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Probability Theory and Related Fields

On the area and perimeter of a random convex hull in a bounded convex set

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Abstract. Suppose K is a compact convex set in $\mathbb{R}^2$ and $X_i,\ 1 \le i \le n$, is a random sample of points in the interior of K. Under general assumptions on K and the distribution of the X_i we study the asymptotic properties of certain statistics of the convex hull of the sample.

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1. Introduction

Consider a compact convex set $K \subset \mathbb{R}^d$ and let $X_i,\ 1 \le i \le n$, be a random sample of points from K. Denote by K_n the convex hull of the points $X_i,\ 1 \le i \le n$, namely the smallest convex set containing the X_i. In Rényi and Sulanke (1963, 1964), the expected area and perimeter of K_n were considered for the case where $d = 2$, K was either a polygon or had a smooth boundary (referred to as polygonal and smooth cases below) and the X_i were assumed to be iid. uniform. In that vein, there have been a series of papers since then investigating the various statistics of K_n for a number of different settings. See Groeneboom (1988) and references therein.

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In the present paper we assume $d = 2$ and that the X_i are iid. with a rather general distribution. Our goals are to investigate the means and asymptotic distributions of the area and perimeter of K_n in the two cases of K mentioned in the previous paragraph. The first paper that successfully treated the asymptotic distribution of a statistic of K_n was Groeneboom (1988), in which the number of vertices was shown to be normally distributed. Hsing (1994) and Cabo and Groeneboom (1994), respectively, derived the asymptotic distributions of the area of K_n for the disk and the polygonal cases. Bräker, Hsing and Bingham (1995) considered the asymptotic distribution of the Hausdorff distance between K and K_n for both the smooth and the polygonal cases.

A novelty of this paper is the derivation and application of two very simple but powerful integral representations in which the area and perimeter of K_n are written as the integrals of certain functions defined on the boundary of K_n. The development of the representations along with the introduction of notation are the topics of Section 2. Another novelty is the introduction of certain point processes which describe the local behavior of sample points close to the boundary and therefore whose limiting behavior is intimately linked to that of the area and perimeter of K_n. This is done in Section 3. The advantages of using these tools will be obvious. Our main results in Sections 4 and 5 either extend existing results to much more general settings or specifically improve upon them. Our primary setting of interest will be the smooth case where the boundary of K has curvature bounded away from 0 and ∞. Under that and the assumption that the density of X_i changes at a geometric rate when approaching the boundary, we show in Section 5 that the area and perimeter of K_n are jointly normally distributed. Moreover we derive the asymptotic behavior of the expected area and perimeter of K_n. We also consider the polygonal case in section 4 where we derive the asymptotic distribution of the perimeter of K_n and the asymptotic behavior of the mean of the perimeter of K_n. In particular, Theorem 5 in Section 4 gives an improved rate of convergence for the expected perimeter over what was given by Buchta (1984).

2. Preliminaries

Throughout, let L and A, respectively, be the perimeter and area of the bounded convex set K in $\mathbb{R}^2$ and let L_n and A_n be the corresponding quantities of K_n, the convex hull of n iid. random points $X_1, \ldots, X_n$ in K. Also write ∂K for the boundary of K. This section is primarily devoted to the derivation of the identities (1), (5) and (8) below for

$L - L_n$ and $A - A_n$ which will be our basis for deriving the asymptotic properties of the two quantities.

First we consider $L - L_n$. For fixed $p, \theta \in \mathbb{R}$ let $\ell(p, \theta)$ be the straight line

$$x \cos \theta + y \sin \theta = p, \qquad x, y \in \mathbb{R} \ .$$

A classical result in integral geometry (cf. Santaló 1976, Chapter 1) states that the perimeter of an arbitrary bounded convex set K can be written as

$$L = \int_{\theta=0}^{\pi} \int_{p \in I_K(\theta)} dp \, d\theta \ ,$$

where $I_K(\theta) = \{p : \ell(p, \theta) \cap K \neq \emptyset\}$. For every fixed θ, the set K has two supporting lines, $\ell(p_1(\theta; K), \theta)$ and $\ell(p_2(\theta; K), \theta)$ say, for some $p_1(\theta; K) < p_2(\theta; K)$. See Fig. 1. Since for every p, θ the lines $\ell(p, \theta)$ and $\ell(-p, \theta + \pi)$ coincide, we have $p_1(\theta + \pi; K) = -p_2(\theta; K)$ and $p_2(\theta + \pi; K) = -p_1(\theta; K)$. Since K_n is also convex, all of the above applies and in fact one has

$$p_1(\theta; K) \leq p_1(\theta; K_n) \leq p_2(\theta; K_n) \leq p_2(\theta; K) \ .$$

Therefore,

$$\begin{aligned}
L - L_n &= \int_{\theta=0}^{\pi} \left(\int_{p \in I_K(\theta)} dp - \int_{p \in I_{K_n}(\theta)} dp \right) d\theta \\
&= \int_0^{\pi} \big(p_2(\theta; K) - p_1(\theta; K) - (p_2(\theta; K_n) - p_1(\theta; K_n)) \big) \, d\theta \\
&= \int_0^{\pi} \big(p_1(\theta; K_n) - p_1(\theta; K) + p_1(\theta + \pi; K_n) - p_1(\theta + \pi; K) \big) \, d\theta \\
&= \int_0^{2\pi} \big(p_1(\theta; K_n) - p_1(\theta; K) \big) \, d\theta \qquad (1)
\end{aligned}$$

Actually, Efron (1965) used the same principle to compute EL_n but, to our knowledge, the approach has been overlooked since then. We also note in passing that $\sup_\theta \big(p_1(\theta; K_n) - p_1(\theta; K) \big)$ is the Hausdorff distance between K and K_n and this was used to study the asymptotic distribution of that measure in Bräker, Hsing and Bingham (1995) (cf. Brozius and De Haan, 1987).

For the rest of this section assume that K has a smooth boundary ∂K in the following sense. First parameterize ∂K as

$$t \mapsto \mathbf{c}(t),$$

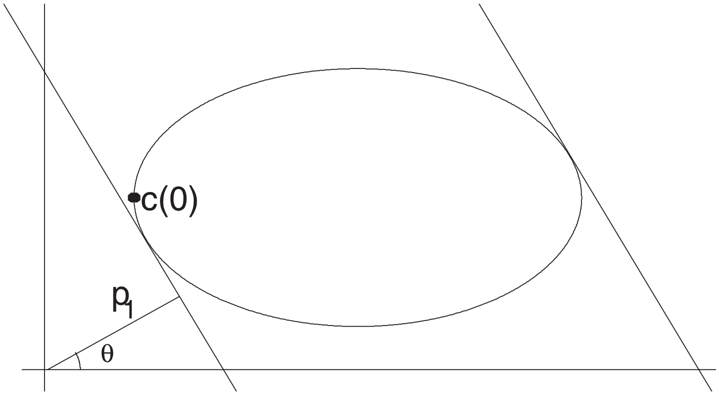

Fig. 1. $\ell(p_i(K;\theta),\theta), i = 1,2$

where t measures the arc length from some starting point $\mathbf{c}(0)$ to $\mathbf{c}(t)$ counterclockwise. Assume that $\mathbf{c}$ is twice continuously differentiable. At $\mathbf{c}(t)$ define the unit tangent vector $\mathbf{e}_1(t) = \dot{\mathbf{c}}(t)$ ($\cdot$ denotes d/dt) and let $\mathbf{e}_2(t)$ be orthogonal to $\mathbf{e}_1(t)$ such that $(\mathbf{e}_1(t),\ \mathbf{e}_2(t))$ form a positively orientated coordinate system. Assume without loss of generality that the starting point $\mathbf{c}(0)$ is such that $\mathbf{e}_2(0)$ points to the positive x-direction on the plane (cf. Fig. 1). By Frenet's equations for plane curves,

$$\dot{\mathbf{e}}_1\ (t) = \kappa(t)\ \mathbf{e}_2(t), \qquad \dot{\mathbf{e}}_2\ (t) = -\kappa(t)\ \mathbf{e}_1(t)\ , \tag{2}$$

where $\kappa(t)$ is the curvature at $\mathbf{c}(t)$. Note that $\kappa(t) \geq 0$ for all $t \in [0, L)$ by convexity and since $\mathbf{c}$ is parameterized counterclockwise. Assume that $\kappa(t)$ is bounded away from 0 and ∞. Also let $\theta(t)$ be the angle between $\mathbf{e}_1(0)$ and $\mathbf{e}_1(t)$. θ and t are related through the equation

$$\dot{\theta}\,(t) = \kappa(t) \tag{3}$$

(cf. Klingenberg 1978, Chapter 1). Observe that the tangent at $\mathbf{c}(t)$ is $\ell(p_1(\theta(t);K),\theta(t))$. For $t \in [0, L)$ and $1 \leq i \leq n$ let $(X_i(t), Y_i(t))$ be the coordinates of the point X_i with respect to the coordinate system $(\mathbf{e}_1(t),\ \mathbf{e}_2(t))$ defined at $\mathbf{c}(t)$, i.e.

$$X_i(t) = \langle X_i -\ \mathbf{c}(t),\ \mathbf{e}_1(t)\rangle, \quad Y_i(t) = \langle X_i -\ \mathbf{c}(t),\ \mathbf{e}_2(t)\rangle \tag{4}$$

where $\langle\cdot,\cdot\rangle$ denotes inner product. Then clearly,

$$p_1(\theta(t);K_n) - p_1(\theta(t);K) = \bigwedge_{i=1}^{n} Y_i(t) =: M_n(t)\ .$$

Therefore by (1) and (3),

$$L - L_n = \int_0^L M_n(t)\kappa(t)\, dt \ . \tag{5}$$

Next we consider $A - A_n$. As in Hsing (1994) we define

$$D_n(t) = \inf\{s > 0 : \ \mathbf{c}(t) + s\ \mathbf{e_2}(t) \in K_n\} \ .$$

Note that it is possible that the line $\mathbf{c}(t) + s\ \mathbf{e}_2(t), s > 0$, does not intersect K_n at all, in which case $D_n(t) = \infty$. However, as we will show below, this event becomes increasing unlikely as n increases. We wish to express $A - A_n$ in terms of the process $\{D_n(t), t \in [0, L)\}$. A crucial issue here is whether we can represent each point in $K \setminus K_n$ uniquely by $\mathbf{c}(t) + s\ \mathbf{e_2}(t)$ for some $t \in [0, L)$ and $s \in [0, D_n(t))$. The answer is clearly negative in general, but we now illustrate how to construct an event E_n which has probability tending to 1 and on which such a one-one correspondence is possible. Define the curve

$$\mathbf{c}_\delta(t) = \ \mathbf{c}(t) + \delta\mathbf{e_2}(t), \quad t \in [0, L) \ ,$$

for fixed $\delta \in (0, 1/\kappa_\infty)$, where $\kappa_\infty = \max\{\kappa(t) : t \in [0, L)\}$, and choose points $0 = \tau_1 < \tau_2 < \tau'_2 \le \tau_3 < \tau'_3 \le \cdots$ such that the linear segment connecting $\mathbf{c}(\tau_j)$ and $\mathbf{c}(\tau'_{j+1})$ is a tangent to $\mathbf{c}_\delta$. Let $k = \sup\{j \ge 1 : \tau'_j \le L\}$ and define $\tau'_1 = \tau_2 \wedge (\tau'_{k+1} - L)$. The fan-shaped region between ∂K and the linear segment connecting $\mathbf{c}(\tau_j)$ and $\mathbf{c}(\tau'_j)$ is denoted by $R(\tau_j, \tau'_j)$. Illustrating graphically, the $R(\tau_j, \tau'_j)$'s are the solid regions in Fig. 2.

