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Probability Theory and Related Fields


# Limiting angles of Γ-martingales

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**Abstract.** Suppose that **M** is a complete, simply connected Riemannian manifold of non-positive sectional curvature with dimension $m \geq 3$ and that, outside a fixed compact set, the sectional curvatures are bounded above by $-c_1/\{r^2 \ln r\}$ and below by $-c_2 r^2$, where $c_1$ and $c_2$ are two positive constants and $r$ is the geodesic distance from a fixed point. We show that, when $\kappa \geq 1$ satisfies certain conditions, the angular part of a $\kappa$-quasi-conformal Γ-martingale on **M** tends to a limit as time tends to infinity and the closure of the support of the distribution of this limit is the entire sphere at infinity. This improves both a result of Le for Brownian motion and also results concerning the non-existence of $\kappa$-quasi-conformal harmonic maps from certain types of Riemannian manifolds into **M**.

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## 1. Introduction

Probabilistic methods have been used in the study of the existence of non-constant bounded harmonic functions on general Riemannian manifolds and the non-existence of non-constant bounded harmonic maps between Riemannian manifolds. Those for the former mainly show the existence of random limiting directions for Brownian motions on the given manifolds and then non-trivial bounded harmonic functions can be constructed in terms of the angular components of Brownian motions. The results obtained are generally for two overlapping classes of Riemannian manifolds, the class of simply connected manifolds of negative curvature and the class of manifolds satisfying Gromov's hyperbolicity criterion. In this paper we shall consider only the former class. For the latter see [2] and the references therein. Some of these probability results have also been generalised to certain classes of $\Gamma$-martingales and various generalisations of Picard's little theorem for harmonic maps then follow.

In the case of simply connected manifolds of negative curvature, a common condition is that the curvature bounds vary with distance $r$ from a reference point. In the case of an $m$-dimensional manifold of negative curvature which is rotationally symmetric with respect to that point, March showed in [13] that there exist non-constant bounded harmonic functions on the manifold if the radial curvatures at any point $x$ are bounded above by $-c/\{r(x)^2 \ln r(x)\}$ for $c > c_m$, where $c_2 = 1$, $c_m = 1/2$ for $m \geq 3$. If, instead, the above bound is the *lower* bound of the radial curvatures, then there exist no non-constant bounded harmonic functions. On the other hand, the known results for general manifolds of negative curvature are relatively weaker. For instance, for a 2-dimensional manifold, Hsu & Kendall in [6] proved that, if the sectional curvatures are bounded above by $-cr^{-2}$ off a compact set, for some $c > 0$, the angular component of Brownian motion converges to a limit as time tends to infinity and the closure of the support of the distribution of this limit is the entire circle of possible directions. For a manifold with dimension at least 3, Hsu & March proved in [5] a similar result if, off a given compact set, the sectional curvatures are bounded above by $-c_1 r^{-2}$ for $c_1 > 2$ and below by $-c_2 r^{2\beta}$ for $c_2 > 0$ and $0 < \beta < 1 - 4/(1 + \sqrt{1 + 4c_1})$. Note that this requires $\beta$ to approach zero as $c_1$ approaches 2. Hsu & March's result has been improved in [11] under more satisfyingly symmetric constraints: we may take $\beta = 1$ irrespective of $c_1$ and, except in dimension 3, $c_1$ itself may be an arbitrary positive number.

In this paper, we improve the constraints in [11] further along the line of those in [13] in order to consider the existence of non-constant bounded harmonic functions on, and the non-existence of non-constant bounded harmonic maps to, a complete, simply connected manifold of negative curvature for which, off a compact set, the sectional curvatures have an upper bound of $-c_1/\{r^2 \ln r\}$ and a lower bound of $-c_2 r^2$. Note that, if a manifold with dimension at least 3 has uncontrolled negative sectional curvatures then Brownian motion upon it may have a non-random limiting direction or no limiting direction at all (c.f. [6]).

