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Stochastic oscillatory integrals with quadratic phase function and Jacobi equations

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Dedicated to Professor Nobuyuki Ikeda on his 70th birthday

Abstract. An evaluation of a stochastic oscillatory integral with quadratic phase function and analytic amplitude function is given by using solutions of Jacobi equations. The evaluation will be obtained as an application of real change of variable formulas and holomorphic prolongations of analytic functions on a real Wiener space. On the way we shall see how a Jacobi equation appears in the evaluation by using the Malliavin calculus.

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Introduction

In [8], an explicit evaluation of stochastic oscillatory integrals with quadratic phase function and analytic amplitude function was established with the help of eigenfunction expansion of the associated Hilbert-Schmidt operator. The evaluation was used in [10] to study a principle of stationary phase on a real abstract Wiener space. In both cases the eigenfunction expansion is indispensable, but such an expansion is a kind of detour when we have a concrete quadratic Wiener functional like Lévy's stochastic area. In this paper, we shall establish another explicit evaluation of stochastic oscillatory integrals with quadratic phase function and analytic amplitude function on a standard Wiener space.

Let $d \in \mathbb{N}$, $\tau > 0$, $\mathscr{W}_\tau$ be a d-dimensional classical Wiener space on $[0, \tau]$,

$$\mathscr{W}_\tau = \left\{ w : [0, \tau] \to \mathbb{R}^d : \text{ continuous and } w(0) = 0 \right\} \ ,$$

and μ_τ be the standard Wiener measure on it. The stochastic oscillatory integral dealt with in the present paper is of the form

$$\int_{\mathscr{W}_\tau} \exp\left[\frac{\zeta}{2} \int_0^\tau \{ \langle \gamma(t) w(t),\ dw(t) \rangle_{\mathbb{R}^d} + \langle \delta(t) w(t),\ w(t) \rangle_{\mathbb{R}^d}\, dt \} \right] \times\ \psi(w) \mu_\tau(dw) \tag{0.1}$$

where $\zeta \in \mathbb{C}$, $\gamma, \delta \in C^0([0, \tau]; \mathbb{R}^d \otimes \mathbb{R}^d)$, $dw(t)$ stands for the Itô integral, and $\psi : \mathscr{W}_\tau \to \mathbb{R}$ is an analytic Wiener functional. For the definition of analytic Wiener functionals, see Sect. 1 In the section, the evaluation of the above integral will be given without proofs (Theorem 1.1). As an application, we shall extend the evaluation of the integral (0.1) to the case when

$$\psi(w) = \delta_\xi(w(\tau)) \ ,$$

where the right hand side of the identity stands for Watanabe's pullback of Dirac's delta function δ_ξ concentrating at $\xi \in \mathbb{R}^d$ through the nondegenerate mapping $\mathscr{W}_\tau \ni w \mapsto w(\tau) \in \mathbb{R}^d$. See Corollary 1.1. The proofs will be given in Sect. 3.

It will be seen that a linear transformation on $\mathscr{W}_\tau$ determined by a Jacobi equation and a holomoprhic prolongation play a key role in the evaluation of such an oscillatory integral. The idea of using Jacobi (or Sturm-Liouville, or Riccati) equations to evaluate Wiener integrals goes back to Cameron-Martin's work in 1945 [1], and was used by several authors (e.g. [4, 6, 9]). We shall come to a Jacobi equation by revisiting Cameron-Martin's idea with recently developed change of variable formulas on $\mathscr{W}_\tau$. For details, see Sect. 2. It should be remarked that the quadratic functionals investigated by the above mentioned authors were of harmonic oscillator type, that is, ones with $\gamma \equiv 0$.

Some remarks on ζ's for which the evaluation of the integral given in (0.1) is possible will be given in Sect. 4.

1. Statements of results

Throughout the paper, $\mathbb{R}^m \otimes \mathbb{R}^n$, $n, m \in \mathbb{N}$, (resp. $\mathbb{C}^m \otimes \mathbb{C}^n$) denotes the space of $m \times n$-matrices $A = (A_{ij})_{1 \le i \le m, 1 \le j \le n}$ with $A_{ij} \in \mathbb{R}$ (resp. $\mathbb{C}$). The adjoint matrix of $A = (A_{ij})$ is denoted by A^*; $(A^*)_{ij} = \overline{A_{ji}}$ (for $A \in \mathbb{R}^m \otimes \mathbb{R}^n$, A^* is just a transporsed matrix). As usual, $\mathbb{R}^m \otimes \mathbb{R}^n \subset \mathbb{C}^m \otimes \mathbb{C}^n$, $\mathbb{R}^m \otimes \mathbb{R}^1 = \mathbb{R}^m$, and $\mathbb{C}^m \otimes \mathbb{C}^1 = \mathbb{C}^m$. For $f =$

$(f_{ij})_{1\le i\le m,1\le j\le n} \in C^2([0,\tau];\mathbb{C}^m\otimes\mathbb{C}^n)$, f' and f'' are used to denote the first and second derivatives of f; $f'(t) = \big((df_{ij}/dt)(t)\big)_{1\le i\le m,\le j\le n}$, and $f''(t) = \big((d^2f_{ij}/dt^2)(t)\big)_{1\le i\le m,1\le j\le n}$.

To state our results, we review briefly on analytic functions on $\mathscr{W}_\tau$. Let H_τ the Cameron-Martin subspace of $\mathscr{W}_\tau$;

$$H_\tau = \left\{ h\in\mathscr{W}_\tau : \begin{array}{l} h \text{ is absolutely continuous and has} \\ \text{a derivative } h' \text{ in } L^2([0,\tau];\mathbb{R}^d) \end{array} \right\}$$

H_τ is a real separable Hilbert space with inner product

$$\langle h,k\rangle = \int_0^\tau \langle h'(t),k'(t)\rangle_{\mathbb{R}^d}\,dt,\qquad h,k\in H_\tau\ ,$$

where $\langle\cdot,\cdot\rangle_{\mathbb{R}^d}$ stands for the inner product on $\mathbb{R}^d$. For a real separable Hilbert space G, $\mathbb{D}^{k,p}(G)$ denotes the space of G-valued k-times differentiable Wiener functionals F: $\mathscr{W}_\tau\to G$ with p-th integrable derivatives of orders up to k in the sense of the Malliavin calculus. Set $\mathbb{D}^{\infty,\infty-}(G) = \bigcap_{k\in\mathbb{N},p\in(1,\infty)}\mathbb{D}^{k,p}(G)$. We shall use ∇ and ∇^* to denote the Malliavin gradient and its adjoint operator, respectively; for $F\in\mathbb{D}^{\infty,\infty-}(G)$, ∇F is an element of $\mathbb{D}^{\infty,\infty-}(G\otimes H_\tau)$, $G\otimes H_\tau$ being the Hilbert space of Hilbert-Schmidt operators of G to H_τ. The n-th Malliavin derivative $\nabla^n F$ is defined successively. For details, see [12].

