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Normal limit theorems for symmetric random matrices

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Abstract. Using the machinery of zonal polynomials, we examine the limiting behavior of random symmetric matrices invariant under conjugation by orthogonal matrices as the dimension tends to infinity. In particular, we give sufficient conditions for the distribution of a fixed submatrix to tend to a normal distribution. We also consider the problem of when the sequence of partial sums of the diagonal elements tends to a Brownian motion. Using these results, we show that if O_n is a uniform random $n \times n$ orthogonal matrix, then for any fixed $k > 0$, the sequence of partial sums of the diagonal of O_n^k tends to a Brownian motion as $n \to \infty$.

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0. Introduction

Let A be a real symmetric $n \times n$ matrix, and consider the random matrix OAO^t, where O is uniformly distributed from the orthogonal group $O(n)$. For n large, and for sufficiently "nice" A, one expects the matrix elements of OAO^t to be approximate Gaussian random variables, independent except as required by symmetry. The aim of the present work is to formalize this statement, in a couple of different ways.

In general, we will be working with a sequence A_i of random matrices, with A_i of dimension n_i tending to infinity, such that

$$OA_iO^t \sim A_i$$

for all $O \in O(n_i)$ (We use the notation $X \sim Y$ to indicate that X and Y have the same distribution). In section 1, we give sufficient conditions for the upper $k \times k$ submatrix of A_i to converge (weakly) to a Gaussian distribution, when each A_i is symmetric. (If weak convergence is replaced by convergence of moments, then the conditions are necessary as well.)

In section 2, we consider a somewhat different problem. For each A_i, we consider the sequence of partial sums of the diagonal elements of A_i. The independence heuristic above suggests that this sequence, scaled appropriately, should converge to a Brownian motion. We give sufficient conditions for this to occur.

In the final section, we give a couple of examples of random matrices satisfying our conditions; in particular, we consider powers of random orthogonal matrices, and show that the partial sum sequence of the diagonal of O^k converges to a Brownian motion.

0.1. Zonal polynomials

We will need several results from the theory of zonal polynomials ([3]; also of interest is [2], which considers some applications to random matrices). The real zonal polynomials are polynomial functions $z_\lambda(A)$ defined on (square) matrices; each z_λ is a symmetric function of the eigenvalues of A. (Here λ ranges over all partitions; see [4] for definition and basic results.) The primary relevance of the zonal polynomials for our purpose is that they satisfy the following identity:

$$E_{O\in O(n)} z_\lambda(AOBO^t) = \frac{z_\lambda(A) z_\lambda(B)}{z_\lambda(1_n)} \quad ,$$

where A and B are any symmetric matrices.

The zonal polynomials can be expanded in terms of power-sum symmetric functions; we will write

$$z_\lambda = \sum_{\mu \vdash |\lambda|} a_\lambda^\mu p_\mu \quad ,$$

with the normalization $a_\lambda^{1^j} = 1$, and note that these coefficients satisfy the following orthogonality conditions:

$$\sum_\nu 2^{\ell(\nu)} Z_\nu a_\mu^\nu a_\kappa^\nu = \delta_{\mu\kappa} \frac{|2\mu|!}{\chi^{2\mu}(1)} \quad ,$$

where Z_ν is the size of the centralizer in $S_{|\nu|}$ of a permutation of cycle-type ν, 2μ is the partition obtained by doubling each element of μ, and

$\chi^\lambda(1)$ is the dimension of the irreducible representation of $S_{|\lambda|}$ corresponding to λ.