## 2. Preminary definitions and results

A $\Gamma$-martingale $X$ on an $m$-dimensional Riemannian manifold $(\mathbf{M}, \mathbf{g})$ is a semimartingale on $\mathbf{M}$ which can be constructed from a local martingale $\hat{X}$ on $\mathbf{R}^m$ via the Stratonovich stochastic differential equations

$$\partial \Xi_t = H_{\Xi_t}\, \partial X_t$$

$$\partial X_t = \Xi_t\, \partial \hat{X}_t \ .$$

Here, for an orthonormal frame $\xi$ in the tangent space $\tau_{\pi(\xi)}(\mathbf{M})$, $H_\xi$ is the horizontal lifting isomorphism, supplied by the Levi-Civita connection, of $\tau_{\pi(\xi)}(\mathbf{M})$ onto the horizontal subspace of the tangent space $\tau_\xi(\mathcal{O}(\mathbf{M}))$ at $\xi$ to the orthonormal frame bundle $\mathcal{O}(\mathbf{M})$. Usually $\hat{X}$ is called the *stochastic development* of $X$ and $\Xi$ on $\mathcal{O}(\mathbf{M})$ is called the *stochastic parallel transport* of $X$. In particular, a Brownian motion on $\mathbf{M}$ is a $\Gamma$-martingale whose stochastic development is Brownian motion on $\mathbf{R}^m$.

**Definition. (Bounded quasi-conformality)** *A $\Gamma$-martingale $X$ is said to be $\kappa$-quasi-conformal, for a fixed $\kappa \geq 1$, if its stochastic development $\hat{X}$ satisfies the condition*

$$[V_1^t\, d\hat{X},\ V_1^t\, d\hat{X}] \leq \kappa\, [V_2^t\, d\hat{X},\ V_2^t\, d\hat{X}]$$

*for all predictable unit vector-valued processes $V_1$ and $V_2$.*

If $\hat{X}_t = \sum_{i=1}^m \hat{X}_t^i\, u_i$, where $\{u_i\ :\ 1 \leq i \leq m\}$ is a fixed basis for $\mathbf{R}^m$ orthonormal with respect to the standard Euclidean metric and $\hat{X}^i$, $1 \leq i \leq m$, are continuous local martingales on $\mathbf{R}$, then the above definition of bounded quasi-conformality is equivalent to the following requirement (c.f. [9]): writing $[X]$ for the *intrinsic time* of $X$ defined by

$$[X]_t = \int_0^t \mathbf{g}(X_s)(dX_s,\ dX_s) = \sum_{i=1}^m [\hat{X}^i,\ \hat{X}^i]_t$$

then, with probability one, the largest and smallest eigenvalues $\lambda^{(1)}$ and $\lambda^{(m)}$ of the matrix process $(d[\hat{X}^i,\ \hat{X}^j]/d[X])$ satisfy the condition that

$$\lambda_1^{(1)} \leq \kappa\, \lambda_t^{(m)} \qquad \text{for } [X]\text{-almost all } t.$$

In particular, if $X$ is a Brownian motion, then it is $\kappa$-quasi-conformal for $\kappa = 1$ and $\kappa > 1$ is required if there are to be any such $\Gamma$-martingales apart from Brownian motions.

We shall need the following two results concerning the behaviour of the distance of a $\Gamma$-martingale on $\mathbf{M}$ from a fixed point $x_0 \in \mathbf{M}$.

**Lemma 1.** *Suppose that* $\mathbf{M}$ *is a complete Riemannian manifold, that $X$ is a $\Gamma$-martingale on* $\mathbf{M}$ *and that $x_0$ is an arbitrarily fixed point in* $\mathbf{M}$ *such that the cut locus of $x_0$ is empty. Then $\rho(X) = \mathrm{dist}(X,\ x_0)$ is a semimartingale* *on* $\mathbf{R}$ *and its stochastic differential equation is given by the following full Itô formula*

$$d\rho(X_t) = \langle \operatorname{grad} \rho(X_t),\ dX_t\rangle + \tfrac{1}{2}\mathrm{Hess}^{\rho}(\partial X_t,\ \partial X_t) + dL_t^0(\rho(X)),$$

*where* $L^0(\rho(X))$ *is the local time of* $\rho(X)$ *at zero.*

**Lemma 2.** *Suppose that* $\mathbf{M}$ *is a complete, simply connected Riemannian manifold of non-positive sectional curvature whose sectional curvatures at* $x$ *are bounded below by* $-cr(x)^2$*, where* $c > 0$ *is a constant and* $r(x)$ *is distance of* $x$ *from a given pole of* $\mathbf{M}$*. If* $X$ *is a* $\Gamma$*-martingale on* $\mathbf{M}$ *such that* $[X]_t = t$ *then there are positive constants* $\tilde{c}_1, \tilde{c}_2$ *and* $r_0$ *such that, for* $r > r_0$ *and for any* $x_0 \in \mathbf{M}$*,*

$$P\left[\sup_{0\le s\le t} \mathrm{dist}(X_s,\ x_0) \ge r\right] \le \tilde{c}_1 \exp\left\{-\frac{\ln r(x_0) + r}{\tilde{c}_2 t}\right\} .$$

Lemma 1 was first obtained by Kendall in [9], later generalised in [10] and [12]. Lemma 2 was obtained in [11]: although the result of Lemma 4 there is stated for Brownian motion it is valid for any $\Gamma$-martingale.

