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Exceptional planes of percolation

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Abstract. Consider the standard continuous percolation in $\mathbb{R}^4$, and choose the parameters so that the induced percolation on a fixed two dimensional linear subspace is critical. Although two dimensional critical percolation dies, we show that there are exceptional two dimensional linear subspaces, in which percolation occurs.

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1 Introduction

This paper presents a natural variant on dynamical percolation, in which, at criticality, there is a nonempty exceptional set of "times" for which percolation occurs. The setup will be two dimensional sections of the continuous percolation process, known as the Poisson or the boolean model.

Start with a homogeneous Poisson point process $\omega = \omega_d$ in $\mathbb{R}^d$ ($d \geq 2$) with intensity 1. Let $\omega(r)$ denote the set of points in $\mathbb{R}^d$ whose distance from ω is at most r; that is, the union of closed balls of radius r with centers in ω. This is often called the *open* or *occupied* set in the percolation jargon. Write P_r for the measure governing the process $\omega(r)$. (For a more formal description and background on continuous percolation, see Meester and Roy [11]). Let A be a two dimensional linear subspace of $\mathbb{R}^d$. The intersection $\omega_d(r) \cap A$ is isomorphic to a Poisson percolation process on A, in which the distribution of the radii of the balls is random, but bounded by r. If $\omega_d(r) \cap A$ contains an unbounded connected component, we say that *percolation occurs in A*.

Let $\mathscr{C}(A)$ denote the event that percolation occurs in A. It is well known that there exists a critical radius $r = r_c^d, 0 < r_c^d < \infty$, such that

$$P_r(\mathscr{C}) = \begin{cases} 1 & \text{if } r > r_c^d, \\ 0 & \text{if } r < r_c^d \end{cases} .$$

Consider the process $\omega(r_c^d) \subset \mathbb{R}^d$, $d \geq 2$. In any fixed two dimensional linear subspace A, with probability one there is no percolation; that is, $P_{r_c^d}(\mathscr{C}(A)) = 0$. See Theorem 4.5 from [11]. Our main result is,

Theorem 1 *For $d \geq 4$ and $r = r_c^d$, with probability one, percolation occurs in some two dimensional linear subspace.*

The proof is based on the second moment method and uses the fact that two generic linear planes in $\mathbb{R}^d, d \geq 4$, intersect only at the origin. There is some technical difficulty in the proof, which stems from the fact that the RSW lemma is still unproven for occupied Poisson percolation with random bounded radii. First, an (almost) proof will be presented, which assumes this still conjectured version of the RSW lemma, and then a more complicated unconditional proof will be given.

It is unknown if the theorem holds for $d = 3$. Consider two distinct linear planes $A, A' \subset \mathbb{R}^3$. The "interaction" of the processes $A \cap \omega_3(r)$ and $A' \cap \omega_3(r)$ is in a strip surrounding the line $A \cap A'$. This means that the processes are independent outside the strip. The width of the strip depends on the angle of intersection of A and A'. What seems to be necessary for proving the theorem in the case $d = 3$ is a good estimate for the probability that a component of $A \cap \omega_3(r)$ contains the origin and has diameter at least R, conditioned on the same happening in $A' \cap \omega_3(r)$. The estimate would be in terms of the angle of intersection of A and A'.

One motivation for Theorem 1 comes from dynamical percolation, as introduced by Häggström, Peres and Steif [8]. Following is a brief description of dynamic percolation; for more information, see the survey of O. Häggström [7]. Consider an infinite, locally finite graph G, and let each edge (bond) be open with probability p and closed with probability $1 - p$, independently of all the other edges. Write P_p for this product measure. Define $\mathscr{C}$ to be the event that there exists an infinite connected component of open edges (open cluster). There is some critical probability $p_c = p_c(G) \in [0, 1]$, such that,

$$P_p(\mathscr{C}) = \begin{cases} 1 & \text{for } p > p_c, \\ 0 & \text{for } p < p_c \end{cases} .$$

In the dynamical version, the edge-configuration at time 0 is distributed according to P_p, and from then on each edge, independently of all other edges, changes its status (open or closed) according to a stationary continuous-time two-state Markov chain. Thus, the edge-configuration is time-stationary, with distribution P_p for any fixed time $t \geq 0$. Write $\widehat{P}_p$ for the probability measure governing this process.

