{X})]$.

From (3.1) and Jensen's inequality, the expectation inside the brackets can be bounded below by

$$\exp\left\{-\frac{\lambda}{n^\gamma}\sum_{i=1}^{n}\tau_{\tilde{X}_i}\right\} \quad . \tag{3.14}$$

The expectation of (3.14) can be expressed as

$$\mathbb{E}\left[\mathbb{E}\left(\exp\left\{-\frac{\lambda}{n^\gamma}\sum_k \tau_k L_{n,k}\right\}\middle|\tilde{X}\right)\right] \tag{3.15}$$

and the expectation inside brackets in (3.15) equals

$$\prod_k \left\{1 - \mathbb{E}\left(1 - \exp\{-\ell\tau_0\}|\tilde{X}\right)\right\} \quad ,$$

where $\ell = \ell(n,k) = \lambda L_{n,k}/n^\gamma$.

It follows from the right hand side inequality of (2.4) that there exists a finite constant c'' such that the last expectation can be bounded above by $c''\ell^\alpha$. Using the facts that given $\delta' > 0$, $1 - x_k \geq \exp\{-(1+\delta')x_k\}$ for small enough nonnegative x_k and that

$$\sup_k L_{n,k}/n^\gamma \leq \sum_k L_{n,k}/n^\gamma = n^{1-\gamma} \to 0 \tag{3.16}$$

as $n \to \infty$ (since $\gamma > 1$), we conclude that (3.15) is bounded below by

$$\mathbb{E}\left[\exp\left\{-\frac{c'''\lambda^\alpha}{n^{\gamma\alpha}}\sum_k L^\alpha_{n,k}\right\}\right] \quad , \tag{3.17}$$

where c''' is a finite constant.

Now by the Law of the Iterated Logarithm for $\tilde{X}$ and the fact that $L_{n,k}$ can be approximated by the Brownian local time, denoted $\mathscr{L}_{n,k}$, within a margin of error of $n^{1/4}$ ([27], Theorem 10.1), (3.17) becomes

$$\mathbb{E}\left[\exp\left\{-\frac{c'''\lambda^\alpha}{\sqrt{n}}\sum_k (\mathscr{L}_{n,k}/\sqrt{n})^\alpha\right\}\right] = \mathbb{E}\left[\exp\left\{-\frac{c'''\lambda^\alpha}{\sqrt{n}}\sum_k \mathscr{L}^\alpha_{1,k/\sqrt{n}}\right\}\right] \tag{3.18}$$

plus an $o(1)$ error term. (The identity in (3.18) follows by a simple change of variables.) But $\frac{1}{\sqrt{n}}\sum_k \mathscr{L}^\alpha_{1,k/\sqrt{n}} \to \int \mathscr{L}^\alpha_{1,x}dx < \infty$ almost surely as $n \to \infty$, since $\mathscr{L}_{1,x}$ is continuous and with bounded support almost surely. (3.13) follows by dominated convergence. □

Remark 3.1. *Arguments like those in the proof of Lemma* 3.1 *show that under the assumption of the left hand side inequality of* (2.4), $\lim_{\lambda\to\infty} \limsup_{n\to\infty} \mathbb{E}\,(\exp\{-\lambda S_n/n^\gamma\}) = 0$. *This and* (3.13) *together imply that if both sides of* (2.4) *are valid*, *then* $\{S_n/n^\gamma,\ n \geq 1\}$ *is a tight sequence and every weak limit is supported on* $(0,\infty)$. *We further note that if* c *and* c' *can be taken arbitrarily close in* (2.4) (*that is*, *if* $t^\alpha \mathbb{P}(\tau_0 > t)$ *has a positive finite limit as* $t \to \infty$), *then* S_n/n^γ *converges weakly as* $n \to \infty$.

3.3. Proof of Lemma 3.2

We will argue that (3.2) implies that for almost every τ and ξ, $\mathbb{E}[\sigma_t(0)|\tau,\xi]$ does not converge as $t \to \infty$. We begin by showing, through a renewal theory argument, that (2.9) holds. It is sufficient to show that

$$\lim_{t\to\infty} \mathbb{P}(X_t = i|X_0 = i, \tau) = 0\ . \tag{3.19}$$

Indeed, denoting the first hitting time of i (starting from 0) by $\mathscr{H}_i$, we have

$$\mathbb{P}(X_t = i|\tau) = \int_0^t \mathbb{P}(X_{t-s} = i|X_0 = i, \tau)\mathbb{P}(\mathscr{H}_i \in ds) \tag{3.20}$$

and (2.9) follows from (3.19) through (3.20).

Now the probability in (3.19), which we now denote by $H(t)$, satisfies the following renewal equation:

$$H(t) = \mathbb{P}(\tau_i T > t|\tau) + \int_0^t H(t-s)\,dF(s)\ , \tag{3.21}$$

where T is an exponential random variable with mean one and F is the distribution function of the sum of the independent random variables $\tau_i T$ and $\mathscr{H}'_i$ (conditional on τ), the latter being the return time to i of X_t (after leaving i).

