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Lifschitz tail in a magnetic field: the nonclassical regime

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Abstract. We obtain the Lifschitz tail, i.e. the exact low energy asymptotics of the integrated density of states (IDS) of the two-dimensional magnetic Schrödinger operator with a uniform magnetic field and random Poissonian impurities. The single site potential is repulsive and it has a finite but nonzero range. We show that the IDS is a continuous function of the energy at the bottom of the spectrum. This result complements the earlier (nonrigorous) calculations by Brézin, Gross and Itzykson which predict that the IDS is discontinuous at the bottom of the spectrum for zero range (Dirac delta) impurities at low density. We also elucidate the reason behind this apparent controversy. Our methods involve magnetic localization techniques (both in space and energy) in addition to a modified version of the "enlargement of obstacles" method developed by A.-S. Sznitman.

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1 Introduction

Magnetic Schrödinger operators with random potentials have been intensively studied by physicists, in particular because of their relevance to the quantum Hall effect. Rigorous mathematical studies of these operators have appeared only recently (e.g. [5], [6], [18], [19]). For a wider background and many references (including physical ones) we refer to [5]. In this paper we focus on the integrated density of the states (IDS) in the low energy limit, i.e. on the so-called

Lifschitz tail. We restrict ourselves to the two-dimensional situation which is the most relevant case for magnetic problems.

The Lifschitz tail is not supposed to depend on the actual shape of the single site obstacle potential if it has short range, and our results are consistent with this expectation. We can treat both soft and hard-core obstacles as well. Our result shows, in particular, that the IDS is continuous at the bottom of the spectrum if the density of the obstacles is positive and if the support of the potential has nonempty interior.

However, the case of zero range (Dirac delta) impurities yields a characteristically different behavior due to a purely magnetic effect. We rigorously justify the prediction of Brézin, Gross and Itzykson [4] that the IDS is discontinuous at the bottom of the spectrum if the impurity density is smaller than the density of states in the unperturbed lowest Landau level.

The conclusion is that while the findings of [4] are correct, it is wrong to extend them to more realistic, nonzero range potentials. The IDS is continuous for typical potentials and the discontinuity in the case of Dirac delta impurities is an exceptional phenomenon.

For most of our work we shall assume that the potential has nonzero range and in Section 9 we investigate the zero range case.

1.1 Definitions

1.1.1 Soft potential

Consider a nonnegative, measurable function $V^{(0)}$ on $\mathbf{R}^2$, which is positive on an open set of positive measure, i.e. $V^{(0)}(x) \geq v$ for $|x| \leq a$ for some $a, v > 0$. For technical simplicity, we assume that $V^{(0)}$ is continuous. Let

$$V(x) = V_\omega(x) = \sum_i V^{(0)}(x - x_i(\omega)) \tag{1.1}$$

be a random potential, where $x_i(\omega)$ is the realization of the Poisson point process $\mathscr{P} = \mathscr{P}_\nu$ on $\mathbf{R}^2$ with density ν (here ω denotes the randomness, but we shall usually omit it from the notations). The expectation with respect to $\mathscr{P}$ is denoted by $\mathscr{E}$. We consider the following magnetic Schrödinger operator with random potential V_ω

$$H(V) = H(V_\omega) = H_\omega = \tfrac{1}{2}[(-i\nabla - A)^2 - B] + V_\omega \ , \tag{1.2}$$

where $A : \mathbf{R}^2 \to \mathbf{R}^2$ is a deterministic vector potential (gauge) generating the constant $B > 0$ magnetic field, i.e. curl $A = B$. The properties we are interested in are independent of the actual gauge choice, so, conveniently, we choose the standard gauge $A(x) := \frac{B}{2}\binom{-x_2}{x_1}$. Here $x = (x_1, x_2) \in \mathbf{R}^2$, i.e.

$$H(V_\omega) = -\frac{1}{2}\Delta_x + \frac{iB}{2}\left(x_1\partial_{x_2} - x_2\partial_{x_1}\right) + \frac{B^2|x|^2}{8} - \frac{B}{2} + V_\omega(x) \ .$$

We subtracted the constant $B/2$ term in the kinetic energy both for physical reasons (spin coupling) and for mathematical convenience. In this way the spectrum of the free operator $H(V \equiv 0)$ is $\{nB : n = 0, 1, 2, \ldots\}$, i.e. it starts at zero. It is well known that each level (Landau band) is infinitely degenerate, and the spectral projection Π_n onto the n^{th} level is an explicit integral operator (see (2.14) in [5]).

We also define $H_Q = H_Q(V)$ as the restriction of H onto a domain $Q \subset \mathbf{R}^2$ (with Dirichlet boundary conditions). In this paper by domain we mean an open, bounded subset of $\mathbf{R}^2$ with regular (C^1) boundary, which is not necessarily connected.

Integration by parts shows that H_Q corresponds to the quadratic form $\langle f, H_Q f\rangle = \frac{1}{2}\int_Q |Tf|^2 + \int_Q V|f|^2$ defined on the core $C_0^\infty(Q)$ with $T := -i\partial_1 + \partial_2 - A_1 - iA_2$.

We shall always assume that $V^{(0)}$ has sufficient decay so that $V_\omega \in L^2_{\text{loc}}$ with probability one, i.e. these operators are almost surely selfadjoint. We always consider ω from this set of full measure. We define the integrated density of states (IDS) as

$$N(E) = \lim_{Q \nearrow \mathbf{R}^2} \frac{1}{|Q|} \mathscr{E}\mathrm{Tr} P_E(H_Q) \ , \tag{1.3}$$

where P_E is the spectral projection onto the half line $(-\infty, E]$, and $Q \nearrow \mathbf{R}^2$ denotes an increasing sequence of nested regular domains, say squares or disks. The trace is over $L^2(Q)$. In fact, the above limit equals to $\mathscr{E}[P_E(H)(x,x)]$, moreover it has the so called self-averaging property, i.e. the random quantity $|Q|^{-1}\mathrm{Tr}P_E(H_Q)$ becomes deterministic in the thermodynamic limit. For details, we refer to [7] and references therein.

The existence of the limit (1.3) is standard by the ergodic technique of superadditive processes. We refer to [15] for the general technique and to [19] for the magnetic case, here we just recall the basic idea for the soft potential case, the same argument applies to the other two cases.

$N(E)$ can be defined via quadratic form as well, using that $\mathrm{Tr}P_E(H_Q) = N_{Q,V_\omega}(E)$ by variational principle, where

$$N_{Q,V_\omega}(E) := \max_{\mathcal{K}\subset C_0^\infty(Q)} \left\{ \dim \mathcal{K} : \frac{1}{2}\int_Q |Tf|^2 + \int_Q V_\omega |f|^2 \leq E|f|^2, \quad \forall f \in \mathcal{K} \right\} . \qquad (1.4)$$

Here $Q \to N_{Q,V_\omega}(E)$ is superadditive, stationary and for $E < B$ it is bounded from above by $\frac{B}{2\pi}\cdot|Q|$, since $N_{Q,V_\omega}(E) \leq \mathrm{Tr}_{L^2(Q)} P_E(H_Q(0)) \leq \int_Q P_E(H(0))(x,x)\,dx = \int_Q \Pi_0(x,x)\,dx = \frac{B}{2\pi}\cdot|Q|$. Hence $\frac{1}{|Q_n|}\mathcal{E}\, N_{Q_n,V_\omega}(E)$ is a bounded, increasing sequence for, say, $Q_n := [-2^n, 2^n]^2$, hence its limit exists. By standard thermodynamic argument one can show that the same limit is obtained for other sequence of regular domains.

1.1.2 Hard-core potential

Let K be a compact set with a non-empty interior and regular boundary, say $B(0,a) \subset K \subset B(0,\overline{a})$ for some $0 < a < \overline{a}$ and $\partial K \in C^1$. $B(x,r)$ denotes the ball of radius r centered at x. Consider the random set

$$\mathcal{T}_\omega = \mathcal{T} = \mathbf{R}^2 \Big\backslash \bigcup_i (K + x_i(\omega)) \qquad (1.5)$$

and let $H_{\mathcal{T}}$ be the operator $\frac{1}{2}[(-i\nabla - A)^2 - B]$ with Dirichlet boundary conditions on $\mathcal{T}$. Since $\{x_i(\omega)\}_i$ is a.s. locally finite, $\mathcal{T}$ is open almost surely. Hence $H_{\mathcal{T}}$ and $H_{\mathcal{T}\cap Q}$ are well defined as a selfadjoint operators. As before, let

$$N(E) := \lim_{Q\nearrow\mathbf{R}^2} \frac{1}{|Q|}\mathcal{E}\mathrm{Tr} P_E(H_{\mathcal{T}\cap Q}) \ . \qquad (1.6)$$

1.1.3 Dirac delta potential

We consider the model introduced by Brézin, Gross and Itzykson [4], where the single site potential is $V^{(0)}(x) = g\delta_0(x)$, $g > 0$, i.e.

$$H_\omega\text{''} = \text{''}\ \tfrac{1}{2}[(-i\nabla - A)^2 - B] + g\sum_i \delta(x - x_i(\omega)) \ . \qquad (1.7)$$

Unfortunately, the mathematically rigorous definition of this operator is problematic even for finite volume. The corresponding quadratic form

$$q_{Q,\omega}(f,f) := \frac{1}{2}\int_Q |Tf|^2 + g\sum_i |f(x_i(\omega))|^2 \ , \tag{1.8}$$

defined on smooth functions vanishing outside of Q, is symmetric but not closable, hence there is no selfadjoint operator associated with it. The reason is that for any $f \in C_0^\infty(Q)$ and for any finite point configuration $\{x_i\}$ in Q, there exists a sequence of cutoff functions ϑ_ϵ such that $\vartheta_\epsilon(x) \to 1$ for $x \neq x_i$, $\vartheta_\epsilon(x_i) = 0$ and $\|T(\vartheta_\epsilon f)\|_2 \to \|Tf\|_2$ as $\epsilon \to 0$. Hence $\|f_\epsilon - f_{\epsilon'}\|_2 + \|T(f_\epsilon - f_{\epsilon'})\|_2 \to 0$ with $f_\epsilon := \vartheta_\epsilon f \to f$, but $q_{Q,\omega}(f_\epsilon, f_\epsilon)$ does not converge to $q_{Q,\omega}(f,f)$ if $f(x_i) \neq 0$. [For example the following function ϑ_ϵ has this property if there is only a single point at the origin, $x_1 = 0$; let $\vartheta_\epsilon(x) = 0$ for $|x| \leq \epsilon$, $\vartheta_\epsilon(x) := \log(a + b\log|x|)$ for $\epsilon \leq |x| \leq \sqrt{\epsilon}$ and $\vartheta_\epsilon(x) = 1$ for $|x| \geq \sqrt{\epsilon}$, where a and b are such that ϑ_ϵ be continuous. The generalization for many centers is straightforward.]

This problem has been widely studied for the nonmagnetic case, for the most comprehensive reference see [3]. In nutshell, the result is that it is possible to identify certain selfadjoint extensions of $-\Delta$ defined on $C_0^\infty(\mathbf{R}^2 \setminus \cup_i\{x_i\})$ as a norm resolvent limit of approximating operators with regularized deltafunctions, but the strength of the coupling g has to be weakened as the regularization goes to zero. However, if the approximating potential is nonnegative, then the limit is always the usual $-\Delta$ on $\mathbf{R}^2$, i.e. the repulsive point centers remain unnoticed. [See Section I.5 in [3] for the one center case in $d = 2$, and the analogous but more elaborated $d = 3$ case for many centers in Section II.1.2.]

To our best knowledge, the magnetic case has not been fully worked out (for partial results see [21]). It is easy to see, however, that if one defines the IDS analogously to (1.4) using the Dirac delta quadratic form (1.8), then the result is trivial, i.e. $N(E)$ is exactly the same as for the free magnetic operator (step function). One could simply use the same orthonormal set of trial functions as for the free case, just one has to infinitesimally cut out the effect of the impurities using the ϑ_ϵ multipliers. As $\epsilon \to 0$, this cutoff does not influence the kinetic energy and the orthogonality. The details are left to the reader.

Despite this triviality, several physical works considered the IDS of the Dirac delta case and obtained a nontrivial result (see [4], [16] and Remark 2.4. (ii) in [5] for a summary). The reason is that these works actually considered $\Pi_0 V_\omega \Pi_0$, the so called *lowest Landau band approximation* of (1.7) which has also been used in the context of Anderson localization (see [10] and references therein).

The corresponding quadratic form, $q^{(0)}_{g,\omega}(f,f) := g\sum_i |(\Pi_0 f)(x_i)|^2 = g\sum_i \langle f, \Pi^{x_i} f\rangle$, is defined for all $f \in L^2(\mathbf{R}^2)$ such that $q^{(0)}_{g,\omega}(f,f) < \infty$, where $\Pi^{x_i} := |P_{x_i}\rangle\langle P_{x_i}|$ is the one dimensional projection onto the subspace spanned by $P_{x_i}(x) := \Pi_0(x_i, x)$. This form is closed in contrast to (1.8), although it is not bounded due to a possible large concentration of points (unlike the analogous operator treated in [10] with impurities on a lattice).

There are several ways to define the corresponding finite volume operator on $Q \subset \mathbf{R}^2$, where Q is a "nice" domain (say disk or square). Dirichlet boundary conditions cannot be imposed as Π_0 is not a local operator, but one can consider $\chi_Q \Pi_0 V_\omega \Pi_0 \chi_Q$, i.e. the almost surely bounded, self-adjoint operator on $L^2(Q)$ corresponding to

$$q^{(0)}_{Q,g,\omega}(f,f) := g \sum_i \big| \big(\Pi_0 \chi_Q f\big)(x_i)\big|^2 \ , \tag{1.9}$$

where χ_Q is the characteristic function of Q. The IDS is defined as

$$N_g(E) := \lim_{Q \nearrow \mathbf{R}^2} \frac{1}{|Q|} \mathscr{E} \mathrm{Tr} P_E(\chi_Q \Pi_0 V_\omega \Pi_0 \chi_Q) \ . \tag{1.10}$$

From physical point of view it seems reasonable to focus only on the lowest Landau band for the low energy ($E \ll B$) behavior of the operator. However, the discussion above shows that this (or similar) approximation is not just convenient but also necessary from the mathematical point of view in order to define a nontrivial operator describing Dirac delta interactions in a magnetic field. Our result remains valid for the finitely many Landau band approximation $\big(\sum_{i=0}^k \Pi_i\big) V_\omega \big(\sum_{i=0}^k \Pi_i\big)$ as well.

Remark. The spectral properties (e.g. IDS) of the magnetic operator are invariant under a *global* gauge transformation on $\mathbf{R}^2$, $A \to A + \nabla \xi$, with $\xi \in C^1(\mathbf{R}^2, S^1)$, hence we can just use the standard gauge (see e.g. [17]). However, for multiply connected domains Ω, that appear in the hard-core and Dirac delta cases, it is also possible to consider *local* gauges, $A \in C^1(\Omega, \mathbf{R}^2)$, curl $A = B$ on Ω, which cannot be extended to a global gauge on $\mathbf{R}^2$ that would generate the constant field everywhere (the holes in the domain may have to carry extra fluxes). If such gauges were allowed, $P_E(H)(x,x)$ or the lowest eigenvalue, for example, would not be gauge invariant, as they would depend on the integer parts modulo 2π of the hole fluxes (this is essentially the Aharonov-Böhm effect [1]). Although we believe that our result on the asymptotic behaviour of $N(E)$ remains valid for any local gauge choice, the current proof does not cover this general case (only the argument in Section 6 uses the specific global gauge). However, in Section 7 we need to introduce local gauges for certain auxiliary operators.

1.2 Results

All our results are equally valid for the soft and hard-core cases, and the proofs are almost identical. We shall focus on the soft potential case, the hard-core case follows with minor formal modifications. However the Dirac delta potential requires a separate treatment.

1.2.1 Soft or hard-core obstacles

First, we recall the relevant results for the nonmagnetic ($B = 0$) case in any dimension d (see [20]). Assume that the single site potential, $V^{(0)}$, is in $L^2_{\rm loc}$ and it has a definite tail behaviour with exponent $\alpha > d$, i.e.

$$\lim_{|x|\to\infty} |x|^\alpha V^{(0)}(x) = \mu, \quad 0 < \mu < \infty \ . \tag{1.11}$$

For slowly decaying potentials, i.e. for $d < \alpha < d + 2$, the asymptotics of $N(E)$ is given by

$$\lim_{E\to 0+0} E^{d/(\alpha-d)} \log N(E) = -C(d, \alpha, \mu, v) \ , \tag{1.12}$$

where the constant is explicit. This behaviour is completely governed by the potential energy, i.e. by classical effects. For $\alpha > d + 2$ (more precisely, if $V^{(0)}(x) = o(|x|^{-d-2})$ at infinity), one has (see [9] and [24])

$$\lim_{E\to 0+0} E^{d/2} \log N(E) = -C(d, v) \ , \tag{1.13}$$

i.e. the tail is independent of the parameters of the single site potential. In fact, we obtain the same behaviour for hard obstacles (formally, it corresponds to $\alpha = \infty$), indicating that the localization properties of the kinetic energy plays the major role in contrast to the previous case of slowly decaying potential. We call this regime nonclassical. Note that the threshold for this transition is at the decay exponent $\alpha = d + 2$. Heuristically, it is obtained from the competition between the kinetic energy within the ball of radius R ($\sim R^{-2}$) and the potential energy within this ball originated from obstacles outside the ball ($\sim \int_{|x|\geq R} \frac{\mu}{|x|^\alpha} v \, dx \sim R^{d-\alpha}$).

In the magnetic case, the situation is different. The magnetic field itself has a strong localization effect, i.e. the kinetic energy is expected to play less role. In fact, it has been proven in [5] that the classical effects dominate for any $d = 2 < \alpha < \infty$, and any $B > 0$, i.e. (compare with (1.12); even the constant is the same)

$$\lim_{E\to 0+0} E^{2/(\alpha-2)} \log N(E) = -C(d = 2, \alpha, \mu, v) \tag{1.14}$$

under the condition (1.11). Note that the strength of the magnetic field does not appear in the asymptotic behaviour of $N(E)$, only the fact that $B > 0$ is used. Analogous result is valid in higher even dimensions as well.

One of the conclusion of [5] is that the low energy tail is always determined by the classical effect. In the present paper, we show that with a more careful analysis one can detect the nonclassical behaviour, but the threshold decay of the potential, where the transition occurs, has to be faster than polynomial. Therefore it is invisible in the regime investigated in [5]. Moreover, we show that the asymptotic behaviour of $N(E)$ does depend on the strength of B in the nonclassical regime. Our theorem is the following [$\mathbf{1}(\cdot)$ or $\mathbf{1}_{(\cdot)}$ denote the characteristic function]:

Theorem 1.1. *In two dimensions let* v *be the intensity of the Poisson point process and let* $B > 0$ *be a constant magnetic field. Let* $V^{(0)}$ *be continuous, compactly supported and* $V^{(0)} \geq v\mathbf{1}_{B(0,a)}$ *for some* $a, v > 0$ *in the soft obstacle case; and assume that the compact set* K *contains a ball* $B(0, a)$ *and has a regular boundary in the hard-core case. Then the integrated density of states of* $H(V)$ *or* $H_{\mathscr{T}}$, *defined in* (1.3) *and* (1.6), *respectively, satisfies*

$$\lim_{E\to 0+0} \frac{\log N(E)}{|\log E|} = -\frac{2\pi v}{B} \quad . \tag{1.15}$$

In particular

$$\lim_{E\to 0+0} N(E) = 0$$

for any $v > 0$, *i.e. the integrated density of states is continuous at the bottom of the spectrum.*

Remark 1. This theorem does not establish the optimal condition on the decay of $V^{(0)}$. In particular, we do not address the question how to improve our result to include potentials with unbounded support (the results of [5] show that the decay of the potential has to be faster than any polynomial). Similarly to the nonmagnetic case, a competition between the kinetic energy and the potential energy suggests that the threshold decay is probably Gaussian.