As was seen in [10], if $F\in\mathbb{D}^{\infty,\infty-}(\mathbb{R})$ admits a $p\in(1,\infty)$ such that

$$\sum_{n=0}^\infty \frac{s^n}{n!}\|\nabla^n F\|_{L^p(\mu_\tau)} < \infty \quad \text{for every } s>0, \tag{1.1}$$

then, choosing suitable μ-versions of $\nabla^n F$, $n=0,1,\ldots$, we arrive at

$$\sum_{n=0}^\infty \frac{s^n}{n!}\|\nabla^n F(w)\|_{H_\tau^{\otimes n}} < \infty \text{ and } \quad F(w+h) = \sum_{n=0}^\infty \frac{1}{n!}\langle\nabla^n F(w),h^{\otimes n}\rangle$$
$$\text{for any } s\in(0,\infty),\ w\in\mathscr{W}_\tau,\ h\in H_\tau\ . \tag{1.2}$$

We shall call $F\in\mathbb{D}^{\infty,\infty-}(\mathbb{R})$ analytic if it satisfies Eq. (1.1) ($F\in C^\omega(\mathscr{W}_\tau)$ in notation). In what follows, we always choose $\nabla^n F$'s appropriately so that every $F\in C^\omega(\mathscr{W}_\tau)$ enjoys the property stated in (1.2).

Write $\mathscr{W}_\tau\oplus\sqrt{-1}\,H_\tau$ for $\mathscr{W}_\tau\times H_\tau$, and $w+\sqrt{-1}\,h$ for $(w,h)\in\mathscr{W}_\tau\times H_\tau$, and think of $\mathscr{W}_\tau\oplus\sqrt{-1}\,H_\tau$ as a complexification of $\mathscr{W}_\tau$. A standard complexification $H_\tau\times H_\tau$ of H_τ is denoted by $H_\tau\oplus\sqrt{-1}\,H_\tau$. For analytic $F:\mathscr{W}_\tau\to\mathbb{R}$, *its holomorphic prolongation* to $\mathscr{W}_\tau\oplus\sqrt{-1}\,H_\tau$, say F again, is defined by

$$F(w+\sqrt{-1}\,h) = \sum_{n=0}^{\infty} \frac{\sqrt{-1}^{\,n}}{n!} \langle \nabla^n F(w), h^{\otimes n} \rangle$$
$$\text{for } w + \sqrt{-1}\,h \in \mathscr{W}_\tau \oplus \sqrt{-1}\,H_\tau \ .$$

For $F \in C^\omega(\mathscr{W}_\tau)$, by Eq. (1.2) and the definition of holomorphic prolongation of F, it holds that

$$F\left((w+h)+\sqrt{-1}\,k\right) = \sum_{n=0}^{\infty} \frac{1}{n!} \langle \nabla^n F(w), (h+\sqrt{-1}\,k)^{\otimes n} \rangle$$
$$\text{for every } w \in \mathscr{W}_\tau,\ h,k \in H_\tau, \tag{1.3}$$

where $\nabla^n F(w)$ is extended to $(H_\tau \oplus \sqrt{-1}\,H_\tau)^{\otimes n}$ as a complex multi-linear mapping (cf. [8, 10]).

We set

$$\mathscr{A}_{\mathbb{C}} = \left\{ A \in C^2([0,\tau]; \mathbb{C}^d \otimes \mathbb{C}^d) : \det A(t) \neq 0 \quad \text{for any } t \in [0,\tau] \right\}$$

For $A \in \mathscr{A}_{\mathbb{C}}$, a transformation $T_A : \mathscr{W}_\tau \to H_\tau \oplus \sqrt{-1}\,H_\tau$ is defined by

$$(T_A w)(t) = -A(t) \int_0^t (A^{-1})'(s) w(s)\, ds, \tag{1.4}$$

where $A^{-1}(t) = A(t)^{-1}$, $t \in [0,\tau]$.

Fix $\alpha \in C^1([0,\tau]; \mathbb{R}^d \otimes \mathbb{R}^d)$ and $\beta \in C^0([0,\tau]; \mathbb{R}^d \otimes \mathbb{R}^d)$, and, for $\zeta \in \mathbb{C}$, let $A_\zeta \in C^2([0,\tau]; \mathbb{C}^d \otimes \mathbb{C}^d)$ be a unique solution to a Jacobi equation;

$$\begin{cases} A_\zeta''(t) - \zeta(\alpha(t) - \alpha(t)^*) A_\zeta'(t) + \zeta(\beta(t) - \alpha'(t)) A_\zeta(t) = 0, \\ A_\zeta(\tau) = I, \quad A_\zeta'(\tau) = \zeta \alpha(\tau). \end{cases} \tag{1.5}$$

It should be emphasized that $\alpha(t)$, $\beta(t)$ are real matrices. Define $\Omega(\alpha,\beta) = \left\{ \zeta \in \mathbb{C} : A_\zeta \in \mathscr{A}_{\mathbb{C}} \right\}$. It is easily seen that $\Omega(\alpha,\beta)$ is open and contains the origin $0 \in \mathbb{C}$ (cf. Lemma 3.4). For $\omega \in C^0([0,\tau] : \mathbb{R}^d \otimes \mathbb{R}^d)$, define the uniform norm of ω by

$$\|\omega\|_\infty = \sup \left\{ |\omega(t)\xi| : t \in [0,\tau],\ \xi \in \mathbb{R}^d,\ |\xi| \le 1 \right\}$$

We denote by $\Omega_0(\alpha,\beta)$ the connected component containing 0 of an open set consisting of $\zeta \in \Omega(\alpha,\beta)$ satisfying that

$$(\Re\zeta)^2 \|\alpha - \alpha^*\|_\infty^2 + 2|\Re\zeta|\|\beta - \alpha'\|_\infty < \frac{1}{\tau^2} \tag{1.6}$$

We are now ready to state our results.

Theorem 1.1. *Let* $\alpha \in C^1([0,\tau]; \mathbb{R}^d \otimes \mathbb{R}^d)$ *and* $\beta \in C^0([0,\tau]; \mathbb{R}^d \otimes \mathbb{R}^d)$. *Suppose that* $\alpha(\tau)^* = -\alpha(\tau)$ *and* $\beta(t)^* = \beta(t)$ *for every* $t \in [0,\tau]$. *Take* $\psi \in C^\omega(\mathscr{W}_\tau)$ *such that*

$$m[\psi, s] \equiv \sum_{n=0}^{\infty} \frac{s^n}{n!} \|\nabla^n \psi\|^2_{H_\tau^{\otimes n}} \in L^1(\mu_\tau) \quad \text{for any } s > 0 \ . \tag{1.7}$$

Then, for every $\zeta \in \Omega_0(\alpha, \beta)$, *it holds that*

$$\int_{\mathscr{W}_\tau} \exp\left[\frac{\zeta}{2}\int_0^\tau \left\{\langle \gamma(t)w(t), dw(t)\rangle_{\mathbb{R}^d} + \langle \delta(t)w(t), w(t)\rangle_{\mathbb{R}^d}\ dt\right\}\right]\psi(w)\mu_\tau(dw)$$
$$= \exp\left[\frac{1}{2}\int_0^\tau \mathrm{tr}\ \left\{A'_\zeta(t)A_\zeta^{-1}(t)\right\} dt\right] \int_{\mathscr{W}_\tau} \psi(w + T_{A_\zeta} w)\mu_\tau(dw)\ , \tag{1.8}$$

where $\gamma(t) = \alpha(t) - \alpha(t)^*$ *and* $\delta(t) = \beta(t) - \alpha'(t)$.

Remark 1.1. As we shall see in Sect. 3, both integrands in Eq. (1.8) are integrable.

