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Monotonic limit theorem of BSDE and nonlinear decomposition theorem of Doob–Meyer's type*

Shige Peng

Department of Mathematics, Shandong University, Jinan, 250100, P. R. China.
e-mail: peng@public.jn.sd.cn and peng@sdu.edu.cn

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Abstract. We have obtained the following limit theorem: if a sequence of RCLL supersolutions of a backward stochastic differential equations (BSDE) converges monotonically up to (y_t) with $\mathbf{E}[\sup_t |y_t|^2] < \infty$, then (y_t) itself is a RCLL supersolution of the same BSDE (Theorem 2.4 and 3.6).

We apply this result to the following two problems: 1) nonlinear Doob–Meyer Decomposition Theorem. 2) the smallest supersolution of a BSDE with constraints on the solution (y, z). The constraints may be non convex with respect to (y, z) and may be only measurable with respect to the time variable t. this result may be applied to the pricing of hedging contingent claims with constrained portfolios and/or wealth processes.

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Introduction

Consider a backward stochastic differential equation (BSDE) of type (all processes mentioned below are $\sigma\{W_s; s \le t\}$ -adapted, where W is a fixed Brownian motion)

$$y_t = y_T + \int_t^T g(y_s, z_s)\, ds + (A_T - A_t) - \int_t^T z_s\, dW_s, \quad t \in [0, T] \;, \quad (1)$$

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where g is assumed to be a Lipschitz function of (y, z). Here A is an RCLL increasing process with $A_0 = 0$ and $\mathbf{E}(A_T)^2 < \infty$. If (y, z) solves (1) then we call (y_t) to be a supersolution of BSDE with generator g, or, more simply, a g-supersolution on $[0, T]$. In particular, when $A \equiv 0$, y is called a g-solution on $[0, T]$.

Since the original work of [EQ], where a penalization method of BSDE is introduced to solve the problem of option pricing for incomplete financial market (more specific explanation was given in [EPQ]), the following limit theorem of g-supersolutions begin to be considered as an essentially important problem: if a sequence of RCLL g-supersolutions converges monotonically up to a process (y_t), with $\mathbf{E}[\sup_t |y_t|^2] < \infty$, can one prove that (y_t) itself is also a RCLL g-supersolution?

In the case where $g = 0$, since a g-supersolution is a classical supermartingale, the answer is affirmative. Other typical case is when g is a linear function. e.g., the Merton's model of an investor's wealth process. In this case, by applying the Girsanov transformation, one can still treat the problem as a classical limit problem of supermartingales (see [CK], [EQ], [FS] and their references).

The first result to treat the case where g is nonlinear was given in [ELal]. Roughly speaking, if $\{y_t^i\}$ and the limit process (y_t) is continuous, then it is a g-supermartingale. But in many useful cases, even $\{y_t^i\}$ are continuous in time, the limit process (y_t) is just a RCLL process (see [EQ], [CK]).

This paper will give a positive answer to this problem: if a sequence of RCLL g-supersolutions converges monotonically up to a process (y_t) with $\mathbf{E}[\sup_t |y_t|^2] < \infty$, then (y_t) itself is also a RCLL g-supersolution (see Theorem 3.6). Furthermore, if $\{y^i\}$ is continuous g-supersolution, then the corresponding martingale parts $\{z^i\}$ converges strongly in L^p sense for $1 \le p < 2$ (Theorem 2.1 and 2.4).

To explain clearly our idea, we consider the case where $\{y_t^i\}$ are continuous but the limit $\{y_t\}$ is RCLL. A main difficulty in this case is that, even in the classical case where $g = 0$, it is known that the strong convergence of the supermartingales (g-supersolution) does not imply the strong convergence of their martingale parts. This seems very serious for the case where $g(y, z)$ is nonlinear since we obviously need a result of strong convergence to pass to the limit. This difficulty is overcome in Theorem 2.1 and Theorem 2.4. Although we are not sure to have any kind of strong convergence for the 'martingale parts' $\int_0^T z_s^i \, dW_s$, of the g-supersolutions, but one can still prove the strong convergence of $\{z_t^i\}$ in L^p sense for $1 \le p < 2$ (Theorem 2.1). This convergence is enough for us to pass to the limit.

An application of this limit theorem is to prove a generalization of Doob–Meyer Decomposition Theorem for a g-supermartingale. Roughly speaking, a process (X_t) is a g-supermartingale on $[0, T]$, if it dominates from above each g-solutions (y_t) on $[0, t_0]$ with the same terminal condition $y_{t_0} = X_{t_0}$ for any $t_0 \leq T$.

If (X_t) is a g-supersolution on $[0, T]$, then it follows from the comparison theorem that (X_t) is also a g-supermartingale. A trivial example is when $g \equiv 0$ (linear case): it just tells us that $X_t = X_0 - A_t + M_t$ ($M_t = \int_0^t z_s \, dW_s$) is a classical supermartingale.

But it is known that the inverse problem is not at all trivial. It is in fact a nonlinear version of Doob–Meyer Decomposition Theorem: can one prove that a g-supermartingale is also a g-supersolution? In other words, does the nonlinear Doob–Meyer Decomposition Theorem hold for g-supermartingales?

Since the classical demonstrations of Doob–Meyer decomposition theorem are essentially based on the fact that the expectation $\mathbf{E}[\cdot]$ is a linear operator, they does not work for this new problem.

This paper will give an affirmative answer to this conjecture. The method of proof is significantly different from the classical proof of Doob–Meyer Decomposition Theorem (se e.g. [DM]): no discretization of time was involved. The main idea is to apply the penalization approach introduced in [ELal] to 'push up' a sequence of g-supersolutions to be above this given g-supermartingale, i.e., the parts of the penalized g-supersolution that are strictly bellow the given g-supermartingale will be heavyly pushed up. An interesting observation is: it happens that these g-supersolutions can never be strictly above this supermartingale (see Lemma 3.4). Thus the limit coincides with the given supermartingale. From the above limit theorem, this limit is also a supersolution.

We also give a notion of nonlinear expectations (called g-expectations) and related g-expectations. Under this notion the corresponding g-martingale as well as g-supermartingales will be defined in the same way as the classical definitions. We would like to show readers that, here, everything works classically except the linearity. The motivation of such new notions is the concept of the "certainty equivalent" in economic theory (see e.g. [DE]). An application of such notion to the study of stochastic geometry was discussed in [Da]. We will see that the option valuation process with constrained wealth process and/or portfolios is in fact such g-supermartingale.

The second application is the existence of the smallest g-supersolution of (or, in other words, the smallest g-supermartingle) subject to a given constraint. This problem was motivated from the pricing of contingent claim with constrained portfolios and/or wealth processes. Since the celebrated papers of Black & Scholes ([BS]) and Merton ([M]), many important progresses have been made for the pricing of contingent claims, see references provided in [K] for complete security markets. For the incomplete markets see e.g. [FS] and recent remarkable works of [EQ], [CK]. This problem may be formulated as to find the smallest g-supersolution of BSDE with constraints imposed on the solution. The results of [EQ] and [CK] are improved: the constraint may be imposed at the same time on the portfolios and/or the wealth processes. We do not need to suppose that the constraint is a convex set. The main argument is to construct a sequence of g-supersolutions that monotonically converges up to a limit which is, due to the above monotonic limit theorem, still a g-supersolution. It is then easy to prove, by our special construction, that this limit g-supersolution is in fact the smallest one subject to the constraint.

This paper is organized as follows: Section 1 provides a comparison theorem which is necessary for the sections followed. The limit theorem of g-supersolutions for the case where the sequence of $\{y^i\}$ is continuous is given in Section 2. Nonlinear Doob–Meyer Decomposition Theorem is introduced and is proved in Section 3.1. We also give the limit theorem of RCLL g-supersolutions in this subsection (Theorem 3.6). We will give a more clear sense of this result, i.e. under the notion of g-expectations in Section 3.2. Section 4 is devoted to the problem of the smallest g-supersolution subject to a given constraint on (y, z). The pricing of contingent claims with constrained portfolios and/or constrained wealth process may be regarded as one of applications of the results of this section.

1. Preliminaries: backward stochastic differential equations

In this section we briefly present the results we need about BSDE.

Let $(\Omega, \mathscr{F}, P)$ be a probability space endowed with a filtration $\{\mathscr{F}_t; 0, \leq t \leq \infty\}$ and let $(W_s)_{s\geq 0}$ be a d-dimensional Brownian motion defined in this space. In order to clarify our interests, we shall not discuss the most general case we only discuss the case where $\mathscr{F}_t$ is the natural filtration generated by the Brownian motion (W_t):

$$\mathscr{F}_t = \sigma\{W_s;\ s \leq t\}\ .$$

All processes mentioned in this paper are supposed to be $\mathscr{F}_t$-adapted. In this section we are interested in the behavior of processes on a given interval $[0, T]$. We use $|\cdot|$ to denote the norm of a Euclidean space $\mathbf{R}^n$. For $p \geq 1$, we denote by $L^p_{\mathscr{F}}(0, T; \mathbf{R}^m)$ the set of all $\mathbf{R}^m$-valued $\mathscr{F}_t$-adapted processes satisfying

$$\mathbf{E}\int_0^T |\phi_s|^p\, ds < \infty\ .$$

A process (ϕ_t) is said to be RCLL if it a.s. has sample path which are right continuous with left limit. A process (A_t) is said to be increasing if its path

$A : t \to A_t(\omega)$ are a.s. non decreasing with $A_0(\omega) = 0$. A is called a finite variation process if the path are of a.s. finite variation on $[0, T]$.