Remark 2. Our method does not work to investigate Lifschitz tails above higher Landau levels. Note that statements analogous to (1.14) were proven in [5] for the restriction of H onto any fixed Landau band.

1.2.2 Dirac delta potential in the lowest Landau band approximation

The results by Brézin, Gross and Itzykson [4], which are based upon nonrigorous supersymmetric functional integrations, show with our notations that for any $g > 0$

$$\lim_{E\to 0+0} \frac{N_g(E)}{|\log E|} = -\frac{2\pi\nu}{B} + 1 \quad \text{if} \quad \frac{2\pi\nu}{B} \geq 1 \tag{1.16}$$

and

$$\lim_{E\to 0+0} N_g(E) = \frac{B}{2\pi}\left(1 - \frac{2\pi\nu}{B}\right) \quad \text{if} \quad \frac{2\pi\nu}{B} < 1 \ . \tag{1.17}$$

In fact the calculations in [4] give more detailed information, as they yield the density of states as well.

The heuristic explanation for these results is that a fraction of the total number of states in the lowest Landau level of the free operator remains unaffected by the Dirac delta potential. This is possible because the ground state magnetic eigenfunctions have many zeros which can neutralize the Dirac delta impurities at low density $\nu < \frac{B}{2\pi}$ (roughly saying the density of zeros of a typical ground state is $\frac{B}{2\pi}$). If the impurity density is high, then there are not enough zeros to accommodate all the obstacles, but the zeros can match $\frac{B}{2\pi}$ obstacles per unit area. Hence the effective density of the obstacles is $\nu_{\text{eff}} := \nu - \frac{B}{2\pi} = \frac{B}{2\pi}\left(\frac{2\pi\nu}{B} - 1\right)$ which accounts for the discrepancy between Theorem 1.1 and (1.16)–(1.17).

Here we rigorously justify the most interesting part of this heuristic explanation by proving the following theorem.

Theorem 1.2. *Let* $\frac{2\pi\nu}{B} < 1$ *and* $g > 0$, *then*

$$\liminf_{E\to 0+0} N_g(E) \geq \frac{B}{2\pi}\left(1 - \frac{2\pi\nu}{B}\right) = \frac{B}{2\pi} - \nu \ . \tag{1.18}$$

In particular, the IDS is discontinuous at the bottom of the spectrum.

Remark. This theorem gives the easier direction of (1.17) and is essentially a trial function calculation. The upper bound would require extending the method of Section 7 to point interactions, but unfortunately the fatness of the obstacle support is heavily used in our technique.

1.3 Idea of the proofs

First we discuss the proof of Theorem 1.1. Similarly to the difference in complexity between the proofs of (1.12) and (1.13), one can expect that the proof of (1.15) requires more detailed analysis than that of (1.14). In particular, the lower bound on the Hamiltonian (upper bound on the IDS) is more complicated; Golden-Thompson and related inequalities used in [5] are insufficient.

First, we translate the problem into a heat kernel estimate. Then, there are three main ingredients in our work. Not unexpectedly, the proof of (1.15) must contain an argument similar to the proof of (1.13), i.e. a Donsker-Varadhan type upper bound on the heat kernel (see [9]). A.-S. Sznitman has worked out an alternative method for proving this upper bound (in fact, his technique, called the method of the "enlargement of obstacles", is applicable to a much wider class of problems), and we found this approach suitable to the magnetic problem. This is the first major ingredient.

Sznitman's method is able to estimate the lowest eigenvalue only. Hence, before applying it, one needs to localize the problem so that the lowest eigenvalue dominate the behavior of the IDS. Here we use a special magnetic localization technique developed in [14]. This is the second ingredient.

Finally, the third ingredient is an isoperimetric inequality. For the nonmagnetic case this boils down to the standard isoperimetric problem; the minimization of the lowest Dirichlet eigenvalue of a domain with a fixed volume. Note that neither Donsker-Varadhan's nor Sznitman's method can avoid this step. Apparently the same is true for the magnetic problem; this was our main motivation to prove the corresponding isoperimetric inequality with a magnetic field in [12]. The necessary results are recalled in Section 4.

The present paper contains a modified version of the "enlargement of obstacles" argument in the magnetic setup and the localization step. Here we explain briefly the strategy.

Sznitman's method relies on heat kernel estimates. This is not directly applicable, since it heavily uses that the transition kernels are positive, in contrast to the oscillatory character of the magnetic problem. It also requires a real valued "free" diffusion process, such that the Feynman-Kac formula for $e^{-tH(V)}$ could be viewed as its perturbation. We were not able to find a suitable diffusion process associated with the free magnetic problem. Diffusion with a drift comes as a natural candidate but we are not aware of any process with a real drift that mimics the free magnetic Schrödinger operator. We are able to replace the effect of the magnetic field by a potential via a variational principle (see Eqs. (6) and (8) in [12]), but this potential changes sign which makes it hard to estimate in the Feynman-Kac formula.

Instead, our key idea is to use the spectral gap in the free magnetic Hamiltonian to separate the lowest Landau band. It turns out, by a simple energy argument, that the contributions from the higher Landau bands are irrelevant as we are investigating unusually low lying eigenvalues. Hence almost the whole principal eigenstate lives in the lowest Landau band. By the proof of the Aharonov-Casher theorem [2], the projection of any state onto the lowest Landau band can be expressed as he^{Bg} where h is analytic, and g solves $\Delta g = -1$ on the domain. We write g in terms of the expected value of the boundary hitting time of a free Brownian motion. It is this stage where we use an argument similar to Sznitman's, applied to a process which is, unlike in Sznitman's papers, completely different from the one generated by the original free Hamiltonian. This is the content of the most technical Section 7.

The IDS in infinite volume contains information about infinitely many (generalized) eigenvalues. However, the crucial part of our analysis (as well as in Sznitman's original work) proves that the lowest eigenvalue of $H(V)$ is comparable to the lowest eigenvalue of another Hamiltonian (the one with the so-called "enlarged obstacles"), whose potential configuration has a smaller entropy factor. This method is able to estimate only the lowest eigenvalue, hence, before applying it, we have to localize the problem onto a small enough box on which the lowest eigenvalue solely determines the low energy asymptotics of the averaged IDS. The choice of the size of this localization box depends on the energy threshold E according to the following two requirements.

On one hand, it turns out that for each energy E the main contribution to the averaged IDS comes from very atypical configurations (where the Poisson cloud has a large clearing) which support an eigenfunction with a very low lying eigenvalue. This eigenfunction has a natural lengthscale, depending only on E. Obviously, the localization box has to be at least as big as this natural lengthscale in order not to destroy these low lying states by the artificial localization.

On the other hand, the actual number of eigenvalues below E should be irrelevant (on logarithmic scale) compared to the large deviation probability that there is any eigenvalue below E at all. This is the case, at least, if the size of the localization box is not much bigger than the natural lengthscale of the typically contributing eigenfunction. The number of the relevant low lying eigenvalues is approximately equal to the volume ratio between the apriori box and the box with side lengths equal to the natural lengthscale.

Using (1.3), the infinite volume IDS can be approximated by a finite volume IDS, but we have to control the error effectively. It is done in two steps. In the first step we localize onto a big "apriori" box to make the problem finite. In the second step we apply the magnetic localization technique of [14]. This enables us to localize onto a constant ($= n_0$) multiple of the natural lengthscale (this will be our localization scale) at the expense of changing the magnetic field by a constant amount 2β. The relation between these constants is that $\beta \to 0$ as $n_0 \to \infty$, but we take this limit only after $E \to 0 + 0$.

The whole proof is actually done in the language of the heat kernel as we explain it in Section 2. We shall take the Laplace transform of $N(E)$ to obtain the heat kernel, whose time parameter t is the conjugate variable to E. The small energy asymptotics of $N(E)$ is related to the large time asymptotics of the heat kernel by a Tauberian type argument, which also establishes the relation between t and E. The apriori localization entails a large deviation estimate of the probability that the Brownian loop (in the Feynman-Kac-Ito representation of the heat kernel) goes extremely far. By the simplest Gaussian tail estimate on this probability one obtains the necessary decay (exponential in t is good enough) if the linear size of the apriori box is of order t. This estimate does not use any extra localization due to the magnetic field (if any); the oscillatory factor in the Feynman-Kac-Ito formula, which supposedly enhances localization, is estimated by absolute value. We remark that it is not clear to what extent the strong Gaussian type off-diagonal decay of the free magnetic heat kernel survives under a general potential perturbation. Warning examples and related phenomena are discussed in [11] and [13].

In the nonmagnetic case the Tauberian argument sets $E \sim t^{-2/(d+2)}$, and the natural lengthscale is of order $E^{-1/2} \sim t^{1/(d+2)}$, d being the space dimension. Since the volume of the apriori box is of order t^d, the number of low lying eigenvalues is around $t^d / t^{d/(d+2)}$, i.e. a power of t. This overcounting is irrelevant compared to the subexponentially small size of the averaged heat kernel ($\sim \exp -t^{d/(d+2)}$).

In the magnetic case, the situation is much tighter. The natural lengthscale is proportional to $\sqrt{|\log E|} \sim \sqrt{\log t}$, and the overcounting is still a power of t if we use only the apriori localization of linear size t. But the actual heat kernel decay is algebraic as well ($t^{-2\pi v/B}$, see (2.7) later). Hence the overcounting error is comparable to the main term. This is the reason why we have to do a second localization which brings us down to a constant multiple of the natural lengthscale from the apriori lengthscale. In fact the second localization (Section 6) is strong enough so that we do not need a particularly effective apriori localization (Section 3).

The proof of Theorem 1.2 is significantly simpler and we present it in Section 9 which is independent of the rest of the paper.

After identifying $\mathbf{R}^2$ with the complex plane $z \in \mathbf{C}$ the idea is to construct many linearly independent analytic functions which all have zeros at the points of the Poisson process. Multiples of the well known Weierstrass product serve as natural candidates. By the Aharonov-Casher theorem, all these functions multiplied by $\exp\big(-B|z|^2/4\big)$ are zero energy eigenfunctions of H_ω if they are in $L^2(\mathbf{C})$. It is well known that the growth rate of the Weierstrass product depends on the density of zeros. It turns out that if the density of zeros is smaller than $\frac{B}{2\pi}$, then the growth is controlled by the factor $\exp\big(-B|z|^2/4\big)$.

To produce the necessary amount of linearly independent functions, all having zeros at the Poisson points, we superimpose the original Poisson process of density ν with another one with density slightly smaller than $\frac{B}{2\pi} - \nu$. The Weierstrass product corresponding to the union of these two Poisson clouds is still controlled by $\exp\big(-B|z|^2/4\big)$ since the union of the two point clouds is also a Poisson process with density slightly smaller than $\frac{B}{2\pi}$. Moreover, this Weierstrass product naturally factorizes into two factors, according to the two processes. Keeping the factor corresponding to the original process fixed (this ensures the vanishing at the points of the original process), we choose the remaining factor by sampling randomly from the second Poisson process. Since this second factor is a random polynomial of a degree essentially $N \sim \big[\frac{B}{2\pi} - \nu\big] \cdot (\text{Volume})$, choosing N samples typically gives linearly independent random polynomials. Including the first factor and $\exp\big(-B|z|^2/4\big)$, we obtain N linearly independent functions, all vanishing at the points of the original process, and typically they are decaying at infinity. The actual proof is done in a finite volume with appropriate cutoffs.

2 Heat kernel, Laplace transform

Let $L(t)$ $(t > 0)$ be the Laplace transform of the density of states $N(E)$:

$$L^{(B)}(t) = L(t) := \int e^{-\lambda t}\, dN(\lambda) \quad . \tag{2.1}$$

We usually omit the B superscript if it refers to the original magnetic field.

2.1.1 Soft potential

Let $B_0 := B/2$ for simplicity. For all ω and $x \in \mathbf{R}^2$ we can define

$$L^x_\omega(t) := (2\pi t)^{-1} e^{B_0 t} \mathbf{E}^{x,t}_{x,0}\left[e^{-i\int_0^t A(W_s)\,dW_s - \int_0^t V_\omega(W_s)\,ds}\right] = e^{-tH_\omega}(x,x) \ , \tag{2.2}$$

where $\mathbf{E}^{x,t}_{y,0}$ stands for the expectation with respect to the probability measure of the two dimensional Brownian bridge W_s $(0 \le s \le t)$ with $W_0 = x$, $W_t = y$. Notice that the heat kernel, $e^{-tH_\omega}(x,y)$ in general exists only for almost all x and y and almost surely. But by Theorem 3.1. from [7] it is continuous almost surely (if $V_\omega \in L^1_{\text{loc}}$), and the Feynman-Kac-Ito formula gives this continuous representation. In particular, the diagonal element is well defined and nonnegative since e^{-tH_ω} is a nonnegative operator.

Notice that $L^x_\omega(t)$ is independent of the gauge choice, in particular its distribution is ergodic with respect to the spatial translations in x, i.e. $\mathscr{E} L^x_\omega(t)$ is independent of the choice of x.

Using the approximation result in [5] (formula (A.10)) we get that

$$L(t) = \mathscr{E} L^x_\omega(t) \tag{2.3}$$

for any x (in particular this shows that $L(t)$ is finite).

2.1.2 Hard-core potential

For all $x \in \mathbf{R}^2$

$$L^x_\omega(t) := (2\pi t)^{-1} e^{B_0 t} \mathbf{E}^{x,t}_{x,0}\left[e^{-i\int_0^t A(W_s)\,dW_s} \mathbf{1}(T_{\mathscr{T}_\omega} > t)\right] = e^{-tH_{\mathscr{T}\omega}}(x,x) \ . \tag{2.4}$$

Here and in the sequel T_Ω denotes the exit time from the domain Ω.

The continuity of the heat kernel of $H_{\mathscr{T}}$ has not been explicitly proven in [7], but it follows immediately from the soft potential case by the following approximation. Since $\mathscr{T} = \mathscr{T}_\omega$ is open (almost surely), by dominated convergence

$$\mathbf{E}^{y,t}_{x,0}\left[e^{-i\int_0^t A(W_s)\,dW_s - \int_0^t V_n(W_s)\,ds}\right] \to \mathbf{E}^{y,t}_{x,0}\left[e^{-i\int_0^t A(W_s)\,dW_s} \mathbf{1}(T_{\mathscr{T}} > t)\right] \tag{2.5}$$

locally uniformly in $x, y \in \mathscr{T}$ and in t, where V_n is an increasing sequence of continuous functions supported on $\mathscr{T}^c$ such that $V_n(x) \to \infty$ for all $x \in \mathscr{T}^c$. Such a sequence exists because $\text{int}(K)$ is non-empty and ∂K is regular.

By continuity of the heat kernel, its diagonal element in (2.4) is well defined and for all x

$$L(t) = \mathcal{E} L^x_\omega(t) \ . \tag{2.6}$$

Again, this relation is proven in [5] only for soft potential, but the proof of (A.10) goes through with obvious changes for the hard-core case as well.

Theorem 2.1. *Under the conditions of Theorem* 1.1, *we have*

$$\lim_{t\to\infty} \frac{\log L(t)}{\log t} = -\frac{2\pi\nu}{B} \tag{2.7}$$

for both the soft and hard-core cases.

By a standard Tauberian argument, Theorem 1.1 immediately follows from Theorem 2.1.

The rest of the paper, apart from Section 9, contains the proof of Theorem 2.1. We shall focus on the proof of the soft potential case, the modifications for the hard-core case are obvious.

3 Apriori localization

Recall that for any box $Q := [-q, q]^2$ and for almost all configurations ω, we defined the self-adjoint operator $H_{Q,\omega} = \frac{1}{2}\left[(-\mathrm{i}\nabla - A)^2 - B\right] + V_\omega$ with Dirichlet boundary condition on Q. Let $\lambda_{Q,\omega}$ be its lowest eigenvalue. For any $x \in Q$, let

$$\begin{aligned} L^x_{Q,\omega}(t) &:= \mathrm{e}^{-tH_{Q,\omega}}(x,x) \\ &= (2\pi t)^{-1} \mathrm{e}^{B_0 t} \mathbf{E}^{x,t}_{x,0}\left[\mathrm{e}^{-\mathrm{i}\int_0^t A(W_s)\,dW_s - \int_0^t V_\omega(W_s)\,ds} \mathbf{1}(T_Q > t)\right] \end{aligned} \tag{3.1}$$

be the diagonal element of the heat kernel by the magnetic Feynman-Kac formula. Recall that for any domain $\Omega \subset \mathbf{R}^2$, T_Ω denotes the exit time from the set Ω.

The heat kernel of $H_{Q,\omega}$ is trace class, therefore

$$L_{Q,\omega}(t) := \frac{1}{|Q|}\int_Q L^x_{Q,\omega}(t)\,dx = \frac{1}{|Q|}\mathrm{Tr} e^{-tH_{Q,\omega}}$$

exists and is finite ($|\cdot|$ denotes the Lebesgue measure of a set). Finally let

$$L_Q(t) := \mathcal{E} L_{Q,\omega}(t) \ .$$

We need the following robust estimates.

Lemma 3.1. (i) *For $t > 2B^{-1}$, for any $x \in \mathbf{R}^2$ and for almost all ω*

$$L^x_\omega(t) = \mathrm{e}^{-tH_\omega}(x,x) \le B \ , \tag{3.2}$$

and for any square Q

$$L^x_{Q,\omega}(t) = \mathrm{e}^{-tH_{Q,\omega}}(x,x) \le B \ . \tag{3.3}$$

(ii) *For any square Q, for any x, $t > 4B^{-1}$ and for almost all ω*

$$L^x_{Q,\omega}(t) = \mathrm{e}^{-tH_{Q,\omega}}(x,x) \le 10B\big(\mathrm{e}^{-t\lambda_{Q,\omega}} + \mathrm{e}^{-B_0 t}\big) \ . \tag{3.4}$$

Proof of Lemma 3.1. (i) The robust diamagnetic estimate in the Feynman-Kac-Ito formula (2.2) gives

$$\mathrm{e}^{-\tau H_\omega}(x,x) \le (2\pi\tau)^{-1}\mathrm{e}^{B_0\tau} \ , \tag{3.5}$$

for any $\tau > 0$. Hence, using $H_\omega \ge 0$, we have

$$\begin{aligned}\mathrm{e}^{-tH_\omega}(x,x) &= \Big(\mathrm{e}^{-B^{-1}H_\omega}(\cdot,x), \mathrm{e}^{-(t-2B^{-1})H_\omega}\mathrm{e}^{-B^{-1}H_\omega}(\cdot,x)\Big)_{L^2(\mathbf{R}^2)} \\ &\le \Big\|\mathrm{e}^{-B^{-1}H_\omega}(\cdot,x)\Big\|^2_{L^2(\mathbf{R}^2)} = \mathrm{e}^{-B^{-1}H_\omega}(x,x) \le B \ .\end{aligned}$$

The proof of (3.3) is identical.