Define the event

$$E_n = \bigcap_{1 \le j \le k} \Big(\{X_i, \ 1 \le i \le n\} \cap R(\tau_j, \tau'_j) \ne \emptyset \Big) \ . \tag{6}$$

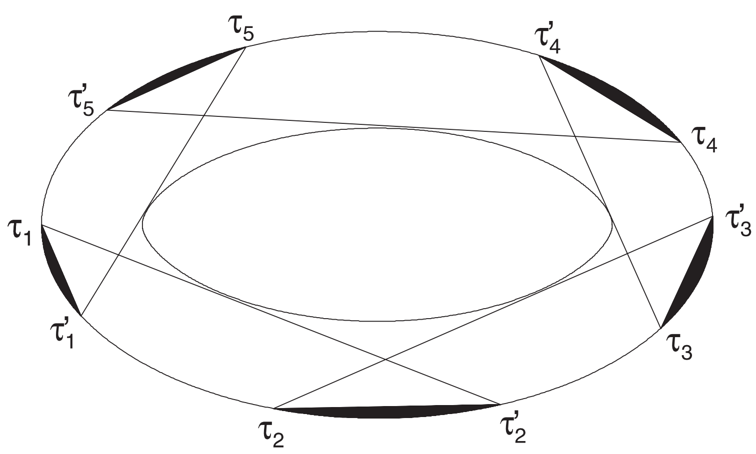

Fig. 2. Construction of E_n

Observe that on E_n, $D_n(t) < 1/\kappa_\infty$ for all $t \in [0, L)$ and if $P(X_1 \in R(\tau_j, \tau'_j)) > 0$ for all $1 \le j \le k$, which we assume, then $P(E_n^c)$ tends to 0 exponentially fast. It follows from Lemma 1 below that on E_n any two linear segments $\mathbf{c}(t_1) + s_1\, \mathbf{e}_2(t_1)$, $s_1 \in [0, D_n(t_1)]$ and $\mathbf{c}(t_2) + s_2\, \mathbf{e}_2(t_2)$, $s_2 \in [0, D_n(t_2)]$ are disjoint, which implies that on the event E_n the transformation

$$(t, s) \mapsto x = \mathbf{c}(t) + s\, \mathbf{e}_2(t), \quad t \in [0, L),\ s \in [0, D_n(t))$$

is one-one. The Jacobian of this transformation is

$$\left|\frac{\partial x}{\partial t} \times \frac{\partial x}{\partial s}\right| = |(1 - \kappa(t)s)\, \mathbf{e}_1(t) \times \mathbf{e}_2(t)| = 1 - \kappa(t)s, \qquad 0 \le s < 1/\kappa(t) \ . \tag{7}$$

Therefore,

$$\begin{aligned}(A - A_n)I_{E_n} &= \int_{t=0}^{L} \int_{s=0}^{D_n(t)} \big(1 - \kappa(t)s\big)\, ds\, dt \\ &= \int_0^L D_n(t)\, dt - \frac{1}{2}\int_0^L \kappa(t) D_n^2(t)\, dt \ .\end{aligned} \tag{8}$$

The fundamental identity (8) will be our basis for considering the asymptotics of $A - A_n$. We now deal with the crucial juncture in the above derivation.

Lemma 1. *Let $t_1, t_2 \in [0, L)$ with $t_1 \ne t_2$ and suppose*

$$\mathbf{c}(t_1) + s_1\, \mathbf{e_2}(t_1) = \mathbf{c}(t_2) + s_2\, \mathbf{e_2}(t_2) \ . \tag{9}$$

Then

$$\max(s_1, s_2) \ge 1/\kappa_\infty \ .$$

Proof. Assume that $\max(s_1, s_2) < 1/\kappa_\infty$. Let

$$\mathbf{c}_i(t) = \mathbf{c}(t) + s_i\, \mathbf{e}_2(t), \quad t \in [0, L),\ i = 1, 2 \ ,$$

be curves parallel to $\mathbf{c}$. First we show that $\mathbf{c}_1$ and $\mathbf{c}_2$ are both convex. Towards that we use the fact that a smooth closed curve with non-negative curvature $\kappa(t)$ is convex if and only if

$$\int_0^L \kappa(t)\, dt = 2\pi$$

(cf. Klingenberg, 1978, Chapter 2). It is easy to see that the curvature of the i-th curve at $\mathbf{c}_i(t)$ is

$$\kappa_i(t) = \frac{\kappa(t)}{1 - s_i\kappa(t)}, \quad i = 1, 2 \ , \tag{10}$$

which by assumption is nonnegative for all $t \in [0, L)$. Moreover, denoting the arc length of the i-th curve from $\mathbf{c}_i(0)$ to $\mathbf{c}_i(t)$ by $u(t)$, we have

$$\int_{u=0}^{u(L)} \kappa_i(t(u))\,du = \int_0^L \kappa_i(t)\,|\dot{\mathbf{c}}_i(t)|\,dt = \int_0^L \kappa_i(t)(1 - s_i\kappa(t))\,dt$$
$$= \int_0^L \kappa(t)\,dt = 2\pi\ . \qquad (11)$$

Now (10) and (11) imply that $\mathbf{c}_i$ is convex.

In the next step we show that $s_1 = s_2$. Assume $s_1 \neq s_2$ and without loss of generality that $s_1 < s_2$. For every fixed $t_0 \in [0, L)$ the tangent at $\mathbf{c}_2(t_0)$ defines two half-planes. Convexity of $\mathbf{c}_2$ and $\kappa_2(t) \geq 0$ imply that $\mathbf{c}_2$ lies in the half-plane which does not contain $\mathbf{c}_1(t_0)$. Since this is true for every $t_0 \in [0, L)$, we conclude $\mathbf{c}_1 \cap \mathbf{c}_2 = \emptyset$, which contradicts $\mathbf{c}_1(t_1) = \mathbf{c}_2(t_2)$. Therefore we must have $s_1 = s_2$, which implies that $\mathbf{c}_1(t) = \mathbf{c}_2(t)$ for all $t \in [0, L)$, in particular

$$\mathbf{c}_1(t_2) = \mathbf{c}_2(t_2) = \mathbf{c}_1(t_1)$$

by (9). Since $\mathbf{c}_1$ is convex this equation can only hold if $t_1 = t_2$, which contradicts the assumption of the lemma. Thus we conclude $\max(s_1, s_2) \geq 1/\kappa_\infty$. $\square$

3. Vertex and boundary point processes

In this section we consider the asymptotic behavior of certain point processes which, together with (1), (5) and (8), provide the basis for obtaining the asymptotic distributions of A_n and L_n. For background on point process theory, see Kallenberg (1983).

Suppose first that K is a closed convex polygon with vertices $v_1, \ldots, v_r$ and angles $\delta_1, \ldots, \delta_r$, ordered counterclockwise. By convexity, $\delta_j \in (0, \pi)$ for each j. Assume also that K has area A. For convenience, we shall restrict ourselves to the case where the points X_i are sampled from the uniform distribution. Other situations are easily adapted. For $1 \leq j \leq r$, let C_j be the cone $\{\lambda_1(v_{j-1} - v_j) + \lambda_2(v_{j+1} - v_j) : \lambda_i \geq 0, i = 1, 2\}$, where v_{r+1} is understood to be v_1 and v_0 as v_r. Let $\mathscr{M}_j$ be the space of locally finite counting measures on C_j endowed with the vague topology and the corresponding Borel σ-field. Define ξ_{nj} to be the $\mathscr{M}_j$-valued point process with points $\sqrt{n}(X_i - v_j),\ 1 \leq i \leq n$.

Theorem 2.

Under the assumptions stated above, as $n \to \infty$,

$$(\xi_{n1}, \dots, \xi_{nr}) \xrightarrow{d} (\xi_1, \dots, \xi_r)$$

in $\mathcal{M}_1 \times \cdots \times \mathcal{M}_r$, where $\xi_1, \dots, \xi_r$ are mutually independent homogeneous Poisson processes each with intensity A^{-1}.

Proof. Since the X_i are independent and uniformly distributed, it suffices (cf. Kallenberg (1983)) to prove that for all Borel sets $B \subset C_j$ and all j,

$$E\xi_{nj}(B) \to E\xi(B) = |B|/A \quad \text{as } n \to \infty \tag{12}$$

where $|B| =$ area of B. For large n,

$$E\xi_{ni}(B) = nP(X_1 \in v_j + n^{-1/2}B) = |B|/A$$

where

$$v_j + n^{-1/2}B = \{v_j + (n^{-1/2}x, n^{-1/2}y) : (x, y) \in B\}$$

and hence (12) is obvious. □

We next consider the smooth case. With the notation introduced in section 2 for the smooth case, assume now that K has a twice continuously differentiable boundary $\mathbf{c}(t)$ with curvature $\kappa(t)$ bounded away from 0 and ∞. We will assume that the density of X_i satisfies

$$\lim_{h \downarrow 0} \sup_{t \in [0,L)} \left| \frac{f(\mathbf{c}(t) + h\ \mathbf{e_2}(t))}{h^\alpha} - g(t) \right| = 0 \tag{13}$$

for some $\alpha > -1$ and some continuous function g bounded away from 0 and ∞. This is a rather general assumption and, in particular, covers the uniform case by $\alpha = 0$.

For any given $t \in [0, L)$, both $D_n(t)$ and $M_n(t)$ are essentially determined by points close to $\mathbf{c}(t)$, and for distinct $t_1, \dots, t_k \in [0, L)$, the point processes that describe the local behavior of points at $\mathbf{c}(t_1), \dots, \mathbf{c}(t_k)$ are asymptotically independent and hence the random vectors $(D_n(t_1), M_n(t_1)), \dots, (D_n(t_k), M_n(t_k))$ are asymptotically independent. Therefore the process $\{(D_n(t), M_n(t)), t \in [0, L)\}$ is weakly dependent in some sense. The important thing is to find the correct time and space scaling factors for the process.

Throughout the rest of this paper write for convenience

$$\gamma = 1/(3 + 2\alpha)\ . \tag{14}$$

For $t \in [0, L)$ and $a \in \mathbb{R}$, let the point process $\xi_{n,t,a}$ consist of the points $(T_{ni}(t, a), U_{ni}(t, a)), 1 \le i \le n$, where

$$\begin{aligned} T_{ni}(t,a) &= n^{\gamma}\sqrt{\kappa(t)/2}X_i(t+an^{-\gamma}),\\ U_{ni}(t,a) &= n^{2\gamma}Y_i(t+an^{-\gamma}), \quad 1 \le i \le n \ . \end{aligned}$$

For fixed $t \in [0,L)$, the process $\{\xi_{n,t,a}, a \in \mathbb{R}\}$ describes the "local" behavior of points close to $\mathbf{c}(t)$. By convexity the points $(T_{ni}(t,a), U_{ni}(t,a)), 1 \le i \le n$, are in $(-\infty,\infty) \times [0,\infty)$. We regard $\xi_{n,t,a}$ as a random element in $\mathscr{M}$, the space of locally finite counting measures on $(-\infty,\infty) \times [0,\infty)$ with the σ-field generated by the vague topology. For $t \in [0,L)$, let $\xi_{t,0}$ be a Poisson process on $(-\infty,\infty) \times [0,\infty)$ with intensity measure

$$\mu_t(B) = g(t)\sqrt{2/\kappa(t)}\iint_B (y-x^2)^{\alpha} 1(x^2 \le y)\, dx\, dy, \qquad B \subset (-\infty,\infty) \times [0,\infty) \ ,$$

and, denoting by (T_i, U_i), $i \ge 1$, the points of $\xi_{t,0}$, let $\xi_{t,a}$ be the point process with points

$$\left(T_i - a\sqrt{\kappa(t)/2}\ ,\ U_i - T_i a\sqrt{2\kappa(t)} + a^2\kappa(t)/2\right), \quad i \ge 1 \ ,$$

for all $a \in \mathbb{R}$. It is straightforward to verify that each $\xi_{t,a}$ is $\mathscr{M}$-valued and has the same distribution as $\xi_{t,0}$.

Theorem 3. *Assume that K has a smooth boundary which has a curvature bounded away from 0 and ∞. Also, assume that (13) holds for some $\alpha > -1$ and continuous function g bounded away from 0 and ∞. Then, as $n \to \infty$, for any $t \in [0,L)$ and $a_1, \ldots, a_k \in \mathbb{R}$,*

$$(\xi_{n,t,a_j}, 1 \le j \le k) \xrightarrow{d} (\xi_{t,a_j}, 1 \le j \le k)$$

in the k-fold product space of $\mathscr{M}$.

