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Construction of asymptotically optimal controls for control and game problems

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Abstract. Recently the connection between control and game problems and Backward Stochastic Differential Equations has been established. This allows us to use an approximation scheme for such equations in order to construct an ε-optimal control.

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1. Introduction

In this paper we deal with the following game problem: r persons are involved in a game whose state is described by the equation

$$dX^u(t) = \sigma(t, X^u)\, dW(t) + b(t, X^u, u(t, X^u))\, dt$$

where W is a d-dimensional Brownian motion and u is a r-dimensional admissible control i.e. a progressively measurable process defined on the canonical path space. The i'th player influences the game by means of the i'th component of the control u which appear in the drift term b. He has to minimize a cost function defined by

$$J_i(u) = E\left(\mathrm{e}^i_\alpha(T, X^u) g^i(X^u) + \int_0^T \mathrm{e}^i_\alpha(s, X^u) h^i(s, X^u, u(s, X^u))\, ds \right) .$$

where g is a final cost, h is a running cost on the game period and $\mathrm{e}^i_\alpha(t, X^u) := \exp(-\alpha^i(t, X^u))$ is an actualization factor.

One looks for a control u^* which is optimal in the sense that it is a saddle point for the problem i.e.

$$J_i(u^*) \leq J_i(u_1^*, \ldots, u_{i-1}^*, u, u_{i+1}^*, \ldots, u_r^*), \quad \forall\, 1 \leq i \leq r \ ,$$

for every admissible control u. This is the well known Nash equilibrium point.

If $r = 1$ this is a classical control problem and if $r = 2$ and $J_1 = -J_2$ this is a zero-sum game problem. Both this situations are essentially one dimensional in the sense that a single cost function appears. In this case the Hamiltonian function associated to the problem is Lipschitz continuous but in the multi-dimensional case it may be even discontinuous. In the one-dimensional case reasonably general sufficient conditions for the existence and unicity of an optimal control are known but in the multidimensional case this is an open problem. In the Markov case and under the hypothesis that the Hamiltonian function is continuous, an existence result has recently been given in [Hamadène, Lepeltier, Peng]. Here we do not deal with optimal controls but construct an ε-optimal control u^ε i.e. a control such that

$$J_i(u^\varepsilon) \leq J_i(u_1^\varepsilon, \ldots, u_{i-1}^\varepsilon, u, u_{i+1}^\varepsilon, \ldots, u_r^\varepsilon) + \varepsilon, \quad \forall\, 1 \leq i \leq r \ .$$

The control u^ε appears as an explicit function of two sequences of independent random variables – Gaussian (respectively Poisson) distributed. This may be viewed as a starting point to construct a numerical algorithm.

2. A priori inequalities

On a probability space (Ω, F, P) a Brownian motion $B = (B^1, \ldots, B^d)$ and a sequence of independent random variables $\sigma_k, k \in N$ are given. Each σ_k is exponentially distributed with parameter λ and is independent of the Brownian motion B. We define $\tau_0 = 0$, $\tau_k = \sigma_1 + \cdots + \sigma_k$ and note that $\tau_k, k \in N$, are the jump times of a Poisson process N which is independent of B. We consider this process as starting from T and going backward to zero. So we define

$$T_k \ := \max(0, T - \tau_k) \ .$$

Since the Brownian motion B runs forward and the Poisson process N runs backward one can not use the stochastic calculus for the two processes simultaneously. Anyway, since the two processes are independent, one can use it twice separately. In order to do this we work on a product probability space $\Omega = \Omega' \times \Omega'', \omega = (\omega', \omega''), B$ depends on ω' and N depends on ω''. $(F_t')_{t \geq 0}$ is the filtration generated by B-(i.e. F_t' is the completion of $\sigma(B_s : s \leq t)$) and $(F_t'')_{t \geq 0}$ is the filtration generated by N-(i.e. F''_t is the completion of $\sigma(N_s : s \le t)$). Finally $G'_t = F'_t \times F''_\infty$ and $G''_t = F'_T \times F''_{T-t}$.

The main ingredient in our calculus is the evaluation in the following lemma. Consider some processes $y(t), \varphi(t), \psi(t), \beta(t) \in R^r$ and $z(t) \in R^{r\times d}$ which are right continuous, adapted to both the filtrations $(G'_t)_{t\ge 0}$ and $(G''_t)_{t\ge 0}$, and satisfy the system of equations

$$\begin{aligned} y(t) + \int_t^T z(s)\,dB(s) &= \xi + \int_t^T (\beta(s) + \varphi(s))\,ds \\ &\quad + \int_0^{T-t} \frac{1}{\lambda}\psi(T-s)\,d\tilde{N}(s) \\ &= \xi + \int_t^T (\beta(s) + \varphi(s) + \psi(s))\,ds \\ &\quad + \sum_{t\le T_k} \frac{1}{\lambda}\psi(T_k)\ . \end{aligned} \tag{1}$$

Here

$$\tilde{N}(s) = N(s) - \lambda s$$

is the compensated Poisson process and the second equality in (1) is due to $\sum_{t\le T_k} \frac{1}{\lambda}\psi(T_k) = \int_0^{T-t} \frac{1}{\lambda}\psi(T-s)\,dN(s)$.