(ii) Similarly to part (i),

$$\begin{aligned}\mathrm{e}^{-tH_{Q,\omega}}(x,x) &= \Big(\mathrm{e}^{-B^{-1}H_{Q,\omega}}(\cdot,x), \mathrm{e}^{-(t-2B^{-1})H_{Q,\omega}}\mathrm{e}^{-B^{-1}H_{Q,\omega}}(\cdot,x)\Big)_{L^2(\mathbf{R}^2)} \\ &\le \mathrm{e}^{-(t-2B^{-1})\lambda_{Q,\omega}}\Big\|\mathrm{e}^{-B^{-1}H_{Q,\omega}}(\cdot,x)\Big\|^2_{L^2(\mathbf{R}^2)} \\ &\le 10B\Big(\mathrm{e}^{-t\lambda_{Q,\omega}} + \mathrm{e}^{-B_0 t}\Big)\end{aligned} \tag{3.6}$$

using that $H_{Q,\omega} \ge \lambda_{Q,\omega}$, $t > 4B^{-1}$ and that $\mathrm{e}^{-\tau H_{Q,\omega}}(x,x) \le (2\pi\tau)^{-1}\mathrm{e}^{B_0\tau}$, for any τ, again by a robust estimate in the Feynman-Kac formula (3.1). □

Proposition 3.2. *Let $Q = [-q,q]^2$ and $M = [-m,m]^2$ be squares. Then for any q and t*

$$L_Q(t) \le L(t) = \liminf_{m\to\infty} L_M(t) \ . \tag{3.7}$$

Proof. We start with the lower bound. Let $M := [-m,m]^2$, $M' := [-m',m']^2$ with $m' < m$ being integer multiples of q. We have

$$
\begin{aligned}
L(t) &\geq \frac{1}{|M|}\int_{M'} \mathscr{E} L^x_\omega(t)\,dx = \frac{1}{|M|}\int_{M'} \mathscr{E} L^x_{M,\omega}(t)\,dx \\
&\quad + \frac{1}{|M|}\int_{M'} \mathscr{E}\left[L^x_\omega(t) - L^x_{M,\omega}(t)\right] dx \\
&\geq \frac{1}{|M|}\int_{M} \mathscr{E} L^x_{M,\omega}(t)\,dx - \frac{B(|M| - |M'|)}{|M|} \\
&\quad - \frac{4e^{B_0 t}|M'|}{2\pi t|M|}\exp\left(-\frac{(m-m')^2}{8t}\right) \; . \qquad (3.8)
\end{aligned}
$$

Here we used (3.3) for $L^x_{M,\omega}$, $x \in M\backslash M'$, and we estimated

$$
\begin{aligned}
&\left|\mathscr{E}\left[\mathrm{e}^{-tH_\omega}(x,x) - \mathrm{e}^{-tH_{M,\omega}}(x,x)\right]\right| \\
&\quad = \left|\mathscr{E}(2\pi t)^{-1} e^{B_0 t}\mathbf{E}^{x,t}_{x,0}\left[\mathrm{e}^{-\mathrm{i}\int_0^t A(W_s)\,ds - \int_0^t V_\omega(W_s)\,ds}\mathbf{1}(T_M \leq t)\right]\right| \\
&\quad \leq 4(2\pi t)^{-1} e^{B_0 t}\exp\left(-\frac{(m-m')^2}{8t}\right) \qquad (3.9)
\end{aligned}
$$

for any $x \in M'$. The last estimate follows from the fact that x is at least at a distance $m - m'$ from the boundary of M and from the standard large deviation estimate for the Brownian loop ([22])

$$
\mathbf{P}^{0,t}_{0,0}\left(\sup_{0\leq s\leq t}|W_s| \geq L\right) \leq 4\exp\left(-\frac{L^2}{8t}\right) \qquad (3.10)
$$

for any $L > 0$.

Since $n = m/q$ is an integer, there are squares $\{Q_i\}_{i=1}^{n^2}$, each being congruent to Q, which partition M. Obviously $H_{M,\omega} \leq \oplus_i H_{Q_i,\omega}$. Since the trace of the heat kernel is operator monotone, we have

$$
\frac{1}{|M|}\int_M L^x_{M,\omega}(t)\,dx = \frac{1}{|M|}\mathrm{Tr}\mathrm{e}^{-tH_{M,\omega}} \geq \frac{1}{n^2}\sum_{i=1}^{n^2}\frac{1}{|Q_i|}\mathrm{Tr}\mathrm{e}^{-tH_{Q_i,\omega}} \; .
$$

Hence (3.8) yields

$$
L(t) \geq L_Q(t) - \frac{2B(m-m')}{m} - \frac{4\mathrm{e}^{B_0 t}}{2\pi t}\exp\left(-\frac{(m-m')^2}{8t}\right) \; .
$$

Since this is true for any m, m', letting $m, m' \to \infty$ such that $m - m' \to \infty$ but $(m - m')/m \to 0$, we obtain the lower bound in (3.7).

The proof of the upper bound in (3.7) is very similar. We again consider the squares M and M'. Then

$$L_{M,\omega}(t) = \frac{1}{|M|}\int_{M'} L^x_\omega(t)\,dx + \frac{1}{|M|}\int_{M\setminus M'} L^x_{M,\omega}(t)\,dx$$
$$+\frac{1}{|M|}\int_{M'}\Big(L^x_{M,\omega}(t) - L^x_\omega(t)\Big)\,dx\ .$$

Taking expectation

$$\begin{aligned} L_M(t) &= \frac{|M'|}{|M|}L(t) + \frac{1}{|M|}\int_{M\setminus M'} \mathscr{E} L^x_{M,\omega}(t)\,dx \\ &\quad + \frac{1}{|M|}\int_{M'} \mathscr{E}\Big(L^x_{M,\omega}(t) - L^x_\omega(t)\Big)\,dx \\ &\geq L(t)\left(1 - \frac{2(m-m')}{m}\right) - \frac{4e^{B_0 t}}{2\pi t}\exp\left(-\frac{(m-m')^2}{8t}\right) \end{aligned} \tag{3.11}$$

using the estimate (3.9). Choosing $m' = m - \sqrt{m}$ and taking $\liminf_{m\to\infty}$ on both sides we obtain the upper bound in (3.7). □

4 Estimates on the lowest eigenvalue on a disk

For any open bounded domain $\Omega \subset \mathbf{R}^2$ with piecewise C^1 boundary we would like to define the lowest magnetic eigenvalue of Ω with constant magnetic field B and with Dirichlet boundary condition. For simply connected domains this eigenvalue is gauge invariant. However, for multiply connected domains, the eigenvalue can depend on the extra fluxes through the holes in the domain, hence on the actual gauge, as we remarked in Section 1.1. If we allow *local* gauges as well, then the lowest eigenvalue might not come from a global gauge on $\mathbf{R}^2$ generating the constant field everywhere. Hence let $\lambda^{(B)}(\Omega) := \inf \mathrm{Spec}\,\frac{1}{2}\big[(-i\nabla - A)^2 - B\big]_\Omega$, and

$$\hat{\lambda}^{(B)}(\Omega) := \inf\left\{ \inf \mathrm{Spec}\frac{1}{2}\left[(-i\nabla - \hat{A})^2 - B\right]_\Omega \ :\ \hat{A} \in \mathscr{A}(\Omega)\cap C^\infty(\overline{\Omega}),\right.$$
$$\left. \mathrm{curl}\,\hat{A} = B \text{ on } \Omega \right\}\ , \tag{4.1}$$

where, as usual, the index Ω refers to Dirichlet boundary condition on Ω and $\mathscr{A}(\Omega)$ is the set of real analytic vectorfields. In general $\hat{\lambda}^{(B)}(\Omega) \leq \lambda^{(B)}(\Omega)$, but if Ω simply connected, then $\hat{\lambda}^{(B)}(\Omega) = \lambda^{(B)}(\Omega)$.

Let B_R be the disk of radius R, then the magnetic isoperimetric inequality [12] states that

$$\hat{\lambda}^{(B)}(\Omega) \geq \lambda^{(B)}(B_R) \tag{4.2}$$

if $|\Omega| = |B_R| = \pi R^2$. Though it is not stated explicitly, notice that the proof given in [12] is valid for any analytic local gauge $\hat{A}$ defined on Ω, i.e. the gauge does not have to be global. The smoothness condition on $\partial\Omega$ can be relaxed to piecewise C^1 by standard approximation.

Moreover, in the Appendix of [12] we gave a lower and an upper bound on $\lambda^{(B)}(B_R)$. The choice of the parameters in the proof given there is not well-suited for our problem, hence we slightly reformulate the statement and the proof. Recall that $B_0 := B/2$.

Proposition 4.1. *For any* $\kappa > 0$, *there exists* $C(\kappa) > 0$ *such that if* $B_0 R^2 \geq C(\kappa)$, *then*

$$e^{-B_0 R^2(1+\kappa)} \leq \lambda^{(B)}(B_R) \leq e^{-B_0 R^2(1-\kappa)} \quad . \tag{4.3}$$

Proof. We slightly modify the proof given in [12]. Recall that it is sufficient to consider the case $B = 1$ by scaling. Then, we identified $\lambda^{(B=1)}(B_R)$ with the smallest eigenvalue of the one dimensional harmonic oscillator $-\partial_x^2 + x^2/4 - 1/2$ with Dirichlet boundary conditions on the interval $[-R, R]$.

To construct a trial function for the upper bound in (4.3), we cut off the Gaussian eigenfunction $\phi(x) = e^{-x^2/4}$ of the unrestricted oscillator by straight line segments on $R - 1 \leq |x| \leq R$. For large R this is more effective than the line segments on $R/2 \leq |x| \leq R$ used in [12].

For the lower bound, we follow the same proof based on the Birman-Schwinger principle, but we choose the following parameters: $E := \exp(\kappa R^2/2)$, $N := \left(\frac{1}{2} + \kappa\right) R^2 E$, $\eta := \exp\left(-\frac{1+\kappa}{2} R^2\right)$. For completeness, we recall the Birman-Schwinger type argument here, since the usual references (e.g. [23]) do not precisely cover our case.

Let $U := (E + \eta)\mathbf{1}_{[-R,R]^c}$ and $H_{\text{osc}} := -\partial_x^2 + x^2/4 - 1/2$ defined on $L^2(\mathbf{R})$. Then $\lambda^{(B=1)}(B_R) \geq \eta$, would follow from $H_{\text{osc}} + U \geq \eta$. To show this latter, it is sufficient to prove that $H_{\text{osc}} - |U - \eta - E|_- \geq -E$. Here $|\cdot|_-$ denotes the negative part.

Suppose there is an eigenvalue $-\lambda$, with eigenfunction f, such that $\lambda > E$, i.e. $\left(H_{\text{osc}} - |U - \eta - E|_-\right) f = -\lambda f$. Then $\left[|U - \eta - E|_-^{1/2} (H_{\text{osc}} + \lambda)^{-1} |U - \eta - E|_-^{1/2}\right] g = g$ with $g := |U - \eta - E|_-^{1/2} f \in L^2(\mathbf{R})$. This means that the bigger operator

$$K_{\eta,E} := |U - \eta - E|_-^{1/2} \frac{1}{H_{\text{osc}} + E} |U - \eta - E|_-^{1/2}$$

(the Birman-Schwinger kernel) has an eigenvalue at least 1. In particular $\text{Tr}(K_{\eta,E})^N \geq 1$.

On the other hand, one can show, exactly as in [12], that $\mathrm{Tr}(K_{\eta,E})^N < 1$. □

5 Proof of the lower bound

Using Proposition 3.2, to give a lower bound on $L(t)$, it is enough to bound $L_Q(t)$ for a convenient Q. Since it corresponds to an upper bound on the Hamiltonian $H_{Q,\omega}$, we have to choose a convenient obstacle configuration and a trial function.

Let $\ell(t) := 10(\log t/B_0)^{1/2}$. This is the natural lengthscale, mentioned in the introduction, on which the eigenfunctions, with eigenvalues substantially contributing to the Lifschitz tail, are supported. Let $\Lambda(t) := [-\ell(t), \ell(t)]^2$.

For the lower bound we choose $Q := \Lambda = \Lambda(t)$. Let supp $V^{(0)} \subset B(0, \overline{a})$ for some $\overline{a}$. Hence

$$
\begin{aligned}
L_\Lambda(t) &= \frac{1}{|\Lambda|}\mathscr{E}\mathrm{Tr}\mathrm{e}^{-tH_{\Lambda,\omega}} \\
&\geq \frac{1}{|\Lambda|}\sup_{\Omega\subset\Lambda}\left\{\mathrm{e}^{-t\lambda(\Omega)}\cdot\mathscr{P}\left(\Omega\cap\bigcup_i B(x_i(\omega),\overline{a}) = \emptyset\right)\right\} \\
&\geq \frac{1}{|\Lambda|}\sup_{\Omega\subset\Lambda}\left\{\mathrm{e}^{-t\lambda(\Omega)}\mathrm{e}^{-v|\Omega+B(0,\overline{a})|}\right\}\ , \qquad (5.1)
\end{aligned}
$$

where the supremum runs through all open sets $\Omega \subset \Lambda$, and recall that $\lambda(\Omega) = \lambda^{(B)}(\Omega)$ denotes the lowest eigenvalue of $H_\Omega := \frac{1}{2}[(-i\nabla - A)^2 - B]$ with Dirichlet boundary conditions on Ω.

For Ω, one can choose a disk with radius $R := (\log t/B_0(1-\kappa))^{1/2} \leq \ell(t)$ with some $0 < \kappa < 1/2$. Then $\lambda(\Omega) \leq \mathrm{e}^{-B_0R^2(1-\kappa)} = t^{-1}$ by Proposition 4.1 for large enough t. Hence, we obtain from (5.1) and (3.7)

$$
L(t) \geq L_{\Lambda(t)}(t) \geq \frac{1}{4\ell(t)^2}\cdot t^{-\frac{2\pi v}{B(1-2\kappa)}} \qquad (5.2)
$$

for large enough $t \geq \tilde{t}(B, v, \overline{a})$. Taking logarithm of (5.2), and considering $\liminf_{t\to\infty}$ first, then $\lim_{\kappa\to 0}$, we immediately obtain the lower bound in (2.7).

6 Second localization for the upper bound

For the upper bound we need a further localization from the uncontrolled lengthscale of the square M to a square S of lengthscale

$s = s(t) \sim (\text{const}) \cdot \ell(t)$ (the precise definition is given later). Here we use the magnetic localization technique developed in [14]. As it is explained there, the usual IMS formula (see, e.g. [8]) makes a localization error of order L^{-2} in the energy, where L is the localization lengthscale. The magnetic kinetic energy can be localized at a price which is of order $\exp[-(\text{const})L^2]$ if one is willing to change the magnetic field by a constant amount.

Fix parameters $0 < \beta < B_0 = B/2$, $1 \le s \le m$ and let $M = [-m, m]^2$, $S = [-s, s]^2$, $\tilde{M} = [-m - s, m + s]^2$, and $\tilde{S} = [-\frac{s}{2}, \frac{s}{2}]^2$. Let $S_z := S + z$ for any $z \in \mathbf{R}^2$. Let $\lambda_{S_z,\omega}^{(B+2\beta)}$ be the lowest eigenvalue of $H_{S_z,\omega}^{(B+2\beta)} := \frac{1}{2}\left[(-i\nabla - (1 + 2\beta B^{-1})A)^2 - (B + 2\beta)\right] + V_\omega$ with Dirichlet boundary conditions on S_z. Recall that $A(x) = (A_1(x), A_2(x)) = (B/2)(-x_2, x_1)$, and the magnetic field in $H_{S_z,\omega}^{(B+2\beta)}$ is $B + 2\beta$.

Proposition 6.1. *Assume that the parameters introduced above satisfy $\beta s^2 \ge 4$. Then for any $z \in \mathbf{R}^2$ there exist functions η_z, supported on S_z, such that for any $f \in H_0^1(M)$*

$$\langle f, H_{M,\omega} f\rangle \ge \frac{\beta}{2\pi}\int_{\tilde{M}} dz \langle f\eta_z, H_{S_z,\omega}^{(B+2\beta)} f\eta_z\rangle - 32\pi\beta^{-1}s^{-2}e^{-\frac{\beta}{8}s^2}\|f\|_{L^2(M)} \ . \tag{6.1}$$

Furthermore $|\eta_z(x)|$ depends only on $x - z$. Hence $\|\eta_z\|^2 = \int |\eta_z|^2$ is independent of z and we denote this common value by $\|\eta\|^2 := \|\eta_z\|^2$, which satisfies

$$\frac{\pi}{2\beta} \le \|\eta\|^2 = \|\eta_z\|^2 \le \frac{\pi}{\beta} \ . \tag{6.2}$$

We shall need the following corollary.

Corollary 6.2. *Under the conditions of Proposition* 6.1 *we have*

$$\mathrm{Tr}\left(e^{-tH_{M,\omega}}\right) \le C(\beta, s, t)\int_{\tilde{M}} dz \mathrm{Tr}_{L^2(S_z)}\left(\exp\left[-\frac{t}{4}H_{S_z,\omega}^{(B+2\beta)}\right]\right) \tag{6.3}$$

with

$$C(\beta, s, t) := \exp\left(32\pi\beta^{-1}s^{-2}e^{-\frac{\beta}{8}s^2}t\right) \ . \tag{6.4}$$

In particular

$$\limsup_{m\to\infty} L_M(t) \le C(\beta, s, t)L_S^{(B+2\beta)}\left(\frac{t}{4}\right) \ . \tag{6.5}$$

In our application, we shall use this corollary together with Proposition 3.2 to obtain

Theorem 6.3. *Let $s = s(t, n_0) := n_0\ell(t)$ and $S = [-s, s]^2$ as usual. For any fixed $\beta > 0$ and any $n_0 \ge (B/\beta)^{1/2}$*

$$\limsup_{t\to\infty} \frac{\log L(t)}{\log t} \leq \limsup_{t\to\infty} (\log t)^{-1} \log \mathscr{E} \exp\left(-t\lambda_{S,\omega}^{(B+2\beta)}\right) \ . \tag{6.6}$$

Proof of Proposition 6.1. We can assume that $\|f\| = 1$. We can think of f to be extended to the whole $\mathbf{R}^2$ by defining it to be zero outside of M, and let $\int$ denote $\int_{\mathbf{R}^2}$ in this section. Then, using integration by parts and that $\int \exp(-\beta(x-z)^2)\,dz = \pi\beta^{-1}$ for all x,

$$\begin{aligned}
\langle f, H_{M,\omega} f\rangle &= \int \left(\tfrac{1}{2}|(-\mathrm{i}\nabla - A)f|^2 - B_0|f|^2 + V_\omega|f|^2\right)\\
&= \int \left(\tfrac{1}{2}|(-\mathrm{i}\partial_1 + \partial_2 - A_1 - \mathrm{i}A_2)f|^2 + V_\omega|f|^2\right)\\
&= \frac{\beta}{\pi}\int dz \int \Big(\tfrac{1}{2}\mathrm{e}^{-\beta(-z)^2}|(-\mathrm{i}\partial_1 + \partial_2 - A_1 - \mathrm{i}A_2)f|^2\\
&\qquad\qquad + V_\omega \mathrm{e}^{-\beta(-z)^2}|f|^2\Big) \ .
\end{aligned} \tag{6.7}$$

Straightforward calculation shows that for any fixed z

$$\begin{aligned}
&(-\mathrm{i}\partial_1 + \partial_2 + (B_0+\beta)x_2 - \mathrm{i}(B_0+\beta)x_1)(\varphi_z f)(x)\\
&\quad = \varphi_z(x)(-\mathrm{i}\partial_1 + \partial_2 + B_0 x_2 - \mathrm{i}B_0 x_1)f(x) \ ,
\end{aligned}$$

where

$$\varphi_z(x) := \mathrm{e}^{-\frac{\beta}{2}(x-z)^2}\mathrm{e}^{\mathrm{i}\beta(x_2z_1 - x_1z_2)} \ .$$

Let

$$T_\beta := -\mathrm{i}\partial_1 + \partial_2 + (B_0+\beta)x_2 - \mathrm{i}(B_0+\beta)x_1 \tag{6.8}$$

for simplicity. Hence from (6.7), using $A(x) = B_0(-x_2, x_1)$

$$\langle f, H_{M,\omega} f\rangle = \frac{\beta}{\pi}\int dz \int \left\{\tfrac{1}{2}|T_\beta(\varphi_z f)|^2 + V_\omega|\varphi_z f|^2\right\} \ . \tag{6.9}$$

Fix a smooth function $\theta(x)$ such that $\theta(x) \equiv 0$ for $x \in \mathbf{R}^2 \setminus S$, $\theta(x) \equiv 1$ on $\tilde{S}$, $0 \leq \theta \leq 1$ and $\|\nabla\theta\|_\infty \leq 4s^{-1}$. Then

$$\frac{\pi}{2\beta} \leq \int \theta^2(x)\mathrm{e}^{-\beta x^2}\,dx \leq \frac{\pi}{\beta} \ , \tag{6.10}$$

using $\beta s^2 \geq 4$. Consider z fixed for a while and let $\theta_z(x) := \theta(x-z)$ and $\eta_z := \theta_z\varphi_z$ for any $z \in \mathbf{R}^2$. Clearly $|\eta_z(x)|$ depends only on $z - x$ and (6.2) is guaranteed by (6.10). Furthermore,

$$\begin{aligned}
&|(1-\theta_z)\varphi_z f|^2 \leq \mathrm{e}^{-\frac{\beta}{8}s^2}|\varphi_z||f|^2 \quad \text{and}\\
&|\varphi_z f|^2 \cdot \mathbf{1}(\operatorname{supp}\nabla\theta_z) \leq \mathrm{e}^{-\frac{\beta}{8}s^2}|\varphi_z||f|^2 \ ,
\end{aligned} \tag{6.11}$$

where $\mathbf{1}(\cdot)$ is the characteristic function of a set as before.