Throughout this paper we assume that $X$ has infinite total intrinsic time, that is, $[X]_\infty = \infty$ so that, in the next section, we may compare its radial part with a Bessel process.

## 3. The radial part of a $\kappa$-quasi-conformal $\Gamma$-martingale

Throughout the remainder of this paper we shall make the following assumptions on the Riemannian manifold $\mathbf{M}$. We require that $\dim(\mathbf{M}) = m > 2$ and that $\mathbf{M}$ be complete, simply connected and of non-positive sectional curvature. We fix any $o \in \mathbf{M}$ as a pole and denote by $(r, \theta)$ the global geodesic polar coordinates with respect to $o$. We further assume that there exist positive constants $r_0 > 1$, $c_1 > 1/2$ and $c_2$ such that the sectional curvatures of $\mathbf{M}$ at $x$ are bounded above by $-c_1/\{r(x)^2 \ \ln r(x)\}$ and below by $-c_2 r(x)^2$ for all $x$ such that $r(x) \ge r_0$. Without loss of generality, we may assume that $r_0$ here is at least as large as that which occurs in Lemma 2 so that the result of that lemma also holds.

**Lemma 3.** *Suppose that* $X$ *is a* $\kappa$*-quasi-conformal* $\Gamma$*-martingale on* $\mathbf{M}$*. If we express* $R_t = r(X_t)$ *in terms of its Doob-Meyer decomposition as* $dR_t = dN_t + d\Lambda_t$ *then, for any constant* $\alpha$ *satisfying the condition that* $\frac{1}{2} < \alpha < c_1$*, there is a constant* $r_1 \ge r_0$*, depending on* $\alpha$*, such that*

$$dR_t \ge \begin{cases} dN_t + \frac{1}{2}\frac{m-1}{\kappa}\frac{1}{R_t} d[N]_t & \text{if } 0 \le R_t \le r_1 \\ dN_t + \frac{1}{2}\frac{m-1}{\kappa}\left\{\frac{1}{R_t} + \frac{\alpha}{R_t \ln R_t}\right\} d[N]_t & \text{if } R_t \ge r_1 \ . \end{cases}$$

*Proof.* Define the smooth function $\varphi : \mathbf{R}_+ \longrightarrow \mathbf{R}$ by

$$\varphi(r) = r\,(\ln r)^\alpha$$

and consider an $m$-dimensional rotationally symmetric manifold $(\tilde{\mathbf{M}},\ \tilde{\mathbf{g}})$ of non-positive sectional curvatures with pole $\tilde{o}$, global geodesic polar coordinates $(\tilde{r},\ \tilde{\theta})$ with respect to $\tilde{o}$ and with Riemannian metric given by

$$d\tilde{s}^2 = d\tilde{r}^2 + f(\tilde{r})^2\, d\tilde{\theta}^2 \ ,$$

where $f$ is a suitably chosen smooth function on $\mathbf{R}$. Writing $K_r$ for the supremum of the sectional curvatures of $\mathbf{M}$ at $x$ with $r(x) = r$, we shall require $f$ to satisfy the following conditions:

1. $f(0) = 0$ and $f'(0) = 1$;
2. for $r < r_1$, $-f''(r)/f(r) \geq K_r$;
3. for $r \geq r_1$, $f(r) = \varphi(r)$,

where the constant $r_1 \geq r_0$ is chosen sufficiently large that, for $r > r_1$,

$$\frac{\alpha}{r^2 \ln r} + \frac{\alpha(\alpha-1)}{r^2(\ln r)^2} < \frac{c_1}{r^2 \ln r} \ .$$

For $\tilde{r}(\tilde{x}) = r > r_1$, the left hand side of this inequality is equal to $f''(\tilde{r}(\tilde{x}))/f(\tilde{r}(\tilde{x}))$ which is the negative of the radial curvature of $\tilde{\mathbf{M}}$ at $\tilde{x}$ (c.f. [4]). It follows that, when $\tilde{r}(\tilde{x}) > r_1$, the radial curvature of $\tilde{\mathbf{M}}$ at $\tilde{x}$ is bounded below by

$$-\frac{c_1}{\tilde{r}(\tilde{x})^2 \ln \tilde{r}(\tilde{x})} \ ,$$

which is an upper bound of the sectional curvatures of $\mathbf{M}$ at points $x$ such that $r(x) = \tilde{r}(\tilde{x})$.