We claim that $\mathbb{E}(\mathscr{H}'_i|\tau) = \infty$ for all τ (with $\tau_i \geq 1$ for all i). This can be argued by coupling (in the natural way) $\mathscr{H}'_i = \mathscr{H}'_i(\tau)$ and $\mathscr{H}'_i(\tilde{\tau})$, with $\tilde{\tau} \equiv 1$, so that $\mathscr{H}'_i(\tau) \geq \mathscr{H}'_i(\tilde{\tau})$. But $\mathscr{H}'_i$ is the return time in the homogeneous continuous time simple symmetric random walk on $\mathbb{Z}$ which is well known to have infinite mean. Going back to (3.21), since $\mathbb{P}(\tau_i T > t|\tau)$ is (directly) Riemann integrable, the Renewal Theorem ([28], Chapter XI) applies and (3.19) follows.

Returning now to $\mathbb{E}[\sigma_t(0)|\tau,\xi]$ and using the representation (2.7), we have (as previously noted in (2.8)) that $\mathbb{E}[\sigma_t(0)|\tau,\xi] = \sum_i \xi_i \mathbb{P}(X_t = i|\tau)$. (2.9) now implies that, for every τ, the convergence of $\mathbb{E}[\sigma_t(0)|\tau,\xi]$ is in the tail sigma-field of the variables in ξ and thus, by the Kolmogorov 0-1

Law, is a trivial event. In other words, if $A_1 = \{\mathbb{E}[\sigma_t(0)|\tau, \xi] \text{ converges}\}$, then $\mathbb{P}(A_1|\tau) = 0$ or 1 for every τ.

Now let $A_2 = \{\mathbb{P}(A_1|\tau) = 1\}$. Notice that A_2 depends only on τ. Our aim is to prove that as a consequence of (3.2), $\mathbb{P}(A_2) = 0$. On A_2, $\mathbb{E}[\sigma_t(0)|\tau, \xi]$ converges to a constant for almost every ξ, again by triviality. The constant has to be 0, since $\mathbb{E}[\sigma_t(0)|\tau, \xi]$ is uniformly bounded (between -1 and 1) and $\mathbb{E}[\sigma_t(0)1_{A_2}|\tau] = 1_{A_2}\mathbb{E}[\sigma_t(0)|\tau] = 0$ for almost every τ.

The uniform boundedness now implies that on A_2,

$$\mathbb{E}\left\{\mathbb{E}^2[\sigma_t(0)|\tau, \xi]\middle|\, \tau\right\} \to 0 \tag{3.22}$$

as $t \to \infty$. But the left hand side of (3.22) equals $\sum_i \mathbb{P}^2(X_t = i|\tau)$, which is greater than or equal to $\sup_i \mathbb{P}^2(X_t = i|\tau)$. Thus (3.22) implies that on A_2, $\sup_i \mathbb{P}(X_t = i|\tau) \to 0$ as $t \to \infty$ and hence (3.2) implies that $\mathbb{P}(A_2) = 0$, as desired. □

4. Convergence for light-tailed τ

4.1. *Proof of Theorem 2*

We will argue only the $d = 1$ and 2 cases here since Theorem 5 includes the result for $d \leq 3$. Let us suppose, without loss of generality, that $\mathbb{E}(\tau_0) = 1$. We will first argue that

$$\mathbb{E}[\sigma_t(0)|\tau, \xi] \to 0 \tag{4.1}$$

almost surely as $t \to \infty$. The key step will be to show that there exists a positive number δ such that almost surely

$$\sup_i \mathbb{P}(X_t = i|\tau) \leq t^{-\delta} \tag{4.2}$$

for all large enough t.

With (4.2) and the representation (2.7)–(2.8), the result will follow after an argument on getting continuous time convergence from convergence along certain discrete subsequences of times, to be presented below. The key step in the argument is to prove a moderate deviations estimate for S_n. We want to argue that there exist numbers $\nu < 1$ and $\delta > 0$ (depending on α) such that for almost all τ

$$\mathbb{P}(|S_n - n| > n^\nu|\tau) < n^{-\delta} \tag{4.3}$$

for all large enough n. In Lemma 4.1 below, we show thatthis implies (4.2).

For now, let us rewrite $S_n - n$ as

$$\sum_{i=1}^{n} \bar{\tau}_{\tilde{X}_i} T_i + \sum_{i=1}^{n} \bar{T}_i \ \ , \tag{4.4}$$

where bars denote centering by the mean and we have used that $E(\tau_0) = 1$. We need only argue the analogue of (4.3) for the former sum, since the result for the latter is well known (see for example Theorem 1.8.8 in [29], which we note is the same as Exercise 1.8.2 in the first edition of [29]).

Let us now reexpress the first term in (4.4) as

$$\sum_{k} \bar{\tau}_k \, G_k(L_{n,k}) \ \ , \tag{4.5}$$

where $G_k(m_k), k \in \mathbb{Z}$, are (conditional on $L_{n,k} = m_k$ for each k) i.i.d. gamma random variables with mean and variance m_k, respectively, and $L_{n,k}$ is the discrete random walk local time.