We now consider the following problem: to find a pair of processes $(y, z) \in L^2_{\mathscr{F}}(0, T; \mathbf{R}^{1+d})$ satisfying

$$y_t = y_T + \int_t^T g(y_s, z_s, s)\, ds + (V_T - V_t) - \int_t^T z_s\, dW_s \quad . \tag{1.1}$$

(1.1) is called backward stochastic differential equation (short for BSDE). By "backward" we mean the condition is given at the final time T. Here the function g, (V_t) and ξ are given such that

(i) $g(y, z, \cdot) \in L^2_{\mathscr{F}}(0, T; \mathbf{R})$, for each $(y, z) \in \mathbf{R}^{1+d}$;

(ii) $y_T \in L^2(\Omega, \mathscr{F}_T, P; \mathbf{R})$; (H1.1)

(iii) $(V_t) \in L^2_{\mathscr{F}}(0, T; \mathbf{R})$ RCLL with $\mathbf{E} \sup_{t \le T} |V_t|^2 < \infty$.

g is assumed to be Lipschitz in (y, z), i.e., there exists a constant μ such that

$$|g(y_1, z_1, s) - g(y_2, z_2, s)| \le \mu(|y_1 - y_2| + |z_1 - z_2|) \quad . \tag{H1.2}$$

We have the following existence and uniqueness theorem.

Proposition 1.1. *We assume* (H1.1) *and* (H1.2). *Then there exists a unique pair of processes* $(y_t, z_t) \in L^2_{\mathscr{F}}(0, T; \mathbf{R}^{1+d})$ *of solution BSDE* (1.1) *such that* $(y_t + V_t)$ *is continuous and that*

$$\mathbf{E} \sup_{0 \le t \le T} |y_t|^2 < \infty \quad . \tag{1.2}$$

Proof. In the case where $V_t \equiv 0$, the proof can be found in [PP1]. Otherwise we can make the change of variable $\bar{y}_t := y_t + V_t$ and treat the equivalent BSDE

$$\bar{y}_t = y_T + V_T + \int_t^T g(\bar{y}_s - V_s, z_s, s)\, ds - \int_t^T z_s\, dW_s \quad .$$

The estimate (1.2) is easy to obtain since we have (H1.1) and

$$\mathbf{E} \sup_{0 \le t \le T} \left| \int_0^t z_s\, dW_s \right|^2 < \infty, \quad \mathbf{E} \int_0^T |g(y_s, z_s, s)|^2\, ds < \infty \quad .$$

□

Let

$$y'_T \in L^2(\Omega, \mathscr{F}_T, P; \mathbf{R}), \tag{H1.3}$$

be given and let $(y', z') \in L^2_{\mathscr{F}}(0, T; \mathbf{R}^{1+d})$ be the solution of

$$y'_t = y'_T + \int_t^T g(y'_s, z'_s, s)\, ds + (V_T - V_t) - \int_t^T z'_s\, dW_s \ . \tag{1.3}$$

We have the following estimate of the difference of the above two solutions. The proof is essentially the same as the one for the uniqueness.

Proposition 1.2. *We suppose* (H1.1), (H1.2) *and* (H1.3). *Then we have the following* "*continuous dependence property*"

$$\mathbf{E} \sup_{0\le t\le T} |y_t - y'_t|^2 + E\int_0^T |z_s - z'_s|^2\, ds \le C\mathbf{E}|y_T - y'_T|^2 \ . \tag{1.4}$$

The following comparison theorem is very useful. It was introduced in [P1]. Two improved versions were given in [EPQ]. The result of strict comparison was established in [P2]. The following formulation is taken from [EPQ].

Comparison Theorem 1.3. *We suppose the assumptions in Proposition* 1.1. *Let* $(\bar{y}, \bar{z})$ *be the solution of the BSDE*

$$\bar{y}_t = \bar{y}_T + \int_t^T \bar{g}_s\, ds + \bar{V}_T - \bar{V}_t - \int_t^T \bar{z}_s\, dW_s \ ,$$

where $(\bar{g}_t), (\bar{V}_t) \in L^2_{\mathscr{F}}(0, T; \mathbf{R})$ *and* $\bar{y}_T \in L^2(\Omega, \mathscr{F}_T, P; \mathbf{R})$ *are given such that*

$$\begin{cases} \hat{y}_T := y_T - \bar{y}_T \ge 0, \quad \hat{g}_t := g(\bar{y}_t, \bar{z}_t, t) - \bar{g}_t \ge 0, \ \textit{a.s., a.e,} \\ \hat{V}_t := V_t - \bar{V}_t \ \textit{is an RCLL increasing process} \ . \end{cases} \tag{1.5}$$

Then we have

$$y_t \ge \bar{y}_t, \quad \textit{a.e., a.s.} \ . \tag{1.6}$$

If, in addition of (1.5), *we assume* $P(\hat{y}_T > 0) > 0$, *then* $P(y_t > \bar{y}_t) > 0$. *In particular*, $y_0 > \bar{y}_0$.

See [EPQ] for the proof.

Remark 1.4. If we replace the deterministic terminal time T by a $\mathscr{F}_t$-stopping time $\tau \le T$, then the above results still hold true.

For a given stopping time, we now consider the following BSDE

$$y_t = \xi + \int_{t\wedge\tau}^{\tau} g(y_s, z_s, s)\,ds + (A_\tau - A_{t\wedge\tau}) - \int_{t\wedge\tau}^{\tau} z_s\,dW_s \quad . \tag{1.7}$$

where $\xi \in L^2(\Omega, \mathscr{F}_\tau, P)$ and A is a given RCLL increasing process with $A_0 = 0$ and $\mathbf{E}(A_\tau)^2 < \infty$. The following terms will be frequently used in this paper.

Definition 1.5. *If (y_t) is a solution of BSDE of form (1.7) then we call (y_t) a g-supersolution on $[0, \tau]$. If $A_t \equiv 0$ in $[0, \tau]$, then we call (y_t) a g-solution on $[0, \tau]$.*

We recall that a g-solution (y_t) on $[0, \tau]$ is uniquely determined if its terminal condition $y_\tau = \xi$ is given, a g-supersolution (y_t) on $[0, \tau]$ is uniquely determined if y_τ and $(A_t)_{0\le t\le\tau}$ are given. If (y_t) is a g-solution on $[0, \tau]$ and (y'_t) is a g-supersolution on $[0, \tau]$ such that $y_\tau \le y'_\tau$ a.s., then for all stopping time $\sigma \le \tau$ we have also $y_\sigma \le y'_\sigma$.

Proposition 1.6. *Given (y_t) a g-supersolution on $[0, \tau]$, there is a unique $(z_t) \in L^2(0, \tau; \mathbf{R}^d)$ and a unique increasing RCLL process (A_t) on $[0, \tau]$ with $A_0 = 0$ and $\mathbf{E}[(A_\tau)^2] < \infty$ such that the triple (y_t, z_t, A_t) satisfies* (1.7).

Proof. If both (y_t, z_t, A_t) and (y_t, z'_t, A'_t) satisfy (1.7), then we apply Itô's formula to $(y_t - y_t)^2 (\equiv 0)$ on $[0, \tau]$ and take expectation:

$$\mathbf{E}\int_0^\tau |z_t - z'_t|^2\,ds + \mathbf{E}\left[\sum_{t\in(0,\tau]} (\Delta(A_t - A'_t))^2\right] = 0 \quad .$$

Thus $z_t \equiv z'_t$. From this it follows that $A_t \equiv A'_t$. □

Thus we can define

Definition 1.7. *Let (y_t) be a supersolution on $[0, \tau]$ and let (y_t, A_t, z_t) be the related unique triple in the sense of BSDE (1.7). Then we call (A_t, z_t) the (unique) decomposition of (y_t).*

2. Basic estimates: limit theorem of *g*-supersolutions

In this section, we first prove a "convergence theorem" by weak convergence method. Then, using this convergence theorem, we study the limit theorem of g-supersolution.