Let $\mathscr{C}(t)$ denote the event that this process exhibits an infinite open cluster at time t. By Fubini's Theorem, if $\widehat{P}_p(\mathscr{C}(0)) = 0$, then $\widehat{P}_p$-a.s., for almost every t, $\mathscr{C}(t)$ does not occur. Häggström, Peres and Steif constructed graphs G for which $\widehat{P}_{p_c}(\mathscr{C}(0)) = 0$, but with probability one there are times t for which $\mathscr{C}(t)$ occurs; that is, $\widehat{P}_p(\cup_{t\geq 0}\mathscr{C}(t)) = 1$. They pose the problem whether the same is true for $G = \mathbb{Z}^2$, the square grid. This problem is analogous to the case $d = 3$ in the following,

Question 1 *Fix some integer $d > 2$, and let A be a linear plane in $\mathbb{R}^d$. Is there a positive probability that there exists a plane A', parallel to A, such that $A' \cap \omega_d(r_c^d)$ has an infinite component?*

It might happen that the answer depends on d.

The method of proof of Theorem 1 also gives the following.

Theorem 2 *If d is sufficiently large, then, with positive probability, there is a linear plane $A \subset \mathbb{R}^d$ such that $A \cap \omega_d(r_c^d)$ contains more than one infinite component.*

This contrasts with the fact that the uniqueness of the infinite component is valid in a very wide class of percolation models, and in supercritical dynamical percolation [12].

As in [8] and in [5], away from criticality there are no exceptions:

Theorem 3 *If $r > r_c^d$, then P_r a.s. there is percolation in every 2-plane $A \subset \mathbb{R}^d$. If $r < r_c^d$, then P_r a.s. there is percolation in no 2-plane $A \subset \mathbb{R}^d$.*

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2 Existence of exceptional sections

Let G_2^4 be the space of all two dimensional linear subspaces of $\mathbb{R}^4$, and let μ be the uniform measure on G_2^4. Given $A, A' \in G_2^4$, let $\alpha(A, A')$ be the angle between them; which is the least angle between a unit vector in A and a unit vector in A'. Our first step is in estimating the tail of $\alpha(A, A')^{-1}$.

Lemma 1 *Fix $A \in G_2^4$, and let $s \in (-1,0)$. Then*

$$\int_{A' \in G_2^4} \alpha(A,A')^s d\mu < \infty \ .$$

Proof. With no loss of generality, assume that A is the x-y plane. Since every linear 2-plane is determined by its intersection with the unit sphere S^3, we think of G_2^4 also as the space of geodesic circles in S^3.

For any point $q \in S^3$, let $\rho(q)$ be the angular distance from q to A, and for any $\rho > 0$, let $M(\rho) = \{q \in S^3 : \rho(q) = \rho\}$. Let $N(\rho)$ be the set of points q in the $x = 0$ hyper-plane with $\rho(q) < \rho$. Observe that $M(\rho)$ is a torus and $N(\rho)$ is a pair of discs with boundary in $M(\rho)$, for every $\rho \in (0, \pi/2)$.

Fix some small δ, and let $A' \in G_2^4$ be a circle which contains points q with $\rho(q) < \delta$. If A' also contains points with $\rho(q) \geq \delta$, then A' must intersect $M(\delta)$. If not, then A' must intersect $N(\delta)$. It is a known, useful, simple fact that the area of a smooth 2 dimensional surface $M \subset S^3$ is proportional to the expected number of intersections of a circle in G_2^4 with M. Consequently, for small $\delta > 0$,

$$\mu\{A' \in G_2^4 : \alpha(A',A) < \delta\} \leq C_1(\operatorname{area} M(\delta) + \operatorname{area} N(\delta)) \ ,$$

where C_1 is some constant. Since $\operatorname{area}(M(\delta)) = O(\delta)$ and $\operatorname{area}(N(\delta)) = O(\delta^2)$, it follows that

$$\mu\{A' \in G_2^4 : \alpha(A',A) < \delta\} = O(\delta) \ . \tag{1}$$

This gives, for all $s \in (-1,0)$,

$$\begin{aligned} \int_{A' \in G_2^4} \alpha(A',A)^s d\mu &= \int_0^\infty \mu\{A' : \alpha(A',A)^s \geq t\} dt \\ &= \int_0^\infty \mu\{A' : \alpha(A',A) \leq t^{1/s}\} dt \\ &\leq \int_0^\infty \min\{1, O(1) t^{1/s}\} dt < \infty \ . \end{aligned}$$ □

One basic observation needed in the proof of Theorem 1 is the following. There is $C > 0$ such that for any two planes intersecting at the origin, with angle α between the two, the induced percolation processes outside a ball of radius $C\alpha^{-1}$ are independent.

