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Probability Theory and Related Fields

Parabolic problems for the Anderson model

II. Second-order asymptotics and structure of high peaks*

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Summary. This is a continuation of our previous work [6] on the investigation of intermittency for the parabolic equation $(\partial/\partial t)u = \mathscr{H}u$ on $\mathbb{R}_+ \times \mathbb{Z}^d$ associated with the Anderson Hamiltonian $\mathscr{H} = \kappa\Delta + \xi(\cdot)$ for i.i.d. random potentials $\xi(\cdot)$. For the Cauchy problem with nonnegative homogeneous initial condition we study the second order asymptotics of the statistical moments $\langle u(t,0)^p \rangle$ and the almost sure growth of $u(t,0)$ as $t \to \infty$. We point out the crucial role of double exponential tails of $\xi(0)$ for the formation of high intermittent peaks of the solution $u(t,\cdot)$ with asymptotically finite size. The challenging motivation is to achieve a better understanding of the geometric structure of such high exceedances which in one or another sense provide the essential contribution to the solution.

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0. Introduction

This paper is a natural continuation of our article [6]. The subject is the same, asymptotic analysis as $t \to \infty$ of the parabolic Anderson problem with homogeneous random potential $\xi(\cdot)$:

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$$\frac{\partial u}{\partial t} = \kappa \Delta u + \xi(x)u, \quad (t,x) \in \mathbb{R}_+ \times \mathbb{Z}^d, \ u(0,x) \equiv 1 \ .$$

In [6] we used rather 'soft' qualitative arguments to prove intermittency for the solution u under minimal conditions on the potential $\xi(\cdot)$. For a general discussion of intermittency and further references see the lectures [7].

Roughly speaking, intermittency means that, in contrast with homogenization, the spatial structure of $u(t,\cdot)$ is highly irregular for large t. In one or another sense the essential part of the solution is believed to consist of islands of high peaks which are located far from each other. The sizes of these islands as well as the heights and shapes of the corresponding peaks (both of the potential $\xi(\cdot)$ and the solution $u(t,\cdot)$) are crucial for different asymptotic questions related to our Anderson problem. A detailed understanding of the geometric structure of intermittent solutions would therefore be extremely useful.

Instead of directly investigating the spatial structure of $u(t,\cdot)$, we will study the second order asymptotics of the moments $\langle u(t,x)^p \rangle$, $p = 1, 2, \ldots$, and the almost sure growth of $u(t,x)$ as $t \to \infty$ for fixed x. We will restrict ourselves to the important case when the potential $\xi(\cdot)$ consists of independent, identically distributed random variables unbounded from above. In this case the solution $u(t,\cdot)$ is known to develop an intermittent behavior as $t \to \infty$, see [6]. Implicitly, our results and their proofs will allow a rather detailed insight into the geometry of the peaks. Let us remark that our method may be modified to study the second order asymptotics of moments for a rather large class of correlated random potentials.

In the i.i.d. case, a crucial role is played by double exponential tails with parameter ϱ, $0 < \varrho < \infty$:

$$\mathrm{Prob}(\xi(0) > r) = \exp\{-\mathrm{e}^{r/\varrho}\}, \quad r \to \infty \ .$$

Such tail behavior leads to islands of asymptotically finite size. In the case of 'heavier' tails (corresponding to $\varrho = \infty$ and including Gaussian potentials) the islands consist of isolated single lattice sites. On the other hand, for 'almost bounded' potentials (with faster decaying tails corresponding to $\varrho = 0$) the optimal peaks form very large flat islands. Qualitatively, the last situation is similar to the picture presented by A.-S. Sznitman in a series of papers on Brownian motion in a Poissonian environment, see e.g. [8] and [9].

In *Section 1* we will prove that

$$\langle u(t,0)^p \rangle = \exp\left\{ H(pt) - 2d\kappa\chi\left(\frac{\varrho}{\kappa}\right)pt + o(t) \right\} \qquad (0.1)$$

as $t \to \infty$, where H is the cumulant generating function of $\xi(0)$ and

$$e^{H(pt)} := \left\langle e^{pt\xi(0)} \right\rangle$$

is supposed to be finite for $t \geq 0$. The last condition guarantees the existence of all statistical moments of the homogeneous and ergodic solution $u(t, \cdot)$. The shapes of the high exceedances of the solution determine the function χ which may be expressed in terms of a variational problem. We will see that $\chi(0) = 0$, $0 < \chi(\varrho) < 1$ for $0 < \varrho < \infty$, and $\chi(\infty) = 1$. In the latter case, the factor $\exp\{-2d\kappa pt\}$ in (0.1) may be easily explained by use of the Feynman-Kac formula

$$u(t, 0) = \mathbb{E}_0 \exp\left\{ \int_0^t \xi(x(s)) ds \right\} , \tag{0.2}$$

where $x(t)$ is simple random walk on $\mathbb{Z}^d$ with generator $\kappa\Delta$. Namely, this factor will appear if the random walk is forced to stay at 0 until time t. Indeed,

$$u(t, 0) \geq \mathbb{E}_0 \exp\left\{ \int_0^t \xi(x(s)) ds \right\} \mathbb{1}(x(s) = 0 \text{ for } s \in [0, t]) = e^{t\xi(0) - 2d\kappa t} ,$$

and therefore

$$\langle u(t, 0)^p \rangle \geq e^{H(pt) - 2d\kappa pt} .$$

Hence, for $\varrho = \infty$ the random walk prefers to stay at one and the same lattice site for almost all the time. This makes it plausible that in this case the islands of high exceedances consist of single lattice sites and that in general $\chi(\varrho/\kappa)$ is closely related to the size of these islands. In fact, the solution to the mentioned variational problem, which is given by a nonlinear difference equation, is expected to determine the nonrandom shape of the relevant peaks. Note also that the simple universal bounds

$$e^{H(pt) - 2d\kappa pt} \leq \langle u(t, 0)^p \rangle \leq e^{H(pt)}$$

are valid for arbitrary homogeneous potentials $\xi(\cdot)$ with finite cumulant generating function H, see [6] for a proof of the upper bound.