We consider $d = 1$ first. Given $\epsilon > 0$ (we will choose its value later, depending on α and ν), we claim that

$$\mathbb{P}(L_{n,k} = 0 \text{ for all } |k| > n^{\frac{1}{2}+\epsilon}) \geq 1 - n^{-\delta'} \tag{4.6}$$

and

$$\mathbb{P}(L_{n,k} \leq n^{\frac{1}{2}+\epsilon} \text{ for all } |k| \leq n^{\frac{1}{2}+\epsilon}) \geq 1 - n^{-\delta'} \tag{4.7}$$

for some $\delta' > 0$ and all large n. (4.6) follows immediately from the Reflection Principle for $\tilde{X}$ and (4.7) follows from Chebyshev's inequality once we prove that, for all positive integers M and integers k, $\mathbb{E}(L_{n,k}^M) \leq c_M \, n^{M/2}$, where c_M is constant in n and k. This in turn follows from

$$L_{n,k}^M = \left(\sum_{i=0}^{n} 1\{\tilde{X}_i = k\} \right)^M \leq M! \sum_{0 \leq i_1 \leq \cdots i_M \leq n} \prod_{j=0}^{M} 1\{\tilde{X}_{i_j} = k\} \ \ , \tag{4.8}$$

and

$$\begin{aligned} &\sum_{0 \leq i_1 \leq \cdots i_M \leq n} \mathbb{P}(\tilde{X}_{i_1} = k) \prod_{j=2}^{M} \mathbb{P}(\tilde{X}_{i_j} = k | \tilde{X}_{i_{j-1}} = k) \\ &\quad = \sum_{0 \leq i_1 \leq \cdots i_M \leq n} \mathbb{P}(\tilde{X}_{i_1} = k) \prod_{j=2}^{M} \mathbb{P}(\tilde{X}_{i_j} = 0 | \tilde{X}_{i_{j-1}} = 0) \\ &\quad = \sum_{0 \leq i_1 \leq \cdots i_M \leq n} \mathbb{P}(\tilde{X}_{i_1} = k) \prod_{j=2}^{M} \mathbb{P}(\tilde{X}_{i_j - i_{j-1}} = 0) \end{aligned}$$

$$\leq \sum_{0\leq i_1\leq n} \sum_{0\leq i_2-i_1\leq n} \cdots \sum_{0\leq i_M-i_{M-1}\leq n} \mathbb{P}(\tilde{X}_{i_1}=k) \prod_{j=2}^{M} \mathbb{P}(\tilde{X}_{i_j-i_{j-1}}=0)$$

$$= \sum_{0\leq i_1\leq n} \mathbb{P}(\tilde{X}_{i_1}=k) \prod_{j=2}^{M} \sum_{0\leq i_j-i_{j-1}\leq n} \mathbb{P}(\tilde{X}_{i_j-i_{j-1}}=0)$$

$$= \left(\sum_{0\leq i\leq n} \mathbb{P}(\tilde{X}_i=k)\right)\left(\sum_{0\leq i\leq n} \mathbb{P}(\tilde{X}_i=0)\right)^{M-1} \tag{4.9}$$

and

$$\sum_{0\leq i\leq n} \mathbb{P}(\tilde{X}_i=k) \leq \sum_{0\leq i\leq n} \mathbb{P}(\tilde{X}_i=0) = O(n^{1/2}) \ . \tag{4.10}$$

The final inequality follows easily from conditioning on the first time $\tilde{X}_i = k$.

Returning to (4.5), by using (4.6) we now only need to be concerned with the sum restricted to $|k| \leq n^{\frac{1}{2}+\epsilon}$. Next, by using (4.7) and a large deviations argument for G_k, we have that

$$\mathbb{P}\left(G_k(L_{n,k}) \leq 2n^{\frac{1}{2}+\epsilon} \text{ for all } |k| \leq n^{\frac{1}{2}+\epsilon}\right) \geq 1 - n^{-\delta'} \tag{4.11}$$

for some $\delta' > 0$ and all large n.

Now, writing the the sum in (4.5) (restricted to $|k| \leq n^{\frac{1}{2}+\epsilon}$), divided by n^{ν}, as

$$\frac{1}{n^{\nu-1/2-\epsilon}} \sum_{|k|\leq n^{1/2+\epsilon}} \bar{\tau}_k \frac{G_k(L_{n,k})}{n^{1/2+\epsilon}} \ , \tag{4.12}$$

restricting attention to the event on the left hand side of (4.11) and following the proof of Theorem 1.8.8 in [29], we have that, for ν close enough to 1 (depending on α), we may choose ϵ small enough (depending on α and ν) so that for almost every τ and then for all n large enough, (4.12) is smaller than 1 (say) and (4.3) follows for $d = 1$.