Let A be some linear plane in $\mathbb{R}^d$. Denote by $\{0 \leftrightarrow_A R\}$ the event that the origin is connected by an open path in A to distance R, and let $\{R' \leftrightarrow_A R\}$, $(0 < R' < R)$, denote the event that there is an open connection in A between the spheres of radii R' and R about 0.

The discrete version of the following lemma is from van den Berg and Kesten [4].

Lemma 2 *Let $A \subset \mathbb{R}^d$ be a linear plane, and set $r = r_c^d$. Then*

$$P_r\{0 \leftrightarrow_A R\} > \text{Const}\, R^{-1/2} \quad ,$$

for all sufficiently large R.

Although the proof is nothing more than a straightforward translation of the proof in [4], it is presented here, because the same argument will be used again below, and for the sake of completeness.

Proof. For simplicity, assume that A is the x-y plane. Let $\mathscr{L}_R$ be the event that there is a left-right crossing of the rectangle $[0, 2R] \times [0, 6R]$. It follows from Lemma 3.3 of [11] that $\inf_R P_r(\mathscr{L}_R) > 0$. Any left-right crossing of the rectangle must pass through the middle vertical segment $\{R\} \times [0, 6R]$. Therefore, conditioned on $\mathscr{L}_R$, there is some point $z = (R, y)$ in the rectangle, such that there are two open paths $\gamma_1, \gamma_2 \subset A$ connecting the sphere of radius $r_1 = 2r$ about z to the sphere of radius R about z, and such that the set of occupied percolation disks that intersect γ_1 is disjoint from that of γ_2. Moreover, by changing r_1 to $3r$, we may assume that y is divisible by r. Let Z be the set of points of the form (R, y), where $y \in [0, 6R]$ is divisible by r, and let $\mathscr{Y}_R(z)$ be the event that there are two paths in A connecting the sphere of radius $3r$ about z to the sphere of radius R about z, and the paths do not meet the same occupied percolation disk. Then

$$P_r\mathscr{L}_R \leq \sum_{z \in Z} P_r\mathscr{Y}_R(z) \quad .$$

Since $\inf_R P_r(\mathscr{L}_R) > 0$ and the cardinality of Z is proportional to R, when R is large, it follows that,

$$P_r\mathscr{Y}_R(0) \geq \text{Const}\, R^{-1} \quad .$$

The BK inequality, which is valid also in the continuous setting [11], now gives,

$$(P_r\{3r \leftrightarrow_A R\})^2 \geq P_r\mathscr{Y}_R(0) \geq \text{Const}\, R^{-1} \quad .$$

By FKG, $P_r\{0 \leftrightarrow_A R\} \geq \text{Const}\, P_r\{3r \leftrightarrow_A R\}$, and the lemma follows. □

We now present an "almost proof" of Theorem 1. Its deficiency is that it assumes the RSW lemma for occupied Poisson percolation, which is currently only proven for constant radii. See K. S. Alexander [2].

Almost proof of Theorem 1. Abbreviate $P = P_{r_c^d}$, take $\mathbb{R}^4 \subset \mathbb{R}^d$, and fix some plane $A_1 \in G_2^4$. Consider the random variable $W_R = \int_{A \in G_2^4} 1_{\{0 \leftrightarrow_A R\}} d\mu$. The event that there is percolation in some linear plane $A \subset \mathbb{R}^d$ is independent of $\omega_d \cap K$ for any bounded set $K \subset \mathbb{R}^d$, and therefore has probability 0 or 1. Moreover, by compactness of G_2^4, this event contains the intersection $\bigcap_{R=1}^{\infty}\{W_R > 0\}$. Consequently, it is enough to show that $\inf_R P(W_R > 0) > 0$. By Cauchy-Schwarz,

$$P(W_R > 0) \geq \frac{(EW_R)^2}{E(W_R^2)} = \frac{(P\{0 \leftrightarrow_{A_1} R\})^2}{E \int_{G_2^4 \times G_2^4} 1_{\{0 \leftrightarrow_A R\}} 1_{\{0 \leftrightarrow_{A'} R\}} d\mu \times d\mu}$$

$$= \frac{(P\{0 \leftrightarrow_{A_1} R\})^2}{P\{0 \leftrightarrow_{A_1} R\} \int_{A' \in G_2^4} P(\{0 \leftrightarrow_{A'} R\} | \{0 \leftrightarrow_{A_1} R\}) d\mu}$$

$$= \frac{P\{0 \leftrightarrow_{A_1} R\}}{\int_{A' \in G_2^4} P(\{0 \leftrightarrow_{A'} R\} | \{0 \leftrightarrow_{A_1} R\}) d\mu} .$$