Proof. Consider first $a = 0$. Let $t \in [0,L)$ be fixed. Again, since the X_i are iid., it is sufficient to show that

$$\lim_{n\to\infty} E\xi_{n,t,0}(B) = \mu_t(B)$$

for arbitrary Borel sets $B \subset (-\infty,\infty) \times [0,\infty)$. The joint density of $(T_{n1}(t,0), U_{n1}(t,0))$ is

$$n^{-3\gamma}\sqrt{2/\kappa(t)}f(\mathbf{c}(t) + n^{-\gamma}\sqrt{2/\kappa(t)}x\ \mathbf{e}_1(t) + n^{-2\gamma}y\ \mathbf{e}_2(t)), \qquad x \in \mathbb{R},\ y > 0 \ .$$

In view of Lemma 1, for large n there exist unique $u \in [0,L)$ and $v \in \mathbb{R}$ such that

$$\mathbf{c}(t) + n^{-\gamma}\sqrt{2/\kappa(t)}x\ \mathbf{e}_1(t) + n^{-2\gamma}y\ \mathbf{e}_2(t) = \mathbf{c}(t+u) + v\ \mathbf{e}_2(t+u) \ . \tag{15}$$

By a Taylor expansion of the right hand side (cf. (2)),

$$\begin{aligned}&\mathbf{c}(t) + n^{-\gamma}\sqrt{2/\kappa(t)}x\ \mathbf{e}_1(t) + n^{-2\gamma}y\ \mathbf{e}_2(t)\\&\quad = \mathbf{c}(t) + u\ \mathbf{e}_1(t) + (u^2\kappa(t)/2 + v)\ \mathbf{e}_2(t) + \mathbf{o}(u^2) + \mathbf{o}(v),\end{aligned}$$

where $\mathbf{o}(h)$ is a vector of length $o(h)$. Therefore,

$$v \sim (n^{-2\gamma}y - u^2\kappa(t)/2) \sim n^{-2\gamma}(y - x^2)\ . \tag{16}$$

Note that if $v < 0$, the left hand side of (15) represents a point outside of K, where $f = 0$. Thus it follows from (13), (15) and (16) that

$$\begin{aligned}E\xi_{n,t,0}(B) &= n\iint_B n^{-3\gamma}f(\mathbf{c}(t) + n^{-\gamma}\sqrt{2/\kappa(t)}x\ \mathbf{e}_1(t) + n^{-2\gamma}y\ \mathbf{e}_2(t))\,dx\,dy\\&\sim n^{1-\gamma(3+2\alpha)}g(t)\sqrt{2/\kappa(t)}\iint_B (y-x^2)^\alpha 1(x^2 \le y)\,dx\,dy\\&= \mu_t(B)\ .\end{aligned}$$

Let now $a \neq 0$. Since

$$\begin{aligned}(T_{ni}(t,a), U_{ni}(t,a)) = \Big(&T_{ni}(t + an^{-\gamma}, 0)\sqrt{\kappa(t)/\kappa(t + an^{-\gamma})},\\&U_{ni}(t + an^{-\gamma}, 0)\Big)\ ,\end{aligned}$$

it follows from the first part and the continuous mapping theorem that

$$\xi_{n,t,a} \xrightarrow{d} \xi_{t,0}\ ,$$

which has the same distribution as $\xi_{t,a}$. Furthermore,

$$\begin{aligned}T_{ni}(t,a) &= n^\gamma\langle X_i - \mathbf{c}(t) - an^{-\gamma}\ \mathbf{e_1}(t) + \mathbf{o}(n^{-\gamma}),\\&\qquad \mathbf{e_1}(t) + an^{-\gamma}\kappa(t)\ \mathbf{e_2}(t) + \mathbf{o}(n^{-\gamma})\rangle\sqrt{\kappa(t)/2}\\&= T_{ni}(t,0) - a\sqrt{\kappa(t)/2} + o_p(1)\end{aligned}$$

and

$$\begin{aligned}U_{ni}(t,a) &= n^{2\gamma}\langle X_i - \mathbf{c}(t) - an^{-\gamma}\ \mathbf{e_1}(t) - a^2/2n^{-2\gamma}\kappa(t)\ \mathbf{e_2}(t) + \mathbf{o}(n^{-2\gamma}),\\&\qquad \mathbf{e_2}(t) - an^{-\gamma}\kappa(t)\ \mathbf{e_1}(t) + \mathbf{o}(n^{-\gamma})\rangle\\&= U_{ni}(t,0) - T_{ni}(t,0)a\sqrt{2\kappa(t)} + a^2\kappa(t)/2 + o_p(1)\ .\end{aligned}$$

Since this representation holds simultaneously for finitely many a's, the theorem follows by another application of the continuous mapping theorem. □

4. Asymptotic properties of L-L_n: polygonal case

Since the asymptotic distribution of $A - A_n$ was already derived by Cabo and Groeneboom (1994), we focus on the asymptotic distribution of $L - L_n$ and the rate of convergence of $E(L - L_n)$. Moreover, as in Section 3, we only consider the uniform case here and more general cases can be treated with modifications.

Let C_j and $\mathscr{M}_j$ be as defined in Section 3. For convenience assume that the edge $v_1 - v_r$ points to the negative y-direction, i.e. it lies on the straight line $\ell(p_1(0;K), 0)$. See section 2 for the definition of ℓ and p. Split the interval $[0, 2\pi)$ into subintervals $[\theta_{j-1}, \theta_j)$, $1 \le j \le r$, where $\theta_0 = 0$ and $\theta_j = \theta_{j-1} + \pi - \delta_j$, $1 \le j \le r$. For $\theta \in [\theta_{j-1}, \theta_j)$ define the mapping $\Xi_\theta : \mathscr{M}_j \to [0, \infty]$ by

$$\Xi_\theta : \eta \mapsto \inf\{p : \eta(H_1(p,\theta)) \cdot \eta(H_2(p,\theta)) > 0\} \ ,$$

where $H_1(p,\theta)$ and $H_2(p,\theta)$ are the two half-planes defined by $\ell(p,\theta)$.

Theorem 4. *Under the assumptions stated above in this section, we have*

$$\sqrt{n}(L - L_n) \xrightarrow{d} \sum_{j=1}^{r} Z_j \ ,$$

with

$$Z_j = \int_{\theta_{j-1}}^{\theta_j} \Xi_\theta(\xi_j)\, d\theta$$

where $\xi_1, \ldots, \xi_r$ are mutually independent homogeneous Poisson processes on $C_1, \ldots, C_r$, respectively, each with intensity A^{-1}.

Proof. By (1) we have

$$L - L_n = \sum_{j=1}^{r} W_{nj} \ ,$$

where

$$W_{nj} = \int_{\theta_{j-1}}^{\theta_j} (p_1(\theta; K_n) - p_1(\theta; K))\, d\theta \ .$$

By choice of the coordinate system, for $\theta \in [\theta_{j-1}, \theta_j)$,

$$\sqrt{n}(p_1(\theta; K_n) - p_1(\theta; K)) = \Xi_\theta(\xi_{nj}) \ ,$$

with ξ_{nj} as defined in Theorem 2. Hence

$$\int_{\theta_{j-1}}^{\theta_j} \Xi_\theta(\xi_{nj})\, d\theta = \sqrt{n} W_{nj} \ .$$

The theorem now follows from Theorem 2 by applying the continuous mapping theorem. □

The following derives the rate of convergence of $E(L - L_n)$. See Buchta (1984) and also Cabo and Groeneboom (1994).

Theorem 5. *Under the assumptions of this section, as $n \to \infty$,*

$$\sqrt{n}(L - EL_n) = \sqrt{\pi A/2} \sum_{j=1}^{r} \int_{-1/\tan\delta_j}^{\infty} \frac{(u + 1/\tan\delta_j)^{1/2}}{(u^2+1)^{3/2}}\, du + \mathrm{O}(1/\sqrt{n}) \ . \tag{17}$$

Proof. We continue to use the notation developed in Section 3 and the proof of Theorem 4. Fix $j = 1, \ldots, r$. For $\theta \in [\theta_{j-1}, \theta_j)$, let

$$h(\theta) = \frac{1}{2A}\big(\tan(\pi/2 - (\theta - \theta_{j-1})) + \tan(\delta_j - \pi/2 + \theta - \theta_{j-1})\big) \ .$$

By making the variable transformation $u = \tan(\pi/2 - (\theta - \theta_{j-1}))$, it is readily seen that

$$\int_{\theta=\theta_{j-1}}^{\theta_j} \int_{x=0}^{\infty} \exp\{-x^2 h(\theta)\}\, dx\, d\theta = \int_{\theta_{j-1}}^{\theta_j} \frac{1}{2}\sqrt{\pi/h(\theta)}\, d\theta$$
$$= \sqrt{\pi A/2} \int_{-1/\tan\delta_j}^{\infty} \frac{(u + 1/\tan\delta_j)^{1/2}}{(u^2+1)^{3/2}}\, du \ .$$

Also it is clear that

$$\sqrt{n} E W_{nj} = \int_{\theta=\theta_{j-1}}^{\theta_j} \int_{x=0}^{\infty} P(\Xi_\theta(\xi_{nj}) > x)\, dx\, d\theta$$

and so it remains to show that

$$\limsup_{n\to\infty} n^{1/2} \int_{\theta=\theta_{j-1}}^{\theta_j} \int_{x=0}^{\infty} |P(\Xi_\theta(\xi_{nj}) > x) - \exp\{-x^2 h(\theta)\}|\, dx\, d\theta < \infty \ . \tag{18}$$

Let $\Delta(x, \theta)$ be the intersection of K and the half-space defined by $\ell(x, \theta)$ that contains v_j and denote its area by $|\Delta(x, \theta)|$. Also write $w(\theta) = p_2(\theta; K) - p_1(\theta; K)$. Then for any θ and $x \in (0, w(\theta))$,

$$P(\Xi_\theta(\xi_{nj}) > x) = P(\xi_{nj}(\Delta(x, \theta)) = 0) = \big(1 - |\Delta(x/\sqrt{n}, \theta)|/A\big)^n \ .$$

Since $\int_{\theta=\theta_{j-1}}^{\theta_j} \int_{x=w(\theta)\sqrt{n}}^{\infty} \exp\{-x^2 h(\theta)\}\, dx\, d\theta$ tends to 0 exponentially fast, (18) follows if we prove

$$\limsup_{n\to\infty} n^{1/2} \int_{\theta=\theta_{j-1}}^{\theta_j} \int_{x=0}^{w(\theta)\sqrt{n}} \big|\big(1 - |\Delta(x/\sqrt{n}, \theta)|/A\big)^n - \exp\{-x^2 h(\theta)\}\big|\, dx\, d\theta < \infty \ . \tag{19}$$

Let

$$x(\theta) = \inf\{x > 0 : v_{j-1} \in \ell(p_1(\theta; K) + x, \theta) \text{ or } v_{j+1} \in \ell(p_1(\theta; K) + x, \theta)\}$$
$$= \min(|v_{j-1} - v_j| \sin(\theta - \theta_{j-1}), |v_{j+1} - v_j| \sin(\theta_j - \theta)) \ .$$

Observe that for $x \in (0, x(\theta))$,

$$n|\Delta(x/\sqrt{n}, \theta)|/A = x^2 h(\theta) \ .$$

Consequently,

$$n^{1/2} \int_{\theta=\theta_{j-1}}^{\theta_j} \int_{x=0}^{w(\theta)\sqrt{n}} \left| \left(1 - |\Delta(x/\sqrt{n}, \theta)|/A\right)^n - \exp\{-x^2 h(\theta)\} \right| \, dx \, d\theta$$
$$= B_n + C_n$$

where

$$B_n = n^{1/2} \int_{\theta=\theta_{j-1}}^{\theta_j} \int_{x=0}^{x(\theta)\sqrt{n}} \left| \left(1 - x^2 h(\theta)/n\right)^n - \exp\{-x^2 h(\theta)\} \right| \, dx \, d\theta \ ,$$

$$C_n = n^{1/2} \int_{\theta=\theta_{j-1}}^{\theta_j} \int_{x=x(\theta)\sqrt{n}}^{w(\theta)\sqrt{n}} \left| \left(1 - |\Delta(x/\sqrt{n}, \theta)|/A\right)^n - \exp\{-x^2 h(\theta)\} \right| dx \, d\theta \ .$$

It is easily seen that $B_n = \mathrm{O}(1)$ and, since both $(1 - |\Delta(x/\sqrt{n}, \theta)|/A)^n$ and $\exp\{-x^2 h(\theta)\}$ are bounded by $\exp\{-n|\Delta(x/\sqrt{n}, \theta)|/A\}$ (as $x^2 h(\theta)/n > |\Delta(x/\sqrt{n}, \theta)|/A$ by convexity), (19) follows from proving

$$\limsup_{n\to\infty} n^{1/2} \int_{\theta=\theta_{j-1}}^{\theta_j} \int_{x=x(\theta)\sqrt{n}}^{w(\theta)\sqrt{n}} \exp\{-n|\Delta(x/\sqrt{n}, \theta)|/A\} \, dx \, d\theta < \infty \ , \quad (20)$$

which we now do. Fix a small $\epsilon > 0$ and write

$$n^{1/2} \int_{\theta=\theta_{j-1}}^{\theta_j} \int_{x=x(\theta)\sqrt{n}}^{w(\theta)\sqrt{n}} \exp\{-n|\Delta(x/\sqrt{n}, \theta)|/A\} \, dx \, d\theta = C_{n,1} + C_{n,2} + C_{n,3}$$

where

$$C_{n,1} = n^{1/2} \int_{\theta=\theta_{j-1}}^{\theta_{j-1}+\epsilon} \int_{x=x(\theta)\sqrt{n}}^{w(\theta)\sqrt{n}} \exp\{-n|\Delta(x/\sqrt{n}, \theta)|/A\} \, dx \, d\theta \ ,$$

$$C_{n,2} = n^{1/2} \int_{\theta=\theta_{j-1}+\epsilon}^{\theta_j-\epsilon} \int_{x=x(\theta)\sqrt{n}}^{w(\theta)\sqrt{n}} \exp\{-n|\Delta(x/\sqrt{n}, \theta)|/A\} \, dx \, d\theta \ ,$$

$$C_{n,3} = n^{1/2} \int_{\theta=\theta_j+\epsilon}^{\theta_j} \int_{x=x(\theta)\sqrt{n}}^{w(\theta)\sqrt{n}} \exp\{-n|\Delta(x/\sqrt{n}, \theta)|/A\} \, dx \, d\theta \ .$$