By a Schwarz inequality, recalling the definition of T_β

$$\theta_z^2|T_\beta(\varphi_z f)|^2 \geq \tfrac{1}{2}|T_\beta(\theta_z\varphi_z f)|^2 - 2\|\nabla\theta\|_\infty^2 \cdot \mathbf{1}(\operatorname{supp}\nabla\theta_z)|\varphi_z f|^2$$
$$\geq \tfrac{1}{2}|T_\beta(\theta_z\varphi_z f)|^2 - 32s^{-2}\mathrm{e}^{-\frac{\beta}{8}s^2}|\varphi_z||f|^2 \ . \tag{6.12}$$

Hence, using $\theta_z^2 \leq 1$, $V_\omega \geq 0$, $\int|\varphi_z(x)|\,dz = 2\pi\beta^{-1}$ (for any x) and $\int|f|^2 = 1$, we have

$$\langle f, H_{M,\omega}f\rangle \geq \frac{\beta}{2\pi}\int dz\left\{\int \tfrac{1}{2}|T_\beta(\eta_z f)|^2 + \int V_\omega|\eta_z f|^2\right\}$$
$$- 32\pi\beta^{-1}s^{-2}\mathrm{e}^{-\frac{\beta}{8}s^2} \ . \tag{6.13}$$

Finally, since f is supported on M and η_z is supported on S_z, we have $\eta_z f \equiv 0$ unless $z \in \tilde{M}$. Therefore the z-integral above can be restricted to $\tilde{M}$. This finishes the proof of Proposition 6.1. □

Proof of Corollary 6.2. Consider the space $\mathscr{M} := \int_{\tilde{M}}^{\oplus} L^2(S_z)\,dz$ defined as a direct integral. Its elements will be denoted by $\mathbf{g} = (g_z)_{z\in\tilde{M}}$ where $g_z \in L^2(S_z)$. Let $\mathbf{H}$ be the following operator defined on $\mathscr{M}$:

$$\mathbf{H} := \int_{\tilde{M}}^{\oplus} H_{S_z,\omega}^{(B+2\beta)}\,dz \tag{6.14}$$

i.e. it acts as

$$\mathbf{H}\mathbf{g} := \left(H_{S_z,\omega}^{(B+2\beta)} g_z\right)_{z\in\tilde{M}} \in \mathscr{M}$$

(strictly speaking it is defined on a dense subspace $\int_{\tilde{M}}^{\oplus} H_0^1(S_z)\,dz$ of $\mathscr{M}$). Let $\mathbf{T} : L^2(M) \to \mathscr{M}$ be defined as

$$\mathbf{T}f := \frac{1}{\|\eta\|}(f\eta_z)_{z\in\tilde{M}} \in \mathscr{M} \ .$$

If S_z is not included in M, then we extend f to be zero outside M to ensure that $f\eta_z$ be defined on S_z. Notice that $\mathbf{T}$ is a partial isometry

$$\begin{aligned}\langle \mathbf{T}f, \mathbf{T}g\rangle_{\mathscr{M}} &= \frac{1}{\|\eta\|^2}\int_{\tilde{M}} dz \int_{S_z} |\eta_z|^2 \overline{f} g \\ &= \frac{1}{\|\eta\|^2}\int_{\tilde{M}} dz \int_{\mathbf{R}^2} |\eta_z|^2 \overline{f} g \\ &= \frac{1}{\|\eta\|^2}\int_{\mathbf{R}^2} dz \int_{\mathbf{R}^2} |\eta_z|^2 \overline{f} g = \langle f, g\rangle_{L^2(M)} \ .\end{aligned}$$

To extend the integrations, we used that η_z is supported on S_z and $\overline{f}g$ is supported on M. In the last step we used that $\int dz|\eta_z(x)|^2 = \|\eta\|^2$ for any x since $|\eta_z(x)|$ depends only on $x - z$.

Let $f_1, f_2, \ldots$ be the normalized eigenfunctions of the (compact) operator $H_{M,\omega}$. Then

$$\begin{aligned}
\operatorname{Tr} \mathrm{e}^{-tH_{M,\omega}} &= \sum_{j=1}^{\infty} \mathrm{e}^{-t\langle f_j, H_{M,\omega} f_j\rangle} \le C(\beta, s, t) \\
&\quad \times \sum_{j=1}^{\infty} \exp\left(-\frac{t\beta}{2\pi} \int_{\tilde{M}} dz \langle f_j \eta_z, H^{(B+2\beta)}_{S_z,\omega} f_j \eta_z \rangle \right) \\
&= C(\beta, s, t) \sum_{j=1}^{\infty} \exp\left(-\frac{t\beta \|\eta\|^2}{2\pi} \langle \mathbf{T} f_j, \mathbf{H} \mathbf{T} f_j \rangle \right) \\
&\le C(\beta, s, t) \sum_{j=1}^{\infty} \exp\left(-\frac{t}{4} \langle \mathbf{T} f_j, \mathbf{H} \mathbf{T} f_j \rangle_{\mathscr{M}} \right)
\end{aligned}$$

using Proposition 6.1 and (6.2). By the Jensen-Peierls inequality and the fact that $\{\mathbf{T} f_j\}_{j=1,2,\ldots}$ is an orthonormal family in $\mathscr{M}$ we can continue this estimate

$$\begin{aligned}
\operatorname{Tr} \mathrm{e}^{-tH_{M,\omega}} &\le C(\beta, s, t) \sum_{j=1}^{\infty} \langle \mathbf{T} f_j, \mathrm{e}^{-\frac{t}{4}\mathbf{H}} \mathbf{T} f_j \rangle_{\mathscr{M}} \le C(\beta, s, t) \operatorname{Tr}_{\mathscr{M}} \left(\mathrm{e}^{-\frac{t}{4}\mathbf{H}} \right) \\
&= C(\beta, s, t) \int_{\tilde{M}} dz \operatorname{Tr}_{L^2(S_z)} \left(\mathrm{e}^{-\frac{t}{4} H^{(B+2\beta)}_{S_z,\omega}} \right)
\end{aligned}$$

to arrive at (6.3).

Finally, to obtain (6.5), we take expectation of (6.3) and divide by $|M|$. Use that $|\tilde{M}|/|M| \to 1$ as $m \to 0$ and notice that $\mathscr{E} L^{(B+2\beta)}_{S_z}(t/4)$ is independent of z by translation invariance. This completes the proof of the Corollary. □

Proof of Theorem 6.3. Using (3.7), (6.5) and (3.4) we obtain

$$L(t) \le 40 B s^2 C(\beta, s, t) \left(\mathscr{E} \mathrm{e}^{-\frac{t}{4} \cdot \lambda^{(B+2\beta)}_{S,\omega}} + \mathrm{e}^{-(B_0+\beta)\frac{t}{4}} \right)$$

if $t \ge 16 B^{-1}$. Taking logarithms and dividing by $\log(t/4)$ we easily obtain (6.6) using the explicit form of $C(\beta, s, t)$ from (6.4). □

7 Enlargement of obstacles

Estimating the IDS of $H_{S,\omega}$ amounts to considering all eigenvalues below E (if any). But, as we mentioned in the introduction, for the Lifschitz tail in a finite box of appropriate size, only a much rougher information is needed, namely the location of the bottom of the spectrum. The main contribution to $L_S(t)$, hence to $N(E)$, comes from few configurations with a large clearing of size of order $\ell(t)$, and the actual number of the eigenvalues below E is irrelevant compared to the large deviation probability that there is any eigenvalue below E at all. We shall see that focusing on the lowest eigenvalue is enough. We really shall need this argument for $H_{S,\omega}^{(B+2\beta)}$, but the actual size of the field does not play much role in this section, so for simplicity we consider $H_{S,\omega} = H_{S,\omega}^{(B)}$.

As before, let $\lambda_{S,\omega}$ be the lowest eigenvalue of $H_{S,\omega}$. Following Sznitman's original idea, we shall estimate this eigenvalue from below by the lowest eigenvalue of another magnetic Hamiltonian which has hard-core potential on the "enlarged obstacles", i.e. on a set which consists of larger balls centered at the points of the Poisson configuration. This part of the analysis yields estimates uniformly in the point configuration, hence the Poisson randomness plays no role.

The enlarged obstacles typically occupy a much larger portion of S than the support of V_ω (in particular their union has a smaller combinatorial complexity), but the enlargement does not substantially influence the largest clearing on which the lowest eigenstate lives if this clearing is really large. In other words, the enlargement does not influence the lowest eigenvalue if it is really small.

We need several definitions.

7.1 Good points, boxes, clearings

This construction follows Sznitman's work [25]. Consider a fixed locally finite set of points $\omega = \{x_i\}_{i=1,2,\dots}$ in $S = [-s,s]^2$. Let $\ell \le s$ and let C_m stand for the cube (square)

$$C_m := \{z \in \mathbf{R}^2 : m_i\ell \le z_i < (m_i+1)\ell, \quad i = 1,2\} \ .$$

Fix two parameters $b > 10a$ and $\varepsilon > 0$ (recall that a is the radius of a disk located fully within the support of $V^{(0)}$). We say that a point $x_i \in C_m$ is good if for all closed balls $C = \overline{B}(x_i, 10^{j+1}b)$, $0 \le j$, and $10^{j+1}b \le \ell/2$

$$\left| C_m \cap C \cap \left(\bigcup_{x_j \in C_m} \overline{B}(x_j, b) \right) \right| \ge \frac{\varepsilon}{9} |C_m \cap C| \ , \tag{7.1}$$

in particular

$$\left| C \cap \left(\bigcup_{x_j \in C_m} \overline{B}(x_j, b) \right) \right| \ge \frac{\varepsilon}{36} |C| \ .$$

Let Good(m) be the set of good points in C_m, the rest is Bad(m), and let $\mathscr{G} := \cup_m \mathrm{Good}(m)$. By a covering argument (see [24]) we know that

$$\left| C_m \cap \left(\bigcup_{x_j \in \mathrm{Bad}(m)} \overline{B}(x_j, b) \right) \right| \leq \varepsilon |C_m| = \varepsilon \ell^2 \ . \tag{7.2}$$

Chop each segment $[k\ell, (k+1)\ell]$ into at most $[\sqrt{2}\ell/b] + 1$ intervals of length $b/\sqrt{2}$ (except the last one). This yields closed boxes with diameter less than b, with union $\overline{C}_m$.

Introduce a number $r > 0$. We define the event Cl_m that "there is a clearing of size r in C_m", i.e.

$$Cl_m := \{\omega : |\tilde{U}_m(\omega)| \geq 9^{-2}\pi r^2 \ell^2\} \ ,$$

where $\tilde{U}_m(\omega)$ is the open subset of int (C_m) obtained by taking the complement in the interior of C_m of the closed boxes where a good point of C_m falls. Let $A^0(\omega)$ be the union of all closed cubes $\overline{C}_m$ where there is clearing of size r:

$$\mathbf{1}_{A^0(\omega)}(z) = \sum_m \mathbf{1}_{\overline{C}_m}(z) \cdot \mathbf{1}_{Cl_m}(\omega) \ .$$

Let $A^1 = A^1(\omega)$ be the open set of points at distance less than ℓ from $A^0(\omega)$. If $A^0(\omega)$ is empty, so is $A^1(\omega)$. Let S_-^ϱ be the open square $(-s-\varrho, s+\varrho)^2$ and $S_+^\varrho := (-s+\varrho, s-\varrho)^2$ for $s > \varrho > 0$. Finally, let

$$\Omega := S \Big\backslash \bigcup_{i \in \mathscr{G}} \overline{B}(x_i, a) \ ,$$

and for any $b > a$, $s > b$

$$\Omega_+^b := \left(S_+^b \cap A^1 \right) \Big\backslash \bigcup_{i \in \mathscr{G}} \overline{B}(x_i, b).$$

Let $\delta := \min\{\frac{1}{200}, \frac{a}{6}\}$ and let

$$\Omega_-^\delta := B(0,\delta) + \Omega \quad \text{ and } \quad \Omega_-^{2\delta} := B(0, 2\delta) + \Omega \ .$$

Notice that $\Omega_+^{2b} \subset \Omega_+^b \subset \Omega \subset \Omega_-^\delta \subset \Omega_-^{2\delta}$ and that there is no A^1 in the definition of Ω, Ω_-^δ and $\Omega_-^{2\delta}$.

Let $\lambda_{S,\omega}$ be the lowest eigenvalue of $\frac{1}{2}\left[(-i\nabla - A)^2 - B\right] + V_\omega$ with Dirichlet boundary conditions on S. For a lower bound on $\lambda_{S,\omega}$ one can replace the true potential V by $\tilde{V} = \tilde{V}_\omega$ which is defined as

$$\tilde{V}_\omega(x) := v \cdot \mathbf{1}\left(x \in \bigcup_{i : x_i(\omega) \in \mathscr{G}} \overline{B}(x_i, a) \right) \ , \tag{7.3}$$

and we can assume that $v \leq B$ and $0 < a < 1$.

7.2 Gauge choice

Since we work on multiply connected domains, we have to define the gauge freedom carefully. In Section 7 we shall allow independent extra fluxes from 0 to 2π through the good points to ensure more gauge freedom on the complementary domain.

Let $\underline{\alpha} = \{\alpha_i\}_{i\in\mathscr{G}} \in [0,2\pi)^{\mathscr{G}}$, $B_{\underline{\alpha}}(x) := B + \sum_{i\in\mathscr{G}} \alpha_i B^*(x-x_i)$ and $A_{\underline{\alpha}}(x) := A(x) + \sum_{i\in\mathscr{G}} \alpha_i A^*(x-x_i)$, where A^* is the (unique) radial gauge generating the uniform radial magnetic field $B^*(x) := 8a^{-2}\mathbf{1}_{B(0,a/2)}(x)$ supported on the disk $B(0,a/2)$ with flux 2π, i.e. $A^*(x) := a(|x|)(-x_2,x_1)$ with $a(r) := (2\pi r^2)^{-1}\int_{|x|\le r} B^*(x)\,dx$. Let

$$\lambda_{b,\underline{\alpha}} = \lambda^{(B)}_{\underline{\alpha}}(\Omega^b_+) := \inf \mathrm{Spec}\,\frac{1}{2}\left[(-i\nabla - A_{\underline{\alpha}})^2 - B_{\underline{\alpha}}\right]_{\Omega^b_+} \tag{7.4}$$

with Dirichlet boundary condition on Ω^b_+ and

$$\tilde\lambda_b = \tilde\lambda^{(B)}(\Omega^b_+) := \inf\left\{\lambda^{(B)}_{\underline{\alpha}}(\Omega^b_+) : \underline{\alpha}\in[0,2\pi)^{\mathscr{G}}\right\} . \tag{7.5}$$

Since $B_{\underline{\alpha}} = \mathrm{curl}\,A_{\underline{\alpha}} = B$ on Ω^b_+ and $A_{\underline{\alpha}}$ is real analytic on Ω^b_+, we have (recall (4.1))

$$\tilde\lambda^{(B)}(\Omega^b_+) \ge \hat\lambda^{(B)}(\Omega^b_+) \tag{7.6}$$

(in fact, one can show that $\tilde\lambda^{(B)}(\Sigma) = \hat\lambda^{(B)}(\Sigma)$ for any domain Σ such that $B_{\underline{\alpha}} = B$ on Σ). Similarly,

$$\lambda_{\underline{\alpha}} = \lambda^{(B)}_{\omega,\underline{\alpha}}(S) := \inf \mathrm{Spec}\,\frac{1}{2}\left[(-i\nabla - A_{\underline{\alpha}})^2 - B_{\underline{\alpha}}\right] + \tilde V_\omega \tag{7.7}$$

with Dirichlet boundary conditions on S and

$$\tilde\lambda = \tilde\lambda^{(B)}_\omega(S) := \inf\left\{\lambda^{(B)}_{\omega,\underline{\alpha}}(S) : \underline{\alpha}\in[0,2\pi)^{\mathscr{G}}\right\} . \tag{7.8}$$

Certainly $\lambda_{S,\omega} \ge \tilde\lambda$. The advantage of these extra fluxes can be seen in the following lemma.

Lemma 7.1. *Let $f\in H^1_0(\Omega)$, $\int|f|^2 = 1$ and $X\in C^1(\Omega,\mathbf{R}^2)$ be a vectorfield that satisfies* $\mathrm{curl}X = B$ *on Ω. Then* $\tilde\lambda \le \int\left|(-i\partial_1 + \partial_2 - X_1 - iX_2)f\right|^2$.

Proof. Any bounded component $\mathscr{C}$ of Ω^c is a union of closed disks; $\mathscr{C} = \cup_{i\in I}\overline{B(x_i,a)}$ with some index set $I = I_{\mathscr{C}}$. For each $\mathscr{C}$, pick one of the element of $I_{\mathscr{C}}$, say i_0, and let $\alpha_i := 0$ for $i\in I_{\mathscr{C}}$, $i\ne i_0$. Let $\alpha_{i_0} := \int_{\partial\mathscr{C}} X - 2\pi\left[\frac{1}{2\pi}\int_{\partial\mathscr{C}} X\right]$, i.e. the fractional part (modulo 2π) of the flux through $\mathscr{C}$. Notice that $\int_{\partial\mathscr{C}} X$ is well defined as $\lim_{\delta\to 0}\int_{\partial(\mathscr{C}+B(0,\delta))} X$ using $\mathrm{curl}\,X = B$ even if X is not defined on $\partial\mathscr{C}$.

Let $\xi \in C^2(\Omega, S^1)$ be a real solution to $\nabla\xi = A_{\underline{\alpha}} - X$ with $\underline{\alpha} := \{\alpha_i\}$. This equation has a solution since $\mathrm{curl}(A_{\underline{\alpha}} - X) = \mathrm{curl}\, A - \mathrm{curl}\, X = 0$ on Ω, and $\int_\gamma (A_{\underline{\alpha}} - X) \in 2\pi\mathbf{Z}$ for every closed curve γ in Ω by the choice of $\underline{\alpha}$. Hence

$$\int \left|\big(-\mathrm{i}\partial_1 + \partial_2 - (A_{\underline{\alpha}})_1 - \mathrm{i}(A_{\underline{\alpha}})_2\big)(\mathrm{e}^{i\xi} f)\right|^2 = \int |(-\mathrm{i}\partial_1 + \partial_2 - X_1 - \mathrm{i}X_2) f|^2$$

proves the claim by the definition of $\tilde{\lambda}$. □

7.3 Green's function, exit time

For any bounded domain Ω with piecewise C^1 boundary we define its Green's function g_Ω as the solution to the following boundary value problem:

$$\Delta g_\Omega = -1 \text{ on } \Omega$$
$$g_\Omega \equiv 0 \text{ on } \partial\Omega \ .$$

By standard elliptic theory, $g_\Omega \in C(\overline{\Omega}) \cap C^\infty(\Omega)$ uniquely exists and is positive and bounded. Let

$$G_\Omega := \max_{x \in \overline{\Omega}} g_\Omega(x) \ . \tag{7.9}$$

Let us consider the standard Brownian motion W_s in Ω, starting from $x \in \Omega$. The probability with respect to this Brownian motion is denoted by $\mathbf{P}_x$, the expectation value by $\mathbf{E}_x$. Let T_Ω be the exit time from Ω. By Ito's formula $g_\Omega(W_s) - g_\Omega(x) - \frac{1}{2}\int_0^s \Delta g_\Omega(W_\tau)\, d\tau$ is a martingale, hence

$$g_\Omega(x) = \tfrac{1}{2}\mathbf{E}_x T_\Omega \ . \tag{7.10}$$

Using these notations, the goal of this section is to prove the following theorem.