Corollary 1.1. *Let* $\alpha, \beta, \gamma, \delta$ *be as in Theorem* 1.1. *Then there exists an* $\varepsilon > 0$ *such that (i)* $U(\varepsilon) \equiv \{\zeta \in \mathbb{C} : |\zeta| < \varepsilon\} \subset \Omega_0(\alpha, \beta)$, *(ii)* $\Re(\det C_\zeta) > 0$ *for any* $\zeta \in U(\varepsilon)$, *and (iii) it holds that, for for every* $\xi \in \mathbb{R}^d$ *and* $\zeta \in U(\varepsilon)$,

$$\int_{\mathscr{W}_\tau} \exp\left[\frac{\zeta}{2}\int_0^\tau \left\{\langle \gamma(t)w(t), dw(t)\rangle_{\mathbb{R}^d} + \langle \delta(t)w(t), w(t)\rangle_{\mathbb{R}^d}\ dt\right\}\right]$$
$$\times \delta_\xi\ (w(\tau))\ \mu_\tau(dw)$$
$$= \exp\left[\frac{1}{2}\int_0^\tau \mathrm{tr}\left\{A'_\zeta(t)A_\zeta^{-1}(t)\right\} dt\right] \frac{1}{\sqrt{2\pi}^d \sqrt{\det C_\zeta}} \exp\left[-\frac{1}{2}\langle C_\zeta^{-1}\xi, \xi\rangle_{\mathbb{R}^d}\right], \tag{1.9}$$

where $\int_{\mathscr{W}_\tau}(\ldots)\delta_\xi(w(\tau))\mu_\tau(dw)$ *stands for the pairing of* $\mathbb{D}^{\infty,\infty-}$ *Wiener functionals and Watanabe's pullback of the Dirac measure* δ_ξ *via the non-degenerate mapping* $\mathscr{W}_\tau \ni w \mapsto w(\tau) \in \mathbb{R}^d$ *(cf.* [7, 12]*), and*

$$C_\zeta = \int_0^\tau \left(A_\zeta^{-1}(s)\right)^T A_\zeta^{-1}(s)\, ds,$$

M^T *being the transposed matrix of* $M \in \mathbb{C}^n \times \mathbb{C}^n$.

Remark 1.2. As we shall see in Sect. 3,

$$\exp\left[\frac{\zeta}{2}\int_0^\tau \{\langle \gamma(t)w(t), dw(t)\rangle_{\mathbb{R}^d} + \langle \delta(t)w(t), w(t)\rangle_{\mathbb{R}^d} dt\}\right] \in \bigcap_{k\in\mathbb{N}} \bigcup_{p\in(1,\infty)} \mathbb{D}^{k,p}(\mathbb{C})$$

for ζ as stated in the assertion. Hence the first generalized integration in the left hand side of Eq. (1.9) is well-defined (cf. [12]).

Due to (ii), C_ζ^{-1} and $(\det C_\zeta)^{-1/2}$ are both well-defined.

2. How one comes to a Jacobi equation

In this section, we see how one comes to Jacobi equations to evaluate stochastic oscillatory integrals with quadratic phase function. Throughout this section, we fix $\alpha \in C^1([0,\tau]; \mathbb{R}^d \otimes \mathbb{R}^d)$ and $\beta \in C^0([0,\tau]; \mathbb{R}^d \otimes \mathbb{R}^d)$, and assume that $\alpha(\tau)^* = -\alpha(\tau)$ and $\beta(t)^* = \beta(t)$, $t \in [0,\tau]$. Let $\lambda \in \mathbb{R}$. We shall see how a Jacobi equation is involved in the evaluation of the Wiener integral in Eq. (1.8) with $\zeta = \lambda$.

Consider a continuous linear transformation $K_\lambda\colon \mathscr{W}_\tau \to H_\tau$, and apply a change of variable formula on $\mathscr{W}_\tau$ (cf. [1, 11]) to $w + K_\lambda w$. Under an integrability condition that $\mathrm{e}^{-\nabla^* K_\lambda} \in \bigcup_{p\in(1,\infty)} L^p(\mu_\tau)$, where, to apply ∇^*, K_λ was thought of as an H_τ-valued Wiener functional, we obtain that

$$\int_{\mathscr{W}_\tau} f\,d\mu = \int_{\mathscr{W}_\tau} f(w + K_\lambda w)\det{}_2\big(I + (K_\lambda|_{H_\tau})\big) \times e^{-\nabla^* K_\lambda(w) - (\langle K_\lambda w, K_\lambda w\rangle/2)}\mu(dw)$$

for any $f \in L^{\infty-}(\mu_\tau)$.

Assuming in addition that $(K_\lambda w)'(t)$ is adapted with respect to the standard filtering on $\mathscr{W}_\tau$ and that $\exp[\|K_\lambda w\|^2_{H_\tau}/2] \in L^1(\mu_\tau)$, and applying Girsanov's and Novikov's theorem, we can conclude that $\det_2(I + K_\lambda|_{H_\tau}) = 1$ (cf. Lemma 3.3 or [13]). Thus our identity (1.8) will be verified once we have found a K_λ so that

$$-\nabla^* K_\lambda(w) - \tfrac{1}{2}\langle K_\lambda w, K_\lambda w\rangle = \frac{\lambda}{2}\int_0^\tau \Big\{\big\langle(\alpha(t) - \alpha(t)^*)w(t),\ dw(t)\big\rangle_{\mathbb{R}^d} + \big\langle(\beta(t) - \alpha'(t))w(t),\ w(t)\big\rangle_{\mathbb{R}^d}\ dt\Big\} \quad (2.1)$$

To find such a K_λ, we compare the second Malliavin gradients of the both sides of Eq. (2.1). We then have that

$$\big\langle\big(-(K_\lambda + K_\lambda^*) - K_\lambda^* K_\lambda\big)h, h\big\rangle = \lambda\int_0^\tau \Big\{\big\langle(\alpha(t) - \alpha(t)^*)h(t), h'(t)\big\rangle_{\mathbb{R}^d} + \big\langle(\beta(t) - \alpha'(t))h(t), h(t)\big\rangle_{\mathbb{R}^d}\Big\}\,dt. \quad (2.2)$$

Suppose that there exists an $X \in C^1([0,\tau]; \mathbb{R}^d \otimes \mathbb{R}^d)$ such that

$$(K_\lambda w)(t) = -\int_0^t X(s)w(s)\,ds.$$

Notice that, for any $G \in C^0([0,\tau]; \mathbb{R}^d \otimes \mathbb{R}^d)$ with $G(t) = G(t)^*$, $t \in [0,\tau]$, and $h \in H_\tau$,

$$\int_0^\tau \langle G(t)h(t), h(t)\rangle_{\mathbb{R}^d}\,dt = 2\int_0^\tau \left\langle \left(\int_t^\tau G(s)\,ds\right) h(t), h'(t)\right\rangle_{\mathbb{R}^d} dt \ .$$

Since $\alpha(\tau)^* = -\alpha(\tau)$ and

$$\big\langle(\beta(t) - \alpha'(t))h(t), h(t)\big\rangle_{\mathbb{R}^d} = \left\langle \left(\beta(t) - \frac{\alpha'(t) + \alpha'(t)^*}{2}\right) h(t), h(t)\right\rangle_{\mathbb{R}^d} ,$$

Eq. (2.2) then implies that

$$\int_0^\tau \left\langle \left(X(t) - \int_t^\tau X(s)^* X(s)\,ds\right) h(t), h'(t)\right\rangle_{\mathbb{R}^d} dt$$
$$= \lambda \int_0^\tau \left\langle \left(\alpha(t) + \int_t^\tau \beta(s)\,ds\right) h(t), h'(t)\right\rangle_{\mathbb{R}^d} dt \ .$$

By Lemma A.1, X satisfies that

$$X(t) - \int_t^\tau X(s)^* X(s)\,ds = \lambda\alpha(t) + \lambda\int_t^\tau \beta(s)\,ds \quad \text{for any } t \in [0, \tau] \ . \tag{2.3}$$

By virtue of Lemma A.2, the solution $X(t)$ gives a rise of Jacobi equation; if we denote by $A_\lambda \in C^2([0, \tau]; \mathbb{R}^d \otimes \mathbb{R}^d)$ a unique solution to an ordinary differential equatiaon (ODE)

$$A'_\lambda(t) = X(t)A_\lambda(t), \quad A_\lambda(\tau) = I \ ,$$

then A_λ solves the Jacobi equation (1.5) with $\zeta = \lambda$.