Because the part of $\omega_d(r_c^d) \cap A'$ that's outside the ball of radius $C\alpha(A_1, A')^{-1}$ around 0 is independent from $\omega_d(r_c^d) \cap A_1$, we have

$$\begin{aligned} P(\{0 \leftrightarrow_{A'} R\} | \{0 \leftrightarrow_{A_1} R\}) &\leq P(\{C\alpha(A_1, A')^{-1} \leftrightarrow_{A'} R\} | \{0 \leftrightarrow_{A_1} R\}) \\ &= P\{C\alpha(A_1, A')^{-1} \leftrightarrow_{A'} R\}. \end{aligned}$$

Consequently,

$$P(W_R > 0) \geq \left(\int_{A' \in G_2^4} \frac{P\{C\alpha(A_1, A')^{-1} \leftrightarrow_{A'} R\}}{P\{0 \leftrightarrow_{A_1} R\}} d\mu \right)^{-1} . \tag{2}$$

We now estimate the integrand. Note that $0 \leftrightarrow_A R$ if $0 \leftrightarrow_A 2R'$ and $R' \leftrightarrow_A R$ and there is an open winding in the annulus $\{x \in A : R' \leq |x| \leq 2R'\}$. By the assumed RSW lemma, the probability for an open winding in this annulus is bounded away from zero.

Hence, FKG implies,

$$cP\{0 \leftrightarrow_A R\} \geq P\{0 \leftrightarrow_A 2R'\}P\{R' \leftrightarrow_A R\} \ , \tag{3}$$

where $c > 0$ is some constant. Applying this with $R' = C\alpha(A_1, A')^{-1}$, and the BK estimate from Lemma 2, and Lemma 1, to (2), we get $\inf_R P(W_R > 0) > 0$, as needed. □

Question 2 *What is the Hausdorff dimension of the set of* $A \in G_2^4$ *such that* $A \cap \omega_d(r_c^d)$ *percolates*?

Proof of Theorem 1. What is missing in the first proof is a good replacement for (3). Our first goal will be to obtain such a replacement.

As above, let $A, A' \subset \mathbb{R}^4 \subset \mathbb{R}^d$ be linear planes, set $r = r_c^d$, and $P = P_r$. Let R be much larger than r. Suppose that ω is such that $R' \leftrightarrow_A R$, where

$$R' \leq \sqrt{R} \ . \tag{4}$$

Then there is a path in $A \cap \omega(r)$ that connects the circles of radii R' and R about 0, and this path must cross the circle of radius $(3/2)R'$. Therefore, the proof of Lemma 2, with slight adjustments, gives,

$$P\{R' \leftrightarrow_A R\} \leq \ \mathrm{Const}\, R' P\{0 \leftrightarrow_A R'\}P\{0 \leftrightarrow_A R - 2R'\} \ . \tag{5}$$

We also know from the RSW lemma for *vacant* percolation (see [11], Chap. 4) that $\sup_{R>0} P\{R \leftrightarrow_A 9R\} < 1$. If $R > R' + 2r$, then the events $\{R \leftrightarrow_A 9R\}$ and $\{R' \leftrightarrow_A 9R'\}$ are independent. By considering a sequence of nested annuli, it follows that

$$P\{0 \leftrightarrow_A R\} \leq R^{-\beta} \tag{6}$$

for all sufficiently large R, and some $\beta > 0$. Hence, we get from (5),

$$\mathrm{Const}\, P\{0 \leftrightarrow_A R - 2R'\} \geq R'^{\beta-1} P\{R' \leftrightarrow_A R\} \ . \tag{7}$$

This is our substitute for (3) from the "almost proof ". The appearance of $R - 2R'$ in place of R on the left hand side will cause annoying difficulties, which are tackled by considering a smooth "gauge", in place of the indicator of the event $\{0 \leftrightarrow_A R\}$.