We first consider the following a family of semi-martingales:

$$y_t^i = y_0^i + \int_0^t g_s^i \, ds - A_t^i + \int_0^t z_s^i \, dW_s, \quad i = 1, 2, \dots \ . \tag{2.1}$$

Here, for each i, the adapted process $g^i \in L^2_{\mathcal{F}}(0, T, \mathbf{R})$ are given, we also assume that, for each i,

$$(A_t^i) \text{ is a continuous and increasing process with } \mathbf{E}(A_T^i)^2 < \infty, \tag{H2.1}$$

We further assume that

$$\begin{cases} \text{(i) } (g_t^i) \text{ and } (z_t^i) \text{are bounded in } L^2_{\mathcal{F}}(0, T)\text{: } \mathbf{E}\int_0^T [|g_s^i|^2 + |z_s^i|^2] \, ds \le C; \\ \text{(ii) } (y_t^i) \text{ increasingly converges to } (y_t) \text{ with } \mathbf{E} \sup_{0 \le t \le T} |y_t|^2 < \infty; \end{cases} \tag{H2.2}$$

It is clear that

$$\begin{aligned} &\text{(i) } \mathbf{E}\left[\sup_{0 \le t \le T} |y_t^i|^2\right] \le C; \\ &\text{(ii) } \mathbf{E}\int_0^T |y_t^i - y_t|^2 \, ds \to 0 \ , \end{aligned} \tag{2.2}$$

where the constant C is independent of i.

Remark. It is not hard to prove that the limit y_t has the following form

$$y_t = y_0 + \int_0^t g_s^0 \, ds - A_t + \int_0^t z_s \, dW_s \ , \tag{2.3}$$

where (g_t^0), (z_t) and (A_t) are respectively the L^2-weak limit of (g_t^i), (z_t^i) and (A_t) is an increasing process. In general, we cannot prove the strong convergence of $\{\int_0^T z_s^i \, dW_s\}_{i=1}^\infty$. Our new observation is: for each $p \in [1, 2)$, $\{z^i\}$ converges strongly in L^p. This observation is crucially important in this paper, since we will treat nonlinear cases.

Theorem 2.1. *Assume* (H2.1) *and* (H2.2) *hold. Then the limit* (y_t) *of* (y_t^i) *has a form* (2.3), *where* $(g_t^0) \in L^2_{\mathcal{F}}(0, T; \mathbf{R})$, (z_t) *is the weak limit of* (z_t^i), (A_t) *is an RCLL square-integrable increasing process. Furthermore, for any* $p \in [0, 2)$, $(z_t^i)_{0 \le t \le T}$ *strongly converges to* (z_t^i) *in* $L^p_{\mathcal{F}}(0, T, \mathbf{R}^d)$, *i.e.,*

$$\lim_{i \to \infty} \mathbf{E}\int_0^T |z_s^i - z_s|^p \, ds = 0, \ \forall p \in [0, 2) \ . \tag{2.4}$$

The following lemma will be applied to prove that the limit processes (y_t) is RCLL.

Lemma 2.2. *Let $\{x^i(\cdot)\}$ be a sequence of (deterministic) RCLL processes defined on $[0, T]$ that increasingly converges to $x(\cdot)$: for each $t \in [0, T]$, $x^i(t) \uparrow x(t)$, with $x(t) = b(t) - a(t)$, where $b(\cdot)$ is an RCLL process and $a(\cdot)$ is an increasing process with $a(0) = 0$ and $a(T) < \infty$. Then $x(\cdot)$ and $a(\cdot)$ are also RCLL processes.*

Proof. Since, for each t, $b(\cdot)$, $a(\cdot)$ and thus $x(\cdot)$ have left and right limits at t, thus we only need to check that $x(\cdot)$ is right-continuous.

Since, for each $t \in [0, T)$, $a(t+) \geq a(t)$, thus

$$x(t+) = b(t) - a(t+) \leq x(t) \ . \tag{2.5}$$

On the other hand, for any $\delta > 0$, there exists a positive integer $j = j(\delta, t)$ such that $x(t) \leq x^j(t) + \delta$. Since $x^j(\cdot)$ is RCLL, thus there exists a positive number $\epsilon_0 = \epsilon_0(j, t, \delta)$ such that $x^j(t) \leq x^j(t + \epsilon) + \delta$, $\forall \epsilon \in (0, \epsilon_0]$. These imply that, for any $\epsilon \in (0, \epsilon_0]$,

$$x(t) \leq x^j(t + \epsilon) + 2\delta \leq x^{i+j}(t + \epsilon) + 2\delta \uparrow\uparrow x(t + \epsilon) + 2\delta \ .$$

Particularly $x(t) \leq x(t+) + 2\delta$ and thus $x(t) \leq x(t+)$. This with (2.5) implies the right continuity of $x(\cdot)$. □

The following lemma tells that, for any given RCLL increasing process, the contribution of the jumps of (A_t) is mainly concentrated within a finite number of left-open right-closed intervals with "sufficiently small total length". Specifically, we have

Lemma 2.3. *Let (A_t) be an increasing RCLL process defined on $[0, T]$ with $A_0 = 0$ and $\mathbf{E}A_T^2 < \infty$. Then, for any $\delta, \epsilon > 0$, there exists a finite number of pairs of stopping times $\{\sigma_k, \tau_k\}$, $k = 0, 1, 2, \ldots, N$ with $0 < \sigma_k \leq \tau_k \leq T$ such that*

$$\text{(i) } (\sigma_j, \tau_j] \cap (\sigma_k, \tau_k] = \emptyset \quad \textit{for each } j \neq k;$$

$$\text{(ii) } \mathbf{E}\sum_{k=0}^{N}[\tau_k - \sigma_k](\omega) \geq T - \epsilon;$$

$$\text{(iii) } \sum_{k=0}^{N}\mathbf{E}\sum_{\sigma_k < t \leq \tau_k}(\Delta A_t)^2 \leq \delta \ .$$

The proof will be given in the Appendix. We now give the

Proof of Theorem 2.1. Since (g^i) (resp. (z^i)) is weakly conpact in $L^2_{\mathscr{F}}(0, T; \mathbf{R})$ (resp. $L^2_{\mathscr{F}}(0, T; \mathbf{R}^d)$), there is a subsequence, still denoted by (g^i) (resp. (z^i)) which converges weakly to (g_t^0) (resp. (z_t)).

Thus, for each stopping time $\tau \le T$, the following weak convergence holds in $L^2(\Omega, \mathscr{F}_\tau, P; \mathbf{R})$.

$$\int_0^\tau z_s^i\, dW_s \rightharpoonup \int_0^\tau z_s\, dW_s, \quad \int_0^\tau g_s^i\, ds \rightharpoonup \int_0^\tau g_s^0\, ds \ .$$

Since

$$A_\tau^i = -y_\tau^i + y_0^i + \int_0^\tau g_s^i\, ds + \int_0^\tau z_s^i\, dW_s$$

thus we also have the weak convergence

$$A_\tau^i \rightharpoonup A_\tau := -y_\tau + y_0 + \int_0^\tau g_s^0\, ds + \int_0^\tau z_s\, dW_s. \tag{2.6}$$

Obviously, $\mathbf{E}A_T^2 < \infty$. For any two stopping times $\sigma \le \tau \le T$, we have $A_\sigma \le A_\tau$ since $A_\sigma^i \le A_\tau^i$. From this it follows that (A_t) is an increasing process. Moreover, from Lemma 2.2, both (A_t) and (y_t) are RCLL. Thus (y_t) has a form of (2.3). Since (y_t) is given, it is clear that (z_t) is uniquely determined. Thus not only a subsequence of (z^i) but also the sequence itself converges weakly to (z).

Our key point is to show that $\{z^i\}$ converges to z in the strong sense of (2.4). In order to prove this we use Itô's formula applied to $(y_t^i - y_t)^2$ on a given subinterval $(\sigma, \tau]$. Here $0 \le \sigma \le \tau \le T$ are two stopping times. Observe that $\Delta y_t \equiv \Delta A_t$ and the fact that y^i and then A^i are continuous. We have

$$\begin{aligned}
&\mathbf{E}|y_\sigma^i - y_\sigma|^2 + \mathbf{E}\int_\sigma^\tau |z_s^i - z_s|^2\, ds \\
&\quad = \mathbf{E}|y_\tau^i - y_\tau|^2 - \mathbf{E}\sum_{t\in(\sigma,\tau]} (\Delta A_t)^2 - 2\mathbf{E}\int_\sigma^\tau (y_s^i - y_s)(g_s^i - g_s^0)\, ds \\
&\qquad + 2\mathbf{E}\int_{(\sigma,\tau]} (y_s^i - y_s)\, dA_s^i - 2\mathbf{E}\int_{(\sigma,\tau]} (y_s^i - y_{s-})\, dA_s \\
&\quad = \mathbf{E}|y_\tau^i - y_\tau|^2 + \mathbf{E}\sum_{t\in(\sigma,\tau]} (\Delta A_t)^2 - 2\mathbf{E}\int_\sigma^\tau (y_s^i - y_s)(g_s^i - g_s^0)\, ds \\
&\qquad + 2\mathbf{E}\int_{(\sigma,\tau]} (y_s^i - y_s)\, dA_s^i - 2\mathbf{E}\int_{(\sigma,\tau]} (y_s^i - y_s)\, dA_s
\end{aligned}$$