The argument for $d = 2$ is similar. (4.6) still holds (for essentially the same reason). The analogue of (4.7) is $\mathbb{P}(L_{n,k} \leq n^{\epsilon} \text{ for all } |k| \leq n^{\frac{1}{2}+\epsilon}) \geq 1 - n^{-\delta'}$, which follows in the same way as did (4.7), upon noticing that the middle expression of (4.10) is now $O(\log n)$. (4.11) takes the form $\mathbb{P}\left(G_k(L_{n,k}) \leq 2n^{\epsilon} \text{ for all } |k| \leq n^{\frac{1}{2}+\epsilon}\right) \geq 1-n^{-\delta'}$. This follows again from a large deviations argument. The analogue of (4.12) will then be

$$\frac{1}{n^{\nu-\epsilon}} \sum_{|k|\leq n^{1/2+\epsilon}} \bar{\tau}_k \frac{G_k(L_{n,k})}{n^{\epsilon}} \tag{4.13}$$

and a similar argument to the one following (4.12) implies (4.3) for $d = 2$.

We next state and prove the lemma yielding (4.2) from (4.3) and after that complete the proof of Theorem 2 (for $d = 1$ and 2).

Lemma 4.1. *Suppose there exist finite positive constants c_1, c_2, δ and $\nu < 1$ such that*

$$\mathbb{P}\,(|S_n - n| > c_1 n^{\nu} | \tau) \leq c_2 n^{-\delta} \tag{4.14}$$

for all large enough n. Then there exist finite positive constants c_3 and δ' such that

$$\sup_i \mathbb{P}\,(X_t = i | \tau) \leq c_3 t^{-\delta'} \tag{4.15}$$

for all large enough t.

Proof. Let N_t denote the number of jumps taken by X up to time t; then we can write X_t as a time-changed discrete time (homogeneous) random walk, $X_t = \tilde{X}_{N_t}$. From (4.14), $\mathbb{P}\,(|N_t - t| > c_4 t^{\nu} | \tau) \leq c_5 t^{-\delta}$ because

$$N_t - t > c_4 t^{\nu} \Longrightarrow N_t > t + c_4 t^{\nu} \Longrightarrow S_{\lfloor t + c_4 t^{\nu} \rfloor} < t \tag{4.16}$$

and

$$N_t - t < -c_4 t^{\nu} \Longrightarrow N_t < t - c_4 t^{\nu} \Longrightarrow S_{\lceil t - c_4 t^{\nu} \rceil} > t \ . \tag{4.17}$$

Thus

$$\begin{aligned} \mathbb{P}\,(X_t = i | \tau) &= \mathbb{P}\left(\tilde{X}_{N_t} = i | \tau\right) \\ &\leq \mathbb{P}\left(\tilde{X}_{N_t} = i, |N_t - t| \leq c_4 t^{\nu} | \tau\right) + c_5 t^{-\delta} \ , \end{aligned} \tag{4.18}$$

where c_5 does not depend on i.

We first argue the case $d = 1$. The last probability in (4.18) is bounded above by

$$\begin{aligned} &\mathbb{P}\left(\tilde{X}_n = i \ \text{ for some } \ n \in [t - c_4 t^{\nu}, t + c_4 t^{\nu}]\right) \\ &\quad \leq \mathbb{P}\left(\tilde{X}_{\lfloor t - c_4 t^{\nu} \rfloor} \in [i - t^{\nu'}, i + t^{\nu'}]\right) \\ &\qquad + \mathbb{P}\left(|\tilde{X}_m - \tilde{X}_{\lfloor t - c_4 t^{\nu} \rfloor}| \geq t^{\nu'} \ \text{ for some } \ m \in [t - c_4 t^{\nu}, t + c_4 t^{\nu}]\right) \\ &\quad \leq \mathbb{P}\left(\tilde{X}_{\lfloor t - c_4 t^{\nu} \rfloor} \in [i - t^{\nu'}, i + t^{\nu'}]\right) + \mathbb{P}\left(\max_{1 \leq m' \leq 2 c_4 t^{\nu} + 1} |\tilde{X}_{m'}| \geq t^{\nu'}\right) \ , \end{aligned} \tag{4.19}$$

where ν' is chosen so that $\nu/2 < \nu' < 1/2$.

The first probability after the last inequality of (4.19) is then bounded above by

$$\sum_{i-t^{\nu'} \le j \le i+t^{\nu'}} \mathbb{P}\left(\tilde{X}_{\lfloor t-c_4 t^{\nu} \rfloor} = j\right) \le c_6 \frac{t^{\nu'}}{\sqrt{t - c_4 t^{\nu}}} \le c_7 t^{\nu'-1/2} \quad . \tag{4.20}$$

and the second by

$$2\mathbb{P}\left(\max_{1 \le m'' \le 2c_4 t^{\nu}+1} \tilde{X}_{m''} \ge t^{\nu'}\right) \le 4\mathbb{P}\left(\tilde{X}_{\lfloor 2c_4 t^{\nu}+1 \rfloor} \ge t^{\nu'}\right) \le c_8 \frac{1}{t^{\delta''}} \tag{4.21}$$

for some $\delta'' > 0$, since $\nu' > \nu/2$ (see (4.6)). The first inequality in (4.21)) is due to the Reflection Principle. Then (4.21) and (4.19) (since $\nu' < 1/2$) together yield (4.15) for $d = 1$.