In *Section 2* we will show under reasonable regularity assumptions that

$$u(t, 0) = \exp\left\{ t\psi(d \log t) - 2d\kappa\chi\left(\frac{\varrho}{\kappa}\right) t + o(t) \right\} \tag{0.3}$$

as $t \to \infty$ for almost all realizations of the random potential $\xi(\cdot)$. Thereby the function ψ is again fully determined by the tail behavior of the distribution of $\xi(0)$. In fact, $\psi(d \log t)$ describes the almost sure asymptotics of the maximum of the potential $\xi(\cdot)$ in a ball of radius t

as $t \to \infty$. Note that, by symmetry, $u(t,0) = \sum_{x\in\mathbb{Z}^d} v(t,x)$, where v is also a solution to our parabolic problem but with initial datum $v(0,x) = 1$ for $x = 0$ and $v(0,x) = 0$ otherwise. Therefore the intermittent peaks of $v(t,\cdot)$ are expected to determine the asymptotics (0.3). For unbounded from above potentials the moments $\langle u(t,0)^p\rangle$ grow much faster than the solution $u(t,0)$ itself, which is one more manifestation of intermittency. Hence, the leading terms in the asymptotic formulas (0.1) and (0.3) are totally different. But the second order correction terms, which contain the essential information about the geometry of the relevant peaks, coincide. This means that in both cases the advantageous peaks have the same shape but different heights.

Conceptionally, the above results are closely related to the spectral analysis of the Anderson Hamiltonian $\mathscr{H}$. One may expect that the high peaks of $u(t,\cdot)$ are generated by high exceedances of $\xi(\cdot)$ and that the height and form of the peaks of $u(t,\cdot)$ are asymptotically given by the principal eigenvalue and the corresponding positive eigenfunction of $\mathscr{H}$ in a neighborhood of the potential peak with Dirichlet boundary conditions, respectively.

Let us finally explain how and why the double exponential tails enter our picture in the i.i.d. case. The fundamental property of the double exponential distribution is that

$$\mathrm{Prob}(\xi(x) > h + \varphi(x), |\mathrm{x}| \le \mathrm{R}) = \exp\left\{ -\mathrm{e}^{h/\varrho} \sum_{|x|\le R} \mathrm{e}^{\varphi(x)/\varrho} \right\} .$$

This means that, independent of their common height h, two local peaks of the potential of the form $h + \varphi(\cdot)$ and $h + \tilde{\varphi}(\cdot)$ occur with the same frequency if and only if

$$\sum_{|x|\le R} \mathrm{e}^{\varphi(x)/\varrho} = \sum_{|x|\le R} \mathrm{e}^{\tilde{\varphi}(x)/\varrho} .$$

We conclude from this that, both for $u(t,0)$ and $\langle u(t,0)^p\rangle$, the shape $\varphi(\cdot)$ of the typical peaks (normalized by $\sum_x \mathrm{e}^{\varphi(x)/\varrho} = 1$) maximizes the principle eigenvalue $\lambda(\tilde{\varphi})$ of the operator $\kappa\Delta + \tilde{\varphi}(\cdot)$ among all shapes $\tilde{\varphi}(\cdot)$ with

$$\sum_x \mathrm{e}^{\tilde{\varphi}(x)/\varrho} = 1 . \tag{0.4}$$

The corresponding positive eigenfunction describes the nonrandom shape of the advantageous peaks of the solution $u(t,\cdot)$ near the relevant local maxima of the potential. We will see in Section 2 that under the constraint (0.4) the maximum of $\lambda(\tilde{\varphi})$ coincides with the term

$-2d\kappa\chi(\varrho/\kappa)$ in (0.1) and (0.3). For technical reasons, in Section 1 we will describe χ by means of a different, but equivalent, variational problem.

1. Asymptotics of the statistical moments

1.1. Statement of the result

This section deals with the random Cauchy problem

$$\frac{\partial u(t,x)}{\partial t} = \kappa\Delta u(t,x) + \xi(x)u(t,x), \quad (t,x) \in \mathbb{R}_+ \times \mathbb{Z}^d,$$
$$u(0,x) = 1, \quad x \in \mathbb{Z}^d \ , \tag{1.1}$$

for the Anderson tight binding Hamiltonian

$$\mathscr{H} := \kappa\Delta + \xi(\cdot) \ .$$

Thereby κ denotes a positive diffusion constant and Δ is the lattice Laplacian:

$$\Delta f(x) := \sum_{y:|y-x|=1} [f(y) - f(x)], \quad x \in \mathbb{Z}^d \ .$$

The potential $\xi(\cdot)$ is supposed to consist of i.i.d. random variables. The underlying probability measure and expectation will be denoted by $\mathrm{Prob}(\cdot)$ and $\langle\cdot\rangle$, respectively.

We will assume throughout that the cumulant generating function H of our random variables is finite on the positive half-axis:

$$H(t) := \log\left\langle \mathrm{e}^{t\xi(0)}\right\rangle < \infty \quad \text{for } t \geq 0 \ .$$

This assumption guarantees that a.s. the Cauchy problem (1.1) admits a unique nonnegative solution u. For each $t \geq 0$, $u(t,\cdot)$ is a spatially homogeneous ergodic random field, and

$$0 < \langle u(t,x)^p\rangle < \infty \quad \text{for } p = 1,2,\ldots \text{ and } (t,x) \in \mathbb{R}_+ \times \mathbb{Z}^d \ .$$

The results stated below remain valid for nonnegative homogeneous initial conditions $u_0(\cdot)$ which are independent of $\xi(\cdot)$ and satisfy

$$0 < \langle u_0(0)^p\rangle < \infty \quad \text{for } p = 1,2,\ldots$$

For technical details we refer to [6], Sections 2 and 3.

The objective of this section is to study the asymptotic behavior of the moments $\langle u(t,x)^p\rangle$, $p = 1,2,\ldots$, as $t \to \infty$ under the following regularity assumption on the cumulant generating function H.

Assumption (H). There exists ϱ, $0 \leq \varrho \leq \infty$, such that

$$\lim_{t\to\infty} \frac{H(ct) - cH(t)}{t} = \varrho c \log c \tag{1.2}$$

for all $c \in (0, 1)$.

In order to explain the meaning of this assumption let us consider the case when the potential is double exponentially distributed with parameter ϱ, $0 < \varrho < \infty$:

$$\mathrm{Prob}(\xi(0) > r) = \exp\{-\mathrm{e}^{r/\varrho}\} \ , \quad r \in \mathbb{R} \ .$$

Then $H(t) = \log \Gamma(\varrho t + 1) = \varrho t \log(\varrho t) - \varrho t + o(t)$, and (1.2) is fulfilled. Therefore, roughly speaking, for $0 < \varrho < \infty$, Assumption (H) tells us that the upper tail of the distribution of $\xi(0)$ behaves like that of a double exponential distribution. In the case $\varrho = \infty$ the tail is 'heavier', i.e. we are 'beyond' the double exponential situation. Finally, $\varrho = 0$ means that the tail decays faster than in the double exponential case, and we will say that the potential $\xi(\cdot)$ is 'almost bounded.'