The radial curvature is one particular value of the sectional curvature and hence the radial curvature of $\tilde{\mathbf{M}}$ at any point $\tilde{x}$ with $\tilde{r}(\tilde{x}) = r$ is greater than or equal to that of $\mathbf{M}$ at any point $x$ with $r(x) = r$. We may thus apply the Hessian Comparison Theorem (c.f. [4]) to $\mathbf{M}$ and $\tilde{\mathbf{M}}$. Since the Hessian of $\tilde{r}$ on $\tilde{\mathbf{M}} \setminus \{\tilde{o}\}$ is given (c.f. [4]) by

$$\mathrm{Hess}^{\tilde{r}} \ : \ (\tilde{v}_1, \tilde{v}_2) \mapsto \frac{f'(\tilde{r})}{f(\tilde{r})} \{\tilde{g}(\tilde{v}_1,\ \tilde{v}_2) - d\tilde{r}(\tilde{v}_1) \otimes d\tilde{r}(\tilde{v}_2)\}$$

we have, for any vector field $v$ on $\mathbf{M}$,

$$\mathrm{Hess}^r(v, v) \geq \frac{f'(r)}{f(r)}\{g(v,\ v) - dr(v) \otimes dr(v)\} \qquad r > 0 \ . \tag{1}$$

On the other hand, since the sectional curvature, and hence also the Ricci curvature, of $\tilde{\mathbf{M}}$ is non-positive we may also apply the Laplacian Comparison Theorem to $\mathbf{R}^m$ and $\tilde{\mathbf{M}}$ (c.f. [4]). However the Laplacian of $\tilde{r}$ on $\tilde{\mathbf{M}} \setminus \{\tilde{o}\}$ is given by

$$\Delta \tilde{r} = \frac{m-1}{2} \frac{f'(\tilde{r})}{f(\tilde{r})}$$

and so

$$\frac{f'(\tilde{r})}{f(\tilde{r})} \geq \frac{1}{\tilde{r}} \qquad \tilde{r} > 0 \ . \tag{2}$$

Finally, noting that

$$\frac{\varphi'(r)}{\varphi(r)} = \frac{1}{r} + \frac{\alpha}{r \ln r} \ , \tag{3}$$

we have from (1), (2) and (3) that, for any vector field $\nu$ on $\mathbf{M}$,

$$\mathrm{Hess}^r(\nu, \nu) \geq \begin{cases} \frac{1}{r}\{g(\nu,\ \nu) - dr(\nu) \otimes dr(\nu)\} & \text{if } 0 < r \leq r_1 \\ \left\{\frac{1}{r} + \frac{\alpha}{r \ln r}\right\}\{g(\nu,\ \nu) - dr(\nu) \otimes dr(\nu)\} & \text{if } r \geq r_1 \ . \end{cases}$$

Thus, Lemma 1 shows that

$$dR_t \geq \begin{cases} dN_t + \frac{1}{2}\frac{m-1}{\kappa}\frac{1}{R_t} d[N]_t + dL_t^0(R) & \text{if } 0 \leq R_t \leq r_1 \\ dN_t + \frac{1}{2}\frac{m-1}{\kappa}\left\{\frac{1}{R_t} + \frac{\alpha}{R_t \ln R_t}\right\} d[N]_t & \text{if } R_t \geq r_1 \ , \end{cases}$$

and the required result follows since the local time $L^0(R)$ is non-decreasing. □