Now for the case $d = 2$. The sup over i of the last probability in (4.18) is bounded above by $\sum_{j=t-c_4 t^{\nu}}^{t+c_4 t^{\nu}} \sup_i \mathbb{P}(\tilde{X}_j = i)$ and the $1/j$ decay of $\sup_i \mathbb{P}(\tilde{X}_j = i)$ in $d = 2$ (a well known fact) yields the desired result since $\nu < 1$. □

4.2. *Completion of Proof of Theorem 2*

The arguments so far can be used to establish (4.1) along a subsequence of times going to infinity and getting dense as it grows, as follows. For $t = 1, 2, \ldots$, let $\mathscr{A}_t = \{i/t,\ i = 0, 1, \ldots, \lfloor t \rfloor\}$. Now define $\mathscr{T} = \cup_{t=1}^{\infty}\{t + \mathscr{A}_t\}$. We argue now, from (4.2), that (4.1) holds along $\mathscr{T}$. First notice that, given a positive integer K, using (2.8)

$$\begin{aligned}
\mathbb{E}\left\{\mathbb{E}^{2K}[\sigma_t(0)|\tau,\xi]\,\middle|\,\tau\right\} &= \sum_{l_1,\ldots,l_{2K}} \prod_{j=1}^{2K} \mathbb{P}(X_t = l_j|\tau) \\
&\le M_K \sum_{i_1,\ldots,i_K} \prod_{j=1}^{K} \mathbb{P}^2(X_t = i_j|\tau) \\
&= M_K \left(\sum_i \mathbb{P}^2(X_t = i_j|\tau)\right)^K \\
&\le M_K \left(\sup_i \mathbb{P}(X_t = i|\tau)\right)^K \le M_K t^{-K\delta}
\end{aligned} \tag{4.22}$$

for almost every τ and then all large t. Here $\sum'$ is the sum over $l_1, \ldots, l_{2K}$ such that the l_i's agree in pairs and M_K is a combinatorial coefficient.

Given $\epsilon > 0$,

$$\mathbb{P}\left(\sup_{s\in t+A_t} \mathbb{E}[\sigma_s(0)|\tau,\xi] > \epsilon \middle| \tau\right) \le t \sup_{s\in t+A_t} \mathbb{P}\left(\mathbb{E}[\sigma_s(0)|\tau,\xi] > \epsilon | \tau\right) \le M_K \epsilon^{-2K} t^{-K\delta+1} \tag{4.23}$$

for almost every τ and then for large t, with the last inequality due to Chebyshev and (4.22). Now (4.1) along $\mathcal{T}$ follows by Borel-Cantelli upon choosing K big enough.

To fill up the gaps, we note (see (2.5) and (2.3)) that for every i

$$\left|\frac{d}{dt}\mathbb{P}(X_t = i|\tau)\right| \le \mathbb{P}(X_t = i|\tau) + \frac{1}{2d}\sum_{j:|i-j|=1} \mathbb{P}(X_t = j|\tau) \tag{4.24}$$

for all τ (with $\tau_i \ge 1$ for all i), which, from (2.8), implies that

$$\left|\frac{d}{dt}\mathbb{E}[\sigma_t(0)|\tau,\xi]\right| \le 2 \tag{4.25}$$

for all τ. Since $\mathbb{E}[\sigma_t(0)|\tau,\xi] \to 0$ as $t \to \infty$ along $\mathcal{T}$, (4.25) implies convergence along all of $\mathbb{R}$.

The argument for convergence of higher correlations is similar. We present the case for the two-point correlation. By (2.7), we have

$$\begin{aligned}\mathbb{E}[\sigma_t(0)\sigma_t(x)|\tau,\xi] &= \sum_{i,j} \xi_i\xi_j \mathbb{P}(X_t(0) = i, X_t(x) = j|\tau) \\ &= \mathbb{P}(X_t(0) = X_t(x)|\tau) \\ &\quad + \sum_{i\ne j} \xi_i\xi_j \mathbb{P}(X_t(0) = i, X_t(x) = j|\tau) \,. \end{aligned} \tag{4.26}$$

The first term in the last expression converges in t for *every* τ by the mononicity in t of the events $\{X_t(0) = X_t(x)\}$ for coalescing random walks.

As for the second term, raising it to the K-th power, for an arbitrary positive integer K, taking the expectation with respect to ξ, and using the fact that for $i \ne j$

$$\mathbb{P}(X_t(0) = i, X_t(x) = j|\tau) \le \mathbb{P}(X_t(0) = i|\tau)\mathbb{P}(X_t(x) = j|\tau),$$

we get exactly the same bound as in (4.22), namely

$$\sum_{l_1,\dots,l_{2K}} \prod_{j=1}^{2K} \mathbb{P}(X_t = l_j|\tau) \le M_K \left(\sup_i \mathbb{P}(X_t = i|\tau)\right)^K \tag{4.27}$$

and we proceed as with $\mathbb{E}[\sigma_t(0)|\tau,\xi]$ to get almost sure convergence to 0. □

Remark 4.1. *From the almost sure coalescence of every pair of random walks of the CRWRR for $d = 1$ and 2 (see Remark 2.3), it is clear that the weak limit of the distribution of σ_t is a fair mixture of the point measures on the two invariant configurations $\sigma^+ \equiv +1$ and $\sigma^- \equiv -1$ for almost all rates and initial configurations.*

5. Convergence for two and higher dimensions

We prove Theorems 4, 5 and 3, in that order.