Consequently, if we expand the power-sum functions (again, see [4]) in terms of the zonal polymials as

$$p\lambda = \sum_{\mu \vdash |\lambda|} A_\lambda^\mu z_\mu \ ,$$

we have

$$A_\lambda^\mu = \frac{\chi^{2\mu}(1)2^{\ell(\lambda)}Z_\lambda}{|2\mu|!} a_\mu^\lambda \ .$$

0.2. A convergence condition

In the sequel, we will use the notation

$$X_n \twoheadrightarrow x$$

to indicate that a sequence of random variables X_n converges to the constant x in the following strong sense:

$$\lim_{n\to\infty} E(X_n^k) = x^k$$

for all k (in particular each X_n must have moments of all orders). The following lemma will be necessary:

Lemma 0.1. *Let X_n and Y_n be two sequences of random variables on the same probability space. If $X_n \twoheadrightarrow x$ and $Y_n \twoheadrightarrow y$, then $X_n Y_n \twoheadrightarrow xy$.*

Proof. It suffices to show

$$\lim_{n\to\infty} E(X_n Y_n) = xy \ ;$$

we can deduce convergence of higher order moments from the fact that $X_n^k \twoheadrightarrow x^k$ and $Y_n^k \twoheadrightarrow y^k$. Moreover, we may assume without loss of generality that x and y are both 0. Then

$$X_n Y_n \leq \max(X_n^2, Y_n^2) \leq X_n^2 + Y_n^2 \ ,$$

and thus by symmetry

$$-(X_n^2 + Y_n^2) \leq X_n Y_n \leq (X_n^2 + Y_n^2) \ .$$

Taking expectations, the lemma follows immediately. □

Thus if a collection of random variables all tend to constants in this sense, then all their joint moments converge to the values one would expect.

We will also need the notation $X_n \Rightarrow X$ for weak convergence.

1. Submatrices

Suppose M_i is a sequence of random symmetric matrices of increasing dimension, where the distribution of each M_i is invariant under orthogonal change of basis. Consider $(M_i)_k$, the top-left $k \times k$ submatrix of M_i, where k is fixed. If the M_i are well-behaved, one would expect this to have a limiting distribution. One clear possibility for this limiting distribution is $N_k = \frac{1}{2}(X + X^t)$, where X is a $k \times k$ matrix with i.i.d. real standard normal entries. Thus the first question we address is when $(M_i)_k$ converges weakly to some multiple of N_k.

We will need the following result on the moments of symmetric normal matrices:

Lemma 1.1. *Let N_k be a sequence of random $k \times k$ matrices, where each N_k is distributed as a symmetric normal matrix. Then as $k \to \infty$,*

$$k^{-1}Tr(N_k) \twoheadrightarrow 0 \ ,$$
$$k^{-2}Tr(N_k^2) \twoheadrightarrow \tfrac{1}{2} \ ,$$

and

$$k^{-j}Tr(N_k^j) \twoheadrightarrow 0 \ ,$$

for all $j > 2$.

Proof. Consider, first, $\mathrm{Tr}(N_k^2)$. This is easily seen to be χ^2-distributed, with $\binom{k+1}{2}$ degrees of freedom. But then we have

$$\frac{1}{\binom{k+1}{2}} \mathrm{Tr}(N_k^2) \twoheadrightarrow 1 \ ,$$

and consequently

$$\tfrac{1}{k^2} \mathrm{Tr}(N_k^2) \twoheadrightarrow \tfrac{1}{2} \ .$$

Now, consider $j \neq 2$. To determine the asymptotics of $\mathrm{Tr}(N_k^j)$, we will first consider $E(\mathrm{Tr}(N_k A)^j)$, as a function of a symmetric matrix A. Since N_k is invariant under conjugation, we may diagonalize A; it follows easily that $\mathrm{Tr}(N_k A)$ has distribution $N(0, \mathrm{Tr}(A^2))$. (Here we use the standard notation $N(\mu, \sigma^2)$ for a Gaussian distribution with mean μ and variance σ^2.) Consequently,

$$E(\mathrm{Tr}(N_k A)^j) = \begin{cases} C_j \mathrm{Tr}(A^2)^{j/2}, & j \text{ even} \\ 0, & \text{otherwise} \end{cases} ,$$

where

$$C_j = \frac{j!}{2^{j/2}(j/2)!}$$

is the jth moment of a $N(0,1)$.