We assume that

$$\begin{aligned} &\text{i) } E\int_t^T (|y(t)|^2 + |z(t)|^2 + |\psi(t)|^2 + |\beta(t)|^2 + |\varphi(t)|^2)\,ds < \infty \\ &\text{ii) } \langle y(t), \varphi(t)\rangle \ \le\ \gamma|y(t)|(|y(t)| + |z(t)|) \\ &\text{iii) } |\beta(t)| + |\varphi(t)| + |\psi(t)| \le c^{1/2}(1 + |y(t)| + |z(t)|)\ . \end{aligned} \tag{2}$$

Lemma 1. *Assume that* $\lambda^{-1/2}\mathrm{c} \le 1/16$. *Then*

$$\begin{aligned} &E|y(t)|^2 + E\int_t^T |z(s)|^2\,ds \\ &\quad \le C\left(E|\xi|^2 + \int_t^T |\beta(s)|^2\,ds + \frac{1}{\sqrt{\lambda}}\left(1 + E\int_t^T |\psi(s)|^2\,ds\right)\right)\ , \end{aligned} \tag{3}$$

with $C = C' \exp\ (3(\gamma+1)^2 T), C'$ *depending on* c *in* (2) iii) *only.*

ii) *For any* $p \in N$ *there are some constants* C, C' *which depend on* p *and* c *in* (2) iii) *such that, for* $\lambda \ge C$,

$$E|y(t)|^{2p} + E\left(\int_t^T |z(s)|^2\,ds\right)^p \le C'\left(E|\xi|^{2p} + 1\right)\ . \tag{4}$$

Remark. In order to get (4) the hypothesis (2) ii) and the control of β in (2) iii) are not necessary.

Remark. The above lemma appears in [Bally] under the stronger assumption $|\varphi(t)| \leq K(|y(t)| + |z(t)|)$ which corresponds to the case where the generator function in the BSDE is Lipschitz continuous. The weakened hypothesis (2) ii) allows to work under monotonicity assumptions only (see (13) iii)).

Proof. One writes (1) for $t \leq S \leq T$ and takes the difference in order to get

$$y(S) = J(S) - I(S) \text{ with}$$

$$J(S) = y(t) + \int_t^S z(s)\,dB(s) - \int_t^S (\beta(s) + \varphi(s))\,ds \tag{5}$$

$$I(S) = \int_{T-S}^{T-t} \frac{1}{\lambda}\psi(T-s)\,d\tilde{N}(s) \quad .$$

The isometry property yields

$$E|I(S)|^2 = \frac{1}{\lambda}E\int_t^S |\psi(s)|^2\,ds \quad , \tag{6}$$

and further, since $J(S) = y(S) + I(S)$,

$$E|J(S)|^2 \leq 2E|y(S)|^2 + \frac{2}{\lambda}E\int_t^S |\psi(s)|^2\,ds \quad . \tag{7}$$

We now look at $S \to J(S), S \geq t$, as a semimartingale with respect to the filtration $(G'_t)_{t\geq 0}$ and use Itô's formula in order to get

$$E|J(T)|^2 = E|y(t)|^2 + E\int_t^T |z(s)|^2\,ds - 2E\int_t^T \langle J(s), \beta(s) + \varphi(s)\rangle\,ds$$

that is

$$E|y(t)|^2 + E\int_t^T |z(s)|^2\,ds = E|J(T)|^2 + 2E\int_t^T \langle J(s), \beta(s) + \varphi(s)\rangle\,ds \quad . \tag{8}$$

By (7)

$$\begin{aligned} 2E\int_t^T |\langle J(s), \beta(s)\rangle|\,ds &\leq E\int_t^T |J(s)|^2\,ds + E\int_t^T |\beta(s)|^2\,ds \\ &\leq 2E\int_t^T |y(s)|^2\,ds + \frac{2T}{\lambda}E\int_t^T |\psi(s)|^2\,ds \\ &\quad + \ E\int_t^T |\beta(s)|^2\,ds \quad . \end{aligned}$$

Further, since $J(s) = y(s) + I(s)$,

$$2E\int_t^T \langle J(s), \varphi(s)\rangle\, ds = 2E\int_t^T \langle y(s), \varphi(s)\rangle\, ds + 2E\int_t^T \langle I(s), \varphi(s)\rangle\, ds := a + b.$$

Using (2) ii) and $2\gamma|y(s)|\ |z(s)| \le 4\gamma^2|y(s)|^2 + \frac{1}{4}|z(s)|^2$ one gets

$$|a| \le (4\gamma^2 + 2\gamma)E\int_t^T |y(s)|^2\, ds + \frac{1}{4}E\int_t^T |z(s)|^2\, ds \ .$$