5.1. Proof of Theorem 4

Two properties of $\tilde{X}$ for $d \geq 3$ related to its transience are

$$\sum_n \mathbb{P}(\tilde{X}_n = i) < \infty \tag{5.1}$$

for all $i \in \mathbb{Z}^d$ and

$$\lim_{|i|\to\infty} \sum_n \mathbb{P}(\tilde{X}_n = i) = 0 \ . \tag{5.2}$$

Let $\Theta_i = \max\{j : \tilde{X}_j = i\}$, that is, the last time $\tilde{X}$ visits i (with $\max \emptyset = -1$). Then (5.1) implies that

$$\Theta_i < \infty \tag{5.3}$$

almost surely.

We have (with N_t as defined in the proof of Lemma 4.1) that $X_t = i \Longrightarrow \Theta_i \geq N_t$ and, since for every τ, $N_t \to \infty$ as $t \to \infty$ almost surely, from this and (5.3)

$$\mathbb{P}(X_t = i|\tau) \to 0 \tag{5.4}$$

as $t \to \infty$ for each i. By (5.2), given $\epsilon > 0$, there exists M (depending on τ) such that

$$\sup_t \sup_{|i|>M} \mathbb{P}(X_t = i|\tau) < \epsilon \ . \tag{5.5}$$

This is because

$$\begin{aligned} \mathbb{P}(X_t = i|\tau) &\leq \mathbb{P}(X_s = i \text{ for some } s|\tau) \\ &= \mathbb{P}(\tilde{X}_n = i \text{ for some } n) \leq \sum_n \mathbb{P}(\tilde{X}_n = i) \ . \end{aligned}$$

Now, by (5.4) and (5.5),

$$\limsup_{t\to\infty} \sup_i \mathbb{P}(X_t = i|\tau) \le \limsup_{t\to\infty} \sup_{|i|\le M} \mathbb{P}(X_t = i|\tau) \vee \epsilon = \epsilon \qquad (5.6)$$

and, since ϵ is arbitrary, it follows that $\limsup_{t\to\infty} \sup_i \mathbb{P}(X_t = i|\tau) = 0$ for every τ and the convergence (to zero) of $E[\sigma_t(0)|\tau, \xi]$ in L^2 (and hence in probability) with respect to ξ then follows from Chebyshev's inequality and

$$\mathbb{E}\left\{ \mathbb{E}^2[\sigma_t(0)|\tau, \xi] \middle| \tau \right\} = \sum_i \mathbb{P}^2(X_t = i|\tau) \le \sup_i \mathbb{P}(X_t = i|\tau) \ . \qquad (5.7)$$

The convergence of higher order correlations is done as in the proof of Theorem 2. □

Remark 5.1. *From Remark* 2.3, *we conclude that the weak limit of* σ_t (*in probability*) *is non-trivial* (*in the sense of not being a mixture of the invariant configurations* $\sigma^+ \equiv +1$ *and* $\sigma^- \equiv -1$). *In fact, it is not difficult to to see that it is the distribution of an i.i.d. assignment of* $+1$ *and* -1 *with equal probabilities to the components of a random partition of* $\mathbb{Z}^d$. *Two sites of* $\mathbb{Z}^d$ *belong to the same component if the walks in the CRWRR starting at those sites eventually coalesce.*

We will need the following result for the proof of Theorem 5; its proof is given below.

Lemma 5.1. *Under the same hypotheses as Theorem* 5, *there exist positive numbers* δ *and* δ' *such that, for almost every* τ, $\mathbb{P}\left(N_t < t^\delta|\tau\right) < t^{-\delta'}$ *for all large* t.

5.2. Proof of Theorem 5

We will argue that there exists a positive number δ such that for almost every τ

$$\sup_i \mathbb{P}(X_t = i|\tau) \le t^{-\delta} \qquad (5.8)$$

for all large enough t. As in the proof of Theorem 2, this is the key step; the result then follows in the same way as did Theorem 2 for $d = 2$. To obtain (5.8), note that

$$\begin{aligned} \mathbb{P}(X_t = i|\tau) &= \sum_k \mathbb{P}(\tilde{X}_k = i, N_t = k|\tau) \\ &\le \mathbb{P}(N_t < t^\delta|\tau) + \sum_{k\ge t^\delta} \mathbb{P}(\tilde{X}_k = i) \end{aligned} \qquad (5.9)$$

so the left hand side in (5.8) is bounded by

$$\mathbb{P}(N_t < t^\delta|\tau) + \sum_{k\geq t^\delta} \sup_i \mathbb{P}(\tilde{X}_k = i) \ . \tag{5.10}$$

It is well known that as $k \to \infty$ the expression inside the sum in (5.10) decays as $1/k^{d/2}$, which is summable if $d \geq 3$. This and Lemma 5.1 yield (5.8). □

5.3. *Proof of Lemma 5.1*

It is enough to show (by arguments like those of (4.16)–(4.17)) that there exist positive finite constants δ and M such that for almost every τ

$$\mathbb{P}\left(S_n > n^M|\tau\right) \leq cn^{-\delta} \ , \tag{5.11}$$

where $c < \infty$ does not depend on n (but may depend on τ). Bound the probability on the left hand side as follows. $\mathbb{P}(n^{-M}S_n > 1|\tau) \leq c'(1 - E[\exp\{-n^{-M}S_n\}|\tau])$, where c' is a constant $(= (1 - \exp\{-1\})^{-1})$.