3. Proofs

We shall give proofs of Theorem 1.1 and Corollary 1.1. Throughout this section, we fix $\alpha \in C^1([0, \tau]; \mathbb{R}^d \otimes \mathbb{R}^d)$ and $\beta \in C^0([0, \tau]; \mathbb{R}^d \otimes \mathbb{R}^d)$ such that $\alpha(\tau)^* = -\alpha(\tau)$ and $\beta(t)^* = \beta(t)$ for any $t \in [0, \tau]$, and put $\gamma(t) = \alpha(t) - \alpha(t)^*$ and $\delta(t) = \beta(t) - \alpha'(t)$. We shall use the same notation $|\cdot|$ to indicate any of Euclidean norms on $\mathbb{R}^d$, $\mathbb{C}^d$, $\mathbb{R}^m \otimes \mathbb{R}^n$, and $\mathbb{C}^m \otimes \mathbb{C}^n$.

To emphasize that real matrices are dealt with, we introduce a family

$$\mathscr{A}_{\mathbb{R}} = \{A \in \mathscr{A}_{\mathbb{C}} : A(t) \in \mathbb{R}^d \otimes \mathbb{R}^d, \quad t \in [0, \tau]\}$$

On account of the observation in Sect. 2, for $A \in \mathscr{A}_{\mathbb{R}}$, we define an operator K_A: $\mathscr{W}_\tau \to H_\tau$ by

$$(K_A w)(t) = -\int_0^t A'(s)A^{-1}(s)w(s)\,ds, \quad t \in [0, \tau] \ .$$

Lemma 3.1. *Let $A \in \mathscr{A}_{\mathbb{R}}$. Then T_A defined in Eq. (1.4) and the above K_A are both continuous linear operators of $\mathscr{W}_\tau$ into itself, and satisfy that*

$$(I + K_A)\,(I + T_A)\,w = (I + T_A)\,(I + K_A)\,w = w \quad \text{for every } w \in \mathscr{W}_\tau.$$

Proof. It is easily seen that K_A and T_A are both continuous linear mappings of $\mathscr{W}_\tau$ to itself. Using integration by parts formulas, we can easily show that the identity holds for $h \in H_\tau$. Since H_τ is dense in $\mathscr{W}_\tau$, the desired identity holds.

Lemma 3.2. *Consider $\kappa \in C^0([0,\tau]; \mathbb{R}^d \otimes \mathbb{R}^d)$ such that $\kappa(t)$ is symmetric for any $t \in [0,\tau]$. Then it holds that*

$$\int_0^\tau \langle \kappa(t)w(t), w(t)\rangle_{\mathbb{R}^d}\, dt = 2\int_0^\tau \left\langle \left(\int_t^\tau \kappa(s)\, ds\right) w(t), dw(t)\right\rangle_{\mathbb{R}^d} + \int_0^\tau \left(\int_t^\tau \operatorname{tr}\kappa(s)\, ds\right) dt \ .$$

Moreover, for $\xi \in C^1([0,\tau]; \mathbb{R}^d \otimes \mathbb{R}^d)$ with $\xi(\tau)^* = -\xi(\tau)$ and $\eta \in C^0([0,\tau]; \mathbb{R}^d \otimes \mathbb{R}^d)$, it holds that

$$\begin{aligned}
&\int_0^\tau \langle \xi(t)w(t),\ dw(t)\rangle_{\mathbb{R}^d} + \int_0^\tau \langle \eta(t)w(t), w(t)\rangle_{\mathbb{R}^d}\ dt \\
&= \frac{1}{2}\int_0^\tau \left\langle (\xi(t) - \xi(t)^*)w(t),\ dw(t)\right\rangle_{\mathbb{R}^d} \\
&\quad + \frac{1}{2}\int_0^\tau \left\langle (\eta(t) + \eta(t)^* - \xi'(t))w(t), w(t)\right\rangle_{\mathbb{R}^d}\, dt - \frac{1}{2}\int_0^\tau \operatorname{tr}\xi(t)\, dt \ .
\end{aligned}$$

Proof. The first assertion can be seen easily by applying Itô's formula to

$$\left\langle \left(\int_t^\tau \kappa(s)\, ds\right) w(t), w(t)\right\rangle_{\mathbb{R}^d}$$

The second one follows from the first by using a decomposition of $\xi(t)$ into symmetric and skew symmetric parts; $\xi(t) = (\xi(t) + \xi(t)^*)/2 + (\xi(t) - \xi(t)^*)/2$.

Lemma 3.3. *Assume that $A \in \mathscr{A}_{\mathbb{R}}$ solves the ODE (1.5) with $\zeta = 1$ and satisfies that*

$$\exp\left[\frac{1}{2}\int_0^\tau \left|A'(t)A^{-1}(t)w(t)\right|^2 dt\right] \in L^1(\mu_\tau) \ . \tag{3.1}$$

Then it holds that

$$\int_{\mathscr{W}_\tau} \exp\left[\frac{1}{2}\int_0^\tau \left\{\langle\gamma(t)w(t),\, dw(t)\rangle_{\mathbb{R}^d} + \langle\delta(t)w(t),\, w(t)\rangle_{\mathbb{R}^d}\, dt\right\}\right] \times f(w)\mu_\tau(dw)$$

$$= \exp\left[\frac{1}{2}\int_0^\tau \mathrm{tr}\,\{A'(t)A^{-1}(t)\}dt\right]\int_{\mathscr{W}_\tau} f(w + T_A w)\mu_\tau(dw) \quad (3.2)$$

for any bounded measurable $f\colon \mathscr{W}_\tau \to \mathbb{R}$.

Proof. Set $X(t) = A'(t)A^{-1}(t)$. Then, K_A can be written as

$$K_A(w)(t) = -\int_0^t X(s)w(s)\,ds.$$

By Lemmas 3.2 and A.2, we obtain that

$$\int_0^\tau \langle X(t)w(t),\, dw(t)\rangle_{\mathbb{R}^d} - \frac{1}{2}\int_0^\tau \langle X(t)w(t),\, X(t)w(t)\rangle_{\mathbb{R}^d}\, dt$$

$$= \frac{1}{2}\int_0^\tau \{\langle\gamma(t)w(t),\, dw(t)\rangle_{\mathbb{R}^d} + \langle\delta(t)w(t),\, w(t)\rangle_{\mathbb{R}^d}\, dt\} - \frac{1}{2}\int_0^\tau \mathrm{tr}\, X(t)\, dt \quad .$$

On account of Eq. (3.1), applying Girsanov's and Novikov's theorems, we obtain from this that

$$\int_{\mathscr{W}_\tau} f(w)\mu_\tau(dw) = \int_{\mathscr{W}_\tau} f(w + K_A w)\exp\left[\frac{1}{2}\int_0^\tau \{\langle\gamma(t)w(t),\, dw(t)\rangle_{\mathbb{R}^d} + \langle\delta(t)w(t),\, w(t)\rangle_{\mathbb{R}^d}\, dt\} - \frac{1}{2}\int_0^\tau \mathrm{tr}\, X(t)\, dt\right]\mu_\tau(dw)$$

for any bounded measurable $f\colon \mathscr{W}_\tau \to \mathbb{R}$. Substituting $f \circ (I + T_A)$ for f in this identity, by virtue of Lemma 3.1, we come to Eq. (3.2).