Using (2) iii) and the hypothesis $4\lambda^{-1/2}\ c \le 1/4$ one gets

$$\begin{aligned} 2|\langle I(s), \varphi(s)\rangle| &\le \lambda^{1/2}|I(s)|^2 + \lambda^{-1/2}|\varphi(s)|^2 \\ &\le \lambda^{1/2}|I(s)|^2 + 4\lambda^{-1/2}c + \tfrac{1}{4}(|y(s)|^2 + |z(s)|^2) \end{aligned}$$

and so, by (6)

$$|b| \le \frac{T}{\sqrt{\lambda}}E\int_t^T |\psi(s)|^2\, ds + 4\lambda^{-1/2}cT + \frac{1}{4}E\int_t^T |y(s)|^2\, ds + \frac{1}{4}E\int_t^T |z(s)|^2\, ds \ .$$

Coming back to (8) and noting that $J(T) = y(T) = \xi$ one gets

$$\begin{aligned} &E|y(t)|^2 + E\int_t^T |z(s)|^2\, ds \\ &\leqslant 2E|\xi|^2 + E\int_t^T |\beta(s)|^2\, ds + \frac{K}{\sqrt{\lambda}}(1 + E\int_t^T |\psi(s)|^2\, ds) \\ &\quad + \left(4\gamma^2 + 2\gamma + \frac{1}{4}\right)E\int_t^T |y(s)|^2\, ds + \frac{1}{2}E\int_t^T |z(s)|^2\, ds \ . \end{aligned}$$

One cancels the term in z and then uses Gronwall's lemma in order to prove (3).

Let us now prove (4). We fix $t \le T$ and, for $S \in (t, T]$ we define

$$\phi(S) = y(t) + \int_t^S z(s)\, dB(s) = y(S) + \int_t^S (\beta(s) + \varphi(s))\, ds + \int_{T-S}^{T-t} \frac{1}{\lambda}\psi(s)\, d\tilde{N}(s) \ .$$

By Ito's formula

$$E|\phi(t+h)|^{2p} = E|y(t)|^{2p} + 2p(2p-1)E\int_t^{t+h} |\phi(s)|^{2p-2}|z(s)|^2\,ds \geq E|y(t)|^{2p} \ .$$

Then, by (1) and (2) iii)

$$\begin{aligned} &E|y(t)|^{2p} \\ &\leq C_1\Big\{E|y(t+h)|^{2p} + E\Big(\int_t^{t+h}(\beta(s)+\varphi(s))\,ds\Big)^{2p} \\ &\quad + E\Big(\int_{T-t-h}^{T-t}\lambda^{-1}\psi(s)\,d\tilde{N}(s)\Big)^{2p}\Big\} \\ &\leq C_1\Big\{E|y(t+h)|^{2p} + h^p \times E\Big(\int_t^{t+h}(|\beta(s)|^2+|\varphi(s)|^2)\,ds\Big)^p \\ &\quad + \lambda^{-p}E\Big(\int_{T-t-h}^{T-t}|\psi(s)|^2\,ds\Big)^p\Big\} \\ &\leq C_2\Big\{E|y(t+h)|^{2p} \\ &\quad + (h^p+\lambda^{-p}) \times E\Big(\int_t^{t+h}(1+|y(s)|^2+|z(s)|^2)\,ds\Big)^p\Big\} \ . \end{aligned} \tag{9}$$

Using Burkholder's inequality and the equation (1) one gets

$$\begin{aligned} E\Big(\int_t^{t+h}|z(s)|^2\,ds\Big)^p &\leq C_3 E\Big(\int_t^{t+h} z(s)\,dB(s)\Big)^{2p} \\ &\leq C_3\Big\{E|y(t)|^{2p} + E|y(t+h)|^{2p} \\ &\quad + E\Big(\int_t^{t+h}(\beta(s)+\varphi(s))\,ds\Big)^{2p} \\ &\quad + E\Big(\int_{T-t-h}^{T-t}\lambda^{-1}\psi(s)\,d\tilde{N}(s)\Big)^{2p}\Big\} \\ &\leq C_4\Big\{E|y(t+h)|^{2p} \\ &\quad + (h^p+\lambda^{-p}) \times E\Big(\int_t^{t+h}(1+|y(s)|^2+|z(s)|^2)\,ds\Big)^p\Big\} \ . \end{aligned}$$

Assume that $C_4(h^p + \lambda^{-p}) \leq 1/2$. Then

$$\frac{1}{2}E\left(\int_t^{t+h} |z(s)|^2\, ds\right)^p \leq C_5\left\{E|y(t+h)|^{2p}\right.$$
$$\left. + (h^p + \lambda^{-p}) \times E\left(\int_t^{t+h} (1 + |y(s)|^2)\, ds\right)^p\right\} \; . \tag{10}$$

We plug this in the right hand side of (9) and get

$$E|y(t|^{2p} \leq C_6\left\{E|y(t+h)|^{2p} + E\left(\int_t^{t+h} (1 + |y(s)|^2)\, ds\right)^p\right\}$$

which by Gronwall's lemma yields

$$E|y(t|^{2p} \leq C_7(E|y(t+h)|^{2p} + 1) \; .$$

Using iteration one gets $E|y(t|^{2p} \leq C_7(E|\xi)|^{2p} + 1)$ and further, using (10) one gets the same inequality for z. □