Let $T(A) = T(A, \omega)$ be the largest $t \geq 0$ such that $0 \leftrightarrow_A t$. Define

$$\phi_R(A,\omega) = \begin{cases} 0, & \text{if } \ \mathrm{T(A,}\omega) \leq R, \\ R^{-1}T(A,\omega) - 1, & \text{if } \ R < T(A,\omega) < 2R, \\ 1, & \text{if } \ \ 2\mathrm{R} \leq T(A,\omega) \end{cases} \tag{8}$$

Set

$$H_R(\omega) = \int_{A \in G_2^4} \phi_R(A,\omega) d\mu \ .$$

As in the "almost proof", is enough to show that $\inf_R P(H_R > 0) > 0$. By Cauchy-Schwarz,

$$\begin{aligned} P(H_R > 0) &\geq \frac{(E_\omega H_R)^2}{E_\omega(H_R^2)} \\ &= \frac{(E_\omega \phi_R(A',\omega))^2}{\int_A \int_{A'} E_\omega(\phi_R(A,\omega)\phi_R(A',\omega)) d\mu\, d\mu} \\ &= \frac{(E_\omega \phi_R(A',\omega))^2}{\int_A E_\omega(\phi_R(A,\omega)\phi_R(A',\omega)) d\mu} \ . \end{aligned} \tag{9}$$

Now fix distinct A and A' in G_2^4. Set

$$R' = C\alpha(A,A')^{-1} \ , \tag{10}$$

where the constant $C > 0$ is sufficiently large so that the processes $A \cap \omega(r)$ and $A' \cap \omega(r)$ are independent outside the ball of radius R' around 0. Assume, for the moment, that (4) holds. Let

$$\widetilde{T} = \widetilde{T}(\omega) = \sup\{t \geq 0 : R' \leftrightarrow_A t\},$$

and define

$$\psi(\omega) = \begin{cases} 0, & \text{if } \widetilde{T} \leq R, \\ R^{-1}\widetilde{T} - 1, & \text{if } R < \widetilde{T} < 2R, \\ 1, & \text{if } 2R \leq \widetilde{T} \ . \end{cases} \tag{11}$$

Note that the definition of $\psi(\omega)$ is the same as that of $\phi_R(A,\omega)$, except that T is replaced by $\widetilde{T}$.

Because $\psi(\omega) \geq \phi_R(A,\omega)$, and outside the ball of radius R' about 0 the processes $\omega(r) \cap A$ and $\omega(r) \cap A'$ are independent, we have,

$$E_\omega(\phi_R(A,\omega)\phi_R(A',\omega)) \leq E_\omega(\psi(\omega)\phi_R(A',\omega)) = E_\omega\psi(\omega) E_\omega \phi_R(A',\omega). \tag{12}$$

In the following estimate of $E_\omega \psi$, the inequalities (7) and (6) are used. (The Const in different rows may mean different constants.)

$$E_\omega \psi = \frac{1}{R} \int_R^{2R} P\{R' \leftrightarrow_A t\}\, dt$$

$$\leq \mathrm{Const}\, R^{-1} R'^{1-\beta} \int_R^{2R} P\{0 \leftrightarrow_A t - 2R'\}\, dt$$

$$\leq \mathrm{Const}\, R^{-1} R'^{1-\beta} \int_R^{2R} P\{0 \leftrightarrow_A t\}\, dt$$

$$+ \ \mathrm{Const}\, R^{-1} R'^{1-\beta} \int_{R-2R'}^{R} P\{0 \leftrightarrow_A t\}\, dt$$

$$\leq \mathrm{Const}\, R'^{1-\beta} E\phi_R(A,\omega) + \ \mathrm{Const}\ R^{-1} R'^{2-\beta} R^{-\beta}\ . \tag{13}$$