Since $(y_t^i - y_t)\, dA_t^i \le 0$,

$$
\begin{aligned}
\mathbf{E}\int_\sigma^\tau & |z_s^i - z_s|^2\,ds \\
&\le \mathbf{E}|y_\tau^i - y_\tau|^2 + \mathbf{E}\sum_{t\in(\sigma,\tau]}(\Delta A_t)^2 + 2\mathbf{E}\int_\sigma^\tau |y_s^i - y_s||g_s^i - g_s^0|\,ds \\
&\quad + 2\mathbf{E}\int_{(\sigma,\tau]} |y_s^i - y_s|\,dA_s \ . \qquad (2.7)
\end{aligned}
$$

The third term on the right side tends to zero since

$$
\mathbf{E}\int_0^T |y_s^i - y_s||g_s^i - g_s^0|\,ds \le C\left[\mathbf{E}\int_0^T |y_s^i - y_s|^2\,ds\right]^{\frac{1}{2}} \to 0 \ . \qquad (2.8)
$$

For the last term, we have, P-almost surely,

$$
|y_s^1 - y_s| \ge |y_s^i - y_s| \to 0, \quad \forall s \in [0,T] \ .
$$

Since

$$
\mathbf{E}\int_0^T |y_s^1 - y_s|\,dA_s \le \left(\mathbf{E}\left[\sup_s \left(|y_s^1 - y_s|^2\right)\right]\right)^{\frac{1}{2}} (\mathbf{E}(A_T)^2)^{\frac{1}{2}} < \infty \ ,
$$

it then follows from Lebesgue's dominated convergence theorem that

$$
\mathbf{E}\int_{(0,T]} |y_s^i - y_s|\,dA_s \to 0. \qquad (2.9)
$$

By convergence (2.8) and (2.9), it is clear from the estimate (2.7) that, once A_t is continuous (thus $\Delta A_t \equiv 0$), then z^i tends to z strongly in $L^2_{\mathscr{F}}(0,T;\mathbf{R}^d)$. But for the general case, the situation becomes complicated.

Thanks to Lemma 2.3, for any $\delta, \epsilon > 0$, there exist a finite number of disjoint intervals $(\sigma_k, \tau_k]$, $k = 0, 1, \ldots, N$, such that $\sigma_k \le \tau_k \le T$ are all stopping times satisfying

$$
\begin{aligned}
&\text{(i) } \mathbf{E}\sum_{k=0}^N [\tau_k - \sigma_k](\omega) \ge T - \frac{\epsilon}{2}; \\
&\text{(ii) } \sum_{k=0}^N \sum_{\sigma_k < t \le \tau_k} \mathbf{E}(\Delta A_t)^2 \le \frac{\delta\epsilon}{3} \ .
\end{aligned} \qquad (2.10)
$$

Now, for each $\sigma = \sigma_k$ and $\tau = \tau_k$, we apply estimate (2.7) and then take the sum. It follows that

$$\sum_{k=0}^{N} \mathbf{E} \int_{\sigma_k}^{\tau_k} |z_s^i - z_s|^2 \, ds \le \sum_{k=0}^{N} \mathbf{E} |y_{\tau_k}^i - y_{\tau_k}|^2 + \sum_{k=0}^{N} \mathbf{E} \sum_{t \in (\sigma_k, \tau_k]} (\Delta A_t)^2$$
$$+ 2\mathbf{E} \int_0^T |y_s^i - y_s||g_s^i - g_s^0| \, ds$$
$$+ 2\mathbf{E} \int_{(0,T]} |y_s^i - y_s| \, dA_s \ .$$

By using the convergence results (2.8) and (2.9) and taking in consideration of (2.10)–(ii), it follows that

$$\overline{\lim_{i \to \infty}} \sum_{k=0}^{N} \mathbf{E} \int_{\sigma_k}^{\tau_k} |z_s^i - z_s|^2 \, ds \le \sum_{k=0}^{N} \mathbf{E} \sum_{t \in (\sigma_k, \tau_k]} (\Delta A_t)^2 \le \frac{\epsilon \delta}{3}$$

Thus there exists an integer $l_{\epsilon\delta} > 0$ such that, whenever $i \ge l_{\epsilon\delta}$, we have

$$\sum_{k=0}^{N} \mathbf{E} \int_{\sigma_k}^{\tau_k} |z_s^i - z_s|^2 \, ds \le \frac{\epsilon \delta}{2}$$

Thus, in the product space $([0, T] \times \Omega, \mathscr{B}([0, T]) \times \mathscr{F}, m \times P)$ (here m stands for the Lebesgue measure on $[0, T]$), we have

$$m \times P \left\{ (s, \omega) \in \bigcup_{k=0}^{N} (\sigma_k(\omega), \tau_k(\omega)] \times \Omega; \ |z_s^i(\omega) - z_s(\omega)|^2 \ge \delta \right\} \le \frac{\epsilon}{2}$$

This with (2.10)–(i) implies

$$m \times P \left\{ (s, \omega) \in [0, T] \times \Omega; \ |z_s^i(\omega) - z_s(\omega)|^2 \ge \delta \right\} \le \epsilon, \quad \forall \quad i \ge l_{\epsilon\delta} \ .$$

From this it follows that, for any $\delta > 0$,

$$\lim_{i \to \infty} m \times P \left\{ (s, \omega) \in [0, T] \times \Omega; \quad |z_s^i(\omega) - z_s(\omega)|^2 \ge \delta \right\} = 0 \ .$$

Thus, on $[0, T] \times \Omega$, the sequence $\{(z^i)\}$ converges in measure to (z_t). Since (z_t^i) is also bounded in $L^2_{\mathscr{F}}(0, T; \mathbf{R}^d)$, then for each $p \in [1, 2)$, it converges strongly in $L^p_{\mathscr{F}}(0, T; \mathbf{R}^d)$. □

Let us now consider the following sequence of g-supersolution (y_t^i) on $[0, T]$ i.e.,

$$y_t^i = y_T^i + \int_t^T g(y_s^i, z_s^i, s) \, ds + (A_T^i - A_t^i) - \int_t^T z_s^i \, dW_s, \quad i = 1, 2, \dots \ . \tag{2.11}$$

Here the function g and the increasing process (A_t^i) are given in (H1.1), (H1.2) and (H2.1). From Proposition 1.1, there exists a unique pair $(y^i, z^i) \in L^2_{\mathscr{F}}(0, T; \mathbf{R}^{1+d})$ satisfying (2.11).

The following theorem prove that the limit of (y_t^i) is still a g-supersolution.

Theorem 2.4. *We assume that g satisfies* (H1.1) *and* (H1.2). *and* (A^i) *satisfies* (H2.1). *Let* (y^i, z^i) *be the solution of BSDE* (2.11), *with* $\mathbf{E}\sup_{0\le t\le T}|y_t^i|^2<\infty$. *If* (y_t^i) *increasingly converges to* (y_t) *with* $\mathbf{E}\sup_{0\le t\le T}|y_t|^2<\infty$. *Then* (y_t) *is a* g*-supersolution. i.e., there exist a* $(z_t)\in L^2_{\mathscr{F}}(0,T;\mathbf{R}^d)$ *and an RCLL square-integrable increasing process* (A_t) *such that the pair* (y_t,z_t) *is the solution of the BSDE*

$$y_t = y_T + \int_t^T g(y_s,z_s,s)\,ds + (A_T - A_t) - \int_t^T z_s\,dW_s, \quad t\in[0,T] \ , \tag{2.12}$$

where $(z_t)_{0\le t\le T}$ *is the weak* (*resp. strong*) *limit of* $\{(z_t^i)\}$ *in* $L^2_{\mathscr{F}}(0,T;\mathbf{R}^d)$ (*resp. in* $L^p_{\mathscr{F}}(0,T;R^d)$, *for* $p<2$) *and, for each* t, A_t *is the weak limit of* $\{A_t^i\}$ *in* $L^2(\Omega,\mathscr{F}_t,P)$.