3. Backward stochastic differential equations

We consider the equation $E(\xi, f)$

$$Y(t) + \int_t^T Z(s)\, dB(s) = \xi + \int_t^T f(s, Y(s), Z(s))\, ds \tag{11}$$

where

$$\xi \in L^2(\Omega')^r \tag{12}$$

and $f : [0, T] \times \Omega' \times R^r \times R^{r\times d} \to R^r$ is such that $(t, \omega') \to f(t, \omega', y, z)$ is progressively measurable with respect to $(F'_t)_{t\geq 0}$ and

$$\begin{aligned} &\text{i) } E\int_0^T |f(t, \omega', 0, 0)|^2\, ds \;\; < \infty \;\; , \\ &\text{ii) } |f(t, \omega', y, z)| \leq C_1(1 + |y| + |z|) \;\; , \\ &\text{iii) } \langle y - y', f(t, \omega', y, z) - f(t, \omega', y', z)\rangle \leq C_2|y - y'|^2 \;\; , \\ &\text{iv) } |f(t, \omega', y, z) - f(t, \omega', y, z')| \leq C_2|z - z'| \;\; . \end{aligned} \tag{13}$$

We also consider the equation $E_\lambda(\bar{\xi}, \bar{f})$:

$$\begin{aligned} \bar{Y}(t) + \int_t^T \bar{Z}(s)\, dB(s) &= \bar{\xi} + \frac{1}{\lambda}\sum_{t\leq T_k} \bar{f}(T_k, \bar{Y}(T_k), \bar{Z}(T_k)) \\ &= \bar{\xi} + \int_t^T \bar{f}(s, \bar{Y}(s), \bar{Z}(s))\, ds \\ &\quad + \frac{1}{\lambda}\int_0^{T-t} \bar{f}(T-s, \bar{Y}(T-s), \bar{Z}(s))\, d\tilde{N}(s) \; . \end{aligned} \tag{14}$$

Theorem 2. i) *Assume that* $\xi, \bar{\xi}$ *satisfy* (12) *and* $f, \bar{f}$ *satisfy* (13). *Then, for* $\lambda \geq C$,

$$E(\sup_{t\leq T}|Y(t)-\bar{Y}(t)|^2)+E\int_0^T |Z(t)-\bar{Z}(t)|^2\,dt$$
$$\leq C'\Big\{E(|\xi-\bar{\xi}|^2)+\frac{1}{\sqrt{\lambda}}+E\int_0^T |\bar{f}(s,\bar{Y}(s),\bar{Z}(s)).$$
$$-f(s,\bar{Y}(s),\bar{Z}(s))|^2\,dt\Big\}\ ,\ (15)$$

where $C' = C''\exp(C'''C_2^2T)$ with C, C'', C''' depending on C_1 in (13)(ii).

ii) For any $p \in N$ there exist constants C, C' depending on p and on C_1 such that

$$\sup_{t\leq T}E(|\bar{Y}(t)|^{2p})+E\left(\int_0^T |\bar{Z}(t)|^2\,dt\right)^p \leq C'(E|\bar{\xi}|^{2p}+1)\ . \tag{16}$$

Proof. (15) follows from (3) with $y = Y - \bar{Y}, z = Z - \bar{Z}$ and (16) follows from (4) with $y = \bar{Y}, z = \bar{Z}$. □

Remark. It is proved in [Pardoux] that under (12), (13) and the additional hypothesis that f is continuous in y, z, the equation $E(\xi, f)$ has a unique solution. Contrary to the initial existance result proved in [Peng, Pardoux], f here is not assumed to be Lipschitz continuous in y-the monotonicity assumption (13) iii) is sufficient. The proof in [Pardoux] is based on an intricate L^2-compactness argument. The inequality (15) (actually an obvious variant of this inequality) provides a simple and natural alternative proof: one constructs the solution (Y, Z) as the limit of the solutions $(\bar{Y}, \bar{Z})$ of the equations $E_\lambda(\xi, f)$, as $\lambda \to \infty$. Note that $(\bar{Y}, \bar{Z})$ may be constructed using the representation theorem on each of the intervals $(T_{k+1}, T_k]$.

On the other hand, if $\bar{\xi}$ and $\omega' \to f(t, \omega', y, z)$ are simple functionals –i.e. depend on a finite number of increments of the Brownian motion B-then $(\bar{Y}, \bar{Z})$ may be explicitly calculated as a function of these increments and of $\sigma_k, k \in N$ (see [Bally]). So one may look at $(\bar{Y}, \bar{Z})$ as an approximation scheme for (Y, Z).

4. The game problem

On the probability space $\Omega = \Omega' \times \Omega''$ are given the Brownian motion B which depends on ω' and a sequence of Poisson processes $N_n, n \in N$ which depend on ω'' and are independent of B.

The admissible controls are functions $u\colon [0,T]\times C\times\Omega'\to U$ where $C := C([0,T];R^d)$ and (U,d) is a metric space. u is assumed to be progressively measurable with respect to $C_t\times F'_t, t\geq 0,$ where $C_t, t\geq 0$, is the standard filtration on C (generated by the projections).