Theorem 7.2. *Given a finite point configuration ω in $S = [-s,s]^2$. Let $\tilde{\lambda}_b$ and $\tilde{\lambda}$ be defined by* (7.5), (7.8). *There exist two universal constants* $0 < \varepsilon_0,\ c_0 < 1$ *and a function* $\tilde{\ell}(\varepsilon, b, B, r) > 0$ *for all* $\varepsilon \le \varepsilon_0$, $b \ge 1$ *such that for all* $\ell \ge \tilde{\ell}(\varepsilon, b, B, r)$

$$\tilde{\lambda}_b^{w(r)} \le \frac{\tilde{\lambda}}{K} \tag{7.11}$$

with

$$K = K(b, B, v, \delta, r, \ell, s) := \left(80b^{-2}v^{-1}\delta^{-2}s^{2+20(1-c_0)^{1/r}}e^{20Bc_0^{-1}r^2\ell^2}e^{30B\ell}\right)^{-w(r)} \tag{7.12}$$

$$w(r) := \frac{1}{1-20(1-c_0)^{1/r}} \tag{7.13}$$

under the following conditions: $\delta := \min\{\frac{1}{200}, \frac{a}{6}\}$, $a < 1 < b$, $r \le 1/4$, $r < \log\left(\frac{1}{1-c_0}\right)/(\log 30)$, $v \le B$, *and*

$$\frac{\tilde{\lambda}}{K} \le 2^{-w(r)} \quad . \tag{7.14}$$

All the estimates are uniform in the point configuration ω.

We shall need this result in the form of the following

Corollary 7.3. *Assume* $a < 1$, $v \le B$ *and fix an integer* n_0. *For any* b *large enough and* r, ε *small enough, for all* ℓ *large enough and* $s := n_0\ell$ *we have* $\tilde{\lambda}_b^{w(r)} \le \tilde{\lambda}/K$ *if* $\tilde{\lambda}/K \le 2^{-w(r)}$, *where*

$$\lim_{r\to 0} w(r) = 1 \tag{7.15}$$

and

$$\limsup_{r\to\infty}\ \limsup_{\substack{b\to\infty \\ \varepsilon\to 0}}\ \limsup_{\ell\to\infty} \frac{\log K}{\ell^2} = 0 \quad . \tag{7.16}$$

Since the relevant size of $\tilde{\lambda}$ is of order $\exp[-(\text{const})B\ell^2]$, we see that $\log\tilde{\lambda} \ll \log K < 0$ for large ℓ and small r. Hence (7.11) tells us that $\log\tilde{\lambda}_b \le (1-o(1))\log\tilde{\lambda} < 0$.

The proof requires several lemmas. The intuitive idea is that the magnetic Dirichlet eigenvalue $\lambda(\Omega)$ of a domain Ω is essentially e^{-BG_Ω} and the corresponding eigenfunction is roughly $e^{B(g_\Omega(x)-G_\Omega)}$ if Ω is large. The increase of the eigenvalue due to the enlargement is determined by the size of the eigenfunction, hence of g_Ω, within a distance of order b from the boundary of Ω (Lemma 7.4). Then, by applying the method of enlargement of obstacles, we show that $g_\Omega(x) \ll G_\Omega$ if Ω is large and x is close to the boundary of Ω (Lemma 7.5). Here we use the probabilistic representation of g_Ω (7.10). For technical reasons, we have to work on a slightly bigger domain, $\Omega_-^{2\delta}$, and we have to compare G_Ω and $G_{\Omega_-^{2\delta}}$ (Lemma 7.6). Finally we have to go back and estimate G_Ω in terms of $\lambda(\Omega)$ (Lemma 7.7). Additional difficulties arise from the facts that $\tilde{\lambda}$ in the theorem is not exactly $\lambda(\Omega)$ because the obstacles are soft and that we have to consider the infimum over the extra gauge freedom.

Lemma 7.4 *With the notations above, if* $\delta \le s$, $v \le B$, *and* $\tilde{\lambda} \le (1/16) v s^{-2} \mathrm{e}^{-2B\eta} \delta^2$, *then*

$$\tilde{\lambda} \le \tilde{\lambda}_b \le 32\tilde{\lambda}\big(1 + 8b^{-2} v^{-1} s^2 \mathrm{e}^{2B\eta} \delta^{-2}\big) \tag{7.17}$$

where

$$\eta := \max\Big\{ g_{\Omega_-^{2\delta}}(z) : z \in \overline{\Omega_-^{2\delta}} \setminus \Omega_+^{2b} \Big\} \ . \tag{7.18}$$

Proof of Lemma 7.4. The first inequality is trivial by variational principle. For the second one, first fix $\underline{\alpha} \in [0, 2\pi)^{\mathscr{G}}$, and let $\varphi_{\underline{\alpha}}$ be a normalized eigenfunction corresponding to $\lambda_{\underline{\alpha}}$. We can assume that $\underline{\alpha}$ is such that $\lambda_{\underline{\alpha}} \le (1/16) v s^{-2} \mathrm{e}^{-2B\eta} \delta^2$, and it is enough to show that $\tilde{\lambda}_b \le 32\lambda_{\underline{\alpha}}\big(1 + 8b^{-2} v^{-1} s^2 \mathrm{e}^{2B\eta} \delta^{-2}\big)$ for all such $\underline{\alpha}$. Then taking the infimum over all these $\underline{\alpha}$'s, (7.17) will follow.

By variational principle

$$\lambda_{b,\underline{\alpha}} = \inf_{\psi \in H_0^1(\Omega_+^b)} \frac{\frac{1}{2}\int_{\Omega_+^b} \Big(|(-i\nabla - A_{\underline{\alpha}})\psi|^2 - B_{\underline{\alpha}} |\psi|^2 \Big)}{\int_{\Omega_+^b} |\psi|^2} = \inf_{\psi \in H_0^1(\Omega_+^b)} \frac{\frac{1}{2}\int_{\Omega_+^b} |T_{\underline{\alpha}} \psi|^2}{\int_{\Omega_+^b} |\psi|^2}$$

using integration by parts with $T_{\underline{\alpha}} := -i\partial_1 + \partial_2 - (A_{\underline{\alpha}})_1 - i(A_{\underline{\alpha}})_2$.

Let θ be a cutoff function such that $\theta \equiv 1$ on Ω_+^{2b}, $\theta \equiv 0$ on $\mathbf{R}^2 \setminus \Omega_+^b$, $0 \le \theta \le 1$ and $|\nabla \theta| \le 4b^{-1}$. Since

$$\begin{aligned} \lambda_{\underline{\alpha}} &= \frac{1}{2}\int_S |T_{\underline{\alpha}} \varphi_{\underline{\alpha}}|^2 + \int_S \tilde{V} |\varphi_{\underline{\alpha}}|^2 \ge \frac{1}{2}\int_S \theta^2 |T_{\underline{\alpha}} \varphi_{\underline{\alpha}}|^2 + \int_S \tilde{V} |\varphi_{\underline{\alpha}}|^2 \\ &\ge \frac{1}{4}\int_S |T_{\underline{\alpha}}(\theta\varphi_{\underline{\alpha}})|^2 - \frac{1}{2}\|\nabla\theta\|_\infty^2 \int_{S \cap \mathrm{supp} \nabla\theta} |\varphi_{\underline{\alpha}}|^2 \end{aligned}$$

by a Schwarz inequality and $\tilde{V} \ge 0$; we can use $\psi := \theta\varphi_{\underline{\alpha}}$ as a trial function to obtain

$$\begin{aligned} \lambda_{b,\underline{\alpha}} &\le \frac{\frac{1}{2}\int_{\Omega_+^b} |T_{\underline{\alpha}} \psi|^2}{\int_{\Omega_+^b} |\psi|^2} \le \frac{2\lambda_{\underline{\alpha}} + 16 b^{-2} \int_{S \cap \mathrm{supp}\nabla\theta} |\varphi_{\underline{\alpha}}|^2}{\int_{\Omega_+^b} \theta^2 |\varphi_{\underline{\alpha}}|^2} \\ &\le \frac{2\lambda_{\underline{\alpha}} + 16 b^{-2} \int_{S \setminus \Omega_+^{2b}} |\varphi_{\underline{\alpha}}|^2}{1 - \int_{S \setminus \Omega_+^{2b}} |\varphi_{\underline{\alpha}}|^2} \end{aligned} \tag{7.19}$$

using $\int_S |\varphi_{\underline{\alpha}}|^2 = 1$ and $\theta \equiv 1$ on Ω_+^{2b}. Hence we have to estimate $\int_{S \setminus \Omega_+^{2b}} |\varphi_{\underline{\alpha}}|^2$ from above.

Let $\Pi_{\underline{\alpha}}$ be the spectral projection onto the lowest (zero energy) Landau level of the unperturbed operator $\frac{1}{2}\big[(-i\nabla - A_{\underline{\alpha}})^2 - B_{\underline{\alpha}}\big] = \frac{1}{2} T_{\underline{\alpha}}^* T_{\underline{\alpha}}$ defined on $\mathbf{R}^2$. Notice that this operator is nonnegative. Extend $\varphi_{\underline{\alpha}}$ on the whole plane to be zero outside S. Then $\varphi_{\underline{\alpha}} = \Pi_{\underline{\alpha}} \varphi_{\underline{\alpha}} + (1 - \Pi_{\underline{\alpha}})\varphi_{\underline{\alpha}}$. Since $T_{\underline{\alpha}} \Pi_{\underline{\alpha}} = 0$, we have

$$\lambda_{\underline{\alpha}} = \frac{1}{2}\int_{\mathbf{R}^2} |T_{\underline{\alpha}}\varphi_{\underline{\alpha}}|^2 + \int_S \tilde{V}|\varphi_{\underline{\alpha}}|^2 = \frac{1}{2}\int_{\mathbf{R}^2} |T_{\underline{\alpha}}(I-\Pi_{\underline{\alpha}})\varphi_{\underline{\alpha}}|^2 + \int_S \tilde{V}|\varphi_{\underline{\alpha}}|^2 \geq B\int_{\mathbf{R}^2} |(I-\Pi_{\underline{\alpha}})\varphi_{\underline{\alpha}}|^2 + \int_S \tilde{V}|\varphi_{\underline{\alpha}}|^2$$

by the gap of size at least B in the spectrum of the free magnetic operator $\frac{1}{2}\left[(-i\nabla - A_{\underline{\alpha}})^2 - B_{\underline{\alpha}}\right]$ defined on $\mathbf{R}^2$. Recall from [8] that $(-i\nabla - A_{\underline{\alpha}})^2 - B_{\underline{\alpha}}$ and $(-i\nabla - A_{\underline{\alpha}})^2 + B_{\underline{\alpha}}$ have the same spectrum apart from 0, and $(-i\nabla - A_{\underline{\alpha}})^2 + B_{\underline{\alpha}} \geq 2B_{\underline{\alpha}} \geq 2B$. Hence, using the form of $\tilde{V}$,

$$\int_{\mathbf{R}^2} |(I-\Pi_{\underline{\alpha}})\varphi_{\underline{\alpha}}|^2 \leq \frac{\lambda_{\underline{\alpha}}}{B} \quad \text{and} \quad \int_{\bigcup_{i\in\mathscr{G}}\overline{B}(x_i,a)} |\varphi_{\underline{\alpha}}|^2 \leq \frac{\lambda_{\underline{\alpha}}}{v} \quad . \tag{7.20}$$

Notice that

$$S\backslash\Omega_+^{2b} \subset \left(\Omega\backslash\Omega_+^{2b}\right) \cup \bigcup_{i\in\mathscr{G}} \overline{B}(x_i,a) \quad .$$

Hence

$$\begin{aligned}\int_{S\backslash\Omega_+^{2b}} |\varphi_{\underline{\alpha}}|^2 &\leq \int_{\Omega\backslash\Omega_+^{2b}} |\varphi_{\underline{\alpha}}|^2 + \frac{\lambda_{\underline{\alpha}}}{v} \\ &\leq 2\int_{\Omega\backslash\Omega_+^{2b}} |\Pi_{\underline{\alpha}}\varphi_{\underline{\alpha}}|^2 + 2\int_{\Omega\backslash\Omega_+^{2b}} |(I-\Pi_{\underline{\alpha}})\varphi_{\underline{\alpha}}|^2 + \frac{\lambda_{\underline{\alpha}}}{v} \\ &\leq 2\int_{\Omega\backslash\Omega_+^{2b}} |\Pi_{\underline{\alpha}}\varphi_{\underline{\alpha}}|^2 + \frac{2\lambda_{\underline{\alpha}}}{B} + \frac{\lambda_{\underline{\alpha}}}{v} \quad .\end{aligned} \tag{7.21}$$

Let $g = g_{\Omega_-^{2\delta}}$ for simplicity (recall that $\Delta g = -1$ on $\Omega_-^{2\delta}$ and $g \equiv 0$ on $\partial\Omega_-^{2\delta}$). Since $T_{\underline{\alpha}}\Pi_{\underline{\alpha}}\varphi_{\underline{\alpha}} = 0$, it implies

$$(-\mathrm{i}\partial_1 + \partial_2 - Y_1 - \mathrm{i}Y_2)h = 0 \tag{7.22}$$

pointwise on $\Omega_-^{2\delta}$, with $Y = (Y_1, Y_2) := \big((A_{\underline{\alpha}})_1 - B\partial_2 g, (A_{\underline{\alpha}})_2 + B\partial_1 g\big)$ and $h := \mathrm{e}^{-Bg}\Pi_{\underline{\alpha}}\varphi_{\underline{\alpha}}$ (we omit the dependence on $\underline{\alpha}$). Notice that curl $Y = 0$ on $\Omega_-^{2\delta}$. From (7.22), by a short calculation

$$\Delta|h|^2 = |(\mathrm{i}\partial_1 + \partial_2 + Y_1 - \mathrm{i}Y_2)h|^2 \quad ,$$

in particular $|h|^2$ is subharmonic on $\Omega_-^{2\delta}$ and $|h|$ satisfies the maximum principle. [The real reason behind these properties is that (7.22) is a Cauchy-Riemann equation in local coordinates on a flat analytic $U(1)$ bundle over $\Omega_-^{2\delta}$, determined by the integer part (mod 2π) of the integrals of Y over noncontractible cycles. Hence its solution is an analytic section. In particular (see [2]), if $\Omega_-^{2\delta}$ is simply connected or

$\int_\gamma Y \in 2\pi\mathbf{Z}$ for all cycles γ, then this bundle is trivial and the solution is an analytic function.]

Let z_0 be the point on $\partial\Omega_-^\delta$ where $|h(z)|$ takes on its maximal value. By the maximum principle $|h(z)| \le |h(z_0)|$ for all $z \in \Omega_-^\delta$.

Since $z_0 \in \partial\Omega_-^\delta$, we have $B(z_0,\delta) \subset \cup_{i\in\mathcal{G}}\overline{B}(x_i,a)\bigcup S^c$ (here S^c is the complement of S) and since $\varphi \equiv 0$ on S^c, we have $\int_{B(z_0,\delta)} |\varphi_{\underline{\alpha}}|^2 \le \lambda_{\underline{\alpha}} v^{-1}$ by (7.20). Again by (7.20) and

$|(I-\Pi_{\underline{\alpha}})\varphi_{\underline{\alpha}}|^2 \ge \frac{1}{2}|\Pi_{\underline{\alpha}}\varphi_{\underline{\alpha}}|^2 - |\varphi_{\underline{\alpha}}|^2$ pointwise, we have

$$\begin{aligned}\frac{\lambda_{\underline{\alpha}}}{B} &\ge \int_{\mathbf{R}^2} |(I-\Pi_{\underline{\alpha}})\varphi_{\underline{\alpha}}|^2 \ge \int_{B(z_0,\delta)} |(I-\Pi_{\underline{\alpha}})\varphi_{\underline{\alpha}}|^2 \\ &\ge \frac{1}{2}\int_{B(z_0,\delta)} |\Pi_{\underline{\alpha}}\varphi_{\underline{\alpha}}|^2 - \int_{B(z_0,\delta)} |\varphi_{\underline{\alpha}}|^2 \\ &\ge \frac{1}{2}\int_{B(z_0,\delta)} |h|^2 \mathrm{e}^{2Bg} - \lambda_{\underline{\alpha}} v^{-1} \ge \frac{\pi}{2}|h(z_0)|^2\delta^2 - \lambda_{\underline{\alpha}} v^{-1}\end{aligned}$$

by subharmoniticity of $|h|^2$ and $g \ge 0$. Hence, by $v \le B$

$$|h(z)|^2 \le |h(z_0)|^2 \le 2\lambda_{\underline{\alpha}}\delta^{-2}v^{-1}$$

for all $z \in \Omega_-^\delta$. Hence, using that $|\Pi_{\underline{\alpha}}\varphi_{\underline{\alpha}}| = \mathrm{e}^{Bg}|h|$, $|\Omega| \le s^2$ and $\Omega\backslash\Omega_+^{2b} \subset \Omega_-^\delta$

$$\int_{\Omega\backslash\Omega_+^{2b}} |\Pi_{\underline{\alpha}}\varphi_{\underline{\alpha}}|^2 \le 2\lambda_{\underline{\alpha}} v^{-1}s^2\mathrm{e}^{2B\eta}\delta^{-2}$$

and therefore

$$\int_{\Omega\backslash\Omega_+^{2b}} |\varphi_{\underline{\alpha}}|^2 \le 4\lambda_{\underline{\alpha}} v^{-1}s^2\mathrm{e}^{2B\eta}\delta^{-2} + \frac{2\lambda_{\underline{\alpha}}}{B} + \frac{\lambda_{\underline{\alpha}}}{v} \le 8\lambda_{\underline{\alpha}} v^{-1}s^2\mathrm{e}^{2B\eta}\delta^{-2}$$

using (7.21), $v \le B$, $\eta \ge 0$ and $\delta \le s$. Hence from (7.19)

$$\lambda_{b,\underline{\alpha}} \le 16\lambda_{\underline{\alpha}} \cdot \frac{1+8b^{-2}v^{-1}s^2\mathrm{e}^{2B\eta}\delta^{-2}}{1-8\lambda_{\underline{\alpha}} v^{-1}s^2\mathrm{e}^{2B\eta}\delta^{-2}} \le 32\lambda_{\underline{\alpha}}\Big(1+8b^{-2}v^{-1}s^2\mathrm{e}^{2B\eta}\delta^{-2}\Big)$$

since we assumed that the denominator is bigger than 1/2. □

Lemma 7.5. *Let* $\Theta := \Omega_-^{2\delta}$ *for simplicity and assume that* $\delta \le \min\{\frac{1}{200}, \frac{a}{6}\}$, $a \le 1 \le b$, $40b \le \ell \le s$, $r \le 1/4$, $G_\Theta \ge \ell$.

(i.) There exist two functions, $\ell_0(\varepsilon,b) > 0$ *and* $1/4 \ge k(\varepsilon,b) > 0$ *for all* b *and* $0 < \varepsilon \le \varepsilon_0$, *where* ε_0 *is a universal constant, such that*

$$g_\Theta(x) \le \left(\frac{\ell}{b}\right)^{-k(\varepsilon,b)} G_\Theta \qquad (7.23)$$

for all $\ell \ge \ell_0(\varepsilon,b)$ *and for all* $x \in \Theta\cap\Sigma$, *where*

$$\Sigma := \left(S_{-}^{2\delta} \setminus (-s+2b, s-2b)^2\right) \cup \bigcup_{i\in\mathcal{G}} \overline{B}(x_i, 2b) \ . \tag{7.24}$$

We can assume that $\ell_0(\varepsilon, b)$ *is increasing and* $k(\varepsilon, b)$ *is decreasing in* b.