The expression inside the parentheses on the right hand side is bounded, using Jensen's inequality, by $1 - E[\exp\{-n^{-M}\sum_k \tau_k L_{n,k}\}|\tau]$. Now the sum in the above expression is bounded (crudely) by $n \max_{|k|\leq n} \tau_k$, since $\sum_k L_{n,k} = n$. But if M' is a sufficiently large number, then $n^{-M'} \max_{|k|\leq n} \tau_k$ converges to 0 as $n \to \infty$ for almost every τ. (5.11) follows upon choosing $M = M' + 1 + \delta$. □

5.4. *Proof of Theorem 3*

We will treat the case $0 < \alpha \leq 1$ only; the case $\alpha > 1$ follows from Theorem 2. The following analogue of Lemma 3.1 (see also Remark 3.1) will be needed.

Lemma 5.2. *Let S_n be as in* (3.1). *Under the hypotheses of Theorem* 3 *with $\alpha \leq 1$ and for any $0 < \epsilon < 1/\alpha$, we have that $S_n/n^\beta \to 0$ or ∞ in probability as $n \to \infty$, depending on whether $\beta = 1/\alpha + \epsilon$ or $1/\alpha - \epsilon$, respectively.*

5.5. *Proof of Lemma 5.2*

We first argue that $S_n/n^\beta \to 0$ in probability as $n \to \infty$ with $\beta = 1/\alpha + \epsilon$. For that, we will show that

$$\lim_{n\to\infty} \mathbb{E}\left(\exp\{-S_n/n^\beta\}\right) = 1 \ . \tag{5.12}$$

Arguing as in the proof of Lemma 3.1, we get the following lower bound for the expectation in (5.12):

$$\mathbb{E}\left(\exp\left\{-\frac{1}{n^\beta}\sum_{i=1}^{n}\tau_{\tilde{X}_i}\right\}\right) \quad . \tag{5.13}$$

Following that proof and using (3.16) with β replacing γ and the right hand side inequality of (2.4), we get the lower bound

$$\mathbb{E}\left(\exp\left\{-\frac{c'}{n^{\beta\alpha}}\sum_{k}L_{n,k}^{\alpha}\right\}\right) \tag{5.14}$$

for (5.13), where c' is a positive constant. (5.12) follows because $\beta\alpha > 1$ and $\sum_k L_{n,k}^{\alpha} \le n$, since $\alpha \le 1$ and $\sum_k L_{n,k} = n$.

Next, we want to show that

$$S_n/n^\beta \to \infty \tag{5.15}$$

in probability as $n \to \infty$ when $\beta = 1/\alpha - \epsilon$. Let $\mathscr{R}_n$ be the range of $\tilde{X}$ up to time n, that is, $\mathscr{R}_n = \{k : L_{n,k} > 0\}$, and set $R_n = |\mathscr{R}_n|$, the cardinality of $\mathscr{R}_n$. Let $\tilde{\tau}_n$ denote the max of τ over the range $\mathscr{R}_n$. That is, $\tilde{\tau}_n = \max_{k\in\mathscr{R}_n}\tau_k$. It is clear that S_n is stochastically bounded below by $\tilde{\tau}_n$ times an exponential random variable of mean one independent of $\tilde{\tau}_n$. Now, Theorem 20.1 in [27] implies that $R_n/n^{1-\delta} \to \infty$ in probability as $n \to \infty$ for any $\delta > 0$. This together with standard results on the asymptotics of the maxima of independent copies of τ_0 (which are easy to derive) implies, using the left hand side inequality of (2.4), that $\tilde{\tau}_n/n^{1/\alpha-\epsilon} \to \infty$ in probability as $n \to \infty$ for any $\epsilon > 0$ and (5.15) follows. □

5.6. Completion of Proof of Theorem 3 for 0< α ≤1.

Arguing as in the beginning of the proof of Lemma 4.1, from Lemma 5.2 we get that for $0 < \alpha \le 1$ and then for $\epsilon > 0$ sufficiently small $\mathbb{P}\big(t^{\alpha-\epsilon} \le N_t \le t^{\alpha+\epsilon}\big) \to 1$ as $t \to \infty$, where N_t is again the number of jumps by X up to time t. This implies that

$$\mathbb{P}\left(t^{\alpha-\epsilon} \le N_t \le t^{\alpha+\epsilon}\,\middle|\,\tau\right) \to 1 \tag{5.16}$$

as $n \to \infty$, in probability (with respect to τ).