The coefficients involved in the problem are

$$\begin{aligned}\sigma\ &: [0,T]\times C\to R^d\times R^d, \quad b\ :\ [0,T]\times C\times U^r\to R^d,\\ \alpha,h\ &: [0,T]\times C\times U^r\to R^r, \quad g\ :\ C\to R^r\ .\end{aligned}$$

On C we consider the seminorms

$$|w|_t=\sup_{s\leq t}|w_t|$$

and assume that

(17) i) σ,α,b,h are progressively measurable, bounded and for each $0\leq t\leq T, \sigma(t,.), \alpha(t,\cdot,\cdot), b(t,\cdot,\cdot), h(t,\cdot,\cdot)$,g $(\cdot)$ are Lipschitz continuous (with respect to $|o|_t$ respectively to $|o|_t+d$).

ii) $\sigma\geq cI$ for some constant $c>0$ (as a consequence σ is invertible and σ^{-1} has the same properties as σ).

Let X be the solution of the equation

$$X(t)=x+\int_0^t\sigma(s,X)\,dB(s)\ . \tag{18}$$

We denote

$$\begin{aligned}\mathrm{e}_{b,u}(t)&=\exp\biggl\{-\int_0^t\langle\sigma^{-1}b(s,X,u(s,X)),\ dB(s)\rangle\\ &\qquad -\frac{1}{2}\int_0^t|\sigma^{-1}b(s,X,u(s,X))|^2\,ds\biggr\}\\ dP_u&=\mathrm{e}_{b,u}(T)\,dP\\ W_u(t)&=B(t)-\int_0^t\sigma^{-1}b(s,X,u(s,X))\,ds\ .\end{aligned}$$

W_u is a Brownian motion under P_u and X solves the equation

$$X(t)=x+\int_0^t\sigma(s,X(s))\,dW_u(s)+\int_0^t b(s,X,u(s,X))\,ds\ . \tag{19}$$

The cost function associated to the r-dimensional control u is defined by

$$J(u)=E_u(\mathrm{e}_{\alpha,u}(T,X)g(X)+\int_0^t\mathrm{e}_{\alpha,u}(s,X)h(s,X,u(s,X))\,ds)$$

with

$$\mathrm{e}_{\alpha,u}(t,X):=\exp\biggl(\int_0^t\alpha(s,X,u(s,X))\,ds\biggr)\ .$$

The interpretation of the cost function is the following: under P_u, X is the controlled diffusion associated to u, $g(X)$ is a final cost and $h(t,\cdot,\cdot)$ is a running cost; $e_{\alpha,u}(t,X)$ is an actualization factor. One looks for a control u^* which is optimal in the sense that

$$J_i(u^*) = \inf_u J_i([u,u^*]_i), \quad 1 \le i \le r \tag{20}$$

where the infimum is taken over all the admissible controls and

$$[x,y]_i := (y_1,\dots,y_{i-1},x,y_{i+1},\dots,y_r) \text{ for } x \in R, y \in R^r.$$

In fact we do not look for an optimal control but for an ε-optimal control i.e. u_ε such that

$$J_i(u_\varepsilon) \le \inf_u J_i([u,u_\varepsilon]_i) + \varepsilon, \quad 1 \le i \le r \ . \tag{21}$$

5. The backward stochastic differential equation associated to the game problem

Following [El-Karoui, Quenez, Peng] and [Hamadéne, Lepeltier] we associate to the above game problem a BSDE in the following way. Define the Hamiltonian function $H\colon [0,T] \times C \times R^r \times R^{r\times d} \times U^r \to R^r$ by

$$H_i(t,w,y,z,u) = h^i(t,w,u) + \langle z_i, \sigma^{-1} b(t,w,u)\rangle + y^i \alpha^i(t,w,u)$$

where $z = (z_1,\dots,z_r), z_i \in R^d$.

Then, for an admissible control u we define

$$f_u(t,\omega,y,z) = H(t,X(\omega),y,z,u(t,X(\omega)))$$

and denote by (Y_u, Z_u) the solution of the equation

$$Y_u(t) + \int_t^T Z_u(s)\,dB(s) = g(X) + \int_t^T f_u(s,Y_u(s),Z_u(s))\,ds \ . \tag{22}$$

Then

$$Y_u(0) = J(u) \ . \tag{23}$$

Proof. We write the i'th equation in the system (22) in terms of W_u

$$\begin{aligned} Y_u^i(t) &+ \sum_{j=1}^d \int_t^T Z_u^{ij}(s)\,dW_u^j(s) \\ &= g^i(X) + \int_t^T h^i(s,X,u(s,X)) + Y_u^i(s)\alpha^i(s,X,u(s,X))\,ds \ . \end{aligned}$$

Then, using Ito's formula

$$e^i_\alpha(T,X)Y^i_u(T) = e^i_\alpha(0,X)Y^i_u(0) + \sum_{j=1}^{d}\int_0^T e^i_\alpha(s,X)Z^{ij}_u(s)\,dW^j_u(s)$$
$$-\int_0^T e^i_\alpha(s,X)h^i(s,X,u(s,X))\,ds \ .$$

One takes expectation with respect to P_u and gets (23).