Expressing $\mathrm{Tr}(N_k A)^j$ in terms of zonal polynomials, we get:

$$p_{1^j}(N_k A) = \sum_{\mu \vdash j} A^{\mu}_{1^j} z_\mu(N_k A) \ .$$

Since $O N_k O^t \sim N_k$, we have:

$$\begin{aligned} E(p_{1^j}(N_k A)) &= \sum_{\mu \vdash j} A^{\mu}_{1^j} E(z_\mu(O N_k O^t A)) \\ &= \sum_{\mu \vdash j} A^{\mu}_{1^j} E(z_\mu(N_k)/z_\mu(1_k)) z_\mu(A) \\ &= \sum_{\mu, \nu \vdash j} A^{\mu}_{1^j} E(z_\mu(N_k)/z_\mu(1_k)) a^{\nu}_{\mu} p_\nu(A) \ . \end{aligned}$$

Since $E(p_{1^j}(N_k A)) = C_j p_{2^{j/2}}(A)$, it follows that

$$\sum_{\mu \vdash j} A^{\mu}_{1^j} E(z_\mu(N_k)/z_\mu(1_k)) a^{\nu}_{\mu}$$

is 0 unless $\nu = 2^{j/2}$, when it equals C_j. Multiplying by A^{κ}_{ν}, and summing over ν, we get:

$$\sum_{\mu \vdash j} A^{\mu}_{1^j} E(z_\mu(N_k)/z_\mu(1_k)) \delta^{\kappa}_{\mu} = C_j A^{\kappa}_{2^{j/2}} \ ,$$

or

$$E(z_\mu(N_k)) = \frac{C_j A^{\mu}_{2^{j/2}}}{A^{\mu}_{1^j}} z_\mu(1_k) \ .$$

We then have

$$\lim_{k\to\infty} E\left(\frac{p_\lambda(N_k)}{k^j}\right) = \lim_{k\to\infty} \sum_{\mu\vdash j} A_\lambda^\mu E\left(\frac{z_\mu(N_k)}{k^j}\right)$$
$$= \lim_{k\to\infty} \sum_{\mu\vdash j} \frac{C_j A_\lambda^\mu A_{2^{j/2}}^\mu}{A_{1^j}^\mu} \frac{z_\mu(1_k)}{k^j}$$
$$= \sum_{\mu\vdash j} \frac{C_j A_\lambda^\mu A_{2^{j/2}}^\mu a_\mu^{1^j}}{A_{1^j}^\mu} \quad .$$

Now, from the orthogonality relations on the $a_\mu^{1^j}$, it follows that

$$F_\lambda = \frac{A_\lambda^\mu a_\mu^{1^j}}{A_{1^j}^\mu a_\mu^\lambda}$$

is independent of μ. Thus, we have

$$\lim_{k\to\infty} E\left(\frac{p_\lambda(N_k)}{k^j}\right) = \sum_{mu\vdash j} \frac{C_j A_\lambda^\mu A_{2^{j/2}}^\mu a_\mu^{1^j}}{A_{1^j}^\mu}$$
$$= C_j F_\lambda \sum_{\mu\vdash j} a_\mu^\lambda A_{2^{j/2}}^\mu$$
$$= C_j F_\lambda \delta_{2^{j/2}}^\lambda \quad .$$

In particular, setting $\lambda = j^l$, we have

$$\lim_{k\to\infty} E((k^{-j}\mathrm{Tr}(N_k^j))^l) = 0 \quad . \qquad \square$$

The referee has pointed out that the above result is well known (for instance, it follows from the semi-circle law for the eigenvalues of N_k); the above proof seems to be new, however.

Theorem 1.2. *Let M_i be a sequence of random symmetric matrices of dimension $n_i \to \infty$, with moments of all orders, where the distribution of each M_i is invariant under orthogonal change of basis, and let $(M_i)_k$ be the top-left $k \times k$ submatrix of M_i, for any fixed k. Then $(M_i)_k$ will converge weakly to cN_k, where N_k is a $k \times k$ symmetric normal matrix, if the following conditions hold*:

$$n_i^{-1}\mathrm{Tr}(M_i) \twoheadrightarrow 0,$$
$$n_i^{-2}\mathrm{Tr}(M_i^2) \twoheadrightarrow \frac{c^2}{2} \quad ,$$

and

$$n_i^{-j}\mathrm{Tr}(M_i^j) \twoheadrightarrow 0$$

for all $j > 2$.