Lemma 3.4. *Let* $\zeta \in \mathbb{C}$. *Consider a unique solution* $A_\zeta \in C^2([0, \tau]; \mathbb{C}^d \otimes \mathbb{C}^d)$ *to the ODE (1.5). Then the mapping* $(t, \zeta) \mapsto A_\zeta(t)$ *is continuous on* $[0, \tau] \times \mathbb{C}$, *and* $\zeta \mapsto A_\zeta(t)$ *is holomorphic on* $\mathbb{C}$. *Moreover, there exists an* $\varepsilon_0 > 0$ *such that (a)* $A_\zeta \in \mathscr{A}_{\mathbb{C}}$ *for* $\zeta \in \overline{U(\varepsilon_0)}$, *where* $U(r) = \{\zeta \in \mathbb{C} : |\zeta| < r\}$, *and (b) there exist* $0 < M_0, M_1 < \infty$ *such that*

$$\sup_{\zeta\in\overline{U(\varepsilon_0)}}\ \sup_{t\in[0,\tau]}\ \max\left\{|A_\zeta(t)|,\ |A'_\zeta(t)|,\ |A_\zeta^{-1}(t)|,\ |(A_\zeta^{-1})'(t)|\right\} \le M_0 \quad , \quad (3.3)$$

$$\sup_{\zeta\in\overline{U(\varepsilon_0)}}\ \sup_{t\in[0,\tau]}\left|\frac{1}{\zeta}A'_\zeta(t)\right| \le M_1 \quad . \quad (3.4)$$

In particular, $\overline{U(\varepsilon_0)} \subset \Omega_0(\alpha, \beta)$.

Proof. The continuity and the holomorphy are elementary facts in the theory of ODE's. Since $A_0(t) \equiv I$, we can find an $\varepsilon_0 > 0$ such that (a) and the estimation (3.3) holds. By (3.3), it follows from Eq. (1.5) that there exists an $M_3 < \infty$ such that

$$\sup_{t\in[0,\tau]} \left|A''_\zeta(t)\right| \le M_3|\zeta| \quad \text{for any } \zeta \in U(\varepsilon_0).$$

This implies that

$$\left|A'_\zeta(t)\right| = \left|A'_\zeta(\tau) - \int_t^\tau A''_\zeta(s)\,ds\right| \le |\zeta|(|\alpha(\tau)| + M_3\tau),$$
$$t \in [0,\tau],\ \zeta \in U(\varepsilon_0),$$

which yields the second estimation (3.4).

Lemma 3.5. *Let* $\psi \in C^\omega(\mathscr{W}_\tau)$ *satisfy* (1.7). *Then* $\psi(* + T_A*) \in L^1(\mu_\tau)$ *for every* $A \in \mathscr{A}_{\mathbb{C}}$.

Proof. By (1.3), we have that, for every $N > 0$,

$$|\psi(w)| \le m[\psi, N]^{1/2} \exp\left[\frac{1}{N}\|T_A w\|^2\right].$$

It is easily seen that $\|T_A w\|^2 \le C \sup_{t\in[0,\tau]} |w(t)|^2$ for some $C > 0$, and hence $\psi(w + T_A w)$ is integrable.

Lemma 3.6. *Let* $\varepsilon_0, M_0, M_1 > 0$ *be as in Lemma* 3.4, *and put*

$$\varepsilon_1 = \{\tau M_0 M_1\}^{-1},$$

and take $\lambda \in \mathbb{R}$ *with* $|\lambda| < \varepsilon_0 \wedge \varepsilon_1$. *Then, Eq. (*1.8*) holds with* λ *instead of* ζ.

Proof. By definition, A_λ obeys Eq. (1.5) with $\zeta = \lambda$. Due to Lemma 3.4, $A_\lambda \in \mathscr{A}_{\mathbb{R}}$. Moreover, it follows from Eqs. (3.3) and (3.4) that

$$\int_0^\tau \left|A'_\lambda(t)A_\lambda^{-1}(t)w(t)\right|^2 dt \le |\lambda|^2(M_0M_1)^2\tau \sup_{t\in[0,\tau]} |w(t)|^2.$$

Remembering that $\exp\left[p\sup_{t\in[0,\tau]} |w(t)|^2\right] \in L^1(\mu_\tau)$ if $p < 1/(2\tau)$, we see that the condition (3.1) is fulfilled with $A = A_\lambda$. As an application of Lemma 3.3, we see that

$$\exp\left[\frac{\lambda}{2}\int_0^\tau \left\{\langle\gamma(t)w(t), dw(t)\rangle_{\mathbb{R}^d} + \langle\delta(t)w(t), w(t)\rangle_{\mathbb{R}^d}\,dt\right\}\right] \in L^{1+}(\mu_\tau),$$

where $L^{1+}(\mu_\tau) = \bigcup_{p\in(1,\infty)} L^p(\mu_\tau)$, and

$$\int_{\mathscr{W}_\tau} \exp\left[\frac{\lambda}{2}\int_0^\tau \left\{\langle \gamma(t)w(t), dw(t)\rangle_{\mathbb{R}^d} + \langle \delta(t)w(t), w(t)\rangle_{\mathbb{R}^d}\, dt\right\}\right] \times \psi_N(w)\mu_\tau(dw)$$

$$= \exp\left[\frac{1}{2}\int_0^\tau \operatorname{tr}\left\{A_\lambda'(t)A_\lambda^{-1}(t)\right\} dt\right]\int_{\mathscr{W}_\tau} \psi_N(w + T_{A_\lambda}w)\mu_\tau(dw),$$

where $\psi_N = ((-N) \vee \psi) \wedge N$. Since $\psi \in L^{\infty-}(\mu_\tau) \equiv \bigcap_{p\in(1,\infty)} L^p(\mu_\tau)$, by Lemma 3.5, applying the dominated convergence theorem, we obtain the desired conclusion.

Lemma 3.7. *Let D be a domain in $\mathbb{C}$ consisting of all $\zeta \in \mathbb{C}$ satisfying Eq.* (1.6) *and set*

$$\Phi(\zeta; w) = \exp\left[\frac{\zeta}{2}\int_0^\tau \left\{\langle \gamma(t)w(t), dw(t)\rangle_{\mathbb{R}^d} + \langle \delta(t)w(t), w(t)\rangle_{\mathbb{R}^d}\, dt\right\}\right].$$

ThenΦ, $\Phi(\zeta; *) \in \bigcap_{k\in\mathbb{N}} \bigcup_{p\in(1,\infty)} \mathbb{D}^{k,p}(\mathbb{C})$ *for any $\zeta \in D$, and the mappings*

$$D \ni \zeta \mapsto \int_{\mathscr{W}_\tau} \Phi(\zeta; w)\psi(w)\mu_\tau(dw), \quad \int_{\mathscr{W}_\tau} \Phi(\zeta; w)\delta_\xi(w(\tau))\mu_\tau(dw)$$

are holomorphic.