Remark 1.1. a) If $0 \leq \varrho < \infty$, then Assumption (H) says that the function $\exp\{H(t)/t\}$ is regularly varying with exponent ϱ.

b) If $0 \leq \varrho < \infty$, then the convergence in (1.2) is uniform on $[0, 1]$. For $\varrho = \infty$, the convergence to $-\infty$ is uniform on each compact subset of $(0, 1)$. This follows from the observation that the function on the left of (1.2) is convex in c.

By $\mathscr{P}(\mathbb{Z})$ we will denote the space of probability measures on $\mathbb{Z}$. We next introduce the Donsker-Varadhan functional $S: \mathscr{P}(\mathbb{Z}) \to \mathbb{R}_+$ and the entropy functional $I: \mathscr{P}(\mathbb{Z}) \to \mathbb{R}_+$ defined by

$$S(p) := \sum_{x\in\mathbb{Z}} \left(\sqrt{p(x+1)} - \sqrt{p(x)}\right)^2, \quad p \in \mathscr{P}(\mathbb{Z}) \ ,$$

and

$$I(p) := -\sum_{x\in\mathbb{Z}} p(x) \log p(x), \quad p \in \mathscr{P}(\mathbb{Z}) \ ,$$

respectively. Note that $S(p)$ is nothing but the Dirichlet form of the one-dimensional lattice Laplacian at $\sqrt{p}$.

Our result will be described in terms of the cumulant generating function H and the function

$$\chi(\varrho) := \frac{1}{2} \inf_{p\in\mathscr{P}(\mathbb{Z})} [S(p) + \varrho I(p)], \quad 0 \leq \varrho < \infty \ . \tag{1.3}$$

One easily checks that χ is strictly increasing and concave and $0 \le \chi < 1$. Moreover, $\chi(0) = 0$ and $\lim_{\varrho\to\infty}\chi(\varrho) = 1$. Set $\chi(\infty) := 1$.

We are now ready to state the main result of this section.

Theorem 1.2. *Let Assumption (H) be satisfied. Then*

$$\langle u(t,0)^p\rangle = \exp\left\{H(pt) - 2d\kappa\chi\left(\frac{\varrho}{\kappa}\right)pt + o(t)\right\} \tag{1.4}$$

as $t \to \infty$ for $p = 1, 2, \ldots$

Remark 1.3. a) It will become obvious from the proof that the same asymptotics holds true for $\langle u(t,x_1)\ldots u(t,x_p)\rangle$, $x_1,\ldots,x_p \in \mathbb{Z}^d$, as well as for the moments of the fundamental solution $q(t,x,y)$ of our Cauchy problem. One only has to check that the large deviation principles of Lemma 1.5 below are also valid for the correspondingly modified measures.

b) For $0 < \varrho < \infty$, the infimum in (1.3) is attained. A probability measure on $\mathbb{Z}$ is a solution to this variational problem if and only if it is of the form const v_ϱ^2, where v_ϱ is a positive solution of the nonlinear difference equation

$$\Delta v_\varrho + 2\varrho v_\varrho \log v_\varrho = 0 \quad \text{on} \quad \mathbb{Z} \tag{1.5}$$

with minimal l^2-norm $\|v_\varrho\|_2$. Moreover,

$$\chi(\varrho) = \varrho \log \|v_\varrho\|_2 \ .$$

For sufficiently large ϱ, the minimal l^2-solution of (1.5) is unique modulo shifts. For small ϱ this is an open problem. As $\varrho \downarrow 0$,

$$\chi(\varrho) = \frac{\varrho}{4}\log\frac{1}{\varrho} + O(\varrho)$$

and v_ϱ has an asymptotically Gaussian shape of width $1/\sqrt{\varrho}$. The proof of these facts may be found in the forthcoming paper [5]. Note also that a similar problem occurs in Bolthausen and Schmock [1] in connection with the investigation of self-attracting random walks. Let us further remark that, for $0 < \varrho < \infty$, the term $2d\kappa\chi(\varrho/\kappa)$ in the expansion (1.4) is concave and strictly increasing as a function of the diffusion constant κ. This is obvious from (1.3).

c) A deeper analysis in the spirit of the almost sure considerations in Section 2 indicates that the typical shapes of the above mentioned high peaks of our solution are (time-dependent) multiples of $v_{\varrho/\kappa} \otimes \ldots \otimes v_{\varrho/\kappa}$.

As a first step towards the proof of Theorem 1.2, we will express the moments of $u(t,0)$ by means of local times of random walks on $\mathbb{Z}^d$. To this end, we exploit the Feynman-Kac representation

$$u(t,x) = \mathbb{E}_x \exp\left\{\int_0^t \xi(x(s))ds\right\} , \tag{1.6}$$

where $(x(t), \mathbb{P}_x)$ denotes simple symmetric random walk on $\mathbb{Z}^d$ with generator $\kappa\Delta$ and $\mathbb{E}_x$ stands for expectation with respect to $\mathbb{P}_x$. Let $p \in \mathbb{N}$ be fixed until the end of the proof. Consider p independent copies $x_1(t), \ldots, x_p(t)$ of the random walk $x(t)$, and denote by $\mathbb{P}_0^p$ and $\mathbb{E}_0^p$ probability and expectation given $x_1(0) = \cdots = x_p(0) = 0$, respectively. Let

$$l_{t,i}(z) := \int_0^t \mathbb{1}(x_i(s) = z)ds$$

be the local time of the i-th random walk spent at $z \in \mathbb{Z}^d$ during the time interval $[0, t]$, and introduce the total local time

$$l_t(z) := \sum_{i=1}^p l_{t,i}(z) \ .$$

It then follows from (1.6) that

$$u(t,0)^p = \mathbb{E}_0^p \exp\left\{\sum_{z\in\mathbb{Z}^d} l_t(z)\xi(z)\right\} \ .$$

Averaging over the random field $\xi(\cdot)$ leads to

$$\langle u(t,0)^p\rangle = \mathbb{E}_0^p \exp\left\{\sum_z H(l_t(z))\right\} \ . \tag{1.7}$$

We next note that the *occupation time measures*

$$L_t(\cdot) := \frac{l_t(\cdot)}{pt}$$

satisfy the *weak* large deviation principle as $t \to \infty$ with rate function being a d-dimensional analogue of the Donsker-Varadhan functional S, cf. Donsker and Varadhan [3]. In the next subsection we will explain how to get appropriate upper and lower bounds for the expectation on the right of (1.7) by 'compactifying' the state space of our random walks and then applying the *full* large deviation principle for the corresponding occupation time measures. After that, in Section 3, we will see how the variational expressions in these upper and lower bounds fit together to arrive at (1.4).