6. The compatibility hypothesis

In order to construct an ε-optimal control we need the following assumption. There exists a function $u_\varepsilon(t,w,y,z)$ such that

$$(C_\varepsilon)\ \text{i)}\ H_i(t,w,y,z,\tilde{u}_\varepsilon(t,w,y,z)) \le H_i(t,w,y,z,[u,\tilde{u}_\varepsilon(t,w,y,z)]_i) + K(1+|y|+|z|)^k\varepsilon^3 \ ,$$
$$\forall u \in U, 1 \le i \le r,$$
$$\text{ii)}\ |\tilde{u}_\varepsilon(t,w,y,z) - \tilde{u}_\varepsilon(t,w',y',z')| \le L_\varepsilon\{|y-y'|+|z-z'| + (1+|y|+|z|)|w-w'|_t\} \ .$$

Remark. L_ε may blow up as $\varepsilon \to 0$, eg. $L_\varepsilon \sim 1/\varepsilon$.

Remark. (C_ε) is a relaxed version of the condition "H admits a saddle point": say that H depends on w by means of $w(t)$ only and assume that there exists a function $u^*(t,x,y,z)$ such that

$$(C^*)\quad H_i(t,x,y,z,u^*(t,x,y,z)) = \inf_{u\epsilon U} H_i(t,x,y,z,[u,u^*(t,x,y,z)]_i) \ .$$

Assume also that u^* is uniformly continuous. Then one may construct the function $\tilde{u}_\varepsilon$ by regularization of u^*.

Finally we give an example in which the Hamiltonian function is discontinuous but the hypothesis (C_ε) holds. Two persons ($r = 2$) are playing a game whose evolution equation is described by the one dimensional (d = 1) controlled diffusion process

$$X^u(t) = x + B(t) + \int_0^t (u_1(s)+u_2(s))\,ds$$

with $u = (u_1,u_2) \in [0,1]^2$. The cost function is $J_i(u) = E(g_i(X^u(T)),$ $i = 1,2$. So $h = \alpha = 0$. Then

$$H_i(t,\omega,y,z_1,z_2,u_1,u_2) = z_i(u_1+u_2), \quad i = 1,2 \ .$$

Note that $\inf_{u_1} z_1(u_1 + u_2) = -|z_1| + z_1 u_2$ and the infimum is achieved in $u_1^*(z) = -sign(z_1)$. From this and from a symmetric relation one gets

$$H(t, \omega, y, z, u^*(z)) = (-|z_1| - z_1 sign(z_2), -|z_2| - z_2 sign(z_1))$$

which is discontinuous on the axes $z_1 = 0, z_2 = 0$.

We construct the control $\tilde{u}_\varepsilon$ which satisfies (C_ε) in the following way: $\tilde{u}_\varepsilon^i(z_i) = f(z_i)$ with $f(x) = \varepsilon x^{-1} + sign(x)$ if $|x| \geq \varepsilon$ and $f(x) = 0$ if $|x| \leq \varepsilon$. Note that $xf(x) \leq \varepsilon - |x|$ and consequently

$$z_1(\tilde{u}_\varepsilon^1(z_2) + \tilde{u}_\varepsilon^2(z_2)) \leq \varepsilon - |z_1| + z_1 \tilde{u}_\varepsilon^2(z_2) \leq \varepsilon + z_1(u + \tilde{u}_\varepsilon^2(z_2))$$

for each $u \in [-1, 1]$. This and a symmetric relation prove (C_ε, i). The Lipschitz constant in (C_ε, ii) is of order ε^{-1}.

7. Construction of an ε-optimal control

Let D be the space of R^d-valued, right continuous with left hand limit functions and let $\sigma_n : [0, T] \times D \to R^d, n \in N$ be a sequence of progressively measurable functions such that

$$\begin{aligned} &\text{i)}\ |\sigma_n(t, w) - \sigma_n(t, w')| \leq K|w - w'|_t \quad \forall w, w' \in D,\ n \in N\ , \\ &\text{ii)}\ |\sigma_n(t, w) - \sigma(t, w)| \leq K/\sqrt{n} \quad \forall w \in C, n \in N\ . \end{aligned} \tag{24}$$

We construct X_n to be the Euler approximation of X i.e.

$$\begin{aligned} &X_n\left(\frac{k+1}{n}\right) = X_n\left(\frac{k}{n}\right) + \sigma_n\left(\frac{k}{n}, X_n\right)\left(B\left(\frac{k+1}{n}\right) - B\left(\frac{k}{n}\right)\right)\ , \\ &X_n(t) = X_n\left(\frac{k}{n}\right) \quad \text{for} \quad \frac{k}{n} \leq t < \frac{k+1}{n}\ . \end{aligned} \tag{25}$$

Standard calculations give

$$E\left(\sup_{t \leq T} |X(t) - X_n(t)|^p\right) \leq K/n^{p/2} \quad \forall p \geq 1\ . \tag{26}$$