Observe that if $\theta \in (\theta_{j-1} + \epsilon, \theta_j - \epsilon)$, then $x(\theta)$ is bounded away from zero and hence

$$\inf_{\substack{\theta\in[\theta_{j-1}+\epsilon,\theta_j-\epsilon]\\ x\in(x(\theta)\sqrt{n},w(\theta)\sqrt{n})}} |\Delta(x/\sqrt{n},\theta)| > \inf_{\theta\in[\theta_{j-1}+\epsilon,\theta_j-\epsilon]} \Delta(x(\theta),\theta) > 0 \ .$$

This implies that $C_{n,3}$ tends to zero exponentially fast. It remains to consider $C_{n,1}$ and $C_{n,2}$. The two can be handled in the same manner and so we focus on $C_{n,1}$. Make the observation that if ϵ is small enough, then there exist $b_1, b_2 > 0$ such that for $\theta \in (\theta_{j-1}, \theta_{j-1}+\epsilon)$,

$$x(\theta) = |v_{j-1} - v_j| \sin(\theta - \theta_{j-1}) \geq b_1(\theta - \theta_{j-1})$$

and

$$\begin{aligned} \inf_{x\in(x(\theta)\sqrt{n},w(\theta)\sqrt{n})} |\Delta(x/\sqrt{n},\theta)| &\geq |\Delta(x(\theta),\theta)| = Ax^2(\theta)h(\theta) \\ &\geq \frac{|v_{j-1}-v_j|^2 \sin(\theta-\theta_{j-1})\cos(\theta-\theta_{j-1})}{2} \\ &\geq b_2(\theta - \theta_{j-1}) \ . \end{aligned}$$

These imply that

$$C_{n,1} \leq n^{1/2} \int_{\theta=\theta_{j-1}}^{\theta_{j-1}+\epsilon} \int_{x=b_1(\theta-\theta_{j-1})\sqrt{n}}^{w(\theta)\sqrt{n}} \exp\{-nb_2(\theta-\theta_{j-1})/A\}\, dx\, d\theta$$

which is clearly bounded. This proves (20) and completes the proof. □

Remark. Buchta (1984) also considered the remainder term in (17). However, the rate he obtained was $o(n^{-1/2+\epsilon})$ for any $\epsilon > 0$, which is less definitive than ours. Moreover, our proof appears to be considerably simpler.

5. Asymptotic properties of $A - A_n$ and $L - L_n$: smooth case

With the notation introduced in Sections 2 and 3 for the smooth case, we will be concerned here with the case where the convex set K has a twice continuously differentiable boundary $\mathbf{c}(t)$ with curvature $\kappa(t)$ bounded away from 0 and ∞. Also we will assume a very weak regularity condition to start with on the distribution of the X_i, i.e., that (13) holds for some $\alpha > -1$ and some continuous function g bounded away from 0 and ∞.

On the subspace

$$\mathscr{S} := \{\eta \in \mathscr{M} : \eta((0,\infty)\times[0,\infty)) \cdot \eta((-\infty,0)\times[0,\infty)) \neq 0\} \ ,$$

define the mapping $\Xi = (\Xi_1, \Xi_2)$, where Ξ_1 maps $\eta \in \mathscr{S}$ to the smallest y-intercept of all lines connecting pairs of points of η whose x-coordinates have opposite signs and Ξ_2 maps η to the smallest y-coordinate of the points of η. Note that the mapping Ξ is finite and continuous at each $\eta \in \mathscr{S}$, and moreover

$$\Xi(\xi_{n,t,a}) = n^{2\gamma}(D_n(t + an^{-\gamma}), M_n(t + an^{-\gamma})), \quad t \in [0, L), \ a \in \mathbb{R} \ . \tag{21}$$

By definition, $P(\xi_{t,a} \in \mathscr{S}^c) = 0$ for all $t \in [0, L)$ and $a \in \mathbb{R}$, and hence by (21) and the continuous mapping theorem we obtain

$$\left(n^{2\gamma}(D_n(t+a_j n^{-\gamma}), M_n(t+a_j n^{-\gamma})), \ 1 \le j \le k\right) \xrightarrow{d} \left(\Xi(\xi_{t,a_j}), \ 1 \le j \le k\right) \ , \tag{22}$$

for fixed $t \in [0, L)$ and $a_1, \ldots, a_k \in \mathbb{R}$.

Theorem 6. *Assume the conditions of Theorem* 3 *and let* $\{\xi_{t,a}\}$ *be as defined there. As* $n \to \infty$, *the distribution of* $n^{5\gamma/2}(A_n - EA_n, L_n - EL_n)$ *converges to the bivariate normal distribution with zero mean and covariance matrix* $\mathbf{\Sigma}$, *where*

$$\mathbf{\Sigma}_{ij} = \int_{t=0}^{L} \int_{a=-\infty}^{\infty} \kappa(t)^{i+j-2} \mathrm{cov}(\Xi_i(\xi_{t,0}), \Xi_j(\xi_{t,a}))\, da\, dt, \quad i, j = 1, 2 \ ,$$

which are well-defined and finite.

Proof. Some technical details of this proof are omitted since they essentially reproduce those in Hsing (1994). The proof comprises two parts. First is the computation of the asymptotic covariance matrix. We will show the computation of $\mathbf{\Sigma}_{12}$ only and the same principle applies for the other entries in $\mathbf{\Sigma}$ in an obvious way. Since

$$\begin{aligned} \mathrm{cov}(A_n, L_n) &= \mathrm{cov}\Big(A - A_n, L - L_n\Big) \\ &= \mathrm{cov}\Big((A - A_n)I_{E_n}, L - L_n\Big) \\ &\quad + \mathrm{cov}\Big((A - A_n)(1 - I_{E_n}), L - L_n\Big) \end{aligned} \tag{23}$$

where E_n is the event constructed in section 2, in view of (5) and (8) we need to handle quantities such as

$$\begin{aligned} &\mathrm{cov}\left(\int_0^L w_1(t) D_n^j(t)\, dt, \int_0^L w_2(s) M_n^k(s)\, ds\right) \\ &\quad = \int_{t=0}^{L} \int_{s=t-L/2}^{t+L/2} w_1(t) w_2(s) \mathrm{cov}(D_n^j(t), M_n^k(s))\, ds\, dt \\ &\quad = n^{-(2(j+k)+1)\gamma} \int_{t=0}^{L} \int_{a=-Ln^{\gamma}/2}^{Ln^{\gamma}/2} w_1(t) w_2(t + an^{-\gamma}) \\ &\qquad \times \mathrm{cov}(n^{2\gamma j} D_n^j(t), n^{2\gamma k} M_n^k(t + an^{-\gamma}))\, da\, dt \end{aligned}$$

for bounded continuous functions w_1, w_2 and positive integers j, k, where w_1, w_2, D_n and M_n are understood to be periodic functions with period L. Now, by a standard uniform integrability argument (cf. Hsing, 1994, Theorem 2.4) and (22), for arbitrary $t \in [0, L)$, $a_1, \ldots, a_k \in \mathbb{R}$ and $r_1, \ldots, r_k, s_1, \ldots, s_k \geq 0$,

$$\lim_{n\to\infty} E\left(\prod_{j=1}^{k}\{n^{2\gamma}D_n(t+a_jn^{-\gamma})\}^{r_j}\{n^{2\gamma}M_n(t+a_jn^{-\gamma})\}^{s_j}\right)$$
$$= E\left(\prod_{j=1}^{k}\{\Xi_1(\xi_{t,a_j})\}^{r_j}\{\Xi_2(\xi_{t,a_j})\}^{s_j}\right) < \infty \ .$$

By this and another uniform integrability argument (cf. Hsing 1994, Lemma 3.5),

$$\lim_{n\to\infty} n^{(2(j+k)+1)\gamma}\mathrm{cov}\left(\int_0^L w_1(t)D_n^j(t)\,dt, \int_0^L w_2(s)M_n^k(s)\,ds\right)$$
$$= \int_{t=0}^{L}\int_{a=-\infty}^{\infty} w_1(t)w_2(t)\mathrm{cov}(\{\Xi_1(\xi_{t,0})\}^j, \{\Xi_2(\xi_{t,a})\}^k)\,da\,dt < \infty \ .$$

So it follows from (23) and the fact that $E(1 - I_{E_n}) \to 0$ exponentially fast, that the asymptotic covariance of A_n and L_n is obtained as that of $\int_0^L D_n(t)\,dt$ and $\int_0^L M_n(t)\kappa(t)\,dt$.

The second step in the proof is to use a blocking argument. The idea is to divide up the interval $[0, L)$ into big and small intervals in such a way that the contribution of $(D_n(t), M_n(t))$ to $(\int_0^L D_n(t)\,dt, \int_0^L M_n(t)\kappa(t)\,dt)$ for t in the small intervals is asymptotically negligible and the contributions of $(D_n(t), M_n(t))$ to $(\int_0^L D_n(t)\,dt, \int_0^L M_n(t)\kappa(t)\,dt)$ for t in distinct big intervals are asymptotically independent. The details of the proof of Theorem 4.1. in Hsing (1994) are readily adapted in the present context and are not reproduced here to conserve space. $\square$

We now turn to the rates of convergence of the expectations. For that we need to have more information about the convergence rate in (13). To illustrate this, in the remaining part of this section assume that there exist $\alpha > -1$, $\eta > 0$ and a function g bounded away from 0 and ∞ such that the density of X_i satisfies

$$\limsup_{h\downarrow 0}\ \sup_{t\in[0,L)} \frac{1}{h^\eta}\left|\frac{f(\mathbf{c}(t) + h\ \mathbf{e_2}(t))}{h^\alpha} - g(t)\right| < \infty \ . \tag{24}$$

Note that for the uniform case $\alpha = 0$ and η may be chosen arbitrarily. The results below extend the original results by Rényi and Sulanke.

Theorem 7. *Assume that* (24) *holds. Then*

$$n^{2\gamma}(L - EL_n) = c + \mathrm{O}\left(n^{-\gamma(1\wedge 2\eta\wedge(\alpha+1))}\right) \quad \text{as} \quad n \to \infty \ , \tag{25}$$

where

$$c = \Gamma(1+2\gamma)\left(2^{3/2}\gamma B(1/2, \alpha+1)\right)^{-2\gamma}\int_0^L \frac{\kappa(t)^{1+\gamma}}{g(t)^{2\gamma}}\, dt \ .$$

Proof. Let

$$a(t) = B(1/2, \alpha+1) g(t)\sqrt{\frac{2}{\kappa(t)}} \ .$$

By calculus,

$$c = \int_{t=0}^{L}\int_{x=0}^{\infty} \kappa(t) \exp\{-2\gamma a(t) x^{\frac{1}{2\gamma}}\}\, dx\, dt \ . \tag{26}$$

On the other hand, by (5),

$$\begin{aligned} n^{2\gamma}E(L - L_n) &= \int_{t=0}^{L} E[n^{2\gamma}M_n(t)]\kappa(t)\, dx\, dt \\ &= \int_{t=0}^{L}\int_{x=0}^{\infty} P(M_n(t) > n^{-2\gamma}x)\kappa(t)\, dx\, dt \ . \end{aligned} \tag{27}$$

Let $x_n = (B\log n)^{2\gamma}$ where B is a finite constant satisfying

$$2B\left(\inf_t a(t)\right) - 2 > \left(1 \wedge 2\eta \wedge (\alpha+1)\right) \ . \tag{28}$$

By (26) and (27),

$$n^{2\gamma}E(L - L_n) - c = \int_{t=0}^{L}\left(B_{n,1}(t) + B_{n,2}(t)\right)\kappa(t)\, dt \tag{29}$$

where

$$B_{n,1}(t) = \int_0^{x_n}\left(P(M_n(t) > n^{-2\gamma}x) - \exp\{-2\gamma a(t) x^{\frac{1}{2\gamma}}\}\right) dx$$

and

$$B_{n,2}(t) = \int_{x_n}^{\infty}\left(P(M_n(t) > n^{-2\gamma}x) - \exp\{-2\gamma a(t) x^{\frac{1}{2\gamma}}\}\right) dx \ .$$