1.2. Compactification and application of large deviations

Given $R \in \mathbb{N}$, let $\mathbb{T}_R^d := \{-R, \ldots, R\}^d$ denote the centered lattice cube of length $2R + 1$. By introducing the periodic distance

$$d_R^{\pi}(x,y) := \min_{z\in(2R+1)\mathbb{Z}^d} |x-y-z|, \quad x,y \in \mathbb{T}_R^d\ ,$$

we may consider $\mathbb{T}_R^d$ as d-dimensional lattice torus.

Let $u^{R,\pi}$ and $u^{R,0}$ denote the solutions to the initial-boundary value problem for the equation

$$\frac{\partial u}{\partial t} = \kappa\Delta u + \xi(x)u \quad \text{on } \mathbb{R}_+ \times \mathbb{T}_R^d$$

with periodic and zero boundary conditions, respectively, and initial datum identically one. The following lemma enables us to reduce the study of the moments to the consideration of a large finite box $\mathbb{T}_R^d$. This has the advantage that the probability laws of the associated occupation time measures live on the *compact* state space $\mathscr{P}(\mathbb{T}_R^d)$.

Lemma 1.4. *Let u be the solution to the Cauchy problem* (1.1). *Then*

$$\left\langle u^{R,0}(t,0)^p \right\rangle \le \langle u(t,0)^p\rangle \le \left\langle u^{R,\pi}(t,0)^p \right\rangle \tag{1.8}$$

for all $R\in\mathbb{N}$, $t\ge 0$, *and* $p = 1,2,\ldots$

The derivation of these bounds relies on probabilistic formulas for the moments and only works for i.i.d. potentials. Rather than directly exploiting the bounds (1.8), we will use later on the corresponding inequalities for their probabilistic representations. But Lemma 1.4 explains the idea in a more analytic language.

We consider the ‘periodized’ local times

$$l_t^R(z) := \sum_{x\in(2R+1)\mathbb{Z}^d} l_t(z+x), \quad z\in\mathbb{T}_R^d\ ,$$

which may be regarded as total local times of p independent random walks on $\mathbb{T}_R^d$ with generator $\kappa\Delta$ and periodic boundary conditions. Let

$$L_t^R(\cdot) := \frac{l_t^R(\cdot)}{pt}$$

be the associated occupation time measures on $\mathbb{T}_R^d$. Let further τ_R^p denote the first time when one of the random walks $x_1(t),\ldots,x_p(t)$ exits $\mathbb{T}_R^d$.

We already know that the moments of the solution u to (1.1) admit the representation (1.7). In analogy with this, we find that

$$\left\langle u^{R,0}(t,0)^p \right\rangle = \mathbb{E}_0^p \exp\left\{\sum_{z\in\mathbb{T}_R^d} H(l_t(z))\right\} \mathbb{1}(\tau_R^p > t) \tag{1.9}$$

and

$$\left\langle u^{R,\pi}(t,0)^p \right\rangle = \mathbb{E}_0^p \exp\left\{ \sum_{z\in\mathbb{T}_R^d} H(l_t^R(z)) \right\} . \tag{1.10}$$

Proof of Lemma 1.4. The lower bound for $\langle u(t,0)^p\rangle$ is obvious from (1.7) and (1.9). Since the cumulant generating function H is convex and $H(0)=0$, we have

$$\sum_{k=1}^{n} H(\lambda_k) \le H\left(\sum_{k=1}^{n} \lambda_k\right)$$

for all $n\in\mathbb{N}$ and $\lambda_1,\ldots,\lambda_n\in\mathbb{R}_+$. Hence,

$$\sum_{z\in\mathbb{Z}^d} H(l_t(z)) \le \sum_{z\in\mathbb{T}_R^d} H(l_t^R(z)) .$$

Using this, we obtain the upper bound for $\langle u(t,0)^p\rangle$ from the probabilistic representations (1.7) and (1.10). □

The main tools for deriving asymptotic formulas for the moments (1.9) and (1.10) are large deviations for the occupation time measures of p independent random walks on $\mathbb{T}_R^d$ with zero and periodic boundary conditions (Lemma 1.5 below). That is, we will consider large deviations for the subprobability measures

$$\mu_t^{R,0}(B) := \mathbb{P}_0^p(L_t(\cdot)\in B, \tau_R^p > t)$$

and the probability measures

$$\mu_t^{R,\pi}(B) := \mathbb{P}_0^p\big(L_t^R(\cdot)\in B\big)$$

on $\mathscr{P}(\mathbb{T}_R^d)$. In this context, we need the Donsker-Varadhan functionals $S_d^{R,0}$ and $S_d^{R,\pi}$ on $\mathscr{P}(\mathbb{T}_R^d)$ defined by

$$S_d^{R,0}(p) := \sum_{\substack{\{x,y\}\subset\mathbb{Z}^d\\ |x-y|=1}} \left(\sqrt{p(x)}-\sqrt{p(y)}\right)^2, \quad p\in\mathscr{P}(\mathbb{T}_R^d) ,$$

and

$$S_d^{R,\pi}(p) := \sum_{\substack{\{x,y\}\subset\mathbb{T}_R^d\\ d_R^\pi(x,y)=1}} \left(\sqrt{p(x)}-\sqrt{p(y)}\right)^2, \quad p\in\mathscr{P}(\mathbb{T}_R^d) ,$$

respectively, where, by convention, in the first formula $p(x):=0$ for $x\notin\mathbb{T}_R^d$. Note that these expressions coincide with the Dirichlet form at $\sqrt{p}$ of the operator $-\Delta$ on $l^2(\mathbb{T}_R^d)$ with either zero or periodic boundary condition.

We intend to apply the following finite dimensional large deviation results which may be derived as particular cases from Donsker and Varadhan [2] or Gärtner [4].