Further, we construct, $(Y_{n,\varepsilon}, Z_{n,\varepsilon})$ to be the solution of

$$Y_{n,\varepsilon}(t) + \int_t^T Z_{n,\varepsilon}(s)\, dB(s) = g(X_n) + \frac{1}{\lambda_n} \sum_{t \leq T_k^n} f_{n,\varepsilon}(T_k^n, Y_{n,\varepsilon}(T_k^n), Z_{n,\varepsilon}, (T_k^n)) \tag{27}$$

where $T_k^n, k \epsilon N$, are the jump times of the Poisson process N_n of parameter λ_n, which starts from T and runs backward to zero (see Section a) and

$$f_{n,\varepsilon}(t,\omega,y,z) := H(t,X_n(\omega),y,z,\tilde{u}_\varepsilon(t,X_n(\omega),y,z)) \ .$$

The ε-optimal control is

$$u_{n,\varepsilon}(t,x,\omega',\omega'') := \tilde{u}_\varepsilon(t,x,Y_{n,\varepsilon}(t,\omega',\omega''),Z_{n,\varepsilon}(t,\omega',\omega'')) \ .$$

Remark. ω'' represents an auxiliary parameter introduced by our approximation scheme. So, for each fixed ω'' the admissible control is $(t,x,\omega') \to u_{n,\varepsilon}(t,x,\omega',\omega'')$ which we denote by $u_{n,\varepsilon}(\omega'')$.

Remark. $Y_{n,\varepsilon}$ and $Z_{n,\varepsilon}$ (and consequently $u_{n,\varepsilon}$) may be calculated explicitly as functions of the independent random variables $B(\frac{k+1}{n}) - B(\frac{k}{n}), k \le nT$ and $\tau_k^n = T_k^n - T_{k+1}^n$ (see [Bally]). In this sense this would be a starting point for an approximation scheme.

Theorem 3. *We assume that* (17), (24) *and* (C_ε) *hold and take n large enough in order that*

$$K\left(\frac{L_\varepsilon^2}{n} + \frac{1}{\sqrt{\lambda_n}}\right) \le \varepsilon^3 \tag{28}$$

where L_ε is the Lipschitz constant of $\tilde{u}_\varepsilon$ (see (C_ε, ii))) and K depends on the constants in (17) and (24).

Then there exists $\Omega''_\varepsilon \subset \Omega''$ such that $P(\Omega''_\varepsilon) \ge 1 - 2\varepsilon$ and, for $\omega'' \in \Omega''_\varepsilon$,

$$J_i(u_{n,\varepsilon}(\omega'')) - \varepsilon \le J_i^*(\omega'') \le J_i(u_{n,\epsilon}(\omega'')) \tag{29}$$

where

$$J_i^*(\omega'') : = \inf_u J_i([u,u_{n,\varepsilon}(\omega'')]_i) \ .$$

Proof. Define

$$\bar{f}(t,\omega,y,z) = H(t,X(\omega),y,z,u_{n,\varepsilon}(t,X(\omega),\omega))$$

and let $(\bar{Y},\bar{Z})$ be the solution of the eq. $E(g(X),\bar{f})$. We first prove that

$$E|Y_{n,\varepsilon}(t) - \bar{Y}(t)|^2 + E\int_0^T |Z_{n,\varepsilon}(s) - \bar{Z}(s)|^2\, ds \le K\left(\frac{L_\varepsilon^2}{n} + \frac{1}{\sqrt{\lambda_n}}\right) \ . \tag{30}$$

We use Theorem 2.: $\bar{f}$ satisfies (13) and by (26), $E|g(X_n) - g(X)|^2 \le K/n$. Moreover

$$E\int_0^T |f_{n,\varepsilon}(s,\omega,Y_{n,\varepsilon}(s),Z_{n,\varepsilon}(s)) - \bar{f}(s,\omega,Y_{n,\varepsilon}(s),Z_{n,\varepsilon}(s))|^2\, ds$$

$$\le KL_\varepsilon^2 E\int_0^T (1+|Y_{n,\varepsilon}(s)|^2+|Z_{n,\varepsilon}(s)|^2)\times|X-X_n|_s^2\,ds$$
$$\le KL_\varepsilon^2\left(E\int_0^T (1+|Y_{n,\varepsilon}(s)|^2+|Z_{n,\varepsilon}(s)|^2)^2\,ds\right)^{1/2}$$
$$\times\left(E\int_0^T |X-X_n|_s^4\,ds\right)^{1/2}\le KL_\varepsilon^2/n$$

the last inequality being a consequence of (4) and of (26).
Now (30) follows from (15).
Define now

$$f_i^*(t,y,z)=\inf_{u\epsilon U} H_i(t,X,y,z,[u,u_{n,\varepsilon}(t,X)]_i)$$

and let (Y^*,Z^*) be the solution of the equation $E(g(X),f^*)$.