We first analyze $B_{n,1}(t)$. Let $Y_i(t)$ be defined by (4). By Lemma 9 below,

$$
\begin{aligned}
&B_{n,1}(t)\\
&\quad = \int_{x=0}^{x_n} \left[\left(1 - P(Y_1(t) \le xn^{-2\gamma})\right)^n - \exp\{-2\gamma a(t) x^{\frac{1}{2\gamma}}\} \right] dx\\
&\quad = \int_{x=0}^{x_n} \left[\left\{ 1 - 2\gamma a(t)(xn^{-2\gamma})^{\frac{1}{2\gamma}} \left(1 + \mathrm{O}\left((xn^{-2\gamma})^{\frac{1}{2}\wedge\frac{\alpha+1}{2}\wedge\eta}\right)\right)\right\}^n \right.\\
&\qquad\qquad \left. - \exp\{-2\gamma a(t) x^{\frac{1}{2\gamma}}\} \right] dx\\
&\quad = \int_{x=0}^{x_n} \left[\exp\left\{ -2\gamma a(t) x^{\frac{1}{2\gamma}} \left(1 + \mathrm{O}\left((xn^{-2\gamma})^{\frac{1}{2}\wedge\frac{\alpha+1}{2}\wedge\eta}\right)\right) + \mathrm{O}\left(n^{-1} x^{\frac{1}{\gamma}}\right)\right\} \right.\\
&\qquad\qquad \left. - \exp\{-2\gamma a(t) x^{\frac{1}{2\gamma}}\} \right] dx
\end{aligned}
$$

and so

$$
|B_{n,1}(t)| = \mathrm{O}\left(n^{-\gamma(1\wedge 2\eta\wedge(\alpha+1))}\right) \text{ uniformly in } t \ . \tag{30}
$$

Next we consider $B_{n,2}(t)$. By the inequality $\log(1-\lambda) < -\lambda, \lambda \in (0,1)$, the fact that $\sup_t \sup_{y>y_0} P(Y_1(t) > y) = 0$ for some $y_0 < \infty$ and the derivations on $B_{n,1}(t)$ above, we obtain

$$
\begin{aligned}
&\int_{x_n}^{\infty} \left(1 - P(Y_1(t) \le xn^{-2\gamma})\right)^n dx\\
&\quad = \int_{x_n}^{y_0 n^{2\gamma}} \exp\left\{ n \log\left(1 - P(Y_1(t) \le xn^{-2\gamma})\right)\right\} dx\\
&\quad \le \int_{x_n}^{y_0 n^{2\gamma}} \exp\left\{ -nP(Y_1(t) \le x_n n^{-2\gamma})\right\} dx\\
&\quad \le y_0 n^{2\gamma} \exp\left\{ -nP(Y_1(t) \le x_n n^{-2\gamma})\right\}\\
&\quad = y_0 n^{2\gamma} \exp\left\{ -2\gamma a(t) x_n^{\frac{1}{2\gamma}} (1 + \mathrm{o}(1))\right\} \quad \text{by Lemma 9 below}\\
&\quad = y_0 n^{2\gamma} \exp\{-2\gamma a(t) B(\log n)(1 + \mathrm{o}(1))\}
\end{aligned}
$$

which tends to 0 faster than $n^{-\gamma(1\wedge(\alpha+1)\wedge 2\eta)}$ by (28). The same can obviously be said for $\int_{x_n}^{\infty} \exp\{-2\gamma a(t) x^{\frac{1}{2\gamma}}\}\, dx$ and so

$$
|B_{n,2}(t)| = \mathrm{O}\left(n^{-\gamma(1\wedge 2\eta\wedge(\alpha+1))}\right) \text{ uniformly in } t \ . \tag{31}
$$

The result therefore follows from (29), (30) and (31). □

Next we consider the expected area.

Theorem 8. *Assume that* (24) *holds. Then we have*

$$\begin{aligned} n^{2\gamma}&(A - EA_n) \\ &= 2^{1-3\gamma}B(1/2, \alpha+1)^{-2\gamma}(1+\gamma)\gamma^{1-2\gamma}\Gamma(2\gamma) \\ &\quad \times \int_0^L \left(\frac{\sqrt{\kappa(t)}}{g(t)}\right)^{2\gamma} dt + \mathrm{O}\left(n^{-\gamma(1\wedge 2\eta\wedge(\alpha+1))}\right) . \end{aligned}$$

Proof. Let E_n be the event defined in (6). By (8) and the fact that $P(E_n)$ tends to 0 exponentially fast, it suffices to prove that uniformly for $t \in [0, L)$,

$$\begin{aligned} n^{2\gamma}ED_n(t) =& 2^{1-3\gamma}B(1/2, \alpha+1)^{-2\gamma}(1+\gamma)\gamma^{1-2\gamma}\Gamma(2\gamma) \\ & \left(\frac{\sqrt{\kappa(t)}}{g(t)}\right)^{2\gamma} + \mathrm{O}\left(n^{-\gamma(1\wedge 2\eta\wedge(\alpha+1))}\right) , \end{aligned}$$

which we now do. Fix $t \in [0, L)$. Let $z_0 > 0$ and $\theta_0 > 0$ be defined by

$$\mathbf{c}(t) + z_0\ \mathbf{e}_2(t) = \ \mathbf{c}(t + \theta_0) \ .$$

Clearly,

$$ED_n(t) = \int_{z=0}^{z_0} P(D_n(t) > z)\ dz$$

Now, focus on the event $(D_n(t) > z)$. For $0 \le \theta < L$, define the set

$$\begin{aligned} G(t, z, \theta) = K \cap & \text{ the right half-plane determined by the directional} \\ & \text{line from } \ \mathbf{c}(t) + z\ \mathbf{e}_2 \text{ to } \ \mathbf{c}(t+\theta) \ . \end{aligned}$$

Ignoring the probability zero event of having points on the line $\{\mathbf{c}(t) + u\ \mathbf{e}_2(t), -\infty < u < \infty\}$, $D_n(t) > z$ if and only if for some $\theta \in [0, \theta_o]$, $G(t, z, \theta)$ contains none of $X_i, 1 \le i \le n$. Since $G(t, z, \theta)$ is monotone in θ, we can write

$$ED_n(t) = \int_{z=0}^{z_0} P(D_n(t) > z)\ dz = \int_{z=0}^{z_0}\int_{\theta=0}^{\theta_0} d_\theta P(\Theta_n(z) \le \theta)\ dz \qquad (32)$$

where $\Theta_n(z) = \sup\{0 \le \theta \le \theta_0 : G(t, z, \theta)$ contains none of $\boldsymbol{X}_i, 1 \le i \le n\}$ (sup(null set) $\equiv \infty$). By (32), for any positive constant c,

$$\begin{aligned} &\left| ED_n(t) - \int_{z=0}^{z_0}\int_{\theta=0}^{\theta_0} I\left(\theta + \frac{2}{\kappa(t)}\frac{z}{\theta} \le cn^{-\gamma}(\log n)^{\gamma}\right) d_\theta P(\Theta_n(z) \le \theta)\ dz \right| \\ &\quad = \left| \int_{z=0}^{z_0}\int_{\theta=0}^{\theta_0} I\left(\theta + \frac{2}{\kappa(t)}\frac{z}{\theta} > cn^{-\gamma}(\log n)^{\gamma}\right) d_\theta P(\Theta_n(z) \le \theta)\ dz \right| \\ &\quad \le E_{n,1} + E_{n,2} + E_{n,3} \end{aligned}$$

where

$$E_{n,1} = \int_{z=0}^{z_0} \int_{\theta=0}^{\theta_0} I\left(z > \frac{c^2\kappa(t)}{8} n^{-2\gamma} (\log n)^{2\gamma}\right) d_\theta P(\Theta_n(z) \le \theta)\ dz \ ,$$

$$E_{n,2} = \int_{z=0}^{z_0} \int_{\theta=0}^{\theta_0} I\left(\theta > \frac{c}{2} n^{-\gamma} (\log n)^{\gamma}\right) d_\theta P(\Theta_n(z) \le \theta)\ dz \ ,$$

$$E_{n,3} = \int_{z=0}^{z_0} \int_{\theta=0}^{\theta_0} I\left(z \le \frac{c^2\kappa(t)}{8} n^{-2\gamma} (\log n)^{2\gamma}, \theta \le \frac{c}{2} n^{-\gamma} (\log n)^{\gamma}, \right.$$
$$\left. \frac{z}{\theta} > \frac{c\kappa(t)}{4} n^{-\gamma} (\log n)^{\gamma}\right) d_\theta P(\Theta_n(z) \le \theta)\ dz \ .$$

By arguments similar to those in Lemma 9 below, for any $\epsilon > 0$ there exists some $c \in (0, \infty)$ such that

$$E_{n,1} = \mathrm{o}(n^{-\epsilon}) \ .$$

Applying Lemma 10 below, it is easy to verify that for any $\epsilon > 0$ there exists some constant $c \in (0, \infty)$ such that

$$E_{n,2} \le z_0 P^n \left(X_1 \notin R(t, t + c2^{-1} n^{-\gamma} (\log n)^{\gamma})\right) = \mathrm{o}(n^{-\epsilon})$$

where $R(\cdot,\cdot)$ was defined in Section 2. By Lemma 13 below and the symmetry of the roles of θ and $\tilde{\theta}$ defined there, we conclude in the same way as for $E_{n,2}$ that for any $\epsilon > 0$ there exists some $c \in (0, \infty)$ such that

$$E_{n,3} = \mathrm{o}(n^{-\epsilon}) \ .$$

Also, observe that

$$d_\theta P(\Theta_n(z) \le \theta) = P^{n-1}(X_1 \notin G(t, z, \theta)) n d_\theta P(X_1 \in H(t, z, \theta))$$

where

$$H(t, z, \theta) = G(t, z, \theta) \cap \text{ the right half-plane determined by the normal of } \partial K \text{ at } \mathbf{c}(t).$$

So for any $\epsilon > 0$ there exists some $c \in (0, \infty)$ such that

$$ED_n(t) = \iint I\left(\theta + \frac{2}{\kappa(t)} \frac{z}{\theta} \le c n^{-\gamma} (\log n)^{\gamma}\right) P^{n-1}(X_1 \notin G(t, z, \theta))$$
$$\times\ n d_\theta P(X_1 \in H(t, z, \theta))\ dz + \mathrm{o}(n^{-\epsilon}) \ .$$

Changing variables from (z, θ) to (x, u) in the integral on the right hand side, where $z(x) = n^{-2\gamma} x$ and $\theta(u) = n^{-\gamma} \sqrt{2/\kappa(t)} u$, we conclude from the preceding line of discussions that there exists some $c > 0$ such that

$$n^{2\gamma}ED_n(t) = \iint I(u + x/u \le c(\log n)^\gamma)P^{n-1}(X_1 \notin G_n(t,x,u))$$
$$\times\, nd_uP(X_1 \in H_n(t,x,u))\; dx + o\left(n^{-\gamma(1\wedge 2\eta\wedge(\alpha+1))}\right) \quad (33)$$

where

$$G_n(t,x,u) = G(t, n^{-2\gamma}x, n^{-\gamma}\sqrt{2/\kappa(t)}u)\ ,$$
$$H_n(t,x,u) = H(t, n^{-2\gamma}x, n^{-\gamma}\sqrt{2/\kappa(t)}u)\ .$$