Remark. Observe that (2.11) can be rewritten in the 'forward formulation':

$$y_t^i = y_0^i - \int_0^t g(y_s^i,z_s^i,s)\,ds - A_t^i - \int_0^t z_s^i\,dW_s \ . \tag{2.11}$$

Similarly, the limit equation (2.12) is

$$y_t = y_0 - \int_0^t g(y_s,z_s,s)\,ds - A_t + \int_0^t z_s\,dW_s \ . \tag{2.12}$$

To prove this theorem, we need following lemma. The lemma says that both $\{z^i\}$ and $\{(A_T^i)^2\}$ are uniformly bounded in L^2:

Lemma 2.5. *Under the assumptions of Theorem 2.4, there exists a constant* C *that is independent of* i *such that*

$$\begin{aligned}&\text{(i) } \mathbf{E}\int_0^T |z_s^i|^2\,ds \le C,\\ &\text{(ii) } \mathbf{E}[(A_T^i)^2] \le C \ .\end{aligned} \tag{2.13}$$

Proof. From BSDE (2.11), we have

$$\begin{aligned}A_T^i &= y_0^i - y_T^i - \int_0^T g(y_s^i,z_s^i,s)\,ds + \int_0^T z_s^i\,dW_s\\ &\le |y_0^i| + |y_T^i| + \int_0^T [\mu|y_s^i| + \mu|z_s^i| + |g(0,0,s)|]\,ds + \left|\int_0^T z_s^i\,dW_s\right| \ .\end{aligned}$$

We observe that $|y_t^i|$ is dominated by $|y_t^1|+|y_t|$. Thus there exists a constant, independent of i, such that

$$\mathbf{E}\left[\sup_{0\le t\le T}[y_t^i|^2\right] \le C \quad . \tag{2.14}$$

It follows that, there exists a constant C_1, independent of i, such that

$$\mathbf{E}|A_T^i|^2 \le C_1 + 2\mathbf{E}\int_0^T |z_s^i|^2\, ds \quad . \tag{2.15}$$

On the other hand, we use Itô's formula applied to $|y_t^i|^2$:

$$\begin{aligned}
&|y_0^i|^2 + \mathbf{E}\int_0^T |z_s^i|^2\, ds \\
&= \mathbf{E}|y_T^i|^2 + 2\mathbf{E}\int_0^T y_s^i g(y_s^i, z_s^i, s)\, ds + 2\mathbf{E}\int_0^T y_s^i\, dA_s^i \\
&\le \mathbf{E}|y_T^i|^2 + 2\mathbf{E}\int_0^T [|y_s^i|(\mu|y_s^i| + \mu|z_s^i| + |g(0,0,s)|)]\, ds \\
&\quad + 2\mathbf{E}\int_0^T |y_s^i|\, dA_s^i \\
&\le \mathbf{E}|y_T^i|^2 + 2\mathbf{E}\int_0^T [(\mu+\mu^2)|y_s^i|^2 + \frac{1}{2}|z_s^i| + |g(0,0,s)|)]\, ds \\
&\quad + 2\mathbf{E}\left[A_T^i \sup_{0\le s\le T}|y_s^i|\right] \\
&\le C_2 + \frac{1}{2}\mathbf{E}\int_0^T |z_s^i|^2\, ds + 2\left[\mathbf{E}\sup_{0\le s\le T}|y_s^i|^2\right]^{1/2}[\mathbf{E}|A_T^i|^2]^{1/2}
\end{aligned}$$

Thus

$$\begin{aligned}
\mathbf{E}\int_0^T |z_s^i|^2\, ds &\le 2C_2 + 4\left[\mathbf{E}\sup_{0\le s\le T}|y_s^i|^2\right]^{1/2}[\mathbf{E}|A_T^i|^2]^{1/2} \\
&\le 2C_2 + 16\mathbf{E}\left[\sup_{0\le s\le T}|y_s^i|^2\right] + \frac{1}{4}\mathbf{E}|A_T^i|^2 \\
&= C_3 + \frac{1}{4}\mathbf{E}|A_T^i|^2 \quad ,
\end{aligned}$$

where, from (2.14), the constants C_2 and C_3 are independent of i. This with (2.15) it follows that (2.13)–(i) and then (2.13)–(ii) hold true. The proof is complete. □

Combining this Lemma with Theorem 2.1, we can easily prove Theorem 2.4.

Proof of Theorem 2.4. In (2.11), we set $g_t^i := -g(y_t^i, z_t^i, t)$; Since $\{z^i\}$ is bounded in $L^2_{\mathcal{F}}(0,T)$, thanks to Theorem 2.1, there exists a $\{z_t\} \in L^2_{\mathcal{F}}(0,T;\mathbf{R})$ such that (z^i) strongly converges to (z) in $L^p_{\mathcal{F}}(0,T); \forall p \in [0,2)$.

As result, $\{g^i\} = \{-g(y^i, z^i, \cdot)\}$ strongly converges in $L^p_{\mathcal{F}}(0,T;\mathbf{R}^d)$ to g^0 and

$$g^0(s) = -g(y_s, z_s, s), \qquad \text{a.s., a.e.}$$

From this it follows immediately that (y_t, z_t) is the solution of the BSDE (2.12). □

3. Nonlinear Doob–Meyer decomposition and limit theorem

In this section we introduce a notion of g-martingales. A special case of such g-martingales, i.e. when g is a linear function and is independent of y, is the classical notion of martingales. A typical example of such g-martingales is the wealth process of an investor in a stocks market (Merton's model). If his consumption is negligible, then this process is a g-martingale. Otherwise, it is a g-supermartingale. An important, and difficult, problem is whether the corresponding Doob–Meyer Decomposition Theorem still holds true. The difficulty is due to the nonlinearity: the classical method is fundamentally based on the fact that the expectation $\mathbf{E}[\cdot]$ is a linear operator. We have found a new method to prove this nonlinear Decomposition Theorem. The idea is to apply the penalization approach given in [ELal] to construct a sequence of g-supersolutions. These g-supersolutions are 'pushed' to be above the g-supermartingale. But an interesting observation is that this sequence can never be above the g-supermartingale. These two effects force the sequence converges to the given g-supermartingale itself. By the limit theorem it follows that this limit is a supersolution.

This section is divided into two subsections. In subsection 3.1 we discuss a general notion of g-martingales and obtain related nonlinear Doob–Meyer Decomposition Theorem. In subsection 3.2, we consider a special but typical case: when $g(y,z)|_{z=0} \equiv 0$. In this case a nonlinear version of expectation: g-expectation is introduced and is related, in a very familiar way, to the corresponding g-martingales. The corresponding nonlinear Doob–Meyer Decomposition Theorem also has a more familiar formulation.

3.1. *Nonlinear decomposition theorem and limit theorem*

We now introduce the notion of g-martingales

Definition 3.1. *A g-martingale on $[0, T]$ is a g-solution on $[0, T]$. An $\mathscr{F}_t$-progressively measurable real-valued process (Y_t) is called a g-supermartingale (resp. g-submartingale) on $[0, T]$ in strong sense if, for each stopping time $\tau \leq T$, $\mathbf{E}|Y_\tau|^2 < \infty$, and the g-solution (y_t) on $[0, \tau]$ with terminal condition $y_\tau = Y_\tau$ satisfies $y_\sigma \leq Y_\sigma$ (resp. $y_\sigma \geq Y_\sigma$) for all stopping time $\sigma \leq \tau$.*

Definition 3.2. *An $\mathscr{F}_t$-progressively measurable real-valued process (Y_t) is called a g-supermartingale on (resp. g-submartingale) $[0, T]$ in weak sense if, for each (deterministic time) $t \leq T$, $\mathbf{E}|Y_t|^2 < \infty$, and the g-solution (y_t) on $[0, t]$ with terminal condition $y_t = Y_t$ satisfies $y_s \leq Y_s$ (resp. $y_s \geq Y_s$) for all deterministic time $s \leq t$.*

Certainly, A g-supermartingale in strong sense is also a g-supermartingale in weak sense. It is already shown that, under assumptions similar to the classical case, a g-supermartingale in weak sense coincides with a g-supermartingale in strong sense (see [CP]). This result corresponds the so-called Optional Stopping Theorem in theory of martingales.

By Comparison Theorem 1.3, it is easy to prove that, a g-supersolution on $[0, T]$ is also a g-supermartingale in both strong and weak sense. In this section we are concerned with the inverse problem: can we say that a right-continuous g-supermartingale is also a g-supersolution? This problem is more difficult since it is in fact a nonlinear version of Doob-Meyer Decomposition Theorem. We claim

Theorem 3.3. *We assume* (H1.1) *and* (H1.2). *Let (Y_t) be a right-continuous g-supermartingale on $[0, T]$ in strong sense with*

$$\mathbf{E}\left[\sup_{0 \leq t \leq T} |Y_t|^2\right] < \infty \ .$$

Then (Y_t) is a g-supersolution on $[0, T]$: there exists a unique RCLL increasing process (A_t) with $A_0 = 0$ and $\mathbf{E}[(A_T)^2] < \infty$ such that (Y_t) coincides with the unique solution (y_t) of the BSDE

$$y_t = Y_T + \int_t^T g(y_s, z_s, s)\, ds + (A_T - A_t) - \int_t^T z_s\, dW_s, \quad t \in [0, T] \ , \tag{3.1}$$

In order to prove this theorem, we now consider the following family of BSDE parameterized by $i = 1, 2, \ldots$.

$$y_t^i = Y_T + \int_t^T g(y_s^i, z_s^i, s)\, ds + i \int_t^T (Y_s - y_s^i)\, ds - \int_t^T z_s^i\, dW_s \ . \tag{3.2}$$

An important observation is that, for each $i > 0$, (y_t^i) is bounded from above by (Y_t). Thus (y^i) is a g-supersolution on $[0, T]$. Under this observation, (3.2) becomes a penalization problem introduced in [ELal].