(ii.) There exists a positive number c_0 *such that if* $x \in \Theta$ *and* $x \notin A^1 \cap S$, *then*

$$g_\Theta(x) \leq \left[(1-c_0)^{1/r} G_\Theta + c_0^{-1} r^2 \ell^2\right] + \sup_{y\in\Sigma\cap\Theta} g_\Theta(y) \ . \tag{7.25}$$

Proof of part (i.). Fix $x \in \Sigma \cap \Theta$. Let T_j be the exit time from $B(x, 2\cdot 10^j b)$ for $j \geq 1$, and $T_0 := 0$. For simplicity, let $T := T_\Theta$ in this proof. Let M be the smallest integer such that $2\cdot 10^{M+1} b > \ell/2$, i.e. $M := \lceil \frac{\log(\ell/4b)}{\log 10} \rceil \geq 1$. Then

$$\begin{aligned}
\mathbf{E}_x T &= \sum_{j=1}^{M} \mathbf{E}_x\left\{T\cdot \mathbf{1}(T_{j-1} \leq T < T_j)\right\} + \mathbf{E}_x\{T\cdot \mathbf{1}(T_M \leq T)\} \\
&= \sum_{j=1}^{M} \mathbf{E}_x\left\{\mathbf{1}(T_{j-1} \leq T)\cdot \left[T_{j-1} + \mathbf{E}_{W_{T_{j-1}}}(T\cdot \mathbf{1}(T<T_j))\right]\right\} \\
&\quad + \mathbf{E}_x\left\{\mathbf{1}(T_M \leq T)\cdot\left[T_M + \mathbf{E}_{W_{T_M}} T\right]\right\} \\
&\leq \sum_{j=1}^{M+1} \mathbf{E}_x\left\{\mathbf{1}(T_{j-1}\leq T)\cdot T_{j-1}\right\} \\
&\quad + \sum_{j=1}^{M} \mathbf{P}_x(T_{j-1}\leq T) \times \sup_{y\in\Theta\cap S(x, 2\cdot 10^{j-1}b)} \mathbf{E}_y(T\wedge T_j) \\
&\quad + \mathbf{P}_x(T_M \leq T) \times \sup_{y\in\Theta\cap S(x,2\cdot 10^M b)} \mathbf{E}_y T
\end{aligned} \tag{7.26}$$

by estimating $T\cdot\mathbf{1}(T\leq T_j) \leq T\wedge T_j$ and using that $T_k < T$ implies $W_{T_k} \in \Theta\cap S(x, 2\cdot 10^k b)$ for $k = 1, 2, \ldots M$ [here $S(z, r)$ denotes the circle of radius r with center z].

Next, we claim that

$$\mathbf{P}_x(T_j \leq T) \leq (1 - m(\varepsilon, b))^j \tag{7.27}$$

for any $0 \leq j \leq M$.

To prove (7.27), let H_F denote the first hitting time of F, for any closed set F, and let

$$c(b) := \inf_{y\in S(0,2b)} \mathbf{P}_y(H_{\overline{B}(0,a-2\delta)} < T_{B(0,6b)}) > 0 \tag{7.28}$$

which is decreasing in b. Since a and δ are considered fixed, they are omitted from the notation. Let

$$m(\varepsilon, b) := c(b) \cdot \inf\left\{ \inf_{|y|=2} \mathbf{P}_y(H_F < T_{B(0,10)}) : F \subset B(0,10), |F| \geq \frac{100\pi\varepsilon}{144} \right\}$$

where the first infimum runs through all closed sets $F \subset B(0,10)$ with relative volume bigger than $\varepsilon/144$. Obviously $m(\varepsilon, b) > 0$ for $\varepsilon > 0$, $m(\varepsilon, b) \to 0$ as $\varepsilon \to 0$ uniformly in b and $m(\varepsilon, b)$ is decreasing in b. These standard statements follow from Lemma 2.1 in [24]. Hence, for small enough $\varepsilon \leq \varepsilon_0$, we can assume that $m(\varepsilon, b) \leq 1/3$. For brevity, let $m = m(\varepsilon, b)$.

First let $x \in \bigcup_{i \in \mathscr{G}} \overline{B}(x_i, 2b)$, i.e. $|x - x_\mu| \leq 2b$ for some $\mu \in \mathscr{G}$ which we fix. Let

$$\Gamma(j) := \bigcup_{\substack{i \in \mathscr{G} \\ |x_i - x_\mu| \leq 10^j b}} \overline{B}(x_i, 2b) \subset B(x, 10^j b + 4b) \subset B(x, 2 \cdot 10^j b)$$

for $j \geq 1$. Then, by the definition of good points,

$$|\Gamma(j)| \geq \frac{\varepsilon}{36} |B(x_\mu, 10^j b)| = \frac{\varepsilon}{144} |B(x, 2 \cdot 10^j b)|$$

if $10^j b \leq \ell/2$. Let W be a Brownian motion, started from x. If $j \geq 1$ and $W_{T_{j-1}} \in \Theta$ (i.e. $T_{j-1} < T$), then

$$\mathbf{P}_{W_{T_{j-1}}} \left[\mathbf{1}(T_j > H_{\Gamma(j)}) \cdot \mathbf{P}_{\widetilde{W}_{H_{\Gamma(j)}}}(T_j > T) \right] \geq m \ , \tag{7.29}$$

using $W_{T_{j-1}} \in S(x, 2 \cdot 10^{j-1} b)$ and that $T < H_{\overline{B}(x_\sigma, a - 2\delta)}$ and $T_{B(x_\sigma, 6b)} \leq T_j$ if $\sigma \in \mathscr{G}$ is such that $|\widetilde{W}_{H_{\Gamma(j)}} - x_\sigma| = 2b$. The first relation is obvious, the second follows from $B(x_\sigma, 6b) \subset B(x_\mu, 10^j b + 10b) \subset B(x, 2 \cdot 10^j b)$. In (7.29) $\widetilde{W}$ denotes another Brownian motion, independent of W. Hence, for $j \geq 1$

$$\begin{aligned} &\mathbf{P}_x(T_j > T | T > T_{j-1}) \\ &\quad \geq \mathbf{P}_x \left\{ \mathbf{P}_{W_{T_{j-1}}} \left[\mathbf{1}(T_j > H_{\Gamma(j)}) \cdot \mathbf{P}_{\widetilde{W}_{H_{\Gamma(j)}}}(T_j > T) \right] \Big| W_{T_{j-1}} \in \Theta \right\} \geq m \ . \end{aligned}$$

By the strong Markov property we obtain (7.27) for any $0 \leq j \leq M$ (recall that for $j \leq M$ we have $10^j b \leq \ell/2$). A similar but easier argument shows that (7.27) is true if $x \in S_-^{2\delta} \setminus (-s + 2b, s - 2b)^2$ as well.

Let $1 \leq N \leq M$ be chosen later. Introduce $A_j := \mathbf{E}_x \left\{ \mathbf{1}(T_j \leq T) \cdot T_j \right\}$, then we can continue the estimate (7.26) as

$$\begin{aligned}\mathbf{E}_x T \le & \sum_{j=1}^{M+1} A_{j-1} + \sum_{j=1}^{N} 2 \cdot 10^{2j} b^2 (1-m)^{j-1} \\ & + 2(M-N) G_\Theta (1-m)^N + 2(1-m)^M G_\Theta \\ \le & \sum_{j=1}^{M+1} A_{j-1} + 8 \cdot 10^{2N} b^2 (1-m)^N + 2MG_\Theta (1-m)^N . \end{aligned} \tag{7.30}$$

Here we estimated

$$\sup_{y \in \Theta \cap S(x, 2\cdot 10^{j-1} b)} \mathbf{E}_y (T \wedge T_j) \le \sup_{y \in B(x, 2\cdot 10^j b)} \mathbf{E}_y T_j = 2 \cdot 10^{2j} b^2$$

for $j \le N$ and

$$\sup_{y \in \Theta \cap S(x, 2\cdot 10^{j-1} b)} \mathbf{E}_y (T \wedge T_j) \le \sup_{y \in \Theta} \mathbf{E}_y T = 2G_\Theta$$

for $j > N$ using (7.9), (7.10) and

$$\sup_{y \in B(0,1)} \mathbf{E}_y T_{B(0,1)} = \frac{1}{2} \ . \tag{7.31}$$

Now we estimate A_j

$$\begin{aligned} A_j &= \mathbf{E}_x \{ \mathbf{1}(T_j \le T) \cdot T_j \} \\ &\le \mathbf{E}_x \Big\{ \mathbf{1}(T_{j-1} \le T) \Big[T_{j-1} \cdot \mathbf{P}_{W_{T_{j-1}}} (T_j \le T) + \mathbf{E}_{W_{T_{j-1}}} T_j \Big] \Big\} \\ &\le (1-m) A_{j-1} + 2 \cdot 10^{2j} b^2 (1-m)^{j-1} \end{aligned}$$

for $1 \le j \le N$ using (7.29), (7.31) and (7.27). Iterating this inequality, and using that $A_1 \le \mathbf{E}_x T_1 \le 200 b^2$ we obtain for $1 \le j \le N$ that

$$A_j \le 2 \cdot 10^{2j} b^2 j (1-m)^{j-1} \ . \tag{7.32}$$

For $N < j \le M$,

$$\begin{aligned} A_j &= \mathbf{E}_x \{ \mathbf{1}(T_j \le T) \cdot T_j \} \\ &\le \mathbf{E}_x \Big\{ \mathbf{1}(T_{j-1} \le T) \Big[T_{j-1} + \mathbf{E}_{W_{T_{j-1}}} (T_j \cdot \mathbf{1}(T_j \le T)) \Big] \Big\} \\ &\le A_{j-1} + 2G_\Theta \mathbf{P}_x (T_{j-1} \le T) \le A_{j-1} + 2G_\Theta (1-m)^N \end{aligned}$$

by (7.27) and $T_N \le T_{j-1}$. By iteration, and using the estimate (7.32) for A_N, we have

$$A_j \le 4 \big(10^{2N} N b^2 + MG_\Theta \big) (1-m)^N \tag{7.33}$$

for $j > N$. Continuing (7.30) we get

$$\mathbf{E}_x T \le 12 M^2 b^2 (1-m)^N \big[20^{2N} + G_\Theta \big] \ .$$

Now choose

$$N := \min\left\{M, \left[\frac{\log G_\Theta}{2\log 20}\right]\right\} \ .$$

Short calculation shows that

$$\mathbf{E}_x T \le 48(\log(\ell/b))^2 b^2 G_\Theta \cdot \max\left\{G_\Theta^{-2k(\varepsilon,b)}, \left(\frac{\ell}{4b}\right)^{-2k(\varepsilon,b)}\right\}$$

with $k(\varepsilon, b) := \frac{1}{4\log 20}\log\frac{1}{1-m(\varepsilon,b)} \le \frac{1}{4}$. Since $G_\Theta \ge \ell$, $b \ge 1$, choosing a large enough function $\ell_0(\varepsilon, b)$ we finish the proof of (7.23) by (7.10).

Proof of part (ii.). Let $x \notin A^1 \cap S$ and $x \in \Theta$. We can assume that $x \notin \Sigma$, in particular $x \in S$. Consider again a Brownian motion W starting from x. Since $S_-^{2\delta}\backslash\Sigma \subset \Theta$, we have $H_\Sigma \le T_\Theta = T$, hence $\mathbf{E}_x T = \mathbf{E}_x\{H_\Sigma + \mathbf{E}_{W_{H_\Sigma}} T\}$. Obviously $\mathbf{E}_{W_{H_\Sigma}} T \le 2\sup_{y\in\Sigma\cap\Theta} g_\Theta(y)$, and it can be estimated by part (i).

To estimate $\mathbf{E}_x H_\Sigma$, let

$$F(n) := \sup_{y\in S:|x-y|\le nr\ell} \mathbf{E}_y H_\Sigma$$

for $n = 1, 2, \dots [1/r] - 1$. Fix $y \in S$ such that $|x - y| \le nr\ell$, then

$$\begin{aligned} \mathbf{E}_y H_\Sigma &= \mathbf{E}_y\{H_\Sigma \cdot \mathbf{1}(H_\Sigma \le T_{B(y,r\ell)})\} \\ &\quad + \mathbf{E}_y\left\{\mathbf{1}(H_\Sigma > T_{B(y,r\ell)}) \cdot \left[T_{B(y,r\ell)} + \mathbf{E}_{W_{T_{B(y,r\ell)}}} H_\Sigma\right]\right\} \\ &\le 2\mathbf{E}_y T_{B(y,r\ell)} + F(n+1) \cdot \mathbf{P}_y(H_\Sigma > T_{B(y,r\ell)}) \quad . \end{aligned} \tag{7.34}$$

To estimate $\mathbf{P}_y(H_\Sigma > T_{B(y,r\ell)})$, notice that the total volume of subboxes containing no good points in $B(x,\ell)$ is at most $3^{-2}\pi r^2\ell^2$. The reason is that $B(x,\ell)$ does not intersect with any clearing box, hence each of the boxes intersecting $B(x,\ell)$ (there are at most nine of them) has at most $9^{-2}\pi r^2\ell^2$ volume of subboxes which contain no good point. Hence, if $B(y, r\ell) \subset B(x,\ell)$, then

$$|\Sigma \cap B(y, r\ell)| \ge \tfrac{1}{4}\pi r^2\ell^2 - 3^{-2}\pi r^2\ell^2 = (2^{-2} - 3^{-2})|B(y, r\ell)|$$

using that if $y \in S \subset S_-^{2\delta}$ then at least one fourth of $B(y, r\ell)$ is in $S_-^{2\delta}$. So let

$$c_0 := \min\left\{\tfrac{1}{2}, \inf_{\substack{F:F\subset B(0,1)\\ |F|\ge(2^{-2}-3^{-2})|B(0,1)|}} \mathbf{P}_0(H_F \le T_{B(0,1)})\right\} > 0$$

be a positive universal constant, then $\mathbf{P}_y(H_\Sigma > T_{B(y,r\ell)}) \le (1 - c_0)$, and by continuing (7.34) we have

$$\mathbf{E}_y H_\Sigma \le r^2\ell^2 + (1-c_0)F(n+1) \ ,$$

by (7.31) if $B(y, r\ell) \subset B(x, \ell)$ which is satisfied as $n \le [1/r] - 1$ and $|x-y| \le nr\ell$.

Taking the supremum for all $y \in S$ such that $|x-y| \le nr\ell$, we have

$$F(n) \le r^2\ell^2 + (1-c_0)F(n+1) \ . \tag{7.35}$$

We also have $F(n_0) \le 2G_\Theta$ for any n_0, since $H_\Sigma \le T_\Theta$. Iterating the inequality (7.35), we get

$$F(0) \le c_0^{-1} r^2\ell^2 + 2(1-c_0)^{n_0} G_\Theta \ .$$

Applying this to $n_0 := [1/r] - 1$ for $r \le 1/4$ we obtain $\mathbf{E}_x T \le 2[c_0^{-1} r^2\ell^2 + (1-c_0)^{1/r} G_\Theta]$ for all $x \notin S \cap A^1$, $x \in \Theta$, which finishes the proof of (7.25). □

Lemma 7.6 *For* $\delta \le \frac{a}{6}$

$$G_\Omega \le G_{\Omega_-^{2\delta}} \le 8G_\Omega + 1000\delta^2 \ .$$

Proof. The first inequality is trivial by $\Omega \subset \Omega_-^{2\delta}$. For the second inequality, let $T_- := T_{\Omega_-^{2\delta}}$ $T_+ := T_\Omega$ for simplicity, and we write

$$\mathbf{E}_x T_- = \mathbf{E}_x \big\{ T_+ + \mathbf{E}_{W_{T_+}} T_- \big\} \le \mathbf{E}_x T_+ + \sup_{y \in \Omega_-^{2\delta} \setminus \Omega} \mathbf{E}_y T_-$$

for any $x \in \Omega$. If $x \in \Omega_-^{2\delta} \setminus \Omega$, then we can drop the first term. Hence we get

$$2G_{\Omega_-^{2\delta}} \le 2G_\Omega + \sup_{y \in \Omega_-^{2\delta} \setminus \Omega} \mathbf{E}_y T_- \ . \tag{7.36}$$

Let us fix $y \in \Omega_-^{2\delta} \setminus \Omega$, let $T := T_{B(y,4\delta)}$ and let the positive number θ to be chosen later. Then

$$\begin{aligned}\mathbf{E}_y T_- = \ &\mathbf{E}_y\{T_- \cdot \mathbf{1}(T > \theta) \cdot \mathbf{1}(T_- > \theta)\} \\ &+ \mathbf{E}_y\{T_- \cdot \mathbf{1}(T \le \theta) \cdot \mathbf{1}(T_- > \theta)\} + \mathbf{E}_y\{T_- \cdot \mathbf{1}(T_- \le \theta)\} \ . \end{aligned} \tag{7.37}$$

For the last term simply use $\mathbf{E}_y\{T_- \cdot \mathbf{1}(T_- \le \theta)\} \le \theta$. For the second term, using that $T \le \theta$ and $\theta < T_-$ implies $W_T \in \Omega_-^{2\delta}$, we obtain

$$\begin{aligned} &\mathbf{E}_y\{T_- \cdot \mathbf{1}(T \le \theta) \cdot \mathbf{1}(T_- > \theta)\} \\ &\quad \le \mathbf{E}_y\big\{\mathbf{1}(T \le \theta) \cdot \mathbf{1}(W_T \in \Omega_-^{2\delta})[T + \mathbf{E}_{W_T} T_-]\big\} \\ &\quad \le \theta + \mathbf{E}_y\big\{\mathbf{1}(W_T \in \Omega_-^{2\delta}) \mathbf{E}_{W_T} T_-\big\} \le \theta + 2G_{\Omega_-^{2\delta}} \cdot \mathbf{P}_y(W_T \in \Omega_-^{2\delta}) \\ &\quad \le \theta + \tfrac{3}{4} \cdot 2G_{\Omega_-^{2\delta}} \end{aligned}$$

since

$$\mathbf{P}_y(W_T \in \Omega_-^{2\delta}) = \frac{|S(y,4\delta) \cap \Omega_-^{2\delta}|}{|S(y,4\delta)|}$$

(the exit measure is the Lebesgue measure on the circle) and this ratio is at most 3/4 by elementary geometry. Here we used that the boundary of $\Omega_-^{2\delta}$ consists of straight line segments and circular arcs of curvature not bigger than $(a-2\delta)^{-1} \leq (4\delta)^{-1}$ by the assumption on a.

Finally for the first term in (7.37) we use

$$\begin{aligned}
&\mathbf{E}_y\{T_- \cdot \mathbf{1}(T>\theta)\cdot \mathbf{1}(T_- > \theta)\} \leq \theta + \mathbf{E}_y\{\mathbf{1}(T>\theta)\mathbf{E}_{W_\theta} T_-\} \\
&\quad \leq \theta + 2G_{\Omega_-^{2\delta}} \cdot \mathbf{P}_y(T>\theta) \leq \theta + 2G_{\Omega_-^{2\delta}}\left((2\pi\theta)^{-1}\int_{\substack{z\in\mathbf{R}^2\\ |z|\leq 4\delta}} \mathrm{e}^{-\frac{z^2}{2\theta}}\,dz\right) \\
&\quad = \theta + 2(1-\mathrm{e}^{-8\delta^2\theta^{-1}})G_{\Omega_-^{2\delta}} \ .
\end{aligned}$$

Putting these three estimates together, we obtain from (7.37)

$$\mathbf{E}_y T_- \leq 3\theta + \left[\tfrac{3}{2} + 2\left(1-\mathrm{e}^{-8\delta^2\theta^{-1}}\right)\right] G_{\Omega_-^{2\delta}} \ .$$

Combining this with (7.36), we have

$$G_{\Omega_-^{2\delta}} \leq G_\Omega + \frac{3\theta}{2} + \left[\frac{3}{4} + \left(1-\mathrm{e}^{-8\delta^2\theta^{-1}}\right)\right] G_{\Omega_-^{2\delta}} \ .$$

Choose θ such that $\mathrm{e}^{-8\delta^2\theta^{-1}} \geq 7/8$, e.g. let $\theta = 80\delta^2$. This gives $G_{\Omega_-^{2\delta}} \leq 8G_\Omega + 1000\delta^2$. □

Finally we bound the Green's function from above in terms of the magnetic eigenvalue.