Proof. By a standard exponential martingale argument, for any $\zeta \in D$ and $\varepsilon > 0$ such that $(1+\varepsilon)\zeta \in D$, we obtain that

$$\int_{\mathscr{W}_\tau} |\Phi(\zeta; w)|^{(1+\varepsilon)}\mu_\tau(dw)$$
$$\le \left\{\int_{\mathscr{W}_\tau} \exp\left[\left(\frac{(1+\varepsilon)^2(\Re\zeta)^2}{2}\|\gamma\|_\infty^2 + (1+\varepsilon)|\Re\zeta|\|\delta\|_\infty\right)\sup_{t\in[0,\tau]}|w(t)|^2\right] \times \mu_\tau(dw)\right\}^{1/2} < \infty.$$

Hence, for any compact $K \subset D$ and $k \in \mathbb{N}$, we can find $p \in (1,\infty)$ such that

$$\sup_{\zeta\in K} \|\Phi(\zeta; *)\|_{\mathbb{D}^{k,p}(\mathbb{C})} < \infty.$$

Now the assertions follows as an application of the dominated convergence theorem and [12,Th. 2.1]

Lemma 3.8. *Let V be an open set in $\mathbb{C}$ satisfying that $A_\zeta \in \mathscr{A}_{\mathbb{C}}$ for any $\zeta \in \overline{V}$ and that*

$$M(R) \equiv \sup_{\zeta \in \overline{V \cap U(R)}} \sup_{t \in [0,\tau]} \max\bigl\{|A_\zeta(t)|,\ |A'_\zeta(t)|,\ |A_\zeta^{-1}(t)|,\ |(A_\zeta^{-1})'(t)|\bigr\} < \infty \tag{3.5}$$

for any $R > 0$. Then, for any $\psi \in C^\omega(\mathscr{W}_\tau)$ satisfying (1.7), the mappings

$$\zeta \mapsto \int_{\mathscr{W}_\tau} \psi(w + T_{A_\zeta} w)\mu_\tau(dw), \quad \exp\left[\frac{1}{2}\int_0^\tau \operatorname{tr}\bigl\{A'_\zeta(t)A_\zeta^{-1}(t)\bigr\}\,dt\right], \quad C_\zeta \tag{3.6}$$

are all holomorphic on V and continuous on the closure $\overline{V}$.

In particular, the mappings

$$\zeta \mapsto \int_{\mathscr{W}_\tau} \psi(w + T_{A_\zeta} w)\mu_\tau(dw), \quad \exp\left[\frac{1}{2}\int_0^\tau \operatorname{tr}\bigl\{A'_\zeta(t)A_\zeta^{-1}(t)\bigr\}\,dt\right], \quad C_\zeta$$

are all holomorphic on $\Omega_0(\alpha, \beta)$.

Proof. Let $R > 0$. It suffices to show that the mappings given in (3.6) are holomorphic on $V \cap U(R)$ and continuous on $\overline{V \cap U(R)}$.

Due to Eq. (1.3), we have that

$$\psi\left(w + T_{A_\zeta} w\right) = \sum_{n=0}^\infty \frac{1}{n!}\langle \nabla^n \psi(w), \left(T_{A_\zeta} w\right)^{\otimes n}\rangle.$$

By (3.5), we can easily see that a mapping $V \cap U(R) \ni \zeta \mapsto \langle \nabla^n \psi(w), \left(T_{A_\zeta} w\right)^{\otimes n}\rangle$ is holomorphic, and that

$$\sup_{\zeta \in \overline{V \cap U(R)}} \|T_{A_\zeta} w\|_{H_\tau \oplus \sqrt{-1} H_\tau} \le M(R)^2 \tau \sup_{t \in [0,\tau]} |w(t)| \quad \text{for any } w \in \mathscr{W}_\tau.$$

Hence a mapping $\zeta \mapsto \psi\left(w + T_{A_\zeta} w\right)$ is holomorphic on $V \cap U(R)$ and continuous on $\overline{V \cap U(R)}$, and

$$\sup_{\zeta \in \overline{V \cap U(R)}} \left|\psi\left(w + T_{A_\zeta} w\right)\right| \le m[\psi, N]^{1/2} \exp\left[\frac{M(R)^4 \tau^2}{N} \sup_{t \in [0,\tau]} |w(t)|^2\right]$$

for any $N \in (0, \infty)$. Applying Lebesgue's theorem, we obtain that a mapping

$$\zeta \mapsto \int_{\mathscr{W}_\tau} \psi\left(w + T_{A_\zeta} w\right) \mu_\tau(dw) \in \mathbb{C}$$

is holomorphic on $V \cap U(R)$ and continuous on $\overline{V \cap U(R)}$.

By Lebesgue's theorem, the mappings

$$\zeta \mapsto \int_0^\tau \operatorname{tr}\big(A'_\zeta(t) A_\zeta^{-1}(t)\big)\, dt, \quad C_\zeta$$

are holomorphic on $V \cap U(R)$ and continuous on $\overline{V \cap U(R)}$.

Proof of Theorem 1.1 The assertion is an immediate consequence of Lemmas 3.6, 3.7, and 3.8

Proof of Corollary 1.1 Let $\xi, \eta \in \mathbb{R}^d$ and set

$$\psi_{\xi,\eta}(w) = \exp\left[\sqrt{-1}\,\langle \eta, w(\tau) - \xi\rangle_{\mathbb{R}^d}\right], \quad w \in \mathscr{W}_\tau.$$

As is easily seen, $\psi_{\xi,\eta} \in C^\omega(\mathscr{W}_\tau)$ and fulfills the condition (1.7). Then, by Theorem 1.1, Eq. (1.8) holds with $\zeta = \lambda$ and $\psi = \psi_{\xi,\eta}$ for $\lambda \in \mathbb{R}$ satisfying the condition (1.6).

Notice that

$$(w + T_{A_\lambda} w)(\tau) = \int_0^\tau A_\lambda(s)^{-1}\, dw(s) \quad \text{(Itô integral)}, \tag{3.7}$$

and hence it is an $\mathbb{R}^d$-valued Gaussian random variable with covariance matrix

$$C_\lambda = \int_0^\tau \big(A_\lambda^{-1}(s)\big)^T A_\lambda^{-1}(s)\, ds.$$

Since $C_0 = \tau I$, for sufficiently small λ's, $\det C_\lambda > 0$, and hence we have that

$$\begin{aligned}
&\int_{\mathscr{W}_\tau} \Phi(\lambda; w)\delta_\xi(w(\tau))\mu_\tau(dw) \\
&= \lim_{\varepsilon\to 0} \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d}\int_{\mathscr{W}_\tau} \Phi(\lambda; w)\psi_{\xi,\eta}(w)\mu_\tau(dw) e^{-\varepsilon|\eta|^2/2}\, d\eta \\
&= \exp\left[\frac{1}{2}\int_0^\tau \operatorname{tr}\big\{A'_\lambda(t)A_\lambda^{-1}(t)\big\}\, dt\right] \frac{1}{\sqrt{2\pi}^d\sqrt{\det C_\lambda}} \exp\left[-\frac{1}{2}\langle C_\lambda^{-1}\xi, \xi\rangle_{\mathbb{R}^d}\right],
\end{aligned}$$

where $\Phi(\lambda; w)$ is the Wiener functional defined in Lemma 3.7. By virtue of Lemmas 3.7 and 3.8, prolongating holomorphically, we obtain the desired assertion.

4. $\boldsymbol{\Omega_0(\alpha, \beta)}$

It is interesting to see if $\Omega_0(\alpha, \beta)$ contains the complex axis $\sqrt{-1}\,\mathbb{R}$. In this section, we shall give three examples where $\sqrt{-1}\,\mathbb{R} \subset \Omega_0(\alpha, \beta)$.