Lemma 3.4. *We have, for each* $i = 1, 2, \ldots$,

$$Y_t \geq y_t^i.$$

Proof. If it is not the case, then there exist $\delta > 0$ and a positive integer i such that the measure of $\{(\omega, t);\, y_t^\epsilon - Y_t - \delta \geq 0\} \subset \Omega \times [0, T]$ is nonzero. We then can define the following stopping times

$$\sigma := \min[T, \inf\{t;\;\; y_t^i \geq Y_t + \delta\}],$$

$$\tau := \inf\{t \geq \sigma;\;\; y_t^i \leq Y_t\}$$

It is seen that $\sigma \leq \tau \leq T$ and $P(\tau > \sigma) > 0$. Since $Y_t - y_t^i$ is right-continuous, we have

$$\begin{aligned} &\text{(i)} \quad y_\sigma^i \geq Y_\sigma + \delta; \\ &\text{(ii)} \; y_\tau^i \leq Y_\tau \;\; . \end{aligned} \tag{3.3}$$

Now let (y_t) (resp. (y_t')) the g-solution on $[0, \tau]$ with terminal condition $y_\tau = y_\tau^i$ (resp. = $y_\tau' = Y_\tau$). By Comparison Theorem (3.3)–(ii) implies $y_\sigma^i \leq y_\sigma \leq y_\sigma'$. On the other hand since (Y_t) is a g-submartingale. Thus

$$Y_\sigma \geq y_\sigma^i.$$

This is in contrary with (3.3)–(i). The proof is complete. □

Remark 3.5. This lemma means that (3.2) is in fact the penalization BSDE introduced in [ELal]. The new observation in this section is the following phenomenon: although the penalized g-supersolutions y^i are pushed up to be above the supermartingale (Y_t), but in fact they can never be strictly above (Y_t). From this fact it follows that, necessarily, this sequence converges to the supermartingale (Y_t) itself. Thus, by Limit Theorem 2.4 (Y) itself is also a g-supersolution. Specifically, we have:

Proof of Theorem 3.3. The uniqueness is due to the uniqueness of g-supersolution i.e. Prop. 1.6. We now prove the existence. We can rewrite BSDE (3.2) as

$$y_t^i = Y_T + \int_t^T g(y_s^i, z_s^i, s)\, ds + A_T^i - A_t^i - \int_t^T z_s^i \, dW_s \quad ,$$

where we denote

$$A_t^i := i \int_0^t (Y_s - y_s^i)\, ds \quad .$$

From Lemma 3.3, $Y_t - y_t^i = |Y_t - y_t^i|$. It follows from the Comparison Theorem that $y_t^i \leq y_t^{i+1}$. Thus $\{y^i\}$ is a sequence of continuous g-supermartingale that is monotonically converges up to a process (y_t). Moreover (y_t) is bounded from above by Y_t. It is then easy to check that all conditions in Theorem 2.4 are satisfied. (y_t) is a g-supersolution on $[0, T]$ of the following form.

$$y_t = Y_T + \int_t^T g(y_s, z_s, s)\, ds + (A_T - A_t) - \int_t^T z_s \, dW_s, \quad t \in [0, T] \quad ,$$

where (A_t) is a RCLL increasing process. It then remains to prove that $y = Y$. From Lemma 2.5–(ii) we have

$$\mathbf{E}|A_T^i|^2 = i^2 \mathbf{E}\left[\int_0^T |Y_t - y_t^i|\, dt\right]^2 \leq C \quad .$$

It then follows that $Y_t \equiv y_t$. The proof is complete. □

The following result is a limit theorem for g-supermartingales or equivalently, g-supersolution, for general (RCLL) situations.

Theorem 3.6. *Let $\{Y^i\}$ be a sequence of RCLL g-supersolutions (or g-supermartingale) on $[0, T]$ that monotonically converges up to (Y) with $\mathbf{E}\sup_{t\in[0,T]} |Y_t|^2] < \infty$. Then (Y) itself is also an RCLL g-supersolution (or g-supermartingale).*

Proof. Like in the proof of Theorem 3.3, we consider the following family of BSDE, for $i = 1, 2, \ldots,$

$$y_t^i = Y_T^i + \int_t^T g(y_s^i, z_s^i, s)\, ds + i \int_t^T (Y_s^i - y_s^i)\, ds - \int_t^T z_s^i \, dW_s \quad .$$

Since, for each i, (Y_t^i) is a g-supermartingale, by Lemma 3.4, $y_t^i \leq Y_t^i (\leq Y_t)$. By comparison theorem $y_t^i \leq y_t^{i+1}$. Thus this sequence of continuous g-supersolutions converges monotonically to $y_t \leq Y_t$. It follows from Theorem 2.4 that (y_t) is an RCLL g-supersolution.

It remains to prove that (y_t) coincides with (Y_t). It suffices to prove that, for each j, $y_t \geq Y_t^j$. To this end we consider the BSDE

$$y_t^{j,i} = Y_T^j + \int_t^T g(y_s^{j,i}, z_s^{j,i}, s)\, ds + i \int_t^T (Y_s^j - y_s^{j,i})\, ds - \int_t^T z_s^{j,i}\, dW_s \ .$$

From comparison theorem, $y_t^i \geq y_t^{j,i}$ whenever $i \geq j$. On the other hand, we have shown in the proof of Theorem 3.3 that, for fixed j, the sequence $\{y^{j,i}\}$ converges monotonically up to the g-supermartingale Y^j. It follows that, for each j, $Y_t \geq y_t \geq Y_t^j$. Thus y_t coincides with Y_t. The proof is complete. □

3.2. *g-expectation and related decomposition theorem*

Since M. Allais's famous paradox, economists begin to look for a nonlinear version of the notion of mathematical expectations and/or non-additive probability in order to improve the notion of 'expected utility' introduced by Von Neumann-Morgenstern, a fundamental concept in modern economic theory. A typical example of such notion is 'certainty equivalence' (see [DE]). But one important problem is: how to find a natural generalization of the classical expectations, e.g., the one which preserves, as many as possible, properties of the expectation except the linearity.

In this subsection we will make an additional assumption on the function g:

$$g(y, 0, \cdot) \equiv 0, \quad \forall y \in \mathbf{R} \ . \tag{3.4}$$

Under this condition one can introduce a notion of g-expectation. This g-expectation preserves all properties of the classical expectation except the linearity. Similarly to the classical case, we can define the related conditional g-expectation with respect to $\mathscr{F}_t$. It also preserves all properties of the classical conditional expectation except the linearity. Thus it is a very natural extension of the classical (or linear) expectations. The above g-supermartingales and the related nonlinear Doob–Meyer decomposition can be stated in a very natural way.

This notion of g-expectation was initially introduced in [P2] in a more general framework. Here we will only state it in $L^2(\Omega, \mathscr{F}_T, P)$ and give some sketch of proofs of its basic properties. Our main objective in this paper is to prove that the decomposition theorem of Doob–Meyer's type still holds true for g-supermartingale. We will also use the notion of g-solution (Def. 1.5) to simplify the statement.

Condition (3.4) is equivalent to

$$g(y1_A, z1_A, \cdot) \equiv 1_A g(y, z, \cdot), \qquad \forall (y, z) \in \mathbf{R} \times \mathbf{R}^n, \quad \forall A \in \mathscr{F}_T. \tag{3.5}$$

Observe that, for a g-solution (y_t) on $[0, T]$, y_0 is a deterministic number that is uniquely determined by its terminal condition y_T. We then define

Definition 3.5. *For each* $X \in L^2(\Omega, \mathcal{F}_T, P; \mathbf{R})$*, let* (y_t) *be the* g*-solution on* $[0, T]$ *with terminal condition* $y_T = X$*. We call* y_0 *the* g*-expectation of* X *and denote it by* $\mathcal{E}_g[X] := y_0$*.*

Remark 3.6. Since X is $\mathcal{F}_T$-measurable, then, for any $T' > T$, X is also $\mathcal{F}_{T'}$-measurable. By Def. 3.5, the g-expectation of X may be also defined by $\mathcal{E}_g[X] := y'_0$, where (y') is the g-solution on $[0, T']$ with terminal condition $y'_{T'} = X$. But, by (3.5), one can check that $y'_t \equiv X$ $(z'_t \equiv 0)$ on $[T, T_1]$. Thus y coincides with y' on $[0, T]$.

Like classical expectations, the "g-expectation" has the following properties.

Lemma 3.7. $\mathcal{E}_g[\cdot]$ *has the following properties:*

(i) (*preserving of constants*): *For each constant* c, $\mathcal{E}_g[c] = c$;
(ii) (*monotonicity*): *If* $X_1 \geq X_2$, *a.s., then* $\mathcal{E}_g[X_1] \geq \mathcal{E}_g[X_2]$*;*
(iii) (*strict monotonicity*): *If* $X_1 \geq X_2$, *a.s., and* $P(X_1 > X_2) > 0$, *then we have* $\mathcal{E}_g[X_1] > \mathcal{E}_g[X_2]$.