Lemma 7.7 *Let* $\Omega = S\setminus\bigcup_{i\in\mathcal{G}}\overline{B}(x_i,a)$ *as usual and recall the definition of* $\tilde\lambda$ *from* (7.8). *Assume* $a \leq 1 \leq \ell \leq s$, $v \leq B$, *then there exists a function* $\ell_1(\varepsilon,B) > 0$ *such that*

$$\mathrm{e}^{BG_\Omega} \leq s^2 \mathrm{e}^{B\ell}\tilde\lambda^{-1} \tag{7.38}$$

for all $\ell \geq \ell_1(\varepsilon,B)$.

Proof. We can assume that $G_\Omega \geq \ell$, otherwise (7.38) is trivial, since $\tilde V \leq v \leq B$, hence $\tilde\lambda \leq B + \lambda^{(B)}(B_1) \leq \ell^2 \leq s^2$ is automatic if $\ell \geq (B + \lambda^{(B)}(B_1))^{1/2}$ (B_1 is the unit ball).

We shall construct a trial function. For any $d>0$ let

$$\Omega^{(d)} := [-s+d, s-d]^2 \Big\backslash \bigcup_{i\in\mathcal{G}} B(x_i,d)$$

and let $g := g_\Omega$ for simplicity. Let $\gamma \geq 0$ be a fixed smooth bounded function with support in the unit ball and $\int_{\mathbf{R}^2} \gamma = 1$, $\|\nabla\gamma\|_1 \leq 4$. Let $\theta := \gamma * \mathbf{1}(\Omega^{(2)} + B(0,1))$, where $*$ denotes the convolution. Notice that $0 \leq \theta \leq 1$, $\theta \equiv 1$ on $\Omega^{(2)}$, $\theta \equiv 0$ on Ω^c and $\|\nabla\theta\|_\infty \leq 4$.

Let $\varphi(x) := \theta(x)\mathrm{e}^{Bg(x)}$ for $x \in S$, obviously $\varphi \in H_0^1(\Omega) \subset H_0^1(S)$, hence we can use it as a trial function for $\tilde{\lambda}$. Notice that φ is zero on the support of $\tilde{V}$. Choose a gauge X corresponding to g, i.e. let $(X_1, X_2) := B(\partial_2 g, -\partial_1 g)$, which is well defined on Ω and curl $X = B$. Since φ is zero outside of Ω, we can use Lemma 7.1 to obtain

$$\tilde{\lambda} \leq \frac{\frac{1}{2}\int_\Omega |(-i\partial_1 + \partial_2 - X_1 - iX_2)\varphi|^2}{\int_\Omega |\varphi|^2} \quad .$$

By a Schwarz inequality

$$\int_\Omega |(-i\partial_1 + \partial_2 - X_1 - iX_2)\varphi|^2 \leq 2\int_\Omega \theta^2|(-i\partial_1 + \partial_2 - X_1 - iX_2)\mathrm{e}^{Bg}|^2 + 2\|\nabla\theta\|_\infty^2 \int_{\mathrm{supp}\nabla\theta} e^{2Bg} \quad .$$

We know that $(-i\partial_1 + \partial_2 - X_1 - iX_2)\mathrm{e}^{Bg(x)} \equiv 0$ for $x \in \Omega$. We can apply part (i) of Lemma 7.5 with $b = b_0 := (a+3)/2 \leq 2$, $\delta := 0$ to obtain

$$g(x) = g_\Omega(x) \leq G_\Omega \left(\frac{\ell}{b_0}\right)^{-k(\varepsilon, b_0)} \leq 2G_\Omega \ell^{-k(\varepsilon, b_0)} \tag{7.39}$$

for $\ell \geq \ell_0(\varepsilon, 2) \geq \ell_0(\varepsilon, b_0)$ and for all $x \in \Omega \backslash \Omega^{(3+a)}$, in particular for all $x \in \mathrm{supp}\nabla\theta$, since $\mathrm{supp}\nabla\theta \subset \Omega\backslash\Omega^{(2)} \subset \Omega\backslash\Omega^{(3+a)}$. Hence

$$\tilde{\lambda} \leq \frac{16s^2 \exp\left(4BG_\Omega \ell^{-k(\varepsilon, b_0)}\right)}{\int_\Omega |\varphi|^2} \tag{7.40}$$

(the extra s^2 comes from volume $\left(\mathrm{supp}\nabla\theta\right) \leq s^2$).

Let $x_0 \in \Omega$ be the point where $g(x)$ takes on its maximum. By (7.39) we know that $x_0 \in \Omega^{(3+a)}$ (in particular $\Omega^{(3+a)}$ is not empty) if $\ell \geq \frac{\log 2}{k(\varepsilon,2)} \geq \frac{\log 2}{k(\varepsilon,b_0)}$.

Moreover, $B(x_0, \pi^{-1/2}) \subset \Omega^{(2)} \subset \Omega$, hence $\theta \equiv 1$ on $B(x_0, \pi^{-1/2})$. Therefore, by Jensen's inequality

$$\int_\Omega |\varphi|^2 \geq \int_{B(x_0, \pi^{-1/2})} \mathrm{e}^{2Bg} \geq \exp\left(2B\int_{B(x_0,\pi^{-1/2})} g\right) \quad . \tag{7.41}$$

Since $\Delta g = -1$, we have

$$\Delta\Big(g + \tfrac{1}{4}|x - x_0|^2\Big) = 0$$

on $B(x_0, \pi^{-1/2})$, hence

$$\int_{B(x_0,\pi^{-1/2})} \Big(g(x) + \tfrac{1}{4}|x-x_0|^2 \Big)\, dx = g(x_0) = G_\Omega \ .$$

Therefore

$$\int_{B(x_0,\pi^{-1/2})} g \geq G_\Omega - \frac{1}{8\pi} \ ,$$

which finally gives

$$\int_\Omega |\varphi|^2 \geq \mathrm{e}^{-B/(4\pi)} \mathrm{e}^{2BG_\Omega}$$

from (7.41). Hence, from (7.40) and $G_\Omega \geq \ell \geq 1$

$$\tilde{\lambda} \leq 16\mathrm{e}^{B/(4\pi)} s^2 \exp\Big[-2BG_\Omega \Big(1 - 2\ell^{-k(\varepsilon,b_0)} \Big) \Big] \leq s^2 \mathrm{e}^{-BG_\Omega}$$

for large enough $\ell \geq \ell_2(\varepsilon, B)$ (recall that $b_0 \leq 2$). Choosing

$$\ell_1(\varepsilon, B) := \max\Big\{ \Big(B + \lambda^{(B)}(B_1) \Big)^{1/2}, \ell_0(\varepsilon, 2), \frac{\log 2}{k(\varepsilon, 2)} + 1, \ell_2(\varepsilon, B) \Big\}$$

we finish the proof of Lemma 7.7. □

Now we are ready to prove Theorem 7.2.

Proof of Theorem 7.2. First notice that $K \leq 1$ and (7.14) imply $\tilde{\lambda} \leq 1$. Then, we need an estimate on η which is defined in (7.18). We can assume that $G_{\Omega_-^{2\delta}} \geq 1$, otherwise $\eta \leq 1$ and Lemma 7.4 immediately implies Theorem 7.2 for large enough ℓ. Hence

$$G_\Omega \leq G_{\Omega_-^{2\delta}} \leq 9 G_\Omega \tag{7.42}$$

by Lemma 7.6 and $\delta \leq \frac{1}{200}$. Furthermore, we can assume that $G_\Omega \geq \ell$, otherwise $\eta \leq 9\ell$ using (7.42), and Theorem 7.2 follows directly from Lemma 7.4 thanks to the factor $e^{20B\ell}$ in the definition of K. In particular $G_{\Omega_-^{2\delta}} \geq \ell$.

Therefore using Lemma 7.5, the fact that $\Omega_-^{2\delta} \setminus \Omega_+^{2b} \subset \Sigma \cup (A^1)^c$ and (7.42) we obtain that

$$\begin{aligned} \eta &\leq \left(\frac{\ell}{b} \right)^{-k(\varepsilon,b)} G_{\Omega_-^{2\delta}} + \Big[(1-c_0)^{1/r} G_{\Omega_-^{2\delta}} + c_0^{-1} r^2 \ell^2 \Big] \\ &\leq 10 \Big[(1-c_0)^{1/r} G_\Omega + c_0^{-1} r^2 \ell^2 \Big] \end{aligned} \tag{7.43}$$

if $\ell \geq \ell_3(\varepsilon, b, r)$. Combining this estimate with (7.38), we obtain

$$\mathrm{e}^{2B\eta} \leq \Big(s^2 \mathrm{e}^{B\ell} \tilde{\lambda}^{-1} \Big)^{20(1-c_0)^{1/r}} \mathrm{e}^{20Bc_0^{-1} r^2 \ell^2} \tag{7.44}$$

if, in addition, $\ell \geq \ell_1(\varepsilon, B)$.

Now we can estimate $\tilde{\lambda}_b$. By Lemma 7.4, (7.44) and the definitions of K and $w(r)$ given in (7.12) and (7.13) we immediately obtain (7.11). The condition on $\tilde{\lambda}$ required in Lemma 7.4 is implied by (7.14) if ℓ is large enough. The function $\tilde{\ell}$ is obtained by taking the maximum of all previous lower bounds on ℓ. □

8 Proof of the upper bound

To complete the upper bound, we shall continue the estimate given in Theorem 6.3, hence we choose $0 < \beta < B_0$, $s = n_0\ell(t)$, with $n_0 := [(B/\beta)^{1/2}] + 1$ and $\ell(t) = 10\sqrt{\frac{\log t}{B_0}}$. We apply Theorem 7.2 to estimate $\lambda_{S,\omega}^{(B+2\beta)}$ from below. The magnetic field is $B + 2\beta$. Let $0 < \varepsilon < \varepsilon_0$, $0 < r \leq 1/4$, $r \leq \log\left(\frac{1}{1-c_0}\right)/(\log 30)$, $b > 1$ and $\delta := \min\{\frac{a}{6}, \frac{1}{200}\}$. With these data, the procedure of Section 7 gives $\tilde{\lambda}_b = \tilde{\lambda}^{(B+2\beta)}(\Omega_+^b)$ and a number K given by (7.12) (replace B by $B + 2\beta$ everywhere) such that (7.11) holds for $\tilde{\lambda} = \tilde{\lambda}_\omega^{(B+2\beta)}(S)$ under the condition (7.14) and that $\ell(t) \geq \tilde{\ell}(\varepsilon_0, b, B + 2\beta, r)$. This latter condition is satisfied if $t \geq t_0(\varepsilon_0, b, B + 2\beta, r)$ with some function $t_0(\varepsilon_0, b, B + 2\beta, r)$. We also have $\tilde{\lambda} \leq \lambda_{S,\omega}^{(B+2\beta)}$. We obtain

$$\limsup_{t\to\infty}(\log t)^{-1}\log\mathscr{E}\mathrm{e}^{-t\lambda_{S,\omega}^{(B+2\beta)}} \leq \limsup_{t\to\infty}(\log t)^{-1}\log\left[\mathscr{E}\exp\left(-tK\tilde{\lambda}_b^{w(r)}\right) + \exp\left(-tK\cdot 2^{-w(r)}\right)\right] \tag{8.1}$$

recalling the condition (7.14) and distinguishing the cases $(\tilde{\lambda}/K)$ is smaller or bigger than $2^{-w(r)}$. Since

$$\limsup_{t\to\infty}(\log t)^{-1}\log\exp\left(-tK\cdot 2^{-w(r)}\right) = -\infty \tag{8.2}$$

we get from Theorem 6.3 and (8.1)

$$\limsup_{t\to\infty}\frac{\log L(t)}{\log t} \leq \limsup_{t\to\infty}(\log t)^{-1}\log\mathscr{E}\exp\left(-tK\tilde{\lambda}_b^{w(r)}\right) . \tag{8.3}$$

Recall (7.6), i.e. $\tilde{\lambda}_b = \tilde{\lambda}^{(B+2\beta)}(\Omega_+^b) \geq \hat{\lambda}^{(B+2\beta)}(\Omega_+^b)$. Using (4.2), $\Omega_+^b \subset \tilde{U}$ and that $\lambda^{(B+2\beta)}(\cdot)$ is monotone function of the domain, we can further estimate $\tilde{\lambda}_b \geq \hat{\lambda}^{(B+2\beta)}(\Omega_+^b) \geq \lambda^{(B+2\beta)}(B(0, \pi^{-1/2}|\Omega_+^b|^{1/2})) \geq \lambda^{(B+2\beta)}(B(0, \pi^{-1/2}|\tilde{U}|^{1/2}))$.

Following Sznitman's construction let

$$D := S \cap \bigcup_{\overline{C}_m \cap A^0 \neq \emptyset} \overline{C}_m \supset S \cap A^1 .$$

Let U (respectively $\tilde{U}$) be the complement in D of the union over $m \in \mathbf{Z}^2$ of closed subboxes intersecting $\mathrm{int}(C_m)$ and containing a point (respectively a good point) from ω. By the definition of good points and the construction of A^0 and D we have (see Eq. (2.17) in [25])

$$U \subset \tilde{U} \subset D, \quad |\tilde{U}| \leq |U| + \varepsilon |D| \ . \tag{8.4}$$

Let $N = |D|/\ell^2$ which is an integer. For fixed N, the number of possibilities for D is at most $(4n_0^2)^N$, since S consists of $4n_0^2$ boxes.

For fixed D the number of possibilities for U and $\tilde{U}$ is at most $2^{2N\cdot([\sqrt{2}\ell/b]+1)^2} \leq 2^{8N(\ell/b)^2}$. Finally $|U| \geq |\tilde{U}| - \varepsilon N \ell^2$, and we know, that once $D, U, \tilde{U}$ is chosen, the total volume which receives no point is at least $\max\{|U|, Nr^2\ell^2\pi/729\} \geq \max\{|\tilde{U}|, Nr^2\ell^2\pi/729\} - \varepsilon N\ell^2$. Here we used the definition of good points and that $|A^0| \geq \frac{1}{9}|D|$ by construction.

Hence

$$\begin{aligned}\mathscr{E} \exp\left(-tK\tilde{\lambda}_b^{w(r)}\right) \leq \sum_{N=0}^{4n_0^2} & (4n_0^2)^N 2^{8N(\ell/b)^2} \mathrm{e}^{v\varepsilon N\ell^2} \\ & \times \exp\left\{-tK\left[\lambda^{(B+2\beta)}\left(B(0, \pi^{-1/2}|\tilde{U}|^{1/2})\right)\right]^{w(r)}\right\} \\ & \times \mathrm{e}^{-v\cdot\max\{|\tilde{U}|, Nr^2\ell^2\pi/729\}} \ .\end{aligned} \tag{8.5}$$

This sum is majorated by

$$\sum_{N=0}^{4n_0^2} (4n_0^2)^N \mathrm{e}^{-N\ell^2 E} \leq \left(1 - 4n_0^2 \mathrm{e}^{-\ell^2 E}\right)^{-1}$$

if

$$E = E(\varepsilon, b, r, v) := \frac{\pi r^2 v}{729} - v\varepsilon - 8(\log 2) b^{-2} > 0$$

and $t \geq t_1(\varepsilon, B + 2\beta, b, r, v, n_0)$.

Fix $\kappa > 0$. We minimize $tK\left[\lambda^{(B+2\beta)}(B(0, \pi^{-1/2}|\tilde{U}|^{1/2}))\right]^{w(r)} + v|\tilde{U}|$ for the number $|\tilde{U}|$ using the lower bound on the lowest eigenvalue of the disk (4.3). The result is

$$\begin{aligned} & tK\left[\lambda^{(B+2\beta)}\left(B\left(0, \pi^{-1/2}|\tilde{U}|^{1/2}\right)\right)\right]^{w(r)} + v|\tilde{U}| \\ & \geq \min_X\left\{tK\mathrm{e}^{-(B_0+\beta)(1+\kappa)\pi^{-1}w(r)X} + vX\right\} \geq F\log(tK/F)\end{aligned}$$

if $t \geq t_2(\kappa, B + 2\beta, v)$, where

$$F = F(\kappa, B + 2\beta, r, v) := \frac{2\pi v}{(B + 2\beta)(1 + \kappa)w(r)} \ .$$

For any $r > 0$ there exists a small enough ε and large enough b such that $E \geq 10^{-3} r^2 v$. In this case (recalling that $n_0 = [(B/\beta)^{1/2}] + 1$)

$$\limsup_{t\to\infty} (\log t)^{-1} \log \mathscr{E} \exp\left(-tK\tilde{\lambda}_b^{w(r)}\right)$$
$$\leq \limsup_{t\to\infty} (\log t)^{-1} \log \frac{\exp(-F\log(tK/F))}{1 - 4([(B/\beta)^{1/2}] + 1)^2 \exp(-10^{-3} r^2 \ell^2 v)} \ . \quad (8.6)$$

Using (7.16), we have

$$\limsup_{r\to 0} \limsup_{\substack{b\to\infty \\ \varepsilon\to 0}} \limsup_{t\to\infty} \frac{-F\log(K/F)}{\log t} = 0 \ ,$$

hence, recalling (7.15), we obtain from (8.3) and (8.6)

$$\limsup_{r\to 0} \limsup_{\substack{b\to\infty \\ \varepsilon\to 0}} \limsup_{t\to\infty} \frac{\log L(t)}{\log t} \leq -\frac{2\pi v}{(B+2\beta)(1+\kappa)} \ . \quad (8.7)$$

Finally, letting $\beta \to 0$ (which is the same as $n_0 \to \infty$) and $\kappa \to 0$, we obtain

$$\limsup_{\beta\to 0} \limsup_{\kappa\to 0} \limsup_{r\to 0} \limsup_{\substack{b\to\infty \\ \varepsilon\to 0}} \limsup_{t\to\infty} \frac{\log L(t)}{\log t} \leq -\frac{2\pi v}{B} \ , \quad (8.8)$$

which completes the proof of the upper bound in (2.7).

9 Discontinuity of the IDS for zero range potentials

In this section we prove Theorem 1.2. Recall the definition of $q^{(0)}_{Q,g,\omega}(f,f)$ from (1.9) and that $\mathscr{P} = \mathscr{P}_v$ is the Poisson process with density v. Let $B_R := B(0,R)$ be the disk of radius R. It is easy to show by a standard limiting argument that Theorem 1.2 follows from

Proposition 9.1. *Assume $v < \frac{B}{2\pi}$. For every $E > 0$ and $0 < \varepsilon < 1 - \frac{2\pi v}{B}$*

$$\lim_{R\to\infty} \mathscr{P}_v\Big(N(B_R, g, E, \omega) \geq N(R,\varepsilon)\Big) = 1 \ , \quad (9.1)$$

where $N(B_R, g, E, \omega)$ is the dimension of the maximal subspace of $L^2(B_R)$ on which $q^{(0)}_{B_R,g,\omega}(f,f) \leq E\|f\|^2$ and

$$N(R,\varepsilon) = N := |B_R| \cdot \frac{B}{2\pi}\left(1 - \frac{2\pi v}{B} - \varepsilon\right) \ .$$

The idea is to present N functions supported on B_R, all having zeros at the points of the Poisson cloud within a disk of radius R, and with total energy less than E.