Example 4.1. By a direct computation, we have that

$$\left((A_{\sqrt{-1}\lambda})^* A_{\sqrt{-1}\lambda}\right)''(t) = \sqrt{-1}\,\lambda \left((A_{\sqrt{-1}\lambda})^* (\alpha - \alpha^*) A_{\sqrt{-1}\lambda}\right)'(t) + 2A'_{\sqrt{-1}\lambda}(t)^* A'_{\sqrt{-1}\lambda}(t),$$

which, in conjunction with the terminal condition, implies that

$$\begin{aligned} &A_{\sqrt{-1}\lambda}(t)^* A_{\sqrt{-1}\lambda}(t) \\ &= I - \sqrt{-1}\,\lambda \int_t^\tau ds\, A_{\sqrt{-1}\lambda}(s)^* (\alpha(s) - \alpha(s)^*) A_{\sqrt{-1}\lambda}(s) \\ &\quad + 2\int_t^\tau ds \int_s^\tau du\, A'_{\sqrt{-1}\lambda}(u)^* A'_{\sqrt{-1}\lambda}(u) \end{aligned}$$

Suppose now that $\alpha \equiv 0$. Then $A_{\sqrt{-1}\lambda}(t)^* A_{\sqrt{-1}\lambda}(t) - I$ is non-negative definite, and hence $\sqrt{-1}\,\mathbb{R} \subset \Omega_0(\alpha, \beta)$.

Example 4.2. Suppose that $\beta \equiv 0$ and that $\alpha \equiv \alpha_0$ for some skew-symmetric $\alpha_0 \in \mathbb{R}^d \otimes \mathbb{R}^d$. Then

$$A_{\sqrt{-1}\lambda}(t) = \frac{1}{2}\left\{I + \mathrm{Exp}\left[\sqrt{-1}\,(2\lambda(t-\tau))\alpha_0\right]\right\},$$

where Exp denotes the exponential mapping on $\mathbb{C}^d \otimes \mathbb{C}^d$. Since the eigenvalues of α_0 are all purely imaginary, those of $\sqrt{-1}\,\alpha_0$ are real numbers. Hence $\mathrm{Exp}\left[\sqrt{-1}\,(2\lambda(t-\tau))\alpha_0\right]$ is non-negative definite, and hence $\sqrt{-1}\,\mathbb{R} \subset \Omega_0(\alpha, \beta)$.

Example 4.3. Let $d = 2$, $a \in C^1([0, \tau]; \mathbb{R})$, $b \in C^0([0, \tau]; \mathbb{R})$, and put

$$\alpha(t) = \begin{pmatrix} 0 & -a(t)/2 \\ a(t)/2 & 0 \end{pmatrix}, \quad \beta(t) = \begin{pmatrix} b(t) & 0 \\ 0 & b(t) \end{pmatrix}.$$

Denote by r_ζ a unique solution to an ODE

$$r''_\zeta + \left(\frac{\zeta^2}{4}a^2 + \zeta b\right) r_\zeta = 0, \quad r_\zeta(\tau) = 1,\ r'_\zeta(\tau) = 0.$$

It is straightforward to see that a unique solution to the ODE (1.5) is given by

$$A_\zeta(t) = r_\zeta(t) \begin{pmatrix} \cos\theta_\zeta(t) & -\sin\theta_\zeta(t) \\ \sin\theta_\zeta(t) & \cos\theta_\zeta(t) \end{pmatrix}, \quad \text{where } \theta_\zeta(t) = -\frac{\zeta}{2}\int_t^\tau a(s)\, ds.$$

By a direct computation, we have that

$$\left(|r_{\sqrt{-1}\lambda}|^2\right)'' = \frac{\lambda^2}{2}a^2|r_{\sqrt{-1}\lambda}|^2 + 2|r'_{\sqrt{-1}\lambda}|^2,$$

from which it follows that $|r_{\sqrt{-1}\lambda}|^2 \geq 1$ and hence that $\sqrt{-1}\,\mathbb{R} \subset \Omega_0(\alpha, \beta)$.

In this case, it should be mentioned that

$$A'_\lambda(t)A_\lambda^{-1}(t) = \begin{pmatrix} r'_\lambda(t)/r_\lambda(t) & -\lambda a(t)/2 \\ \lambda a(t)/2 & r'_\lambda(t)/r_\lambda(t) \end{pmatrix}.$$

Hence we have that

$$\exp\left[\frac{1}{2}\int_0^\tau \operatorname{tr}\left(A'_\zeta(t)A_\zeta^{-1}(t)\right)dt\right] = \frac{1}{r_\zeta(0)},$$

which, in conjunction with Eq. (1.8), leads us to an identity

$$\int_{\mathscr{W}_\tau} \exp\left[\frac{\zeta}{2}\int_0^\tau \left\{a(t)\left(w^1(t)\,dw^2(t) - w^2(t)\,dw^1(t)\right) + b(t)|w(t)|^2\,dt\right\}\right]$$
$$\times\psi(w)\mu_\tau(dw) = \frac{1}{r_\zeta(0)}\int_{\mathscr{W}_\tau} \psi\left(w + T_{A_\zeta}w\right)\mu_\tau(dw)$$

for $\zeta \in \Omega_0(\alpha, \beta)$.

Remark 4.1. The Wiener space $\mathscr{W}_\tau$ has another complexification (more standard one than our $\mathscr{W}_\tau \oplus \sqrt{-1}\,H_\tau$) given by

$$\mathscr{W}_\tau^{\mathbb{C}} = \left\{w : [0, \tau] \to \mathbb{C}^d : w \text{ is continuous and } w(0) = 0\right\}.$$

In the study of a principle of stationary phase on $\mathscr{W}_\tau$ in [10], one of key facts is that a Wiener functional $(I - 2\sqrt{-1}\,\lambda A)^{-1/2}w$ with values in $\mathscr{W}_\tau^{\mathbb{C}}$ converges to 0 in probability in the space, where A is a Hilbert-Schmidt operator on H_τ uniquely determined from the considered quadratic Wiener functional. For details, see [10]. $(I - 2\sqrt{-1}\,\lambda A)^{-1/2}w$ comes from a change of coordinate system on $\mathscr{W}_\tau$ based on the eigenfunction expansion. Thus it is natural to ask if our change of coordinate $w + T_{A_{\sqrt{-1}\lambda}}w$ has a similar asymptotic as $(I - 2\sqrt{-1}\,\lambda A)^{-1/2}w$. The answer is negative in general. Namely, as in the case of Example 4.3, let $d = 2$, and take $a \equiv 1$ and $b \equiv 0$. Then, we have that

$$A_{\sqrt{-1}\lambda}(t) = \cosh\left(\frac{\lambda(t-\tau)}{2}\right)\begin{pmatrix} \cosh\left(\frac{\lambda(t-\tau)}{2}\right) & -\sqrt{-1}\,\sinh\left(\frac{\lambda(t-\tau)}{2}\right) \\ \sqrt{-1}\,\sinh\left(\frac{\lambda(t-\tau)}{2}\right) & \cosh\left(\frac{\lambda(t-\tau)}{2}\right) \end{pmatrix}.$$

Hence

$$\left(w + T_{A_{\sqrt{-1}\lambda}} w\right)(\tau)$$
$$= \int_0^\tau \begin{pmatrix} 1 & \sqrt{-1}\,\tanh\left(\frac{\lambda(t-\tau)}{2}\right) \\ -\sqrt{-1}\,\tanh\left(\frac{\lambda(t-\tau)}{2}\right) & 1 \end{pmatrix} dw(t)$$

(cf. Eq. (3.7)). This implies that

$$\left(w + T_{A_{\sqrt{-1}\lambda}} w\right)(\tau) \longrightarrow \begin{pmatrix} w^1(\tau) \mp \sqrt{-1}\,w^2(\tau) \\ w^2(\tau) \pm \sqrt{-1}\,w^1(\tau) \end{pmatrix} \quad \text{as } \lambda \to \pm\infty \quad \text{in law}$$

Thus $w + T_{A_{\sqrt{-1}\lambda}} w$ does not converge to 0 in law on $\mathscr{W}_\tau^{\mathbb{C}}$ as $\lambda \to \pm\infty$.