Proof. By (3.5), the g-solution (y_t) on $[0, T]$ with terminal condition $y_T = c$ is identically equal to c. Thus (i) holds. (ii) and (iii) are a direct consequence of the comparison theorem 1.3. □

For each $t \leq T$, we now introduce the notion of the conditional g-expectation with respect to $\mathcal{F}_t$. Similarly to the classical case, for a given $X \in L^2(\Omega, \mathcal{F}_T, P)$, we look for a random variable $\eta \in L^2(\Omega, \mathcal{F}_t, P)$ satisfying

$$\mathcal{E}_g[1_A X] = \mathcal{E}_g[1_A \eta], \quad \text{for all } A \in \mathcal{F}_t \tag{3.6}$$

We have

Proposition 3.8. *For each* $X \in L^2(\Omega, \mathcal{F}_T, P)$*, let* (y_s) *be the* g*-solution on* $[0, T]$ *with terminal condition* $y_T = X$*. Then, for each* $t \leq T$, $\eta = y_t$ *is the unique element in* $L^2(\Omega, \mathcal{F}_t, P)$ *such that* (3.6) *is satisfied.*

It is then reasonable to define y_t *as the conditional* g*-expectation with respect to* $\mathcal{F}_t$*.*

Definition 3.9. *Let* (y_s) *be the* g*-solution on* $[0, T]$ *with terminal condition* $y_T = X \in L^2(\Omega, \mathcal{F}_T, P)$*. For each* $t \leq T$*, We call* y_t *the conditional* g*-expectation of* X *with respect to* $\mathcal{F}_t$ *and denote it by* $\mathcal{E}_g[X|\mathcal{F}_t]$*.*

Remark 3.10. For a stopping time $\tau \leq T$, we can similarly define the conditional g-expectation of X with respect to $\mathcal{F}_\tau$ by $\mathcal{E}_g[X|\mathcal{F}_\tau] =: y_\tau$.

Proof of Proposition 3.8. 1) *Uniqueness*: If both η_1 and η_2 satisfy (3.6), then

$$\mathscr{E}_g[\eta_1 1_A] = \mathscr{E}_g[\eta_2 1_A], \quad \text{for all } A \in \mathscr{F}_t \ .$$

Particularly, when $A = \{\eta_1 \geq \eta_2\}$, we have

$$\mathscr{E}_g[\eta_1 1_{\{\eta_1 \geq \eta_2\}}] = \mathscr{E}_g[\eta_2 1_{\{\eta_1 \geq \eta_2\}}]$$

But $\eta_1 1_{\{\eta_1 \geq \eta_2\}} \geq \eta_2 1_{\{\eta_1 \geq \eta_2\}}$. Thus, by the strict monotonicity of g-expectation (Lemma 3.7 (iii)) it follows that

$$\eta_1 1_{\{\eta_1 \geq \eta_2\}} = \eta_2 1_{\{\eta_1 \geq \eta_2\}}, \quad \text{a.s.}$$

Similarly,

$$\eta_1 1_{\{\eta_1 \leq \eta_2\}} = \eta_2 1_{\{\eta_1 \leq \eta_2\}} \eta \quad \text{a.s.}$$

These two relations implies $\eta_1 = \eta_2$, a.s..

2) *Existence*: Since (y_s) is the g-solution on $[0, T]$ with terminal condition $y_T = X$, then by (3.5), for each $A \in \mathscr{F}_t$,

$$1_A y_s = 1_A X + \int_s^T g(1_A y_r, 1_A z_r, r)\, dr - \int_s^T 1_A z_r\, dW_r, \quad s \in [t, T].$$

Thus the g-solution $(\hat{y}_s)$ on $[0, T]$ with terminal condition $\hat{y}_T = 1_A X_T$ satisfies $\hat{y}_t = 1_A y_t$. It follows that $(\hat{y}_s)$ is also the g-solution on $[0, t]$ with terminal condition $\hat{y}_t = 1_A y_t$. According to the definition of $\mathscr{E}_g[\cdot]$ (see Remark 3.6) it follows immediately that

$$\mathscr{E}_g[1_A X] = \mathscr{E}_g[1_A y_t], \quad \text{for all } A \in \mathscr{F}_t.$$

The proof is complete. □

The conditional g-expectation preserves essential properties (except linearity) of the classical expectations.

Lemma 3.10. *$\mathscr{E}_g[\cdot|\mathscr{F}_t]$ has the following properties*

(i) *If X is $\mathscr{F}_t$-measurable, then $\mathscr{E}_g[X|\mathscr{F}_t] \equiv X$*
(ii) *For each t and r, then $\mathscr{E}_g[\mathscr{E}_g[X|\mathscr{F}_t]|\mathscr{F}_r] = \mathscr{E}_g[X|\mathscr{F}_{t \wedge r}]$;*
(iii) *If $X_1 \geq X_2$, then $\mathscr{E}_g[X_1|\mathscr{F}_t] \geq \mathscr{E}_g[X_2|\mathscr{F}_t]$. If, moreover $P(X_1 > X_2) > 0$, then we have $P(\mathscr{E}_g[X_1|\mathscr{F}_t] > \mathscr{E}_g[X_2|\mathscr{F}_t]) > 0$.*
(iv) *For each $B \in \mathscr{F}_t$, $\mathscr{E}_g[1_B X|\mathscr{F}_t] = 1_B \mathscr{E}_g[X|\mathscr{F}_t]$.*

The proof of these properties is similar to this of the classical case: We omit it.

With the notion g-expectations, a g-martingale $(M_t)_{0\le t\le T}$ defined in Def. 3.1 can be written in the form

$$M_t = \mathscr{E}_g[\xi|\mathscr{F}_t] \ . \tag{3.7}$$

The following property provides another, and more familiar definition of g-supermartingales.

Proposition 3.11. *We assume* (3.4). *Then a process* $(Y_t)_{0\le t\le T}$ *satisfying* $\mathbf{E}[|Y_t|^2] < \infty$ *is a* g*-martingale* (*resp.* g*-supermartingale*) *in weak sense iff*

$$\mathscr{E}_g[Y_t|\mathscr{F}_s] = Y_s, \text{ (resp. } \mathscr{E}_g[Y_t|\mathscr{F}_s] \le Y_s), \quad \forall s \le t \le T \ .$$

They are g*-martingale (resp.* g*-supermartingale) in strong sense iff, in the above relations,* $s \le t$ *are stopping times.*

Corollary 3.12. *We make the same assumptions as in Theorem* 3.1. *We assume furthermore that g is independent of y and that* (3.4) *hold. Let* (X_t) *be a* g*-submartingale on* $[0, T]$ *in the strong sense satisfying* $\mathbf{E}[\sup_{t\le T} |X_t|^2] < \infty$. *Then* (X_t) *has the following decomposition*

$$X_t = M_t - A_t \ .$$

Here (M_t) *is a* g*-martingale of the form* (3.7) *and* (A_t) *is an RCLL increasing process with* $A_0 = 0$ *and* $\mathbf{E}[(A_T)^2] < \infty$. *Moreover such decomposition is unique.*

Proof. By Theorem 3.3, this g-submartingale (X_t) on $[0, T]$ has the following form: there exists an RCLL increasing process $(A_t) \in L^2_{\mathscr{F}}(0, T)$ such that

$$X_t = X_T + \int_t^T g(z_s, s)\, ds + A_T - A_t - \int_t^T z_s\, dW_s, \quad t \in [0, T] \ .$$

We set $M_t = X_t + A_t$, then

$$M_t = (X_T + A_T) + \int_t^T g(z_s, s)\, ds - \int_t^T z_s\, dW_s, \quad t \in [0, T] \ .$$

It follows from the above definitions that (M_t) is a g-martingale and that

$$M_t = \mathscr{E}_g[\xi + A_T|\mathscr{F}_t], \quad t \in [0, T] \ .$$

The proof is complete. □

4. BSDE with constraints on (y, z)

Let $\phi : \mathbf{R}^{1+d} \times [0, T] \times \Omega \to \mathbf{R}^+$ be a given nonnegative function such that, for each $(y, z) \in \mathbf{R}^{1+d}$, $\phi(y, z, \cdot) \in L^2_{\mathcal{F}}(0, T; \mathbf{R}^+)$ and such that ϕ is globally Lipschitz with respect to (y, z). In this section we consider BSDE of type (2.12) with constraints imposed to the solution. i.e., the solution (y_t, z_t) must be inside of the zone given by

$$K_t(\omega) := \{(y, z) \in \mathbf{R}^{1+d};\ \phi(y, z, t, \omega) = 0\}\ . \tag{4.1}$$

The problem consists of finding the smallest g-supersolution on $[0, T]$, i.e., a solution of BSDE of the form

$$y_t = \xi + \int_t^T g(y_s, z_s, s)\, ds + (A_T - A_t) - \int_t^T z_s\, dW_s \tag{4.2}$$

such that

$$\phi(y_t, z_t, t) = 0, \quad \text{a.e., a.s.} \tag{4.3}$$

Definition 4.1. *A g-supersolution (y_t) on $[0, T]$ with the decomposition (z_t, A_t) is said to be the smallest g-supersolution, given $y_T = \xi$, subject to the constraint (4.3) if (4.3) is satisfied and $y_t \le y'_t$, a.e., a.s., for any g-supersolution (y'_t) on $[0, T]$ with $y'_T = \xi$ and the decomposition (z'_t, A'_t) subject to $\phi(y'_t, z'_t, t) \equiv 0$.*

We need to introduce the following conditions to ensure the existence of the upper bound.