We proceed with an argument similar to that of (the rest of) the proof of Lemma 4.1. By (5.16)

$$\mathbb{P}(X_t = i|\tau) = \mathbb{P}(\tilde{X}_{N_t} = i|\tau)$$

$$\leq \mathbb{P}\left(\tilde{X}_{N_t} = i,\ t^{\alpha-\epsilon} \leq N_t \leq t^{\alpha+\epsilon} \middle| \tau\right) + o(1) \ , \tag{5.17}$$

where $o(1)$ denotes a random variable (with respect to τ) tending to 0 *in probability*. Let $B_\alpha = B_\alpha(i,t)$ denote the ball $\{j \in \mathbb{Z}^2 : |j-i| \leq t^{\alpha/2-\epsilon}\}$ and let H_α denote the hitting time of B_α by $\tilde{X}$ after time $t^{\alpha-\epsilon}$. That is, $H_\alpha = \min\{n \geq t^{\alpha-\epsilon} : |\tilde{X}_n - i| \leq t^{\alpha/2-\epsilon}\}$. The last probability in (5.17) is then seen to be bounded above by

$$\mathbb{P}\left(|\tilde{X}_{\lfloor t^{\alpha-\epsilon}\rfloor} - i| \leq t^{\alpha/2-\epsilon}\right) + \mathbb{P}\left(|\tilde{X}_{\lfloor t^{\alpha-\epsilon}\rfloor} - i| > t^{\alpha/2-\epsilon},\ L^*_{t^{\alpha+\epsilon},\,i} > 0\right) \tag{5.18}$$

where $L^*_{\cdot,\cdot}$ is the local time of $\tilde{X}$ from H_α on, that is, for $n \leq H_\alpha$, $L^*_{n,k} = 0$ and, for $n > H_\alpha$, $L^*_{n,k} = \sum_{m=H_\alpha+1}^{n} 1\{\tilde{X}_m = k\}$.

By the $1/n$ decay of the max over i of $\mathbb{P}(\tilde{X}_n = i)$ (a well known result, see Lemma 16.4 in [27]), the sup over i of the first probability in (5.18) goes to 0 as $n \to \infty$. By the strong Markov property of $\tilde{X}$, the second probability in (5.18) can be written as

$$\sum_{j:||i-j||>t^{\alpha/2-\epsilon}} \ \sum_{l=\lfloor t^{\alpha-\epsilon}\rfloor+1}^{\lfloor t^{\alpha+\epsilon}\rfloor} \ \sum_{k\in\partial B_\alpha} \mathbb{P}\left(\tilde{X}_{\lfloor t^{\alpha-\epsilon}\rfloor} = j,\ H_\alpha = l,\ \tilde{X}_l = k\right) \times \mathbb{P}\left(L_{\lfloor t^{\alpha+\epsilon}\rfloor - l,\, i-k} > 0\right) \ ,$$

where ∂B_α is an appropriately defined boundary of B_α. The sup over i of the above sum is thus bounded above by

$$\sup_{i,j:||i-j||>t^{\alpha/2-\epsilon}} \ \sup_{\lfloor t^{\alpha-\epsilon}\rfloor<l\leq\lfloor t^{\alpha+\epsilon}\rfloor} \ \sup_{k\in\partial B_\alpha(i,t)} \mathbb{P}\left(L_{\lfloor t^{\alpha+\epsilon}\rfloor - l,\, i-k} > 0\right) \leq \sup_{k\in\partial B_\alpha(0,t)} \mathbb{P}\left(L_{\lfloor t^{\alpha+\epsilon}\rfloor - \lfloor t^{\alpha-\epsilon}\rfloor,\, k} > 0\right) \ . \tag{5.19}$$

Now, Theorem 19.3 in [27] tells us that the lim sup of (5.19) as $t \to \infty$ is $O(\epsilon)$. Since ϵ is arbitrary, it follows that $\sup_i \mathbb{P}(X_t = i|\tau) \to 0$ in probability. Then, as in the proof of Theorem 4 (compare with (5.7)), $E[\sigma_t(0)|\tau,\xi] \to 0$ in L^2 and hence in probability (with respect to τ and ξ). The convergence of higher order correlations is obtained as in the proof of Theorem 2 or by using the almost sure coalescence of random walks for $d = 2$ (see Remark 5.2). □

Remark 5.2. *From the almost sure coalescence of all random walks of the CRWRR for* $d = 2$ (*see Remark* 2.3), *it is clear that the weak limit of the distribution of* σ_t (*in probability with respect to* τ *and* ξ) *is a fair mixture* *of the point measures on the two invariant configurations* $\sigma^+ \equiv +1$ *and* $\sigma^- \equiv -1$. *This means that the correlations of* σ_t *converge in probability with respect to* τ *and* ξ *to those of* $\frac{1}{2}\delta_{\sigma^+} + \frac{1}{2}\delta_{\sigma^-}$, *where* $\delta_{\sigma'}$ *is the degenerate probability measure supported on the single state* σ'.

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