From Lemma 2, we have,

$$E_\omega \phi_R(A,\omega) \geq \mathrm{Const}\ R^{-1/2}\ . \tag{14}$$

Provided (4) holds, it now follows from (12)–(14) that

$$\frac{E_\omega(\phi_R(A,\omega)\phi_R(A',\omega))}{(E_\omega \phi_R(A,\omega))^2} \leq \ \mathrm{Const}\, R'^{1-\beta} + \mathrm{Const}\, R^{-\beta-1/2} R'^{2-\beta}$$

$$\leq \ \mathrm{Const}\, R'^{1-\beta}\ . \tag{15}$$

Let X be the set of $A \in G_2^4$ such that R' as given by (10) is at most $\sqrt{R}$. By (1), we have,

$$\mu(G_2^4 - X) \leq \ \mathrm{Const}\, R^{-1/2}\ . \tag{16}$$

Lemma 1 and (15) give,

$$\int_{A\in X} \frac{E_\omega(\phi_R(A,\omega)\phi_R(A',\omega))}{(E_\omega \phi_R(A,\omega))^2} d\mu \leq \mathrm{Const} \int_A \alpha(A,A')^{\beta-1} d\mu < \infty\ .$$

On the other hand, by (16) and (14),

$$\int_{A\in G_2^4 - X} \frac{E_\omega(\phi_R(A,\omega)\phi_R(A',\omega))}{(E_\omega \phi_R(A,\omega))^2} d\mu \ \leq \int_{A\in G_2^4 - X} \frac{E_\omega(\phi_R(A,\omega))}{(E_\omega \phi_R(A,\omega))^2} d\mu$$

$$\leq \ \mathrm{Const}\, \mu(G_2^4 - X) R^{1/2} \leq \ \mathrm{Const}\ .$$

By (9), the proof is now complete. □

Proof of Theorem 2. First observe that, at $r = r_c^d$ and for large R, the probability of $\{0 \leftrightarrow R\}$ in a quadrant of a fixed plane, is bounded from below by R^{-a}, for some constant $a > 0$, and that holds both for occupied and for vacant percolation. (This would be an easy consequence of the RSW lemma, but, again, the RSW lemma is not known yet for variable size disk occupied percolation.) To prove this, recall that Lemma 3.3 of [11] shows that the probability for a left-right crossing of an $L \times (3L)$ rectangle is bounded from below. Therefore, it follows that the probability that 0 is connected to the segment $\{L\} \times [0, 3L]$ inside the rectangle $[0, L] \times [-3L, 3L]$ is at least $\mathrm{Const}\, L^{-1}$. An application of FKG now shows that the probability that 0 is connected to $(L, 0)$ inside the rectangle $[0, L] \times [-3L, 3L]$ is at least $\mathrm{Const}\, L^{-2}$. From this, it is easy to deduce an asymptotic lower bound of the form R^{-a} for $\{0 \leftrightarrow R\}$ in a quadrant. Similarly, one concludes that the probability to join the two boundary rays of the quadrant inside the intersection of the quadrant with an annulus of the form $R \le |z| \le 2R$ is asymptotically bounded below by $R^{-a'}$, for some constant $a' > 0$.

Let A be a 2-plane and let $A = Q_1 \cup Q_2 \cup Q_3 \cup Q_4$ be a dissection of A into quadrants, where Q_1, Q_2, Q_3, Q_4 are in cyclic order. The probability that there is an open crossing $\{3r \leftrightarrow R\}$ in each of Q_1, Q_3 and there is a closed crossing $\{0 \leftrightarrow R\}$ in each of Q_2, Q_4 is at least $\mathrm{Const}\, R^{-4a}$. If that holds for arbitrarily large R, then there are at least two unbounded open clusters in A. The "almost proof" of Theorem 1 adapted to this situation shows that it is enough to prove that

$$\int_{G_2^d} \alpha(A, A')^{-4a-4a'} d\mu < \infty \ ,$$

where G_2^d is the space of all linear 2-planes in $\mathbb{R}^d$, and μ denotes the uniform measure on G_2^d. A higher dimensional analog of Lemma 1 shows that this is valid once $d > 3 + 4a + 4a'$. □

Here are some heuristic bounds for the least d satisfying Theorem 2. The actual estimate obtained from RSW (in the discrete setting) for the constant a is about 9.09. Of course, the constant a' should be zero. It is known that in the discrete setting the probability for $\{0 \leftrightarrow R\}$ in a half-plane is at least $\mathrm{Const}\, R^{-1/2}$ for large R. One way to see this is to observe that there is probability bounded away from zero that the lowest crossing of an $R \times R$ square will hit the left edge inside its middle third. Hence, with probability at least Const / R there is both an open and a closed crossing in a half-plane from 0 to distance $R/3$.