Proof. By the method of moments, and the fact that Gaussian random variables are determined by their moments, we see that $(M_i)_k \Rightarrow cN_k$ if for all symmetric matrices B and for all j,

$$\lim_{i\to\infty} E(\mathrm{Tr}((M_i)_k B)^j) = c^j E(\mathrm{Tr}(N_k B)^j) \ .$$

Clearly, the distribution of $(M_i)_k$ is invariant under orthogonal change of basis, so we can expand the left hand side in terms of zonal polynomials as above:

$$\lim_{i\to\infty} E(\mathrm{Tr}((M_i)_k B)^j) = \sum_{\mu} A^{\mu}_{1^j} \lim_{i\to\infty} E(z_\mu((M_i)_k)) \frac{z_\mu(B)}{z_\mu(1_k)} \ .$$

Consequently, we need only show that

$$\lim_{i\to\infty} E(z_\mu((M_i)_k)) = E(z_\mu(cN_k)) \ .$$

Now, $(M_i)_k$ can be written as $\Pi_k M_i \Pi_k^t$, where Π_k is orthogonal projection onto the first k coordinates. This allows us to write:

$$\begin{aligned} E(z_\mu((M_i)_k)) &= E(z_\mu(\Pi_k O A_i O^t \Pi_k^t)) \\ &= \frac{z_\mu(\Pi_k^t \Pi_k) E(z_\mu(M_i))}{z_\mu(1_{n_i})} \ . \end{aligned}$$

and similarly for N_k. Thus we must show that

$$\lim_{i\to\infty} \frac{E(z_\lambda(cN_{n_i})) - E(z_\lambda(M_i))}{z_\lambda(1_{n_i})} = 0 \ .$$

Now, for $\lambda \vdash j$, $\lim_{n\to\infty} n^{-j} z_\lambda(1_n) = a_\lambda^{1^j} \neq 0$; consequently, we can replace $z_\mu(1_{n_i})$ in the limit with n_i^j. But then, by taking an appropriate linear combination, the condition becomes

$$\lim_{i\to\infty} \frac{E(p_\lambda(cN_{n_i})) - E(p_\lambda(M_i))}{n_i^j} = 0$$

From the lemma, this is equivalent to the desired conditions. □

This result can be generalized in a number of ways; for instance, by using the complex or quaternionic zonal polynomials, the result easily generalizes to complex or quaternionic Hermitian matrices. One can also prove a similar result for antisymmetric matrices, using the mixed zonal polynomials of [6]. Finally, we note that the only step in the above proof which is not reversible is the deduction of weak convergence from convergence of moments; thus the given conditions are necessary for the convergence of the moments of $(M_i)_k$ to those N_k for all k.

In the complex case, [5] gives convergence conditions for a larger class of limiting distributions. It is likely that the above arguments can be generalized to cover these distributions (as well as analogous real and quaternionic distributions); all that is needed is an analogue of Lemma 1.1 (which should be a straightforward application of Theorem 1.1 of [2]), and a proof that these distributions are determined by their moments.

2. Brownian motion

As before, let M_i be a sequence of random symmetric matrices of increasing dimension, with distribution invariant under orthogonal change of basis. Consider, now, the random walk given by partial sums of the diagonal of M_i. Again, assuming the M_i are reasonably well-behaved, one would expect this process to tend to a limit; here the expected limit is Brownian motion.

To be precise, define, for a symmetric matrix, X, a partial trace function $\mathrm{tr}(x;X)$, by:

$$\mathrm{tr}(x;X) = \left(\sum_{1\le i\le \lfloor \dim(X)x \rfloor} X_{ii} \right) + (\dim(X)x - \lfloor \dim(X)x \rfloor) X_{\lceil \dim(X)x \rceil \lceil \dim(X)x \rceil} \ .$$