Write $\gamma = (m-1)/\kappa$ and $\mathrm{BES}_r^a$ for the Bessel process of index $a$ starting from $r$ and let $\psi(t) = \sqrt{t}\,(\ln t)^{-\beta}$, where $\beta > 1/(\gamma-1) = \kappa/(m-1-\kappa)$. Then since, for $t_0 > 1$ and $\gamma > 1$,

$$\int_{t_0}^{\infty} (\ln t)^{-\beta(\gamma-1)} \frac{dt}{t} < \infty$$

we have (c.f. [14]) that, for $\gamma > 1$,

$$P[\mathrm{BES}_0^{\gamma+1}(t) < \psi(t), \ \text{ infinitely often as } t \uparrow \infty] = 0 \ .$$

Then, comparing $\mathrm{BES}_r^{\gamma+1}$ with $\mathrm{BES}_0^{\gamma+1}$, we have

$$P[\mathrm{BES}_r^{\gamma+1}(t) > \psi(t) \text{ for sufficiently large } t] = 1 \ . \tag{4}$$

**Lemma 4.** *Suppose that $X$ is a $\kappa$-quasi-conformal $\Gamma$-martingale on* $\mathbf{M}$, *that $R_t = r(X_t)$ and that the constants $\alpha$ and $r_1$ are as in the statement of Lemma* 3. *Then, for $r > e^{r_1}$,*

$$P[R_t \geq \max\{\ln r,\ \psi([R]_t)\},\ \forall t \mid R_0 = r] \ \geq\ p(r)$$

*where*

$$\lim_{r\to\infty} p(r) = -\lim_{r\to\infty} \frac{(\ln\ln r)^{\alpha\gamma}}{r^{\gamma-1}(\ln r)^{\gamma(\alpha-1)}} + \lim_{r\to\infty} P[\mathrm{BES}_1^{\gamma+1}(t) \geq \psi(t),\ \textit{for all } t \textit{ s.t. } \psi(t) \geq \ln r]\ .$$

*Proof.* Consider the 1-dimensional diffusion $\hat{R}$ on $\mathbf{R}$ given by

$$d\hat{R}_t = dB_t + \frac{1}{2}\gamma\mu(\hat{R}_t)\,dt \quad \text{and} \quad \hat{R}_0 = r$$

where

$$\mu(r) = \begin{cases} 1/r & \text{if } r \leq r_1 \\ 1/r + \alpha/(r\ \ln r) & \text{if } r > r_1\ . \end{cases}$$

It follows from Lemma 3 that, if $R_0 = r(X_0) = r$, then $R_t \geq \hat{R}_{[N]_t}$. However $[R]_t = [N]_t$ and hence we have

$$P[R_t \geq \max\{\ln r,\ \psi([R]_t)\}, \quad \forall t \mid R_0 = r] \geq P[\hat{R}_t \geq \max\{\ln r,\ \psi(t)\}, \quad \forall t \mid \hat{R}_0 = r]\ .$$

It is obvious that $\hat{R}_t \geq \mathrm{BES}_r^{\gamma+1}(t)$ and hence that

$$\begin{aligned} P[\hat{R}_t &\geq \max\{\ln r,\ \psi(t)\}, \quad \forall t \mid \hat{R}_0 = r] \\ &\geq P[\hat{R}_t \geq \psi(t),\ \text{for } t \text{ s.t. } \psi(t) \geq \ln r \mid \hat{R}_0 = r] \\ &\quad -P[\hat{R} \text{ hits the level } \ln r \text{ for some } t \mid \hat{R}_0 = r]\ . \end{aligned} \tag{5}$$

If $r \geq 1$, then $\mathrm{BES}_r^{\gamma+1}(t) \geq \mathrm{BES}_1^{\gamma+1}(t)$ so that

$$\begin{aligned} P[\hat{R}_t &\geq \psi(t),\ \text{for } t \text{ s.t. } \psi(t) \geq \ln r \mid \hat{R}_0 = r] \\ &\geq P[\mathrm{BES}_1^{\gamma+1}(t) \geq \psi(t),\ \text{for all } t \text{ s.t. } \psi(t) \geq \ln r]\ . \end{aligned} \tag{6}$$

To estimate the second term of the right hand side of (5) we first look at the expression for the scale function of $\hat{R}$

$$S(r) = \int_1^r s(u)\,du$$

where

$$s(r) = \exp\left\{-\gamma \int_1^r \mu(u)\,du\right\} = \frac{1}{r_1^\gamma} \exp\left\{-\gamma \int_{r_1}^r \left\{\frac{1}{u} + \frac{\alpha}{u\,\ln u}\right\} du\right\}$$

$$= \frac{1}{r_1^\gamma} \exp\left\{-\gamma \int_{r_1}^r \frac{\varphi'(u)}{\varphi(u)}\,du\right\} = \left\{\frac{\varphi(r_1)}{r_1 \varphi(r)}\right\}^\gamma \quad ,$$

that is

$$S(r) = \int_1^r \left\{\frac{\varphi(r_1)}{r_1 \varphi(u)}\right\}^\gamma du \quad .$$