Then the bound $\text{Const}\,R^{-1/2}$ follows from FKG. The map $z \to z^2$ takes a quadrant conformally to a half space. Conjecturally, crossing probabilities for percolation are asymptotically conformally invariant (see [10] and [3]). With this conjecture, the above bound translates to a lower bound of $\text{Const}\,R^{-1}$ for $\{0 \leftrightarrow R\}$ in a quadrant. The exponent -1 is probably not sharp; that is, $a < 1$. Therefore, presumably, Theorem 2 is valid already for $d = 7$.

Question 3 *Is there a dimension d such that for any finite n a.s. there is a 2 dimensional section of $\omega_c(r_c^d)$ with at least n distinct unbounded open clusters?*

3 Away from criticality

We turn to the proof of Theorem 3, starting with the sub-critical case. Assume $r < r_c^d$, and let $r' = (r + r_c^d)/2$. Write $\{0 \leftrightarrow_B^r R\}$, for the event that there is an open connection from 0 to the R-sphere in $\omega_d(r) \cap B$.

The following two standard definitions are needed below. Given two planes $A, B \in G_2^d$, the maximal angle between A and B is

$$\sup_v \inf_u \{\text{angle between } v \textit{ and } u\}\ ,$$

where u and v are unit vectors in A and B, respectively. The maximal angle provides a metric on G_2^d. Recall that an ϵ-net N in a metric space X is a set of points $N \subset X$, such that the distance between any two points in N is bounded below by ϵ, but any point in X is within distance 2ϵ from N.

Lemma 3 *There is $m > 0$, so that for any large R, there is a set N of at most R^m 2-planes, so that if for some A, $\{0 \leftrightarrow_A^r R\}$, then there is a $B \in N$ satisfying $\{0 \leftrightarrow_B^{r'} R\}$.*

Proof. Consider two planes, A, B, and suppose that the maximal angle between them is at most $(r' - r)/2R$. Then $\{0 \leftrightarrow_A^r R\}$ implies $\{0 \leftrightarrow_B^{r'} R\}$, because for any point $x \in A$, with distance less than R to 0, there is a point $y \in B$, with distance less than $r' - r$ to x. The space G_2^d, of two dimensional linear subspaces in $\mathbb{R}^d$, were the distance between two planes is the maximal angle between them, is a compact finite dimensional Riemannian manifold. Hence for any small ϵ, any ϵ-net in G_2^d has less then ϵ^{-m} elements for some constant $m > 0$. Take N to be an $(r' - r)/4R$-net. The Lemma follows. □

Proof of Theorem 3. As before, assume $r < r_c^d$, and set $r' = (r + r_c^d)/2$. Since $r' < r_c^d$, for any fixed plane A, the probability $P\{0 \leftrightarrow_A^{r'} R\}$ decays exponentially in R. (See Theorems 2.4 and 3.5 from [11].) Because the size of N from Lemma 3 is bounded by polynomial in R, the sub-critical case follows.

So assume now that $r > r_c^d$. Fix a plane, A, and look at the ball $\{z \in A : |z| \le R\} \subset A$. If this R-ball does not intersect an unbounded open cluster in $A \cap \omega(r)$, then there is a vacant cut-set in A which intersects some ball of minimal radius $R' > R$ centered at 0, and has diameter bigger than R'. This has probability which is exponentially small in R', since the vacant process is sub-critical. (See Lemma 4.1 and Theorem 4.3 from [11].) The proof is now completed as in the sub-critical case. □

Note that with only minor modifications, the same proof shows that away from criticality, there are no exceptional affine planes.

Problem 4 *The proof of the supercritical case uses planarity in an essential way. Show that the analogous result holds for higher dimensional sections.*

4 Extensions and remarks

1. It is believed that critical percolation dies in any dimension greater than 1. It is possible to show that for some $b = b(n)$,

$$\frac{P_r\{R' \leftrightarrow_A R\}}{P_r\{0 \leftrightarrow_A R\}} \le \text{Const}\, R'^b,$$

where A is an n-dimensional linear subspace of $\mathbb{R}^d$, and $r = r_c^d(n)$ is the critical radius for percolation in a fixed n dimensional linear subspace. Therefore, the proof of Theorem 1 can be modified to show that given any integer $n > 1$, there is some d such that a.s. $\omega_d(r_c^d(n))$ percolates in some n dimensional linear subspace $A \subset \mathbb{R}^d$. An interesting problem is to understand the dependence of d on n.