In other words, $\mathrm{tr}(x;X)$ is a linear interpolation of the partial sums of the diagonal of X on the interval $[0,1]$. In particular, $\mathrm{tr}(0;X)=0$ and $\mathrm{tr}(1;X)=\mathrm{Tr}(X)$. We would like to find conditions for $\mathrm{tr}(x;M_i)$ to converge weakly to Brownian motion. One obstacle that arises immediately is that the traces of the M_i might not converge to a normal distribution. As long as they do converge to *some* distribution, however, we can always subtract that contribution, and ask when the result tends to a "Brownian bridge" (that is, $B(x)-B(1)x$, where B is Brownian motion). In other words, we can consider instead $\mathrm{tr}(x;M_i-\mathrm{Tr}(M_i)/\dim(M_i))$. Consequently, we will restrict our attention to traceless matrices.

Theorem 2.1. *Let M_i be a sequence of random traceless symmetric matrices of dimension $n_i \to \infty$, having moments of all orders, where the distribution of each M_i is invariant under orthogonal change of basis, and let* $\mathrm{tr}(x;M_i)$ *be as above. Then* $\mathrm{tr}(x;M_i)$ *converges weakly to*

$c(B(x) - B(1)x)$, *where B is Brownian motion*, *if the following conditions hold*:

$$n_i^{-1}\mathrm{Tr}(M_i^2) \twoheadrightarrow \frac{c^2}{2} \ ,$$

and

$$n_i^{-j+1}\mathrm{Tr}(M_i^j) \twoheadrightarrow 0 \ ,$$

for all $j > 2$.

Proof. Again, we use the method of moments. First, note that $\mathrm{tr}(x; M_i)$ can be written as $\mathrm{Tr}(M_i P(x, n_i))$, for suitable diagonal matrices $P(x, n_i)$ (with the first $\lfloor n_i x \rfloor$ entries 1, etc.). The theorem follows, therefore, if we can show that

$$E(\mathrm{Tr}(M_i F(n_i))^j)$$

tends to the correct value for all j, and all fixed linear combinations F of the $P(x)$.

For each k, define a random matrix

$$B_k = c(k^{-1/2}N_k - k^{-3/2}\mathrm{Tr}(N_k)) \ .$$

Clearly, B_k has the right convergence properties; we therefore need to show that

$$E(\mathrm{Tr}(M_i F(n_i))^j) - E(\mathrm{Tr}(B_{n_i} F(n_i))^j) \to 0 \ .$$

Again, we expand things in terms of zonal polynomials:

$$\begin{aligned} E(\mathrm{Tr}(M_i F(n_i))^j) &= \sum_{\mu \vdash j} A^{\mu}_{1^j} E(z_\mu(M_i F(n_i))) \\ &= \sum_{\mu \vdash j} \frac{A^{\mu}_{1^j}}{z_\mu(1_{n_i})} E(z_\mu(M_i)) z_\mu(F(n_i)) \\ &= \sum_{\mu,\lambda,\kappa \vdash j} \frac{A^{\mu}_{1^j} a^{\lambda}_{\mu} a^{\kappa}_{\mu}}{z_\mu(1_{n_i})} E(p_\lambda(M_i)) p_\kappa(F(n_i)) \end{aligned}$$

Unfortunately, we can no longer replace $z_\mu(1_{n_i})$ by n_i^j; we need higher-order information in this case. To be precise, we need a better asymptotic understanding of

$$\sum_{\mu \vdash j} \frac{A^{\mu}_{1^j} a^{\lambda}_{\mu} a^{\kappa}_{\mu}}{z_\mu(1_{n_i})} \tag{2.1}$$

First, some information about the a^{μ}_{λ}:

Lemma 2.2. *For any partitions $\lambda, \kappa \vdash j$, there exist coefficients $C_\nu^{\lambda\kappa}$ such that*

$$a_\mu^\lambda a_\mu^\kappa = \sum_{\nu \vdash j} C_\nu^{\lambda\kappa} a_\mu^\nu \ .$$

Moreover, $C_\nu^{\lambda\kappa}$ is 0 unless

$$|\ell(\lambda) - \ell(\kappa)| \le j - \ell(\nu) \le (j - \ell(\lambda)) + (j - \ell(\kappa)) \ , \qquad (2.2)$$

where $\ell(\nu)$ is the number of parts of ν.