Lemma 1.5. *Given $R \in \mathbb{N}$, the following holds true as $t \to \infty$.*

a) *The subprobability measures $\mu_t^{R,0}$ satisfy the full large deviation principle with scale pt and rate function $\kappa S_d^{R,0}$.*

b) *The probability measures $\mu_t^{R,\pi}$ satisfy the full large deviation principle with scale pt and rate function $\kappa S_d^{R,\pi}$.*

Remark 1.6. Since the formulation of a large deviation principle for unnormalized measures may appear to be unconventional, let us remark that assertion a) of Lemma 1.5 may be rephrased as follows. The probability measures $\mu_t^{R,0}(\cdot)/\mu_t^{R,0}(\mathbb{T}_R^d)$ satisfy the full large deviation principle with scale pt and rate function $\kappa S_d^{R,0} - \min(\kappa S_d^{R,0})$ and

$$\lim_{t\to\infty} \frac{1}{pt} \log \mu_t^{R,0}(\mathbb{T}_R^d) = -\min(\kappa S_d^{R,0}) \ .$$

We are now in a position to derive the desired asymptotic formulas for $\langle u^{R,0}(t,0)^p\rangle$ and $\langle u^{R,\pi}(t,0)^p\rangle$. Passing from the description by local times to the description by occupation time measures, we may rewrite (1.10) in the form

$$\left\langle u^{R,\pi}(t,0)^p \right\rangle = \mathrm{e}^{H(pt)} \mathbb{E}_0^p \exp\left\{ pt \sum_{z\in\mathbb{T}_R^d} \frac{H(L_t^R(z)pt) - L_t^R(z)H(pt)}{pt} \right\} \ . \quad (1.11)$$

Let us first consider the case when $0 \le \varrho < \infty$. Then Remark 1.1b) to Assumption (H) implies that the expression under the last sum becomes uniformly close to $\varrho L_t^R(z) \log L_t^R(z)$ as $t \to \infty$, and we arrive at

$$\left\langle u^{R,\pi}(t,0)^p \right\rangle = \mathrm{e}^{H(pt)+o(t)} \mathbb{E}_0^p \exp\left\{ -pt\varrho I_d^R(L_t^R(\cdot)) \right\} \ , \quad (1.12)$$

where I_d^R is the entropy functional on $\mathscr{P}(\mathbb{T}_R^d)$:

$$I_d^R(p) := -\sum_{z\in\mathbb{T}_R^d} p(z)\log p(z), \quad p \in \mathscr{P}(\mathbb{T}_R^d) \ .$$

We may now apply the Laplace-Varadhan method for the large deviation probabilities of Lemma 1.5 b) to see that

$$\mathbb{E}_0^p \exp\left\{ -pt\varrho I_d^R(L_t^R(\cdot)) \right\} = \exp\left\{ -pt \min\left[\kappa S_d^{R,\pi} + \varrho I_d^R \right] + o(t) \right\} \ . \quad (1.13)$$

Combining (1.12) with (1.13), we arrive at assertion b) of the next lemma.

Lemma 1.7. *Let Assumption (H) be satisfied. Then the following holds true as $t \to \infty$ for arbitrary $R \in \mathbb{N}$ and $p = 1, 2, \ldots$*

a) *If* $0 \le \varrho < \infty$, *then*

$$\langle u^{R,0}(t,0)^p \rangle \ge \exp\left\{H(pt) - pt \min\left[\kappa S_d^{R,0} + \varrho I_d^R\right] + o(t)\right\} .$$

b) *If* $0 \le \varrho < \infty$, *then*

$$\langle u^{R,\pi}(t,0)^p \rangle \le \exp\left\{H(pt) - pt \min\left[\kappa S_d^{R,\pi} + \varrho I_d^R\right] + o(t)\right\} .$$

c) *If* $\varrho = \infty$, *then*

$$\langle u^{R,0}(t,0)^p \rangle = \exp\{H(pt) - 2d\kappa pt + o(t)\} , \tag{1.14}$$

and the same asymptotics is valid for $\langle u^{R,\pi}(t,0)^p \rangle$.

The proof of assertion a) follows the same lines as that of b). Instead of (1.10) and Lemma 1.5 b), one has to use (1.9) and Lemma 1.5 a), respectively. To prove assertion c) assume that $\varrho = \infty$. The expression on the right of (1.14) is a trivial lower bound for $\langle u^{R,0}(t,0)^p \rangle$ which is obtained from (1.9) by forcing all random walks $x_1(t), \ldots, x_p(t)$ to stay at 0 during the whole time interval $[0, t]$. In view of Lemma 1.4, it now only remains to show that the expression on the right of (1.14) may also serve as an upper bound for $\langle u^{R,\pi}(t,0)^p \rangle$. From (1.11) and Remark 1.1 b) we conclude that

$$\langle u^{R,\pi}(t,0)^p \rangle \le e^{H(pt)} \left[\mathbb{P}_0^p\left(L_t^R(z) \notin (\varepsilon, 1-\varepsilon) \text{ for all } z \in \mathbb{T}_R^d\right) + o(e^{-\gamma t})\right]$$

for any $\varepsilon \in (0,1)$ and arbitrarily large γ. Here we have also used that the expression under the sum on the right of (1.11) is always non-positive. But the large deviation principle for $L_t^R(\cdot)$ (Lemma 1.5b)) tells us that the probability on the right behaves like

$$\exp\left\{ -pt \min\left\{\kappa S_d^R(p) : p(z) \notin (\varepsilon, 1-\varepsilon) \text{ for all } z \in \mathbb{T}_R^d\right\} + o(t)\right\} .$$

Since the minimum in the exponent tends to $2d\kappa$ as $\varepsilon \to 0$, this yields the correct upper bound.

A combination of Lemma 1.4 with Lemma 1.7 c) proves Theorem 1.2 in the case when $\varrho = \infty$. To complete the proof for $0 \le \varrho < \infty$ one has to show that the minima in the exponents on the right of the assertions a) and b) of Lemma 1.7 converge to the same limit as $R \to \infty$ and that this limit equals $2d\kappa\chi(\varrho/\kappa)$. This final step will be carried out in the next subsection in Lemma 1.10.

1.3. Properties of associated variational problems

We first consider the d-dimensional Donsker-Varadhan functional S_d and the d-dimensional entropy functional I_d defined by

$$S_d(p) := \sum_{\substack{\{x,y\}\subset\mathbb{Z}^d \\ |x-y|=1}} \left(\sqrt{p(x)} - \sqrt{p(y)}\right)^2, \quad p \in \mathscr{P}(\mathbb{Z}^d) \ ,$$

and

$$I_d(p) := -\sum_{x\in\mathbb{Z}^d} p(x)\log p(x), \quad p \in \mathscr{P}(\mathbb{Z}^d) \ ,$$

respectively. Note that S_1 and I_1 coincide, respectively, with the functionals S and I introduced in Section 1.1.

We claim that our d-dimensional variational problems split into the sum of d one-dimensional problems.

Lemma 1.8. *For* $0 \le \varrho < \infty$,

$$\inf\,[S_d + \varrho I_d] = d \inf\,[S + \varrho I] \ .$$

If $0 < \varrho < \infty$, *then the infimum on the left is attained at* $p \in \mathscr{P}(\mathbb{Z}^d)$ *if and only if* p *is a product measure*,

$$p = \bigotimes_{i=1}^{d} p_i \ ,$$

and the infimum on the right is attained at all $p_i \in \mathscr{P}(\mathbb{Z})$, $i = 1, \ldots, d$.