This implies (c.f. [7]) that

$$P[\hat{R} \text{ hits the level} \ln r \text{ for some } t \mid \hat{R}_0 = r]$$

$$= \lim_{\hat{r}\to\infty} P[\hat{R} \text{ hits the level} \ln r \text{ before the level } \hat{r} \mid \hat{R}_0 = r]$$

$$= \lim_{\hat{r}\to\infty} \frac{S(\hat{r}) - S(r)}{S(\hat{r}) - S(\ln r)} = \frac{\int_r^\infty \varphi(u)^{-\gamma}\,du}{\int_{\ln r}^\infty \varphi(u)^{-\gamma}\,du} \quad . \tag{7}$$

Thus, using the equality

$$\lim_{r\to\infty} \frac{\int_r^\infty \varphi(u)^{-\gamma}\,du}{\int_{\ln r}^\infty \varphi(u)^{-\gamma}\,du} = \lim_{r\to\infty} \frac{r\,\varphi(r)^{-\gamma}}{\varphi(\ln r)^{-\gamma}} = \lim_{r\to\infty} \frac{(\ln\ln r)^{\alpha\gamma}}{r^{\gamma-1}(\ln r)^{\gamma(\alpha-1)}} \quad ,$$

the required result follows from (5), (6) and (7). □

## 4. Main result

Our main theorem in this paper concerns the limiting angle of $\Gamma$-martingales on our manifold $\mathbf{M}$. In order to prove it we shall need firstly to apply Lemma 4 to the angular part of such processes and for that purpose we require the following result which is a direct consequence of the Rauch Comparison Theorem (c.f. [1]) applied to $\mathbf{M}$ and $\tilde{\mathbf{M}}$ (c.f. [11]). To state it we denote by $\omega(\theta_1,\, \theta_2)$ the distance, measured on the unit tangent sphere at the pole $o$, between two of its points $\theta_1$ and $\theta_2$.

**Lemma 5.** *For any constant* $\epsilon > 0$, *there exists a constant* $r_\epsilon > r_0$ *such that, if* $r(x_1) > r_\epsilon$ *and* $\operatorname{dist}(x_1, x_2) \le (\ln r(x_1))^\epsilon$, *then*

$$\omega(\theta(x_1),\ \theta(x_2)) \leq \frac{1}{2\, r(x_1)\, (\ln r(x_1))^{\alpha-\epsilon}} \ .$$

We shall also need the following sequence of stopping times $\{T_n \ : \ n \geq 0\}$. Writing $\tau$ for the inverse map of $[X]$, that is, $[X]_{\tau_t} = \tau_{[X]_t} = t$ and $Y_t = X_{\tau_t}$, we define $T_0 = 0$ and

$$\begin{aligned} T_{n+1} &= \inf\{t > T_n : \mathrm{dist}(Y_t,\ Y_{T_n}) = (\ln r(Y_{T_n}))^{\epsilon}\} \\ &= \inf\{[X]_t > T_n : \mathrm{dist}(X_t,\ Y_{T_n}) = (\ln r(Y_{T_n}))^{\epsilon}\} \ . \end{aligned}$$

Then $Y$ is a Γ-martingale on $\mathbf{M}$, $[Y]_t = t$ and hence, by Lemma 2, when $R_{T_n} > r_0$,

$$P[T_{n+1} - T_n \geq 2\ln(r(Y_{T_n}) + (\ln r(Y_{T_n}))^{\epsilon}) \mid \mathscr{F}_{T_n}] \geq c > 0$$

for some constant $c > 0$ independent of $n$ and $R_{T_n}$. Thus, from the sequence $\{T_n \ : \ n \geq 0\}$ and a sequence of *iid* $[0,\ 1]$-uniformly distributed random variables which are independent of $\mathscr{F}_\infty = \bigvee_n \mathscr{F}_n$, we can construct a sequence of *iid* $\{0,\ 1\}$-valued random variables $\{U_n \ : \ n \geq 1\}$ such that, for some constant $\tilde{\delta} > 0$, we have $T_{n+1} - T_n \geq 2\tilde{\delta} U_{n+1}$ with probability at least $p(r)$ (c.f. [11]). Then, by the Strong Law of Large Numbers, there exist a constant $\delta > 0$ and a natural number $N$ such that, when $n > N$, $T_n \geq \delta n$ with probability at least $p(r)$.