Temporarily we use complex notation and identify $\mathbf{C}$ with $\mathbf{R}^2$ as $z = x_1 + ix_2 \equiv (x_1, x_2)$. Let the random function $F(z)$ be defined as

$$F(z) = F_{R,\omega}(z) := \prod_{j:|u_j(\omega)|\le R} \left(1 - \frac{z}{u_j}\right) \ .$$

In other words $F(z) := \mathrm{e}^{X(z)}$, with

$$X(z) = X_{R,\omega}(z) := \int_{|u|\le R} \log\left(1 - \frac{z}{u}\right) P_\omega(du)$$

where $\{u_j(\omega)\}_{j=1,2,\dots}$ is the realization of the Poisson process and

$$P(du) = P_\omega(du) = \sum_j \delta(u - u_j(\omega))du$$

is the corresponding random Poisson measure. In most cases we shall omit ω from the notation. Let $Y(z) := 2\mathrm{Re}\, X(z)$ and $\chi(z) = \chi_{B_R}(z)$. Finally let

$$G_\omega(z) := F_\omega(z)\mathrm{e}^{-\frac{B|z|^2}{4}} \text{ and } f(z) = f_\omega(z) := \chi(z)G_\omega(z) \ .$$

The basic result is the following

Lemma 9.2 *For any* $g > 0$, $E > 0$ *and* $v < \frac{B}{2\pi}$

$$\lim_{R\to\infty} \mathscr{P}_v\left[\frac{q^{(0)}_{B_R,g,\omega}(f_\omega, f_\omega)}{\|f_\omega\|_2^2} \le E\right] = 1 \ . \qquad (9.2)$$

Postponing the proof of Lemma 9.2 we first show that it implies Proposition 9.1. For simplicity we let $q_{R,\omega}(\cdot,\cdot) := q^{(0)}_{B_R,g,\omega}(\cdot,\cdot)$.

Proof of Proposition 9.1. Consider two independent Poisson processes, $\mathscr{P}^{(1)}$ and $\mathscr{P}^{(2)}$, on a common probability space with densities $v_1 := v$ and $v_2 := \frac{B}{2\pi}\left(1 - \frac{2\pi v}{B} - \frac{\varepsilon}{2}\right)$. Let $\mathscr{S}_1$ and $\mathscr{S}_2$ be the corresponding σ-algebras. The sum of these two processes (as the sum $P^{(1)}(\mathrm{d}u) + P^{(2)}(\mathrm{d}u)$ of random point measures on $\mathbf{R}^2$) is again a Poisson process $\mathscr{P}^+ := \mathscr{P}^{(1)} \otimes \mathscr{P}^{(2)}$ with density $v_+ := v_1 + v_2 = \frac{B}{2\pi}(1 - \varepsilon/2)$. We can apply Lemma 9.2 to this process to obtain for any $E > 0$ that

$$\lim_{R\to\infty} \mathscr{P}^+\left[\frac{q^+_{R,\omega}(f_\omega, f_\omega)}{\|f_\omega\|_2^2} \le E\right] = 1 \ , \qquad (9.3)$$

where $q^+_{R,\omega}$ is the quadratic form with obstacle configuration ω from the process $\mathscr{P}^+$.

Let $q^{(1)}_{R,\omega}$ be the quadratic form with obstacles only from the $\mathscr{P}^{(1)}$ process, then the distribution of $q^{(1)}_{R,\omega}$ is the same as that of $q_{R,\omega}$ in Proposition 9.1, and certainly $q^{(1)}_{R,\omega} \le q^{+}_{R,\omega}$.

Taking conditional probability, we have

$$\lim_{R\to\infty} \mathscr{P}^{(2)}\left\{ \mathscr{P}^{(1)}\left[\frac{q^{(1)}_{R,\omega}(f_\omega, f_\omega)}{\|f_\omega\|_2^2} \le E \middle| \mathscr{S}_2 \right] \right\} = 1 \ . \tag{9.4}$$

For any $\eta > 0$ let $\mathscr{Q} = \mathscr{Q}(R, \eta)$ be the following event

$$\mathscr{Q} := \left\{ \omega : \mathscr{P}^{(1)}\left[\frac{q^{(1)}_{R,\omega}(f_\omega, f_\omega)}{\|f_\omega\|_2^2} \le E \middle| \mathscr{S}_2 \right] \ge 1 - \eta \right\} \ ,$$

which is $\mathscr{S}_2$-measurable, and clearly

$$\lim_{R\to\infty} \mathscr{P}^{(2)}(\mathscr{Q}(R,\eta)) = 1 \ . \tag{9.5}$$

The function $f = f_\omega$ factorizes as $f(z) = \tilde{f}(z)h(z)$ where

$$\tilde{f}(z) = \tilde{f}_\omega(z) := \chi(z)e^{-\frac{B|z|^2}{4}} \exp\left[\int_{|u|\le R} \log\left(1 - \frac{z}{u}\right) P^{(1)}_\omega(du) \right]$$

and

$$h(z) = h_\omega(z) = \exp\left[\int_{|u|\le R} \log\left(1 - \frac{z}{u}\right) P^{(2)}_\omega(du) \right] = \prod_{\substack{u:|u|\le R \\ P^{(2)}_\omega(\{u\})=1}} \left(1 - \frac{z}{u}\right) \ .$$

Notice that $\tilde{f}$ is $\mathscr{S}_1$-measurable and h is $\mathscr{S}_2$-measurable. Consider the following $\mathscr{S}_2$-measurable event

$$\mathscr{R} = \mathscr{R}(R, \varepsilon) := \left\{ \omega : \left| P^{(2)}_\omega(B_R) - v_2|B_R| \right| < \frac{\varepsilon |B_R| B}{4\pi} \right\} \ ,$$

then clearly

$$\lim_{R\to\infty} \mathscr{P}^{(2)}(\mathscr{R}(R,\varepsilon)) = 1 \tag{9.6}$$

for any fixed $\varepsilon > 0$. On the event $\mathscr{R}$, the function $h(z)$ is a polynomial of degree at least $v_2|B_R| - \frac{\varepsilon|B_R|B}{4\pi} = N(R,\varepsilon)$.

Finally consider the event $\mathscr{Q}(R,\eta) \cap \mathscr{R}(R,\varepsilon)$ which is $\mathscr{S}_2$-measurable and its probability tends to 1 as $R \to \infty$ by (9.5) and (9.6). Pick independently $N(R,\varepsilon)$ functions $h = h_\omega$ with $\omega \in \mathscr{Q}(R,\eta) \cap \mathscr{R}(R,\varepsilon)$ using the distribution induced by $\mathscr{P}^{(2)}$. We obtain $N(R,\varepsilon)$ polynomials of degree at least $N(R,\varepsilon)$. Let $h_1, h_2, \ldots, h_N$ denote them. We claim that almost surely these are linearly independent.

For the proof, we can focus on fixed degree, since polynomials with different degree are always linearly independent. Hence consider the conditional measures

$$\mathscr{P}_n^{(2)}[\cdot] = \mathscr{P}^{(2)}\Big[\cdot \Big| P_\omega^{(2)}(B_R) = n\Big] \tag{9.7}$$

with $n \geq N$ (this is in fact the n-fold product of uniform measures on B_R). Let $h(z) = h_\omega(z) = 1 + a_1(\omega)z + \ldots a_n(\omega)z^n$ i.e, let $a_j = a_j(\omega)$ be the (random) coefficients, and notice that the (random) roots of $h(z)$ are the Poisson points $\{u_j(\omega)\}_{j=1,2,\ldots n}$ taken from the process $\mathscr{P}_n^{(2)}$. Since there is a continuous one-to-one mapping between the sets of the n roots and the n-tuples of coefficients by the fundamental theorem of algebra, and since the joint distribution of the Poisson points under the conditioning (9.7) is the Lebesgue measure, we obtain that the distribution of n-tuples of coefficients of the polynomials $h(z)$, induced by $\mathscr{P}_n^{(2)}$, is absolutely continuous. But then N random n-tuples ($n \geq N$) are almost surely linearly independent, so are the corresponding polynomials.

Summarizing, we have shown that for any $\varepsilon, \eta > 0$, with a $\mathscr{P}^{(2)}$-probability tending to 1 as $R \to \infty$ and with $\mathscr{P}^{(1)}$-probability bigger than $1 - \eta$, there exist at least $N = N(R, \varepsilon)$ functions, $\tilde{f}h_1, \tilde{f}h_2, \ldots \tilde{f}h_N$ with energy below E. Letting first $R \to \infty$, then $\eta \to 0$ we obtain Proposition 9.1. □

Finally we have to prove Lemma 9.2. As a preparatory step we prove some results about the random variable $Y(z)$.

Lemma 9.3 *Suppose that $R \leq |z|$ and R is large enough. Then*

(i)

$$\mathscr{E}Y(z) \leq \pi v|z|^2 \ , \tag{9.8}$$

$$\mathscr{E}\Big[Y(z) - \mathscr{E}Y(z)\Big]^2 \leq cv|z|^2 \ . \tag{9.9}$$

(ii) *If, in addition,* $|w| \geq R + 1$, *then*

$$\mathscr{E}\Big[e^{Y(z)-Y(w)}\Big] \leq \exp\Big(cv\Big\{|z-w|^2 \log R + |z-w|R\Big\}\Big) \tag{9.10}$$

Proof of Lemma 9.3 (i)

$$\mathscr{E}Y(z) = 2v\int_{|u|\leq R} \log\Big|1 - \frac{z}{u}\Big|\, du = 2v|z|^2 \int_{|y|\geq |z|/R} \frac{\log|1-y|}{|y|^4}\, dy$$

(here $\int du$ and $\int dy$ refer to the two dimensional Lebesgue integration on $\mathbf{C} \equiv \mathbf{R}^2$). By Newton's theorem $\int_{|y|=r} \log|1-y| = 2\pi r \log r$ if $r \geq 1$, hence

$$\mathscr{E}Y(z) = 4\pi v|z|^2 \int_{|z|/R}^{\infty} \frac{\log r}{r^3}\, dr \le 4\pi v|z|^2 \int_1^{\infty} \frac{\log r}{r^3}\, dr = \pi v|z|^2 \ ,$$

using $|z| \ge R$. Similarly

$$\begin{aligned}\mathscr{E}\Big[Y(z) - \mathscr{E}Y(z)\Big]^2 &= 4v \int_{|u|\le R} \log^2 \left|1 - \frac{z}{u}\right| \mathrm{d}u \\ &= 4v|z|^2 \int_{|y|\ge|z|/R} \frac{\log^2 |1-y|}{|y|^4} \mathrm{d}y \le cv|z|^2 \ .\end{aligned}$$

(ii) Write $Y(z) - Y(w) = \Omega_1 + \Omega_2$ with

$$\Omega_1 := 2 \int_{\substack{|u-z|\ge 2|z-w| \\ |u|\le R}} \left(\log\left|1 - \frac{z}{u}\right| - \log\left|1 - \frac{w}{u}\right|\right) P_\omega(du) \ ,$$

$$\Omega_2 := 2 \int_{\substack{|u-z|< 2|z-w| \\ |u|\le R}} \left(\log\left|1 - \frac{z}{u}\right| - \log\left|1 - \frac{w}{u}\right|\right) P_\omega(du) \ .$$

Notice that Ω_1 and Ω_2 are independent, hence

$$\mathscr{E}\left[\mathrm{e}^{Y(z)-Y(w)}\right] = \left[\mathscr{E}\mathrm{e}^{\Omega_1}\right]\left[\mathscr{E}\mathrm{e}^{\Omega_2}\right] \ .$$

For Ω_1, we have that

$$\left| \log\left|1 - \frac{z}{u}\right| - \log\left|1 - \frac{w}{u}\right|\right| \le \frac{c|z-w|}{|z-u|}$$

since $\left|\nabla_\zeta \log|1 - \frac{\zeta}{u}|\right| \le |\zeta - u|^{-1}$ and $|\zeta - u|$ is comparable to $|z-u|$ for any ζ on the segment $[z, w]$. Therefore (recall that c denotes universal constants, whose values can change from line to line)

$$\begin{aligned}\mathscr{E}\mathrm{e}^{\Omega_1} &\le \mathscr{E}\exp\left(\int_{\substack{|u-z|\ge 2|z-w| \\ |u|\le R}} \frac{c|z-w|}{|z-u|} P_\omega(du)\right) \\ &= \exp\left(v \int_{\substack{|u-z|\ge 2|z-w| \\ |u|\le R}} \left[\mathrm{e}^{\frac{c|z-w|}{|z-u|}} - 1\right] \mathrm{d}u \right) \\ &\le \exp\left(v \int_{\substack{|u-z|\ge 2|z-w| \\ |u|\le R}} \frac{c|z-w|}{|z-u|}\, du \right) \le \mathrm{e}^{cv|z-w|R} \ .\end{aligned}$$

For Ω_2, we have

$$\begin{aligned}\mathscr{E}\mathrm{e}^{\Omega_2} &= \exp\left(v \int_{\substack{|u-z|< 2|z-w| \\ |u|\le R}} \left[\mathrm{e}^{2\log\left|1-\frac{z}{u}\right| - 2\log\left|1-\frac{w}{u}\right|} - 1\right] du \right) \\ &= \exp\left(v \int_{\substack{|u-z|< 2|z-w| \\ |u|\le R}} \left[\left|1 + \frac{z-w}{w-u}\right|^2 - 1\right] du \right) \le e^{cv|z-w|^2 \log R} \ ,\end{aligned}$$

where we used that $|w-u| \geq 1$. This completes the proof of Lemma 9.3. □

Proof of Lemma 9.2. Let $\mathscr{A}(R)$ be the following event

$$\mathscr{A}(R) := \left\{\omega : P_\omega\left(B(0,R^{-1})\right) = 0, |P_\omega(B_R) - \nu|B_R|| < \nu|B_R|\right\} \ .$$

Clearly

$$\lim_{R\to\infty} \mathscr{P}\big(\mathscr{A}(R)\big) = 1, \tag{9.11}$$

hence we can assume that there is no Poisson point in the R^{-1} neighborhood of the origin and the total number of points in B_R is less than twice its expectation.

To estimate the norm from below, we notice that $f_\omega(0) = 1$ and for all $|z| \leq (2R)^{-1}$

$$|\nabla|f_\omega(z)|| \leq |\nabla f_\omega(z)| \leq |f_\omega(z)| \left(\int_{R^{-1}\leq|u|\leq R} \frac{1}{|z-u|} P_\omega(du) + \frac{B|z|}{2} \right)$$
$$\leq c\nu R^3 |f_\omega(z)|$$

with some universal constant c if R is large enough. Here we used the condition that the total number of obstacles is bounded. Therefore $|\nabla \log |f_\omega(z)|| \leq c\nu R^3$, hence $|f_\omega(z)| \geq \exp(-c\nu R^3|z|)$ for $|z| \leq (2R)^{-1}$, using $f_\omega(0) = 1$. Hence $|f_\omega(z)| \geq \frac{1}{2}$ for all $|z| \leq (c\nu R^3)^{-1}$ and for large enough R, and this means for $\omega \in \mathscr{A}(R)$ that

$$\|f_\omega\|_2^2 \geq c\nu^{-2}R^{-6} \ . \tag{9.12}$$

To estimate $q_{R,\omega}(f_\omega, f_\omega)$, we notice that it would be exactly zero if there were no cutoff $\chi = \chi_{B_R}$ since $\Pi_0 G_\omega = 0$. Hence, for $\omega \in \mathscr{A}(R)$

$$\begin{aligned} q_{R,\omega}(f_\omega, f_\omega) &= g \int_{|u|\leq R} |(\Pi_0(1-\chi)G_\omega)(u)|^2 P_\omega(du) \\ &\leq \frac{Bg}{2\pi} \left\|(1-\chi)G_\omega\right\|_2^2 P_\omega(B_R) \\ &\leq Bg\nu R^2 \int_{|z|\geq R} |F_\omega(z)|^2 \mathrm{e}^{-\frac{B|z|^2}{2}}\, dz \ , \end{aligned}$$

where we used that $\|\Pi_0\|^2_{L^2\to L^\infty} = \frac{B}{2\pi}$ (see (2.11) in [5]).

For each $k = 1, 2, \ldots$, fix $M_k := \left[2\pi\sqrt{R+k}\right] + 1$ equidistant points $\{z_{k,i}\}_{i=1,\ldots M_k}$ on the circle $|z| = R + k$. For any point $|z| \geq R$ there is a pair of indices (k, i), $i \leq M_k$ such that $|z - z_{k,i}| \leq |z_{k,i}|^{1/2}$. Let $D_{k,i} := B\left(z_{k,i}, |z_{k,i}|^{1/2}\right)$, then $B_R^c \subset \cup_{k,i} D_{k,i}$. Therefore

$$q_{R,\omega}(f_\omega, f_\omega) \leq Bg\nu R^2 \sum_{k=1}^\infty \sum_{i=1}^{M_k} \mathrm{e}^{Y(z_{k,i})} \int_{z\in D_{k,i}} \mathrm{e}^{Y(z)-Y(z_{k,i})} \mathrm{e}^{-\frac{B|z|^2}{2}}\, dz \ . \tag{9.13}$$

Using Lemma 9.3 and Chebyshev's inequality

$$\mathscr{P}\Big(Y(z_{k,i}) \geq \pi v|z_{k,i}|^2 + \kappa|z_{k,i}|^2\Big) \leq \frac{cv}{\kappa^2|z_{k,i}|^2}$$

for any $\kappa > 0$. Since $\sum_{k=1}^{\infty}\sum_{i=1}^{M_k}|z_{k,i}|^{-2} \to 0$ as $R \to \infty$, we have

$$\lim_{R\to\infty} \mathscr{P}\Big(\mathscr{B}(R,\kappa)\Big) = 1 \ , \tag{9.14}$$

where $\mathscr{B}(R,\kappa)$ is the following event:

$$\mathscr{B}(R,\kappa) := \left\{\omega : \max_{\substack{k=1,2,\ldots \\ i=1,2,\ldots M_k}} Y(z_{k,i}) \leq (\pi v + \kappa)|z_{k,i}|^2\right\} \ .$$

Hence, using (9.13) and part (ii) of Lemma 9.3, we obtain for R large enough

$$\begin{aligned}
&\mathscr{E}\left[\ q_{R,\omega}(f_\omega, f_\omega)\mathbf{1}\Big(\mathscr{B}(R,\kappa)\cap\mathscr{A}(R)\Big)\ \right] \qquad (9.15)\\
&\quad \leq BgvR^2\sum_{k=1}^{\infty}\sum_{i=1}^{M_k}\exp\left[\left(\pi v + 2\kappa - \frac{B}{2}\right)|z_{k,i}|^2\right]\int_{z\in D_{k,i}}\mathscr{E}\left(\mathrm{e}^{Y(z)-Y(z_{k,i})}\right)dz\\
&\quad \leq BgvR^2\sum_{k=1}^{\infty}\sum_{i=1}^{M_k}|D_{k,i}|\exp\left[\left(\pi v + 2\kappa - \frac{B}{2}\right)|z_{k,i}|^2.\right.\\
&\qquad\qquad\qquad\qquad\qquad\qquad \left. + cv\Big(|z_{k,i}|\log R + |z_{k,i}|^{1/2}R\Big)\right]\\
&\quad \leq cBgvR^2\exp\left[\left(\pi v + 3\kappa - \frac{B}{2}\right)R^2\right] \ .
\end{aligned}$$

We also used that $|z_{k,i}| \geq R+1$, $|z - z_{k,i}| \leq |z_{k,i}|^{1/2}$ for $z \in D_{k,i}$, hence $\frac{B}{2}|z|^2 \geq \left(\frac{B}{2} - \kappa\right)|z_{k,i}|^2$ for $R \geq R(\kappa, B)$. Combining this with (9.12), we conclude that

$$\lim_{R\to\infty}\mathscr{E}\left[\frac{q_{R,\omega}(f_\omega,f_\omega)}{\|f_\omega\|_2^2}\cdot\mathbf{1}(\mathscr{A}(R)\cap\mathscr{B}(R,\kappa))\right] = 0$$

for any $\kappa < \frac{1}{3}\left(\frac{B}{2} - \pi v\right)$. Together with (9.11) and (9.14) this completes the proof of Lemma 9.2. □

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