Write

$$\iint I(u + x/u \le c(\log n)^\gamma)$$
$$\times\, P^{n-1}(\boldsymbol{X}_1 \notin G_n(t,x,u))nd_uP(X_1 \in H_n(t,x,u))\; dx$$
$$= g(t)\sqrt{2/\kappa(t)}\int_{x=0}^{\infty}\int_{u=0}^{\infty}\int_{y=0}^{u}\exp\{-\beta(u+x/u)^{1/\gamma}\}\frac{y(u^2+x)}{u^2}$$
$$\times\left(\frac{y(u^2-x)}{u} + x - y^2\right)^\alpha dy\; du\; dx + \Delta_{n,1} + \Delta_{n,2} + \Delta_{n,3} \quad (34)$$

where $\beta = g(t)\sqrt{\frac{2}{\kappa(t)}}\gamma 2^{1-1/\gamma}B(1/2, \alpha+1)$ and

$$\Delta_{n,1} = \iint I(u + x/u \le c(\log n)^\gamma)P^{n-1}(X_1 \notin G_n(t,x,u))nd_u$$
$$\times\, P(\boldsymbol{X}_1 \in H_n(t,x,u))\; dx - g(t)\sqrt{2/\kappa(t)}$$
$$\times \int_x\int_u\int_{y=0}^{u} I(u + x/u \le c(\log n)^\gamma)P^{n-1}(X_1 \notin G_n(t,x,u))$$
$$\times \frac{y(u^2+x)}{u^2}\left(\frac{y(u^2-x)}{u} + x - y^2\right)^\alpha dy\; du\; dx,$$

$$\Delta_{n,2} = g(t)\sqrt{2/\kappa(t)}\int_x\int_u\int_{y=0}^{u} I(u + x/u \le c(\log n)^\gamma)$$
$$\times\, P^{n-1}(X_1 \notin G_n(t,x,u)) \times \frac{y(u^2+x)}{u^2}\left(\frac{y(u^2-x)}{u} + x - y^2\right)^\alpha dy\, du\, dx$$
$$- g(t)\sqrt{2/\kappa(t)}\int_x\int_u\int_{y=0}^{u} I(u + x/u \le c(\log n)^\gamma)$$
$$\times \exp\{-\beta(u+x/u)^{1/\gamma}\} \times \frac{y(u^2+x)}{u^2}\left(\frac{y(u^2-x)}{u} + x - y^2\right)^\alpha dy\, du\, dx\ ,$$

$$\Delta_{n,3} = g(t)\sqrt{2/\kappa(t)} \int_x \int_u \int_{y=0}^{u} I(u + x/u > c(\log n)^\gamma)$$
$$\times \exp\{-\beta(u + x/u)^{1/\gamma}\} \frac{y(u^2+x)}{u^2} \left(\frac{y(u^2-x)}{u} + x - y^2\right)^\alpha dy\, du\, dx \ .$$

In view of Lemma 11, (33) and (34), it suffices to show that $\Delta_{n,i} = \mathrm{O}\big(n^{-\gamma(1\wedge 2\eta\wedge(\alpha+1))}\big)$ for $i = 1, 2, 3$. The proof for $i = 3$ can be seen from that of Lemma 11 and is therefore omitted. The proofs for $i = 1$ and $i = 2$ are given in Lemmas 16 and 15, respectively. □

The lemmas below provide some of the details required in the proofs of Theorems 7 and 8. The assumptions and notations of the theorems will therefore be assumed without further mention.

Lemma 9. *Let $Y_1(t)$ be defined by (4). Uniformly for $t \in [0, L)$, the density of $Y_1(t)$ has the form*

$$h_{Y_1(t)}(y) = a(t) y^{\alpha+1/2}(1 + \mathrm{O}(y^{\eta'})) \qquad as\ y \downarrow 0 \ ,$$

where

$$a(t) = B(1/2, \alpha+1) g(t)\sqrt{2/\kappa(t)}$$

and

$$\eta' = \min\left(\frac{1}{2}, \frac{\alpha+1}{2}, \eta\right) \ .$$

Proof. Let $X_1(t)$, $Y_1(t)$ be defined by (4). Write $X_1(t) = Z_1(t)\sqrt{2Y_1(t)/\kappa(t)}$. Then the joint density of $Y_1(t)$ and $Z_1(t)$ is

$$h_{Y_1(t),Z_1(t)}(y, z) = f(\mathbf{c}(t) + z\sqrt{2y/\kappa(t)}\ \mathbf{e}_1(t) + y\ \mathbf{e}_2(t))\sqrt{2y/\kappa(t)} \ .$$

Integrating z out gives

$$h_{Y_1(t)}(y) = \int_{z_1}^{z_2} h_{Y_1(t),Z_1(t)}(y, z)\, dz \ ,$$

where $z_i = z_i(y),\ i = 1, 2$ are defined through

$$\mathbf{c}(t) + z_i\sqrt{2y/\kappa(t)}\ \mathbf{e}_1(t) + y\ \mathbf{e}_2(t) \in \partial K \ .$$

Write

$$\begin{aligned} &\mathbf{c}(t) + z\sqrt{2y/\kappa(t)}\ \mathbf{e}_1(t) + y\ \mathbf{e}_2(t) \\ &\quad = \mathbf{c}(t+u) + v\ \mathbf{e}_2(t+u) \\ &\quad = \mathbf{c}(t) + (u - uv\kappa(t))\ \mathbf{e}_1(t) + (u^2\kappa(t)/2 + v)\ \mathbf{e}_2(t) + \mathrm{O}(u^3 + vu^2) \ , \end{aligned} \tag{35}$$

where $u = u(y,z)$ and $v = v(y,z)$. Observe that

$$u \sim z\sqrt{2y/\kappa(t)} \ ,$$

which implies

$$u = \mathrm{O}(y^{1/2}) \ , \tag{36}$$

uniformly in z. Therefore by assumption and since $v \leq y$,

$$h_{Y_1(t),Z_1(t)}(y,z) = g(t)\sqrt{2/\kappa(t)}(1 + \mathrm{O}(y^{1/2}))y^{1/2}v(y,z)^{\alpha}(1 + O(y^{\eta})) \ .$$

Hence the proof is complete upon showing

$$y^{-\alpha}\int_{z_1}^{z_2} v(y,z)^{\alpha}\, dz = B(1/2, \alpha + 1) + O(y^{1/2} + y^{(\alpha+1)/2}) \ . \tag{37}$$

From (35) we obtain

$$\begin{aligned} z^2 y &= u^2\kappa(t)/2 + \mathrm{O}(u^4 + vu^2), \\ y &= u^2\kappa(t)/2 + v + \mathrm{O}(u^3 + vu^2) \ . \end{aligned}$$

Hence

$$(1 - z^2)y = v + \mathrm{O}(u^3 + vu^2) \ , \tag{38}$$

which implies

$$v(y,z) = (1 - z^2 + \Delta(y,z))y$$

with

$$\Delta(y,z) = \mathrm{O}(y^{1/2}) \ ,$$

uniformly in z. Since $v(y, z_i) = 0,\ i = 1, 2$, it is readily seen from (38) and (36) that

$$1 - z_i^2 = \mathrm{O}(y^{1/2}),\ \ i = 1, 2 \ .$$

Now in (37) consider the range of integration $z \in [0, z_2]$. Write $\Delta_1(y) = \min\{\Delta(y,z) : 0 \leq z \leq z_2\}$ and $\Delta_2(y) = \max\{\Delta(y,z) : 0 \leq z \leq z_2\}$. Then

$$\begin{aligned} &\int_0^{z_2 \wedge \sqrt{1+\Delta_1(y)}} (1 - z^2 + \Delta_1(y))^{\alpha}\, dz \\ &\quad \leq \int_0^{z_2} (1 - z^2 + \Delta(y,z))^{\alpha}\, dz = y^{-\alpha}\int_0^{z_2} v(y,z)^{\alpha}\, dz \\ &\quad \leq \int_0^{z_2} (1 - z^2 + \Delta_2(y))^{\alpha}\, dz \ . \end{aligned} \tag{39}$$

First consider the upper bound in (39). Writing $w = z/\sqrt{1 + \Delta_2(y)}$ and using the fact that $z_2 \leq \sqrt{1 + \Delta_2(y)}$, we obtain

$$
\begin{aligned}
&\int_0^{z_2} (1 - z^2 + \Delta_2(y))^\alpha \, dz \\
&= \int_0^{z_2/\sqrt{1+\Delta_2(y)}} (1 + \Delta_2(y))^{\alpha+1/2}(1 - w^2)^\alpha \, dw \\
&\quad \le (1 + \mathrm{O}(y^{1/2})) \int_0^1 (1 - w^2)^\alpha \, dz \\
&= \frac{1}{2} B(1/2, \alpha + 1) + \mathrm{O}(y^{1/2}) \ .
\end{aligned}
$$

Next consider the lower bound in (39).

$$
\begin{aligned}
&\int_0^{z_2 \wedge \sqrt{1+\Delta_1(y)}} (1 - z^2 + \Delta_1(y))^\alpha \, dz \\
&= \int_0^{z_2/\sqrt{1+\Delta_1(y)} \wedge 1} (1 + \Delta_1(y))^{\alpha+1/2}(1 - w^2)^\alpha \, dw \\
&= (1 + \mathrm{O}(y^{1/2}))\Big(\frac{1}{2} B(1/2, \alpha + 1) - \int_{z_2/\sqrt{1+\Delta_1(y)} \wedge 1}^1 (1 - w^2)^\alpha \, dw\Big) \\
&\ge \frac{1}{2} B(1/2, \alpha + 1) + \mathrm{O}(y^{1/2}) - \left|1 - \frac{z_2}{\sqrt{1 + \Delta_1(y)}}\right| \left|1 - \frac{z_2^2}{1 + \Delta_1(y)}\right|^\alpha \\
&= \frac{1}{2} B(1/2, \alpha + 1) + \mathrm{O}(y^{1/2} + y^{(\alpha+1)/2}) \ .
\end{aligned}
$$

By symmetry we also have

$$
y^{-\alpha} \int_{z_1}^0 v(y, z)^\alpha \, dz = \frac{1}{2} B\Big(1/2, \alpha + 1) + \mathrm{O}(y^{1/2} + y^{(\alpha+1)/2}\Big)
$$

which concludes the proof. □

Lemma 10. *Let $R(\cdot,\cdot)$ be as defined in Section 2. Then*

$$
P(\boldsymbol{X}_1 \in R(w', w)) = b(t)(w - w')^{1/\gamma}(1 + \mathrm{O}((w - w')^\nu))
$$
$$
\text{as } w', w \to t \ (w' \le t \le w),
$$

where $b(t) = B(1/2, \alpha + 1)\gamma g(t)(\kappa(t)/2)^{\frac{1}{2\gamma} - \frac{1}{2}} 2^{1-1/\gamma}$ *and* $\nu = \min(1, \alpha + 1, 2\eta)$.

Proof. Let $\tilde{t} \in (w', w)$ be such that $\mathbf{e}_1(\tilde{t})$ is parallel to $\mathbf{c}(w) - \mathbf{c}(w')$, i.e.

$$
\mathbf{c}(w) = \mathbf{c}(\tilde{t}) + y \, \mathbf{e}_2(\tilde{t}) + p \, \mathbf{e}_1(\tilde{t})
$$

for some y and p. Clearly, $\tilde{t} - t \le w - w'$. By Lemma 9 we have

$$
P(X_1 \in R(w', w)) = P(Y(\tilde{t}) \le y) = 2\gamma a(t) y^{\frac{1}{2\gamma}}(1 + \mathrm{O}(y^{\eta'})) \ .
$$

Along similar lines as in Lemma 9 it is seen that $w - \tilde{t}$ and $\tilde{t} - w'$ can both be written as

$$\sqrt{\frac{2y}{\kappa(\tilde{t})}}\left(1 + \mathrm{O}\left(y^{1/2}\right)\right)$$

and therefore

$$w - w' = 2\sqrt{\frac{2y}{\kappa(\tilde{t})}}\left(1 + \mathrm{O}\left(y^{1/2}\right)\right) \ .$$

From this we conclude

$$y = \frac{\kappa(\tilde{t})}{8}(w - w')^2(1 + \mathrm{O}(w - w'))$$

and, since κ and g have bounded derivatives,

$$\begin{aligned} P(\boldsymbol{X}_1 \in R(w', w)) \ &= 2\gamma a(\tilde{t})\left(\frac{\kappa(\tilde{t})}{8}\right)^{\frac{1}{2\gamma}}(w - w')^{1/\gamma}(1 + \mathrm{O}(w - w')) \\ &\quad (1 + \mathrm{O}((w - w')^{2\eta'})) \\ &= b(t)(w - w')^{1/\gamma}(1 + \mathrm{O}(w - w') + \mathrm{O}((w - w')^{2\eta'})). \end{aligned}$$

□

Lemma 11. *For each* $t \in [0, L)$,

$$\begin{aligned} &g(t)\sqrt{2/\kappa(t)}\int_{x=0}^{\infty}\int_{u=0}^{\infty}\int_{y=0}^{u}\exp\{-\beta(u + x/u)^{1/\gamma}\}\frac{y(u^2 + x)}{u^2} \\ &\times\left(\frac{y(u^2 - x)}{u} + x - y^2\right)^{\alpha} dy\, du\, dx \\ &\quad = 2^{1-3\gamma}B(1/2, \alpha + 1)^{-2\gamma}(1 + \gamma)\gamma^{1-2\gamma}\Gamma(2\gamma)\left(\frac{\sqrt{\kappa(t)}}{g(t)}\right)^{2\gamma}. \end{aligned}$$