Proof. Given $d \ge 1$ and ϱ with $0 \le \varrho < \infty$, abbreviate

$$F_d := S_d + \varrho I_d \ .$$

We will show that

$$\inf F_{d+1} = \inf F_d + \inf F_1 \ . \tag{1.15}$$

First observe that

$$S_{d+1}(p_d \otimes p_1) = S_d(p_d) + S_1(p_1) \tag{1.16}$$

and

$$I_{d+1}(p_d \otimes p_1) = I_d(p_d) + I_1(p_1) \tag{1.17}$$

for all $p_d \in \mathscr{P}(\mathbb{Z}^d)$ and $p_1 \in \mathscr{P}(\mathbb{Z})$. This implies that the expression on the left of (1.15) does not exceed that on the right. To obtain the opposite inequality fix $p \in \mathscr{P}(\mathbb{Z}^{d+1})$ arbitrarily. Denote by p_d and p_1 the marginals of the first d and the last component of p, respectively, and consider the conditional laws

$$p_d(x|y) := \frac{p(x,y)}{p_1(y)} \text{ and } p_1(y|x) := \frac{p(x,y)}{p_d(x)}, \quad (x,y) \in \mathbb{Z}^d \times \mathbb{Z} \ .$$

Then

$$S_{d+1}(p) = \sum_{y \in \mathbb{Z}} p_1(y) S_d(p_d(\cdot|y)) + \sum_{x \in \mathbb{Z}^d} p_d(x) S_1(p_1(\cdot|x))$$

and

$$\begin{aligned} I_{d+1}(p) = &\sum_{y \in \mathbb{Z}} p_1(y) I_d(p_d(\cdot|y)) + \sum_{x \in \mathbb{Z}^d} p_d(x) I_1(p_1(\cdot|x)) \\ &+ \sum_{x \in \mathbb{Z}^d} \left[\sum_{y \in \mathbb{Z}} \Big(p_1(y) p_d(x|y) \log p_d(x|y) \Big) - p_d(x) \log p_d(x) \right] . \end{aligned}$$

Since the function $x \log x$, $x \geq 0$, is strictly convex, an application of Jensen's inequality shows that the expression in the square brackets is nonnegative and vanishes identically if and only if $p = p_d \otimes p_1$. Hence,

$$F_{d+1}(p) \geq \sum_{y \in \mathbb{Z}} p_1(y) F_d(p_d(\cdot|y)) + \sum_{x \in \mathbb{Z}^d} p_d(x) F_1(p_1(\cdot|x)) \ . \tag{1.18}$$

This yields the desired lower bound. Moreover, if $0 < \varrho < \infty$, then in (1.18) equality holds only if $p = p_d \otimes p_1$. Together with (1.16) and (1.17), this shows that the infimum on the left of (1.15) is attained at p if and only if p has the form $p_d \otimes p_1$ and the infima on the right are attained at p_d and p_1, respectively. □

Remark 1.9. It is obvious from the above proof that assertions analogous to Lemma 1.8 are valid for the functionals $S_d^{R,\pi} + \varrho I_d^R$ and $S_d^{R,0} + \varrho I_d^R$ considered in Section 1.2.

Recall that the function χ has been defined in (1.3). The next lemma fills the outstanding gap in the proof of Theorem 1.2.

Lemma 1.10. *For* $0 \leq \varrho < \infty$ *and each* $R \in \mathbb{N}$,

$$\min \left[S_d^{R,\pi} + \varrho I_d^R \right] \leq \inf \left[S_d + \varrho I_d \right] \leq \min \left[S_d^{R,0} + \varrho I_d^R \right] \ . \tag{1.19}$$

Moreover,

$$\begin{aligned} \lim_{R \to \infty} \min \left[S_d^{R,\pi} + \varrho I_d^R \right] &= \lim_{R \to \infty} \min \left[S_d^{R,0} + \varrho I_d^R \right] \\ &= \inf \left[S_d + \varrho I_d \right] = 2d\chi(\varrho) \ . \end{aligned} \tag{1.20}$$

Proof. Because of Lemma 1.8 and Remark 1.9, it will be enough to consider the case $d = 1$. For convenience, we will suppress the dimension index in our notation.

The right inequality in (1.19) is obvious. To derive the left inequality, we fix $p \in \mathscr{P}(\mathbb{Z})$ arbitrarily and consider the 'periodized' measure

$$p_R(z) := \sum_{x \in (2R+1)\mathbb{Z}} p(z+x), \quad z \in \mathbb{T}_R \ .$$

It will then be enough to check that

$$S^{R,\pi}(p_R) \le S(p) \tag{1.21}$$

and

$$I^R(p_R) \le I(p) \ . \tag{1.22}$$

As a consequence of the Cauchy-Schwarz inequality, we have

$$\left(\sqrt{p_R(y)} - \sqrt{p_R(z)}\right)^2 \le \sum_{x \in (2R+1)\mathbb{Z}} \left(\sqrt{p(y+x)} - \sqrt{p(z+x)}\right)^2$$

for all $y, z \in \mathbb{T}_R$. This yields (1.21). Inequality (1.22) follows from the fact that the function $\varphi(x) := -x \log x$, $x \ge 0$, is concave and $\varphi(0) = 0$ and therefore

$$\varphi(p_R(z)) \le \sum_{x \in (2R+1)\mathbb{Z}} \varphi(p(z+x))$$

for all $z \in \mathbb{T}_R$.

To prove (1.20), let $p \in \mathbb{T}_R$ be a measure at which the minimum of $S^{R,\pi} + \varrho I^R$ is attained. Because of shift invariance, we may assume without loss of generality that

$$p(-R) + p(R) \le \frac{2}{2R+1} \ .$$

Then

$$S^{R,0}(p) - S^{R,\pi}(p) = 2\sqrt{p(-R)}\sqrt{p(R)} \le \frac{2}{2R+1} \ .$$

This implies that

$$\min\left[S^{R,0} + \varrho I^R\right] - \min\left[S^{R,\pi} + \varrho I^R\right] \le \frac{2}{2R+1} \ .$$

Together with (1.19), this proves the convergence relations in (1.20). For $d = 1$ the last equality on the right of (1.20) is the definition of χ. □

2. Almost sure asymptotics

2.1. Statement of the result

In this section we will study the almost sure behavior of the solution $u(t,x)$ to our basic Cauchy problem (1.1) as $t \to \infty$ for fixed $x \in \mathbb{Z}^d$.

We will assume throughout that the potential $\xi(\cdot)$ consists of independent, identically distributed random variables with *continuous* distribution function F satisfying $F(r) < 1$ for all r (i.e. $\xi(\cdot)$ is unbounded from above a.s.).