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A. Appendix

Lemma A.1. *Let $G \in C^0([0,\tau]; \mathbb{R}^d \otimes \mathbb{R}^d)$. If*

$$\int_0^\tau \langle G(t)h(t), h'(t)\rangle_{\mathbb{R}^d}\, dt = 0 \quad \textit{for every } h \in H_\tau, \tag{A.1}$$

then $G \equiv 0$.

Proof. We shall show the assertion by induction on the dimension d of $\mathscr{W}_\tau$.

If $d = 1$, (A.1) implies that

$$\int_0^\tau \left(\int_t^\tau G(s)\,ds\right)(h(t))^2\,dt = 0 \quad \text{for any } h \in H_\tau.$$

Thus the assertion holds for $d = 1$.

Suppose that the assertion holds for $d-1$. By the hypothesis of induction, we see that $G_{ij}(t) \equiv 0$ if $(i,j) \notin \{(1,d),(d,1)\}$. Consider $g \in C^1([0,\tau]; \mathbb{R})$ and $h(t)$ of the form $h(t) = (g(t), 0, \ldots, 0, g(t))$. Then, Eq. (A.1) leads us to

$$\int_0^\tau \left(\int_t^\tau \{G_{1d}(t) + G_{d1}(t)\}\right)(g(t))^2\,dt = 0,$$

from which we can then conclude that

$$G_{1d}(t) + G_{d1}(t) = 0 \quad \text{for every } t \in [0,\tau]. \tag{A.2}$$

Now consider h of the form $h(t) = (g(t), 0, \ldots, 0, k(t))$. By virtue of Eq. (A.2), Eq. (A.1) reads as

$$\int_0^\tau (gk' - kg')(t)\ G_{1d}(t)\,dt = 0.$$

Due to the arbitrariness of g, k, we see that

$$G_{1d}(t) = 0, \quad t \in [0, \tau],$$

which completes the proof.

Lemma A.2. *Let* $\alpha, \beta, \gamma, \delta$ *be as in Theorem* 1.1. *Suppose that* $A \in \mathscr{A}_{\mathbb{R}}$ *obeys the Jacobi equation* (1.5) *with* $\zeta = 1$. *Set* $X(t) = A'(t)A^{-1}(t)$. *Then it holds that*

$$X(t) - \int_t^\tau X(s)^* X(s)\,ds = \alpha(t) + \int_t^\tau \beta(s)\,ds \quad \textit{for any } t \in [0, \tau]. \quad \text{(A.3)}$$

Conversely, if $X \in C^1([0, \tau]; \mathbb{R}^d \otimes \mathbb{R}^d)$ *obeys Eq.* (A.3)*, then a unique solution* A *to an ODE*

$$A'(t) = X(t)A(t), \quad A(\tau) = I,$$

belongs to $\mathscr{A}_{\mathbb{R}}$ *and enjoy Eq.* (1.5) *with* $\zeta = 1$.

Proof. It is easily seen that

$$X'(t) = \gamma(t)X(t) - \delta(t) - X(t)^2, \quad t \in [0, \tau] \quad \text{and} \quad X(\tau) = \alpha(\tau). \tag{A.4}$$

Putting $S(t) = X(t) - (\gamma(t)/2)$, by a straightforward computation, we see that S obeys the ODE

$$\begin{cases} S'(t) = \frac{1}{2}(\gamma(t)S(t) - S(t)\gamma(t)) + \frac{\gamma(t)^2}{4} \\ \qquad\quad -\beta(t) + \frac{1}{2}\left(\alpha'(t) + \alpha'(t)^*\right) - S(t)^2, \quad t \in [0, \tau], \\ S(\tau) = 0. \end{cases} \tag{A.5}$$

Since $\gamma(t)$ is skew symmetric and $\beta(t)$ is symmetric, it holds that $S(t) = S(t)^*$, $t \in [0, \tau]$, which implies that

$$X(t) - X(t)^* = \gamma(t) \quad \text{for any } t \in [0, \tau]. \tag{A.6}$$

In conjunction with Eq. (A.4), this implies that

$$X'(t) + X(t)^* X(t) = -\delta(t) = \alpha'(t) - \beta(t).$$

Integrating both side over $[t, \tau]$ and substituting $X(\tau) = \alpha(\tau)$, we obtain Eq. (A.3). Thus the first assertion has been verified.

In the second assertion, it is obvious that $\det A(t) \neq 0$ for any $t \in [0, \tau]$ and that $A'(\tau) = \alpha(\tau)$. Since Eq. (A.3) implies that $X' + X^* X = \alpha' - \beta$ and $X - X^* = \alpha - \alpha^*$, the Jacobi equation can be also derived easily by noting that

$$A'' A^{-1} = \left(A' A^{-1}\right)' + \left(A' A^{-1}\right)^2 .$$

References

[1] Cameron, R.H., Martin, W.T.: Evaluation of various Wiener integrals by use of certain Sturm-Liouville differential equations. Bull. A.M.S., **51**, 73–89 (1945)
[2] Ikeda, N., Watanabe, S.: Stochastic differential equations and diffusion processes. North-Holland/Kodan-sha, Amsterdam/Tokyo, 1981
[3] Ince, E.L.: Ordinary differential equations. Dover, New York, 1956
[4] Kac, M.: Integration in function spaces and some of its applications. Accademia Nazionale dei Lincei, Scula Normale Superiore, Pisa, 1980
[5] Kuo, H.-H.: Gaussian measures in Banach spaces. Lect. Notes Math. vol. 463, Springer-Verlag, Berlin Heidelberg New York, 1975
[6] Liptser, R.S., Shiryaev, A.N.: Statistics of random processes, Vol. I: General theory. Springer-Verlag, Berlin Heidelberg New York, 1977
[7] Malliavin, P.: Stochastic analysis. Springer-Verlag, Berlin Heidelberg New York, 1997
[8] Malliavin, P., Taniguchi, S.: Analytic functions, Cauchy formula and stationary phase on a real abstract Wiener space. Jour. Funct. Anal. **143**, 470–528 (1997)
[9] Rogers, L.C.G., Shi, Z.: Quadratic functionals of Brownian motion, optimal control, and the "Colditz" example. Stoch. and Stoch. Reports. **41**, 201–218 (1992)
[10] Sugita, H., Taniguchi, S.: Oscillatory integrals with quadratic phase function on a real abstract Wiener space. Jour. Funct. Anal. **155**, 229–262 (1998)
[11] Üsünel, A.S., Zakai, M.: Applications of the degree theorem to absolute continuity on Wiener space. Probab. Theory Relat. Fields **95**, 509–520 (1993)
[12] Watanabe, S.: Analysis of Wiener functionals (Malliavin calculus) and its applications to heat kernels. Ann. Probab. **15**, 1–39 (1987)
[13] Zakai, M. and Zeitouni, O.: When does the Ramer formula look like the Girsanov formula. Ann. Probab. **20**, 1436–1440 (1992)