Proof. Write

$$k = k(t) = g(t)\sqrt{\frac{2}{\kappa(t)}} \ .$$

Then

$$
\begin{aligned}
&k\int_0^u \frac{y(u^2+x)}{u^2}\left(\frac{u^2-x}{u}y+x-y^2\right)^{\alpha}dy\\
&\quad= k\frac{u^2+x}{2u^2}\int_0^u\left(\left(\frac{u^2+x}{2u}\right)^2-\left(y-\frac{u^2-x}{2u}\right)^2\right)^{\alpha}2y\,dy\\
&\quad= k\left(\frac{u^2+x}{2u}\right)^{2\alpha+1}\frac{1}{u}\int_0^u\left(1-\left(\frac{y-(u^2-x)/(2u)}{(u^2+x)/(2u)}\right)^2\right)^{\alpha}2y\,dy\\
&\quad= k\left(\frac{u^2+x}{2u}\right)^{2\alpha+2}\frac{1}{u}\int_{(x-u^2)/(x+u^2)}^1(1-v^2)^{\alpha}2\left(\frac{u^2+x}{2u}v+\frac{u^2-x}{2u}\right)dv\\
&\quad= k\left(\frac{u^2+x}{2u}\right)^{2\alpha+3}\frac{1}{u}\int_{(x-u^2)/(x+u^2)}^1(1-v^2)^{\alpha}2v\,dv\\
&\qquad+k\left(\frac{u^2+x}{2u}\right)^{2\alpha+2}\frac{u^2-x}{u^2}\int_{1-2u^2/(x+u^2)}^1(1-v^2)^{\alpha}\,dv\\
&\quad=\frac{k}{\alpha+1}\frac{u^2+x}{2u^2}x^{\alpha+1}+k2^{-(2\alpha+2)}\left(\frac{u^2+x}{u}\right)^{2\alpha+2}\\
&\qquad\times\frac{u^2-x}{u^2}\int_{1-2u^2/(x+u^2)}^1(1-v^2)^{\alpha}dv\ .
\end{aligned}
$$

Write

$$
\begin{aligned}
&k\int_{x=0}^{\infty}\int_{u=0}^{\infty}\int_{y=0}^{u}\exp\left\{-\beta(u+x/u)^{1/\gamma}\right\}\\
&\quad\times\frac{y(u^2+x)}{u^2}\left(\frac{y(u^2-x)}{u}+x-y^2\right)^{\alpha}dy\,du\,dx\\
&\quad= I_1+I_2\ ,
\end{aligned}
$$

where

$$
\begin{aligned}
I_1=k\int_{x=0}^{\infty}\int_{u=0}^{\sqrt{x}}\int_{y=0}^{u}&\exp\left\{-\beta(u+x/u)^{1/\gamma}\right\}\\
&\times\frac{y(u^2+x)}{u^2}\left(\frac{y(u^2-x)}{u}+x-y^2\right)^{\alpha}dy\,du\,dx
\end{aligned}
$$

and

$$
\begin{aligned}
I_2=k\int_{x=0}^{\infty}\int_{u=\sqrt{x}}^{\infty}\int_{y=0}^{u}&\exp\left\{-\beta(u+x/u)^{1/\gamma}\right\}\\
&\times\frac{y(u^2+x)}{u^2}\left(\frac{y(u^2-x)}{u}+x-y^2\right)^{\alpha}dy\,du\,dx\ .
\end{aligned}
$$

Let $s=u+x/u$. Then

$$u = \tfrac{1}{2}\left(s \pm \sqrt{s^2 - 4x}\right) \quad \text{and} \quad du = \tfrac{1}{2}\left(1 \pm s/\sqrt{s^2 - 4x}\right) ds \ ,$$

where the $-$ and $+$ signs apply for I_1 and I_2, respectively. First we compute I_1.

$$I_1 = \int_{x=0}^{\infty} \int_{s=2\sqrt{x}}^{\infty} \exp\{-\beta s^{1/\gamma}\} \left[\frac{k}{2(\alpha+1)} \frac{s}{s - \sqrt{s^2 - 4x}} x^{\alpha+1} \right.$$
$$+ k 2^{-(2\alpha+3)} s^{2\alpha+2} \left(\frac{2s}{s - \sqrt{s^2 - 4x}} - \frac{8x}{(s - \sqrt{s^2 - 4x})^2}\right)$$
$$\left. \int_{v=\sqrt{s^2-4x}/s}^{1} (1 - v^2)^{\alpha} dv \right] \times \frac{s - \sqrt{s^2 - 4x}}{\sqrt{s^2 - 4x}} \ ds \ dx \ .$$

Now interchanging the order of the two integrations and writing $w = \sqrt{s^2 - 4x}$, we obtain

$$I_1 = \int_{s=0}^{\infty} \int_{w=0}^{s} \exp\left\{ - \beta s^{1/\gamma}\right\} \left[\frac{k}{4(\alpha+1)} s \left(\frac{s^2 - w^2}{4}\right)^{\alpha+1} \right.$$
$$\left. + k 2^{-1/\gamma} s^{2\alpha+2} \left(s - \frac{s^2 - w^2}{s - w}\right) \int_{v=w/s}^{1} (1 - v^2)^{\alpha} dv \right] dw \ ds \ .$$

Similar calculations give

$$I_2 = \int_{s=0}^{\infty} \int_{w=0}^{s} \exp\left\{ - \beta s^{1/\gamma}\right\} \left[\frac{k}{4(\alpha+1)} s \left(\frac{s^2 - w^2}{4}\right)^{\alpha+1} \right.$$
$$\left. + k 2^{-1/\gamma} s^{2\alpha+2} \left(s - \frac{s^2 - w^2}{s + w}\right) \int_{v=-w/s}^{1} (1 - v^2)^{\alpha} dv \right] dw \ ds \ .$$

Combining the two gives

$$I_1 + I_2 = k 2^{-1/\gamma} \left(\int_{s=0}^{\infty} \int_{w=0}^{s} \frac{1}{\alpha+1} \exp\{-\beta s^{1/\gamma}\} s^{2\alpha+3} (1 - w^2/s^2)^{\alpha+1} dw \ ds \right.$$
$$+ \int_{s=0}^{\infty} \int_{w=0}^{s} w s^{2\alpha+2} \exp\left\{ - \beta s^{1/\gamma}\right\}$$
$$\left. \times \left[\int_{v=-w/s}^{1} (1 - v^2)^{\alpha} dv - \int_{v=w/s}^{1} (1 - v^2)^{\alpha} dv \right] dw \ ds \right) \ .$$

It is easy to check that

$$\int_{w=0}^{s} (1 - w^2/s^2)^{\alpha+1} dw = \gamma(\alpha+1) s B(1/2, \alpha+1) \ .$$

Moreover, by partial integration

$$\int_{w=0}^{s}\int_{v=-w/s}^{1} w(1-v^2)^{\alpha}\,dv\,dw - \int_{w=0}^{s}\int_{v=w/s}^{1} w(1-v^2)^{\alpha}\,dv\,dw$$

$$= \frac{s^2}{2}B(1/2,\alpha+1) - \int_{w=0}^{s}\frac{w^2}{2}\left(1-\frac{w^2}{s^2}\right)^{\alpha}\frac{1}{s}\,dw$$

$$- \int_{w=0}^{s}\frac{w^2}{2}\left(1-\frac{w^2}{s^2}\right)^{\alpha}\frac{1}{s}\,dw$$

$$= \frac{s^2}{2}B(1/2,\alpha+1) - \int_{w=0}^{s}\frac{w^2}{s}\left(1-\frac{w^2}{s^2}\right)^{\alpha}dw$$

$$= \frac{s^2}{2}B(1/2,\alpha+1)(1-\gamma) \ .$$

Therefore,

$$I_1 + I_2 = k2^{-1/\gamma}B(1/2,\alpha+1)\left(\gamma+\frac{1-\gamma}{2}\right)\int_{s=0}^{\infty}\exp\left\{-\beta s^{1/\gamma}\right\}s^{2\alpha+4}\,ds$$

$$= k2^{-1-1/\gamma}B(1/2,\alpha+1)(1+\gamma)\gamma\beta^{-1-2\gamma}\Gamma((2\alpha+5)\gamma)$$

$$= 2^{1-3\gamma}B(1/2,\alpha+1)^{-2\gamma}(1+\gamma)\gamma^{1-2\gamma}\Gamma(2\gamma)\left(\frac{\sqrt{\kappa(t)}}{g(t)}\right)^{2\gamma} \ . \qquad \square$$

For small positive θ and z define $\tilde{\theta} = \tilde{\theta}(\theta, z)$ to be the point such that $\mathbf{c}(t-\tilde{\theta})$, $\mathbf{c}(t+\theta)$ and $\mathbf{c}(t) + z\,\mathbf{e}_2(t)$ lie on the same line.

Lemma 12. *Suppose θ and z are positive and such that $\theta + z/\theta$ is small enough. We have*

$$\frac{1}{\kappa_\infty}\frac{z}{\theta} < \tilde{\theta}(\theta,z) < \frac{4}{\kappa_0}\frac{z}{\theta} \ , \tag{40}$$

where $\kappa_0 = \inf\kappa(t)$ and $\kappa_\infty = \sup\kappa(t)$, and moreover,

$$\tilde{\theta}(\theta,z) = \frac{2}{\kappa(t)}\frac{z}{\theta} + \mathrm{O}\left(\left(\theta+\frac{z}{\theta}\right)^2\right) \ . \tag{41}$$

Proof. Consider the parabola $y = ax^2$, where $a > 0$, and for given positive x and z define x' to be the point such that (x', ax'^2), (x, ax^2) and $(0, z)$ are on the same line. It is easy to show that

$$x' = \frac{1}{a}\frac{z}{x} \ . \tag{42}$$

Now by Taylor expansion and (2),

$$\begin{aligned} \langle \mathbf{c}(t+\theta) - \mathbf{c}(t),\ \mathbf{e}_1(t)\rangle &= \theta + \mathrm{O}(\theta^3), \\ \langle \mathbf{c}(t+\theta) - \mathbf{c}(t),\ \mathbf{e}_2(t)\rangle &= \frac{\kappa(t)}{2}\theta^2 + \mathrm{O}(\theta^3) \end{aligned} \tag{43}$$

for small θ. Hence (40) follows at once from (42) and the fact that the curve $\mathbf{c}$ is sandwiched locally between the two parabolas $\{(x, (\kappa_0/4)x^2)\}$ and $\{(x, \kappa_\infty x^2)\}$. Next, observe that

$$\frac{\langle \mathbf{c}(t+\theta) - \mathbf{c}(t-\tilde{\theta}),\ \mathbf{e}_2(t)\rangle}{\langle \mathbf{c}(t+\theta) - \mathbf{c}(t-\tilde{\theta}),\ \mathbf{e}_1(t)\rangle} = \frac{\langle\ \mathbf{c}(t+\theta) - \mathbf{c}(t),\ \mathbf{e}_2(t)\rangle - z}{\langle\ \mathbf{c}(t+\theta) - \mathbf{c}(t),\ \mathbf{e}_1(t)\rangle} \ .$$

By (43), this gives

$$\frac{\frac{\kappa(t)}{2}(\theta^2 - \tilde{\theta}^2) + \mathrm{O}(\theta^3 + \tilde{\theta}^3)}{\theta + \tilde{\theta} + \mathrm{O}(\theta^3 + \tilde{\theta}^3)} = \frac{\frac{\kappa(t)}{2}\theta^2 - z + \mathrm{O}(\theta^3)}{\theta + \mathrm{O}(\theta^3)} \ .$$

Using (40) and collecting terms give (41). □

Lemma 13. *For large n and $c > 0$,*

$$\inf\Big\{\tilde{\theta}(\theta, z) : 0 < z \le \frac{c\kappa(t)}{8} n^{-2\gamma}(\log n)^{2\gamma},$$
$$0 < \theta \le \frac{c}{2} n^{-\gamma}(\log n)^{\gamma}, \frac{z}{\theta} > \frac{c\kappa(t)}{4} n^{-\gamma}(\log n)^{\gamma}\Big\} \ge \frac{\kappa_0}{\kappa_\infty}\frac{c}{4} n^{-\gamma}(\log n)^{\gamma} \ . \tag{44}$$

Proof. Fix z in $(0, \frac{c\kappa(t)}{8} n^{-2\gamma}(\log n)^{2\gamma}]$ and then observe that

$$\inf\Big\{\tilde{\theta}(\theta, z) : \frac{z}{\theta} > \frac{c\kappa(t)}{4} n^{-\gamma}(\log n)^{\gamma}\Big\} \ge \tilde{\theta}\Big(z\frac{4}{c\kappa(t)} n^{\gamma}(\log n)^{-\gamma}, z\Big) \ .$$

Since both $z\frac{4}{c\kappa(t)} n^{\gamma}(\log n)^{-\gamma}$ and $\frac{c\kappa(t)}{4} n^{-\gamma}(\log n)^{\gamma}$ are small for large n and for z in the described set, the preceding right hand side is bounded below by the right hand side of (44) according to Lemma 12. □