Proof. The existence of the $C_\nu^{\lambda\kappa}$ follows immediately from the fact that the a_μ^λ are spherical functions on S_{2j}/B_j, and from elementary results on spherical functions. The coefficients can be expressed in terms of multiplication of double cosets $B_j \backslash S_{2j}/B_j$, as in [2]. In particular, it suffices to show that the product of elements of double cosets with type λ and κ can be in a double coset of type ν only when the stated condition is satisfied. (It is worth noting that the same condition applies to cycle types of permutations; essentially the same proof can be used here.) □

Corollary 2.3. For any partition $\lambda \vdash j$,

$$\sum_{\mu \vdash j} \frac{A_\lambda^\mu}{z_\mu(1_n)} = O(n^{-2j+\ell(\lambda)}) \ .$$

Proof. First, consider the function f_μ: $n \mapsto n^{-j} z_\mu(1_n)$. $f_\mu(1/x)$ is polynomial in x; the coefficient of x^k is the sum of a_μ^ν for all ν such that $\ell(\nu) = j - k$. If we compute the power series of $1/f_\mu(1/x)$ in x, it is easy to see that the coefficient of x^k in $1/f_\mu(1/x)$ is a polynomial in the a_μ^ν (with coefficients independent of μ), and further that the sum of $j - \ell(\nu)$ for all the factors of any term in the polynomial will be k. It follows from the lemma that the coefficient of x^k in $1/f_\mu(1/x)$ is a linear combination, again independent of μ, of terms of the form a_μ^ν, where $j - \ell(\nu)$ is at most k. Multiplying by A_λ^μ and summing over $\mu \vdash j$, we conclude that the coefficient of n^{-k} in the asymptotic expansion of $\sum_{\lambda \vdash j} A_\lambda^\mu / f_\mu(n)$ can be nonzero only if $k \ge j - \ell(\lambda)$. Dividing through by n^j, the result follows. □

Consider, now, equation (2.1). By the lemma, $a_\mu^\lambda a_\mu^\kappa$ can be written as a linear combination of a_μ^ν, subject to the constraint (2.2). But

$$A_{1^j}^\mu a_\mu^\nu = \frac{\chi^{2\mu}(1) 2^j j!}{|2\mu|!} a_\mu^\nu = \frac{2^j j!}{2^{\ell(\nu)} Z_\nu} A_\nu^\mu \ .$$

Dividing by $z_\mu(1_n)$ and summing in μ, we get something of order $O(n^{-2j+\ell(v)})$. By (2.2), this is, in turn, of order $O(n^{-j-|\ell(\lambda)-\ell(\kappa)|})$.

Recall that we have shown

$$E(\mathrm{Tr}(M_iP(n_i))^j) = \sum_{\lambda,\kappa,\mu\vdash j} \frac{A^\mu_{1^j}a^\lambda_\mu a^\kappa_\mu}{z_\mu(1_{n_i})} E(p_\lambda(M_i))p_\kappa(P(n_i)) \ .$$

Summed over μ, the first factor has order $O(n^{-j-|\ell(\lambda)-\ell(\kappa)|})$, as we have just shown. Since the eigenvalues of $P(n_i)$ are bounded, $p_\kappa(P(n_i)) = O(n_i^{\ell(\kappa)})$. From the hypotheses, it is clear that $E(p_\lambda(M_i)) = O(n_i^{j-\ell(\lambda)})$, and that $E(p_\lambda(M_i)) - E(p_\lambda(B_{n_i})) = o(n_i^{j-\ell(\lambda)})$, assuming that B_{n_i} also satisfies the hypotheses. Thus

$$E(\mathrm{Tr}(M_iP(n_i))^j) = \sum_{\lambda,\kappa\vdash j} O\left(n_i^{-j-|\ell(\lambda)-\ell(\kappa)|}\right) O\left(n_i^{j-\ell(\lambda)}\right) O\left(n_i^{\ell(\kappa)}\right) = O(1) \ ,$$