2. There are ways to get an analogous model that provides similar properties for bond percolation on the $\mathbb{Z}^n$ lattice. Consider a random coloring of $\mathbb{R}^d$ by two colors (black and white), so that the probability of any point to be white is p, and if the distance between two points is bigger than a half, then their colors are independent. Now embed isometrically the lattice $\mathbb{Z}^2$ in $\mathbb{R}^d$, and decide whether a bond is open according to the color of its midpoint. The theorems above translate easily to this model.

We now describe such a coloring with additional useful properties.

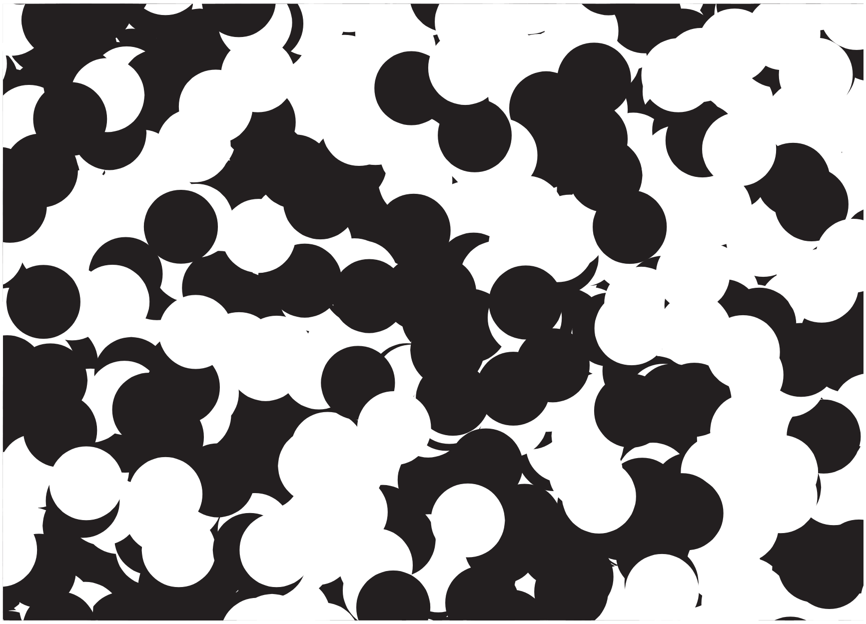

Fig. 1.

Confetti percolation. Before the formal definition, here is a loose description. Throw confetti discs on $\mathbb{R}^d$; each piece is either white or black. The color of a point will be determined by the color of the top piece. (See Fig. 1.) Planar confetti percolation is an isotropic self dual model, as the Voronoi percolation model. (See [3] for a description of the Voronoi model.) Yet it has the advantage that the colors of points with distance bigger then the radius of the confetti discs are independent. An interesting problem might be to prove some of the known properties of other models for confetti percolation. The confetti model is also known as the **dead leaves** model; see [9].

Problem 5 *Prove the RSW-lemma for confetti percolation on* $\mathbb{R}^2$, *and show that* $p_c = 1/2$.

Here is a more formal description of confetti percolation. Perform a Poisson process in $\mathbb{R}^{d+1} = \mathbb{R}^d \times \mathbb{R}$ with some intensity. Color each site in the process black with probability p or white otherwise, independently. To decide the color of a point $x \in \mathbb{R}^d$, move the d-dimensional disc of radius $1/2$ in $\mathbb{R}^d$, centered at x, in the direction orthogonal to $\mathbb{R}^d$. If the first site of the Poisson process it intersects is white, color x white, and otherwise black.

Note that for each translation of $\mathbb{R}^d$ inside $\mathbb{R}^{d+1}$, we get a coloring. This gives another natural dynamic percolation model.

3. The paper of Adelman, Burdzy and Pemantle [1] studies exceptional planes for projections of three dimensional Brownian motion. The paper of van den Berg, Meester and White [5] contains another continuous variant on dynamic percolation.

4. We end with a question that arose in trying to understand the case $d = 3$ in Theorem 1. Let G be the graph obtained by identifying two copies of $\mathbb{Z}^2$ along the x-axis. Denote the two $\mathbb{Z}^2$ copies by A and B, and consider critical percolation on G.

Question 6 *Is percolation to distance R in A and B asymptotically independent*; *that is*, *does*

$$\lim_{R\to\infty} \frac{P(\{0 \leftrightarrow_A R\} \cap \{0 \leftrightarrow_B R\})}{(P\{0 \leftrightarrow_A R\})^2} = 1?$$

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