Lemma 14. If

$$u + \frac{x}{u} \le c(\log n)^{\gamma} \ ,$$

then uniformly for $t \in [0, L)$,

$$P(X_1 \in G_n(t,x,u)) = \frac{\beta(t)}{n}\left(u + \frac{x}{u}\right)^{1/\gamma}\left(1 + \mathrm{O}\left(\left[\left(u + \frac{x}{u}\right)n^{-\gamma}\right]^{v}\right)\right) \ ,$$

with $v = \min(1, \alpha + 1, 2\eta)$ *and*

$$\beta(t) = \left(\frac{2}{\kappa(t)}\right)^{\frac{1}{2\gamma}} b(t) = B(1/2, \alpha+1) g(t)\sqrt{2/\kappa(t)}2^{1-1/\gamma} \ .$$

Proof. Clearly

$$G_n(t,x,u) = R\left(t - \tilde{\theta}\left(n^{-\gamma}\sqrt{\frac{2}{\kappa(t)}}u, n^{-2\gamma}x\right), t + n^{-\gamma}\sqrt{\frac{2}{\kappa(t)}}u\right) \ .$$

By (41) of Lemma 12,

$$\begin{aligned} n^{-\gamma}\sqrt{\frac{2}{\kappa(t)}}u &+ \tilde{\theta}\left(n^{-\gamma}\sqrt{\frac{2}{\kappa(t)}}u, n^{-2\gamma}x\right) \\ &= \sqrt{\frac{2}{\kappa(t)}}n^{-\gamma}\left(u + \frac{x}{u}\right) + n^{-2\gamma}O\left(\left(u + \frac{x}{u}\right)^2\right) \ . \end{aligned}$$

The result now follows easily from Lemma 10. □

Lemma 15. *Uniformly in* $t \in [0, L)$,

$$\begin{aligned} g(t)\sqrt{2/\kappa(t)}\int_x\int_u\int_{y=0}^{u} & I(u + x/u \le c(\log n)^{\gamma})\Big(P^{n-1}(X_1 \notin G_n(t,x,u)) \\ & - \exp\{-\beta(u + x/u)^{1/\gamma}\Big)\frac{y(u^2+x)}{u^2}\left(\frac{y(u^2-x)}{u} + x - y^2\right)^{\alpha} dy\, du\, dx \\ & = \mathrm{O}\left(n^{-\gamma(1\wedge(\alpha+1)\wedge 2\eta)}\right) \ . \end{aligned}$$

Proof. The proof follows as a straightforward application of Lemma 14. □

Lemma 16. *Uniformly in* $t \in [0, L)$,

$$\begin{aligned} \int_x\int_u & I(u + x/u \le c(\log n)^{\gamma}) P^{n-1}(\boldsymbol{X}_1 \notin G_n(t,x,u)) \\ & \Bigg(n\frac{d}{du}P(X_1 \in H_n(t,x,u)) - g(t)\sqrt{2/\kappa(t)} \\ & \quad \times \int_{y=0}^{u}\frac{y(u^2+x)}{u^2}\left(\frac{y(u^2-x)}{u} + x - y^2\right)^{\alpha} dy\Bigg)\, du\, dx \\ & = \mathrm{O}\left(n^{-\gamma(1\wedge 2\eta\wedge(\alpha+1))}\right) \ . \end{aligned}$$

Proof. Define $l_n(x,u,v)$ to be the length of the linear segment

$$\{\ \mathbf{c}(t+v)+q\ \mathbf{e}_2(t+v) : q \geq 0\} \cap H_n(t,u,x)\ \ ,$$

i.e.

$$\begin{aligned}\mathbf{c}(t+v)+l_n(x,u,v)\ \mathbf{e}_2(t+v) &= z\ \mathbf{c}(t+n^{-\gamma}\sqrt{2/\kappa(t)}u) \\ &\quad + (1-z)(\ \mathbf{c}(t)+n^{-2\gamma}x\ \mathbf{e}_2(t))\end{aligned} \tag{45}$$

for some $z \in [0,1]$. First, it is clear that (cf. (7))

$$\begin{aligned}&P(X_1 \in H_n(t,x,u)) \\ &\quad = \int_{v=0}^{n^{-\gamma}\sqrt{2/\kappa(t)}u} \int_{q=0}^{l_n(x,u,v)} f\big(\mathbf{c}(t+v)+q\ \mathbf{e}_2(t+v)\big)(1-q\kappa(t+v))\, dq\, dv\end{aligned}$$

and so

$$\begin{aligned}&\frac{d}{du}P(\boldsymbol{X}_1 \in H_n(t,x,u)) \\ &\quad = \int_{v=0}^{n^{-\gamma}\sqrt{2/\kappa(t)}u} \frac{\partial}{\partial u} l_n(x,u,v) f\big(\ \mathbf{c}(t+v)+l_n(x,u,v)\ \mathbf{e}_2(t+v)\big) \\ &\qquad\qquad (1-l_n(x,u,v)\kappa(t+v))\ dv \\ &\quad = g(t)\int_{v=0}^{n^{-\gamma}\sqrt{2/\kappa(t)}u} \frac{\partial}{\partial u} l_n(x,u,v)(l_n(x,u,v))^{\alpha} \\ &\qquad\qquad (1+\mathrm{O}(v+(l_n(x,u,v))^{\eta\wedge 1}))\ dv\ \ ,\end{aligned} \tag{46}$$

where we used (24). Now solve (45) for $l_n(x,u,v)$. By Taylor expansion with (2), we obtain

$$\begin{aligned}&\mathbf{c}(t)+\Big(v-l_n v\kappa(t)\Big)\ \mathbf{e}_1(t)+\left(\frac{v^2}{2}\kappa(t)+l_n\right)\ \mathbf{e}_2(t)+\mathrm{O}(v^3+l_n v^2) \\ &\quad = z\Big(\mathbf{c}(t)+n^{-\gamma}\sqrt{2/\kappa(t)}u\ \mathbf{e}_1(t)+n^{-2\gamma}u^2\ \mathbf{e}_2(t)+\mathrm{O}(n^{-3\gamma}u^3)\Big) \\ &\qquad + (1-z)(\mathbf{c}(t)+n^{-2\gamma}x\ \mathbf{e}_2(t))\ \ ,\end{aligned}$$

from which it follows

$$z = n^{\gamma}\sqrt{\kappa(t)/2}vu^{-1}(1+\mathrm{O}(l_n+n^{-2\gamma}u^2))$$

and

$$\frac{v^2}{2}\kappa(t)+l_n+\mathrm{O}(v^3+l_n v^2) = zn^{-2\gamma}(u^2-x)+n^{-2\gamma}x+\mathrm{O}(n^{-3\gamma}u^3)\ \ .$$

Now let $y = n^{\gamma}\sqrt{\kappa(t)/2}v$. Then

$$
\begin{aligned}
&l_n(x,u,n^{-\gamma}\sqrt{2/\kappa(t)}y)\\
&\quad = n^{-2\gamma}[-y^2+z(u^2-x)+x]+\mathrm{O}\big(n^{-3\gamma}u^3+n^{-2\gamma}y^2l_n\big)\\
&\quad = n^{-2\gamma}[-y^2+y(u-x/u)+x]+n^{-2\gamma}\mathrm{O}(l_n+n^{-2\gamma}u^2)\\
&\qquad \times(u^2-x)+\mathrm{O}(n^{-3\gamma}u^3+n^{-2\gamma}y^2l_n)\ .
\end{aligned}
$$

Observe that $y\le u$ and the expression in brackets is bounded by $(u+x/u)^2/4$. Therefore

$$
l_n(x,u,n^{-\gamma}\sqrt{2/\kappa(t)}y)=n^{-2\gamma}[y(u-x/u)-y^2+x+R_1]\ ,
$$

where

$$
R_1=\mathrm{O}\big(n^{-\gamma}(u+x/u)^3\big) \tag{47}
$$

and similarly by differentiating (45),

$$
\frac{\partial}{\partial u}l_n\big(x,u,n^{-\gamma}\sqrt{2/\kappa(t)}\ y\big)=n^{-2\gamma}\left(\frac{y(u^2+x)}{u^2}+R_2\right) \tag{48}
$$

where

$$
R_2=\mathrm{O}\big(n^{-\gamma}(u+x/u)^2\big)\ . \tag{49}
$$

Combining (46) and (48), we obtain

$$
\begin{aligned}
&n\frac{d}{du}P(\boldsymbol{X}_1\in H_n(t,x,u))-g(t)\sqrt{\frac{2}{\kappa(t)}}\\
&\times\int_{y=0}^{u}\frac{y(u^2+x)}{u^2}\left(\frac{y(u^2-x)}{u}+x-y^2\right)^{\alpha}dy\\
&=g(t)\sqrt{\frac{2}{\kappa(t)}}\int_{y=0}^{u}\left(\frac{y(u^2+x)}{u^2}+R_2\right)\left(\frac{y(u^2-x)}{u}+x-y^2+R_1\right)^{\alpha}\\
&\quad\times\mathrm{O}\Big(n^{-\gamma}y+l_n(x,u,n^{-\gamma}\sqrt{2/\kappa(t)}y)^{\eta\wedge 1}\Big)dy\\
&\quad+g(t)\sqrt{\frac{2}{\kappa(t)}}\int_{y=0}^{u}R_2\left(\frac{y(u^2-x)}{u}+x-y^2+R_1\right)^{\alpha}dy\\
&\quad+g(t)\sqrt{\frac{2}{\kappa(t)}}\int_{y=0}^{u}\frac{y(u^2+x)}{u^2}\left[\left(\frac{y(u^2-x)}{u}+x-y^2+R_1\right)^{\alpha}\right.\\
&\quad\left.-\left(\frac{y(u^2-x)}{u}+x-y^2\right)^{\alpha}\right]dy\\
&=:E_1+E_2+E_3\ .
\end{aligned}
$$

By Lemma 14, it suffices to show that

$$\int_x \int_u I(u + x/u \le c(\log n)^\gamma) \exp\{-\beta(u + x/u)^{1/\gamma}\} \, E_i \, du \, dx$$
$$= \mathrm{O}\big(n^{-\gamma(1 \wedge 2\eta \wedge (\alpha+1))}\big) \tag{50}$$

for $i = 1, 2, 3$. For $i = 1, 2$ this follows in a straightforward manner using (47) and (49). For $i = 3$ we use a variable transformation similar to the one in Lemma 9 Let $R_{11} = \min\{R_1 : 0 \le y \le u\}$ and $R_{12} = \max\{R_1 : 0 \le y \le u\}$. Since $R_1(y = 0) = 0$, it is clear that $R_{11} \le 0 \le R_{12}$. First we find an upper bound for E_3. Let

$$w(y) = \left(y - \frac{u^2 - x}{2u}\right)\left(1 + \frac{4R_{12}}{(u + x/u)^2}\right)^{-1/2} + \frac{u^2 - x}{2u} \ .$$

Then

$$\int_{y=0}^{u} y\left(\frac{y(u^2 - x)}{u} + x - y^2 + R_1\right)^{\alpha} dy$$
$$\le \int_{y=0}^{u} y\left(\frac{y(u^2 - x)}{u} + x - y^2 + R_{12}\right)^{\alpha} dy$$
$$= \left(1 + \frac{4R_{12}}{(u + x/u)^2}\right)^{\alpha+1/2} \int_{w=w(0)}^{w(u)} \left[\left(w - \frac{u^2 - x}{2u}\right)\right.$$
$$\left. \times \left(1 + \frac{4R_{12}}{(u + x/u)^2}\right)^{1/2} + \frac{u^2 - x}{2u}\right]\left(\frac{w(u^2 - x)}{u} - w^2 + x\right)^{\alpha} dw \ . \tag{51}$$

Observe that

$$\frac{4R_{12}}{(u + x/u)^2} = \mathrm{O}\big(n^{-\gamma}(u + x/u)\big)$$

and so

$$w(0) = \mathrm{O}\big(n^{-\gamma}(u + x/u)^2\big).$$

Also observe that $w(u) \le u$. Hence (51) is bounded by

$$(1 + \mathrm{O}(n^{-\gamma}(u + x/u)))\left(\int_{w=0}^{u} w\left(\frac{w(u^2 - x)}{u} - w^2 + x\right)^{\alpha} dw\right.$$
$$\left. + \mathrm{O}(n^{-2\gamma}(u + x/u)^4 x^{\alpha})\right) \ .$$

Similar calculations for the lower bound show that E_3 has the same order of magnitude as

$$n^{-\gamma}(u+x/u)\int_0^u w\Big(w(u-x/u)-w^2+x\Big)^{\alpha}dw+n^{-2\gamma}(u+x/u)^4x^{\alpha}$$

which suffices to verify (50). □

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