We are now in a position to state and prove our main result. We recall the basic feature of our assumptions in section 3 that, for some constant $c_1 > 1/2$, the sectional curvatures of $\mathbf{M}$ at $x$ are bounded above by $-c_1/\{r(x)^2 \ln r(x)\}$ for $r(x) > r_0$, where $r$ is the distance of $x$ from the pole $o$.

**Theorem.** *Suppose that* $\mathbf{M}$ *satisfies the hypotheses given at the beginning of section* 3 *and that $X$ is a* Γ*-martingale on* $\mathbf{M}$ *with infinite life time. If $X$ is $\kappa$-quasi-conformal, where $\kappa$ satisfies the conditions that* (i) $1 \leq \kappa < m-1$ *and* (ii) $\kappa/(m-1-\kappa)+1/2 < c_1$, *then the limiting angle of $X$ as $t$ tends to infinity exists and the closure of the support of the distribution of this limit is the entire sphere at infinity.*

*Proof.* Suppose that $\kappa$ satisfies the given conditions. We choose positive constants $\alpha$ and $\beta$ such that $\kappa/(m-1-\kappa)+1/2 < \alpha < c_1$ and $\beta < \kappa/(m-1-\kappa)$ and then choose $\epsilon$ sufficiently small that $\alpha - \beta - \epsilon > 1/2$.

Define, for each $n \geq 0$, $S_n = \tau_{T_n}$. Then by Lemmas 4 and 5 there is a constant $r_\epsilon \geq r_1 \geq r_0$ such that, if $(r,\ \theta) = (r(X_0),\ \theta(X_0))$ with $r > e^{r_\epsilon}$, we have that, with probabiliy at least $p(r)$,

$$\begin{aligned}4\sup_{t\ge 0}\omega^2(\theta,\theta(X_t)) &\le 4\sum_{n\ge 0}\sup_{S_n\le t\le S_{n+1}}\omega^2(\Theta_{S_n},\Theta_t)\\ &\le \sum_{n\ge 0} R_{S_n}^{-2}\,(\ln R_{S_n})^{-2(\alpha-\epsilon)}\\ &\le \sum_{n\ge 0}\left\{\max\{\ln r,\ \psi([R]_{S_n})\}\right\}^{-2}\\ &\quad\times\left\{\ln(\max\{\ln r,\ \psi(([R]_{S_n})\})\right\}^{-2(\alpha-\epsilon)}\quad .\end{aligned}$$

However, $X$ is $\kappa$-quasi-conformal and so

$$d[R]_t \ge \frac{1}{1+(m-1)\kappa}d[X]_t\quad .$$

In particular,

$$[R]_{S_n} \ge \frac{1}{1+(m-1)\kappa}[X]_{S_n} = \frac{1}{1+(m-1)\kappa}T_n\quad .$$

Thus, when $r$ is sufficiently large,

$$\begin{aligned}&4\sup_{t\ge 0}\omega^2(\theta,\theta(X_t))\\ &\quad=\sum_{n\ge 0}\left\{\max\left\{\ln r,\ \psi\left(\frac{1}{1+(m-1)\kappa}T_n\right)\right\}\right\}^{-2}\\ &\qquad\times\left\{\ln\left(\max\left\{\ln r,\ \psi\left(\frac{1}{1+(m-1)\kappa}T_n\right)\right\}\right)\right\}^{-2(\alpha-\epsilon)}\\ &\quad\le\left(N+\frac{1+(m-1)\kappa}{\delta}\right)(\ln\ln r)^{-2(\alpha-\epsilon)}\\ &\qquad+4^{2(\alpha-\epsilon)}\int_{(\ln r)^2+1}^{\infty}\frac{(\ln(x-1))^{2(\epsilon+\beta-\alpha)}}{(x-1)}dx\\ &\quad=\left(N+\frac{1+(m-1)\kappa}{\delta}\right)(\ln\ln r)^{-2(\alpha-\epsilon)}\\ &\qquad+\frac{2^{2(\alpha+\beta-\epsilon)+1}}{2(\alpha-\beta-\epsilon)-1}(\ln\ln r)^{2(\epsilon+\beta-\alpha)+1}\\ &\quad\le 4\,C^2(\ln\ln r)^{-2\lambda}\end{aligned}$$

where

$$C^2 = \frac{1}{4}\left\{N + \frac{1+(m-1)\kappa}{\delta} + \frac{2^{2(\alpha+\beta-\epsilon)+1}}{2(\alpha-\beta-\epsilon)-1}\right\}$$

and $\lambda = \min\{\alpha - \epsilon,\ \alpha - \beta - \epsilon - 1/2\} > 0$.