Let us introduce the non-decreasing function

$$\varphi(r) := \log \frac{1}{1 - F(r)}, \quad r \in \mathbb{R} \ ,$$

and its left-continuous inverse

$$\psi(s) := \min\{r : \varphi(r) \geq s\}, \quad s > 0 \ .$$

Note that ψ is strictly increasing and $\varphi(\psi(s)) = s$ for all $s > 0$. The function ψ has been determined in such a way that the distribution of the field $\xi(\cdot)$ coincides with that of $\psi(\eta(\cdot))$, where $\eta(\cdot)$ is a field of independent, exponentially distributed random variables with mean 1. Hence, we may and will assume without loss of generality that $\xi(\cdot) = \psi(\eta(\cdot))$. This will allow us to study the high peaks of $\xi(\cdot)$ by investigating those of the 'standard' field $\eta(\cdot)$.

We next formulate our crucial restriction on the tail behavior of the distribution function F.

Assumption (F). There exists ϱ, $0 \leq \varrho \leq \infty$, such that

$$\lim_{s \to \infty} [\psi(cs) - \psi(s)] = \varrho \log c \tag{2.1}$$

for all $c \in (0, 1)$. If $\varrho = \infty$, then we demand in addition that

$$\lim_{s \to \infty} [\psi(s + \log s) - \psi(s)] = 0 \ . \tag{2.2}$$

Roughly speaking, if $0 < \varrho < \infty$, then assumption (2.1) requires that the upper tail of F behaves like that of a double exponential distribution with parameter ϱ. The case $\varrho = 0$ is that of an 'almost bounded' potential. If $\varrho = \infty$, then we are 'beyond' the double exponential tails, and (2.2) mainly restricts the tails to be not as 'heavy' as for exponentially distributed variables. In particular, problem (1.1) admits a unique nonnegative solution u, and this solution is given by the Feynman-Kac formula

$$u(t, 0) = \mathbb{E}_0 \exp\left\{ \int_0^t \xi(x(s)) ds \right\} \ , \tag{2.3}$$

cf. [6], Sections 2 and 3. This sounds very similar to what was assumed in the previous section. In fact, we will see later (Lemma 2.3 below) that Assumption (F) is slightly stronger than Assumption (H). The latter was imposed on the cumulant generating function in Section 1.1.

Remark 2.1. The following assertions are easily verified.

a) For $0 \le \varrho < \infty$, (2.1) says that e^{ψ} is regularly varying with exponent ϱ.

b) Condition (2.2) is equivalent to

$$\lim_{s\to\infty} [\psi(s + c \log s) - \psi(s)] = 0 \quad \text{for all } c \in \mathbb{R} \ .$$

If $0 \le \varrho < \infty$, then (2.2) follows from (2.1). As a consequence of (2.2), $\psi(s) = o(s/\log s)$.

c) Assumption (F) implies that

$$\lim_{r\to\infty} \frac{\varphi(r+\beta)}{\varphi(r)} = e^{\beta/\varrho} \quad \text{for all } \beta \in \mathbb{R}$$

(with the obvious definition of $e^{\beta/\varrho}$ for $\varrho = 0$ and $\varrho = \infty$) and

$$\varphi(r) + \log \varphi(r) \le \varphi(r+\beta)$$

for each $\beta > 0$ and all sufficiently large r.

d) If $0 \le \varrho < \infty$, then

$$\psi(\varphi(r)) = r + o(1) \quad \text{as } r \to \infty \ .$$

The almost sure asymptotics of $u(t,x)$ as $t \to \infty$ will now be characterized in terms of the function ψ and the function χ which was introduced in Section 1.1 by means of the Donsker-Varadhan functional S and the entropy functional I.

Theorem 2.2. *Let Assumption (F) be satisfied. If $d = 1$, suppose in addition that $\langle \log(1 + \xi(0)^-)\rangle < \infty$. Then almost surely*

$$u(t,0) = \exp\left\{\psi(d \log t)t - 2d\kappa\chi\left(\frac{\varrho}{\kappa}\right)t + o(t)\right\} \quad \textit{as } t \to \infty \ . \tag{2.4}$$

The leading term in this asymptotic expansion is related to the maximum of the potential $\xi(\cdot)$ along those paths of the random walk $x(t)$ which give the main contribution to the Feynman-Kac formula (2.3). As we will see,

$$\max_{|x|\le t} \xi(x) = \psi(d \log t) + o(1) \quad \text{a.s} \ .$$

For 'heavy' tails violating assumption (2.2), this non-random asymptotics breaks down and the second order term in (2.4) is expected to be superimposed by random fluctuations.

Our proof of Theorem 2.2 indicates the following interpretation. Assume that $0 < \varrho < \infty$ and that the minimal l^2-solution v_ϱ of equation (1.5) is unique modulo shifts. Let the initial total mass $\sum u(0,x)$ be finite a.s. (instead of $u(0,x) \equiv 1$). Then, as $t \to \infty$, the main contribution to the total mass of $u(t,\cdot)$ will be given by widely spaced high peaks the local shapes of which consist of (time-dependent) multiples of $v_{\varrho/\kappa} \otimes \ldots \otimes v_{\varrho/\kappa}$. These peaks of $u(t,\cdot)$ correspond to high exceedances of the potential $\xi(\cdot)$ of the form

$$\psi(d \log t) - 2d\kappa\chi(\varrho/\kappa) + 2\varrho \log\bigl(v_{\varrho/\kappa} \otimes \ldots \otimes v_{\varrho/\kappa}\bigr) \ .$$

The proof of Theorem 2.2 will be broken down into several steps. In the Sections 2.2–2.4 we will collect all the ingredients necessary for the proof which will then be put together in Section 2.5.

We close this subsection by revealing without proof the relationship between Assumption (F) and Assumption (H) from Section 1.1. As before, assume that the distribution function F is continuous and $F(r) < 1$ for all r. Let H denote the associated cumulant generating function.

Lemma 2.3. *Assume that $H(t) < \infty$ for all $t > 0$ or, equivalently, $\psi(s) = o(s)$ as $s \to \infty$. If $0 \le \varrho < \infty$, then the following two conditions are equivalent*:

$$\lim_{s\to\infty} [\psi(cs) - \psi(s)] = \varrho \log c \quad \text{for all } c \in (0,1) \tag{2.5}$$

and

$$\lim_{t\to\infty} \left[\frac{H(ct)}{ct} - \frac{H(t)}{t}\right] = \varrho \log c \quad \text{for all } c \in (0,1) \ . \tag{2.6}$$

Moreover, either of them implies that

$$\frac{H(t)}{t} = \psi(t) + \varrho \log \varrho - \varrho + o(1) \quad \text{as } t \to \infty \ . \tag{2.7}$$

2.2. Percolation bounds

In dimension $d \ge 2$, there is no need to impose any restrictions on the lower tail of the distribution function F. As a consequence of a percolation effect, in the Feynman-Kac representation (2.3) the random walk is able to bypass clusters of extremely negative peaks of the potential $\xi(\cdot)$. This subsection contains the related notation and auxiliary results.