(H4.1) We assume that there exists g-supersolution $(\hat{y}_t)$ on $[0, T]$ with terminal condition $\hat{y}_T = \xi$ such that the constraint (4.3) is satisfied: Let $(\hat{z}_t, \hat{A}_t)$ be the decomposition of $(\hat{y}_t)$. We have

$$\phi(\hat{y}_t, \hat{z}_t, t) = 0, \quad \text{a.e., a.s.} \tag{4.4}$$

In order to construct the smallest solution of BSDE (4.2) with constraints (4.3), we need to introduce the following BSDE

$$y^i_t = \xi + \int_t^T g(y^i_s, z^i_s, s)\, ds + A^i_T - A^i_t - \int_t^T z^i_s\, dW_s\ . \tag{4.5}$$

where

$$A^i_t := i \int_0^t \phi(y^i_s, z^i_s, s)\, ds\ . \tag{4.6}$$

We can claim

Theorem 4.2. *We assume* (H1.1), (H1.2) *as well as* (H4.1). *Then the sequence of g-supersolutions $\{y^i\}$ converges monotonically up to (y) with*

$$\mathbf{E}\left[\sup_{0\le t\le T}|y_t|^2\right]<\infty. \tag{4.7}$$

$\{z^i\}$ *and* $\{A^i\}$ *converges to* (z) *and* (A) *respectively in the sense of Theorem* 2.4. *Furthermore,* (y,z,A) *is the smallest g-supersolution of BSDE* (4.2) *subject to constraints* (4.3).

Proof. We first observe that, according to Proposition 1.2,

$$\mathbf{E}\left[\sup_t|\hat{y}_t|^2\right]<\infty. \tag{4.8}$$

From the Comparison Theorem 1.3, we have

$$y_t^i\le\hat{y}_t.$$

The second relation is due to the fact that, since the $(\hat{y}_t)$ is a g-supersolution with decomposition $(\hat{A}_t,\hat{z}_t)$ such that the constraint (4.4) is satisfied, it can be regarded as the solution of the following BSDE

$$\hat{y}_t=\xi+\int_t^T[g(\hat{y}_s,\hat{z}_s,s)+i\phi(\hat{y}_s,\hat{z}_s,s)]\,ds+\hat{A}_T-\hat{A}_t-\int_t^T\hat{z}_s\,dW_s\ . \tag{4.9}$$

It follows from the Comparison Theorem that $y_t^i\le\hat{y}_t$. But, always from comparison theorem, we have $y_t^i\le y_t^{i+1}$. Thus $y_t^i\uparrow\uparrow y_t\le\hat{y}_t$.

We observe that the above argument implies that (y_t) is dominated from above by any g-supersolution subject to the constraint (4.3).

Clearly (y_t) satisfies (4.7). It follows from Theorem 2.4 and 2.5 that (y_t) is a g-supersolution and that there exists a constant C such that

$$\mathbf{E}(A_T^i)^2=i^2\mathbf{E}\left[\int_0^T\phi(y_s^i,z_s^i,s)\,ds\right]^2\le C\ . \tag{4.10}$$

This theorem also tell us that $z^i\to z$ in $L^p_{\mathcal{F}}(0,T)$ for $p<2$. It then follows that

$$\phi(y_s^i,z_s^i,s)\to\phi(y_s,z_s,s)\quad\text{in } L^p_{\mathcal{F}}(0,T)\ .$$

This with (4.10) implies that the constraint (4.3) is also satisfied.

The proof is complete. □

Appendix

We now give the proof of Lemma 2.3. To this end we need the following somewhat classical lemmas: for any given square-integrable increasing RCLL process on [0,T], the principal contribution of its jumps can be limited within a finite number of (random) points.

Lemma A.1. *Let* (A_t) *be an increasing RCLL process defined on* $[0, T]$ *with* $A_0 = 0$ *and* $\mathbf{E}(A_T)^2 < \infty$*. Then, for any* $\epsilon > 0$*, there exists a finite number of stopping times* τ_k, $k = 0, 1, 2, \ldots, N+1$ *with* $\tau_0 = 0 < \tau_1 \leq \cdots \leq \tau_k \leq T = \tau_{N+1}$ *and with disjoint graphs on* $(0, T)$ *such that*

$$\sum_{k=0}^{N} \mathbf{E} \sum_{t \in (\tau_k, \tau_{k+1})} (\Delta A_t)^2 \leq \epsilon \ . \tag{A.1}$$

Sketch of Proof. For each $\nu > 0$, we denote

$$A_t(\nu) = A_t - \sum_{s \leq t} \Delta A_s 1_{\{\Delta A_s > \nu\}} \ .$$

It has jumps smaller than ν. Thus there is a sufficiently small $\nu > 0$ such that

$$\mathbf{E}\left[\sum_{s \leq T} (\Delta A_s(\nu))^2\right] \leq \frac{\epsilon}{2} \ .$$

Now let σ_k, $k = 1, 2, \ldots$ be the succesive times of jumps of A with size bigger than ν; they are stopping times, and there is N such that

$$\mathbf{E}\left(\sum_{s \in (\sigma_N, T)} (\Delta A_s)^2\right) \leq \frac{\epsilon}{2} \ .$$

Then $\tau_k = \sigma_k \wedge T$ for $k \leq N$, and $\tau_{N+1} = T$ satisfies (A.1). □

For applying the formula of the integral by part to the process (y_t) (with jumps), the above open intervals (σ_k, σ_{k+1}) is not so convenient. Thus we will cut a sufficiently small part and only work on the remaining subintervals $(\sigma_k, \tau_k]$. This is possible since our filtration is continuous. In fact we have:

Lemma A.2. *Let* $0 < \sigma \leq T$ *be a stopping time. Then there exists a sequence of stopping times* $\{\sigma_i\}$ *with* $0 < \sigma_i < \sigma$*, a.s. for each* $i = 1, 2, \ldots$ *, such that* $\sigma_i \uparrow \sigma$*.*

This lemma is quite classical. The proof is omitted.

We now give

Proof of Lemma 2.3. We first apply Lemma A.1 to construct a sequence of non-decreasing stopping times $\{\sigma_k\}_{k=0}^{N+1}$ with $\sigma_0 = 0$ and $\sigma_{N+1} = T$ such that, $\sigma_k < \sigma_{k+1}$ whenever $\sigma_k < T$ and that

$$\sum_{k=0}^{N} \mathbf{E} \sum_{t\in(\sigma_k,\sigma_{k+1})} (\Delta A_t)^2 \le \delta \ .$$

Then for each $0 \le k \le N$, we apply Lemma A.2 to construct a stopping time $0 < \tau'_k < \sigma_{k+1}$, such that

$$\mathbf{E} \sum_{k=0}^{N} (\sigma_{k+1} - \tau'_k) \le \epsilon \ .$$

Finally we set

$$\tau_0 = \tau'_0, \quad \tau_1 = \sigma_1 \vee \tau'_1, \quad \dots, \quad \tau_N = \sigma_N \vee \tau'_N$$

It is clear that $\tau_k \in [\sigma_k, \sigma_{k+1}) \cap [\tau'_{k+1}, \sigma_{k+1}]$. We have also $\tau_k < \sigma_{k+1}$ whenever $\sigma_k < T$. Thus $(\sigma_k, \tau_k] \in (\sigma_k, \sigma_{k+1})$. It follows that

$$\mathbf{E} \sum_{k=0}^{N} (\sigma_{k+1} - \tau_k) \le \epsilon \ ,$$

or

$$\mathbf{E} \sum_{k=0}^{N} (\tau_k - \sigma_k) \ge T - \epsilon \ ,$$

and

$$\sum_{k=0}^{N} \mathbf{E} \sum_{t\in(\sigma_k,\tau_k]} (\Delta A_t)^2 \le \sum_{k=0}^{N} \mathbf{E} \sum_{t\in(\sigma_k,\sigma_{k+1})} (\Delta A_t)^2 \le \delta \ .$$

Thus the above conditions (i)–(iii) are satisfied. □

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