and

$$E(\mathrm{Tr}(M_iP(n_i))^j) - E(\mathrm{Tr}(B_{n_i}P(n_i))^j) = o(1) \ .$$

It remains only to show that B_k satisfies the hypotheses. From Lemma 1.1, it follows that the first term has eigenvalues of order $O(k^{1/2})$, while the trace term has eigenvalues of order $O(k^{-1})$. It follows that $E(\mathrm{Tr}(B_k^j)) = O(k^{j/2})$; thus the conditions are satisfied for $j > 2$. For $j = 2$,

$$\begin{aligned} k^{-1}E(\mathrm{Tr}(B_k^2)) &= k^{-2}E(\mathrm{Tr}(N_k^2)) + O(k^{-1/2}) \\ &= \tfrac{c^2}{2} + O(k^{-1/2}) \ . \end{aligned}$$

□

3. Examples

Theorem 3.1. *Let M_i be a sequence of random traceless symmetric matrices of dimension $n_i \to \infty$, with moments of all orders, where the distribution of each M_i is invariant under orthogonal change of basis. If all moments of a random eigenvalue of M_i converge, then the sequence $\sqrt{n_i}M_i$ satisfies the hypotheses of Theorem 1.2, and the sequence M_i satisfies the hypotheses of Theorem 2.1; in each case, c is the limiting standard deviation of the eigenvalue distribution.*

Proof. Clearly, $n_i^{-1}\mathrm{Tr}(M_i^j)$ converges to the same limit as the j-th moment of the eigenvalue distribution, for each j. The theorem follows immediately. □

One way to get matrices satisfying the hypotheses of Theorem 3.1 is to generate diagonal matrices with i.i.d. entries, then apply a random orthogonal change of basis.

A less contrived example is this:

Theorem 3.2. *For each $n > 0$, let O_n be a uniform random element of the orthogonal group $O(n)$ (chosen with respect to Haar measure). For any fixed $k, l > 0$, the matrices $\frac{1}{2}n^{1/2}(O_n^l + O_n^{-l} - 2\mathrm{Tr}(O_n^l)/n)_k$ converge weakly to N_k. Moreover, $\mathrm{tr}(x; O_n^l)$ converges weakly to the process $N(0, l-1)x + B(x)$ if l is odd and $N(1, l-1)x + B(x)$ if l is even.*

Proof. The eigenvalues of O_n are bounded; it follows that the trace of any power of $O_n^l + O_n^{-l} - 2\mathrm{Tr}(O_n^l)/n$ will be of order $O(n)$. Moreover,

$$\begin{aligned} E(\mathrm{Tr}(O_n^l + O_n^{-l} - 2\mathrm{Tr}(O_n^l)/n)) &= 0 \\ E(\mathrm{Tr}((O_n^l + O_n^{-l} - 2\mathrm{Tr}(O_n^l)/n)^2)) &= 2n + 2E(\mathrm{Tr}(O_n^{2l})) \\ &\quad - 4E(\mathrm{Tr}(O_n^l)^2)/n \\ &= 2n + O(1) \ , \end{aligned}$$

since $E(\mathrm{Tr}(O_n^{2l})) = 2l$ for sufficiently large n [1]. The first claim follows by Theorem 1.2.

The second claim follows by Theorem 2.1 applied to $\frac{1}{2}(O_n^l + O_n^{-l})$ (which clearly does not change the partial traces); we find

$$\mathrm{tr}(x; O_n^l) \Rightarrow N((l+1) \bmod 2, l)x + (B(x) - B(1)x)$$

(note that $\mathrm{Tr}(O_n^l) \Rightarrow N((l+1) \bmod 2, l)$ [1]). But this has the same distribution as

$$\mathrm{tr}(x; O_n^l) \Rightarrow N((l+1) \bmod 2, l-1)x + B(x). \qquad \square$$

For $l = 1$, these results were already known; a stronger version of the first claim, as well as unitary and sympletic versions, was proved in [7], while a simple proof of the second claim was given by P. Diaconis (personal communication). This latter was the motivation for section 2. As before, Theorem 3.2 has analogues for unitary and symplectic matrices.

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