On the other hand, by (4), for any $n > 0$, there is a $k$ such that

$$P[\mathrm{BES}_1^{\gamma+1}(t) \leq \psi(t), \quad \forall t \geq k] \geq 1 - e^n \ .$$

For each $r > 0$ we define $\sigma_r = \inf\{t :\ R_t = r\}$ and for each $n$ define

$$k_n = \max\{n,\ \min\{k :\ \text{for } k \text{ which satisfy the above condition}\}\}$$

and

$$E_n = \{\text{there are } s, t > \sigma_{k_n} \text{ such that } \omega(\theta(X_s),\ \theta(X_t)) > 2C(\ln k_n)^{-\lambda}\} \ .$$

Then, by Lemma 4, when $n$ is large enough,

$$\begin{aligned} P[E_n] &\leq 1 - P\left[\sup_{t \geq \sigma_{k_n}} \omega(\theta(X_t), \theta(X_{\sigma_{k_n}})) \leq C\,(\ln k_n)^{-\lambda}\right] \\ &\leq 1 - p(e^{k_n}) \leq e^{-n} + e^{-n(\gamma-1)} \ , \end{aligned}$$

so that $\sum P[E_n] < \infty$. The theorem then follows by the Borel-Cantelli Lemma. □

**Corollary 1.** *Suppose that* **M** *satisfies the hypotheses given at the beginning of section* 3 *with* $c_1 > 1/2 + 1/(m-2)$. *Then the limiting angle of Brownian motion on* **M** *always exists and the closure of the support of the distribution of this limit is the entire sphere at infinity.*

**Corollary 2.** *Suppose that* **M** *satisfies the hypotheses given at the beginning of section* 3 *with* $c_1 > 1/2 + 1/(m-2)$. *Then* **M** *supports non-constant bounded harmonic functions.*

**Corollary 3.** *Suppose that* **M** *satisfies the hypotheses given at the beginning of section* 3 *and that* $\mathbf{M}^*$ *is a Riemannian manifold supporting no non-constant bounded harmonic functions. If* $h :\ \mathbf{M}^* \longrightarrow \mathbf{M}$ *is harmonic and* $\kappa$*-quasi-conformal, where* $\kappa$ *satisfies the conditions in the Theorem, then* $h$ *is constant.*

*Proof.* Let $B$ be a Brownian motion on $\mathbf{M}^*$ and define $X = h(B)$. Then $X$ is a $\kappa$-quasi-conformal Γ-martingale on **M** (c.f. [9]).

The condition that $\mathbf{M}^*$ supports no non-constant bounded harmonic functions is equivalent to the fact that all time-invariant behaviour of Brownian motion on $\mathbf{M}^*$ satisfies the zero-one law. Since $X_\infty$ is measurable with respect to the invariant $\sigma$-field of $B$, all time-invariant behaviour of $X$ on $\mathbf{M}$ also satisfies the zero-one law. In particular, this implies that $X_\infty$ is non-random and that the total intrinsic time of $X$ is infinite with probability one or zero.

If $[X]_\infty = \infty$ a.s. then $h$ must be non-constant and the Theorem shows that then the limiting angle of $X$ would be random, contradicting the fact that $X_\infty$ is non-random. Thus we have $[X]_\infty < \infty$ a.s. This implies that $X_\infty$ is finite and constant a.s. Then we take $x_0 = X_\infty$. By Lemma 3, $\mathrm{dist}(X,\ x_0)$ is a submartingale and so, $\forall t > 0$,

$$0 \le \mathrm{dist}(X_t,\ x_0) \le E[\mathrm{dist}(X_\infty,\ x_0) \mid \mathscr{F}_t] = 0\ ,$$

that is, $X = h(B)$ is constant a.s. so that $h$ is constant. □

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