We use Lemma 1.i) for $y=\bar{Y}-Y^*, z=\bar{Z}-Z^*$ which satisfy (1) with

$$\begin{aligned}\varphi_t&=f^*(t,\bar{Y}(t),\bar{Z}(t))-f^*(t,Y^*(t),Z^*(t))\\ \beta_t&=\bar{f}(t,\bar{Y}(t),\bar{Z}(t))-f^*(t,\bar{Y}(t),\bar{Z}(t))\\ \xi&=0,\ \psi=0\ .\end{aligned}$$

Clearly φ satisfies (2) ii). Further we dominate $|\beta|\le\sum_{i=1}^3|\beta_i|$ with

$$\begin{aligned}\beta_t^1&=\bar{f}(t,\bar{Y}(t),\bar{Z}(t))-\bar{f}(t,Y_{n,\varepsilon}(t),Z_{n,\varepsilon}(t))\\ \beta_t^2&=f^*(t,\bar{Y}(t),\bar{Z}(t))-f^*(t,Y_{n,\varepsilon}(t),Z_{n,\varepsilon}(t))\\ \beta_t^3&=\bar{f}(t,Y_{n,\varepsilon}(t),Z_{n,\varepsilon}(t))-f^*(t,Y_{n,\varepsilon}(t),Z_{n,\varepsilon}(t))\ .\end{aligned}$$

Since $\bar{f}$ and f^* are Lipschitz continuous we may use (30) in order to get

$$E\int_0^T |\beta_s^1|^2+|\beta_s^2|^2\,ds\le K\left(\frac{L_\varepsilon^2}{n}+\frac{1}{\sqrt{\lambda_n}}\right)\ . \tag{31}$$

Further, using the definition of $\tilde{u}_\varepsilon$ one gets

$$\begin{aligned}&H_i(t,X,Y_{n,\varepsilon}(t),Z_{n,\varepsilon}(t),\tilde{u}_\varepsilon(t,X,Y_{n,\varepsilon}(t),Z_{n,\varepsilon}(t)))\\ &\quad\le\inf_u H_i(t,X,Y_{n,\varepsilon}(t),Z_{n,\varepsilon}(t),[u,\tilde{u}_\varepsilon(t,X,Y_{n,\varepsilon}(t),Z_{n,\varepsilon}(t))]_i)+\varepsilon^3\\ &\quad\le H_i(t,X,Y_{n,\varepsilon}(t),Z_{n,\varepsilon}(t),\tilde{u}_\varepsilon(t,X,Y_{n,\varepsilon}(t),Z_{n,\varepsilon}(t)))+\varepsilon^3\ .\end{aligned}$$

So

$$|\bar{f}^i(t,Y_{n,\varepsilon}(t),Z_{n,\varepsilon}(t))-f^{*,i}(t,Y_{n,\varepsilon}(t),Z_{n,\varepsilon}(t))|\le\varepsilon^3,\quad\forall\ 1\le i\le r\ .$$

This, together with (31) yields

$$E\int_0^T |\beta_s|^2 \quad ds \le K\left(\frac{L_\varepsilon^2}{n} + \frac{1}{\sqrt{\lambda_n}}\right) + \varepsilon^3 \ .$$

Now by Lemma 1. we get

$$E|Y^*(t) - \bar{Y}(t)|^2 + E\int_0^T |Z^*(s) - \bar{Z}(s)|^2 \ ds \le K\left(\frac{L_\varepsilon^2}{n} + \frac{1}{\sqrt{\lambda_n}}\right) + \varepsilon^3 \ . \tag{32}$$

Finally we define (recall that $\bar{Y}(0)$ and $Y^*(0)$ depend on ω'' but not on ω')

$$\Omega''_\varepsilon = \{\omega'' : |Y^*(0, \omega'') - \bar{Y}(0, \omega'')| \le \varepsilon\} \ .$$

Use Chebyshev's inequality in order to get

$$P(\Omega \backslash \Omega''_\varepsilon) \le \varepsilon^{-2} E|Y^*(t) - \bar{Y}(t)|^2 \le \varepsilon^{-2} K\left(\frac{L_\varepsilon^2}{n} + \frac{1}{\lambda_n}\right) + \varepsilon \le 2\varepsilon \ .$$

Let us now prove (29). We fix $u \epsilon U, \omega'' \in \Omega''_\varepsilon$ and $1 \le i \le r$, define $\hat{u}^i(\omega'') = [u, u_{n,\varepsilon}(\omega'')]_i$ and take $(\hat{Y}, \hat{Z})$ to be the solution of the equation (22) associated to the control $\hat{u}^i(\omega'')$. We look to the i'th equation of the system of equations (22) and use the comparison theorem in order to get $\hat{Y}^i(0, \omega'') \le Y^{*,i}(0, \omega'')$. Since $\omega'' \in \Omega''_\varepsilon$ it follows that

$$\begin{aligned} J_i(u_{n,\varepsilon}(\omega'')) &= \bar{Y}^i(0, \omega'') \le Y^{*,i}(0, \omega'') + \varepsilon \le \hat{Y}^i(0, \omega'') + \varepsilon \\ &= J_i([u, u_{n,\varepsilon}(\omega'')]_i) + \varepsilon \ . \end{aligned}$$

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