Given $x \in \mathbb{Z}^d$ and $r \ge 0$, let $B_r(x) := \{y \in \mathbb{Z}^d : |y - x| \le r\}$ denote the closed ball in $\mathbb{Z}^d$ with center x and radius r. Here and in the sequel $|\cdot|$ stands for the lattice norm on $\mathbb{Z}^d$. We will abbreviate $B_r(0)$ by B_r.

Given a natural number R, we will say that $x, y \in \mathbb{Z}^d$ are *R-neighbors* if $|x - y| \le R$. A subset W of $\mathbb{Z}^d$ will be called *R-connected* if any two sites x, y of W may be joined by a path $x = z_0 \to z_1 \to \ldots \to z_n = y$ of R-neighbors in W. The minimum of the lengths $\sum |z_k - z_{k-1}|$ of all such paths will be denoted by $d_W^R(x,y)$. Hence, $d_W^R(x,y)$ measures the distance of x and y inside the R-connected set W. Each subset of $\mathbb{Z}^d$ splits into *R-connected components.*

For each $R \in \mathbb{N}$, consider the random variables

$$\xi_R(z) := \min_{x \in B_R(z)} \xi(x), \quad z \in (2R+1)\mathbb{Z}^d \ .$$

Note that $\xi_R(\cdot)$ is a field of i.i.d. random variables on the sublattice $(2R+1)\mathbb{Z}^d$. Define the level sets

$$A_\alpha^+(R) := \{z \in (2R+1)\mathbb{Z}^d : \xi_R(z) > \alpha\}, \quad \alpha \in \mathbb{R} \ .$$

Lemma 2.4. *Suppose that* $d \geq 2$. *Then, for each* $R \in \mathbb{N}$, *one finds a level* $\alpha = \alpha_R$ *such that the following holds true.*

a) *A.s. there exists a unique infinite* $(2R+1)$*-connected component* $W^+ = W^+(R)$ *of* $A_\alpha^+(R)$, *and* $\mathrm{Prob}(0 \in W^+) > 0$.

b) *There exists* $\vartheta_R > 1$ *such that a.s.*

$$\limsup_{|y| \to \infty, y \in W^+} \frac{d_{W^+}^{2R+1}(x,y)}{|x-y|} \leq \vartheta_R$$

for all $x \in W^+$.

Proof. This repeats the proof given in [6], Section 2.4, with the random field $\xi(\cdot)$ on $\mathbb{Z}^d$ replaced by $\xi_R(\cdot)$ on the sublattice $(2R+1)\mathbb{Z}^d$. □

We will assume from now on that, for each $R \in \mathbb{N}$, a level $\alpha = \alpha_R$ has been chosen as in Lemma 2.4 and the level set $A_\alpha^+(R)$ and the infinite percolation cluster $W^+(R)$ are defined accordingly.

As before, let $(x(t), \mathbb{P}_x)$ denote random walk on $\mathbb{Z}^d$ with generator $\kappa\Delta$. By τ_x and $\tau(r)$ we denote the first hitting times of the site $x \in \mathbb{Z}^d$ and the complement of the ball B_r, respectively.

Lemma 2.5. a) *For arbitrary* $r > 0$ *and* $t > 0$, *we have*

$$\mathbb{P}_0(\tau(r) \leq t) \leq 2^{d+1} \exp\left\{-r \log \frac{r}{d\kappa t} + r\right\} \ .$$

b) *Suppose that* $d \geq 2$. *Fix* $R \in \mathbb{N}$ *arbitrarily. Then there exists* $\vartheta_R > 1$ *such that for each* $t > 0$ *a.s.*

$$\mathbb{E}_0 \exp\left\{\int_0^{\tau_x} \xi(x(s))ds\right\} \mathbb{1}(\tau_x \leq t) \geq \exp\{-\vartheta_R |x| \log |x|\} \qquad (2.8)$$

for all sufficiently large $x \in \bigcup_{z\in W^+(R)} B_R(z)$. *In dimension* $d = 1$, *a corresponding estimate is valid a.s. for all sufficiently large* $|x|$ *provided that* $\langle \log(1 + \xi(0)^-)\rangle < \infty$.

This is a slight modification of Lemma 4.3 in [6]. The proof of part b) relies on the percolation bound in Lemma 2.4 b).

2.3. High exceedances of the random potential

In this subsection we consider random fields $\xi(\cdot)$ which satisfy Assumption (F) for some $\varrho \in [0,\infty]$. We will show that almost surely as $t \to \infty$ the high exceedances of the field $\xi(\cdot)$ in the ball B_t are of order $\psi(d\log t)$ and, for $\varrho \in (0,\infty]$, form islands of bounded size which are separated from each other by an arbitrarily large distance. After that we will prove that almost surely the set of local peaks of the 'vertically' shifted potential $\xi(\cdot) - \psi(d\log t)$ in B_t is asymptotically described by the class of profiles $h(\cdot)$ for which

$$\sum_x e^{h(x)/\varrho} \le 1 \quad .$$

Our results will first be formulated for the field $\eta(\cdot)$ of independent, exponentially distributed random variables with mean 1. As a corollary, we will then obtain the corresponding statements for the transformed field $\xi(\cdot) = \psi(\eta(\cdot))$.

Let us begin with the almost sure behavior of the maxima of the field $\eta(\cdot)$.

Lemma 2.6. *We have*

$$\limsup_{t\to\infty} \frac{\left|\max\limits_{x\in B_t} \eta(x) - \log|B_t|\right|}{\log\log|B_t|} \le 1 \quad a.s \quad .$$

The proof of this classical result will be given for the sake of completeness only.

Proof of Lemma 2.6. Fix $\theta > 1$ and an increasing sequence (t_n) of positive numbers so that $|B_{t_n}| \sim \theta^n$ as $n \to \infty$. Since

$$\log|B_{t_{n+1}}| - \log|B_{t_n}| = o(\log\log|B_{t_n}|)$$

and

$$\log\log|B_{t_{n+1}}| \sim \log\log|B_{t_n}| \quad ,$$

it will be enough to prove the statement for the sequence (t_n) instead of t. For each $c > 1$, we get

$$\begin{aligned}&\mathrm{Prob}\left(\max_{x\in B_{t_n}}\eta(x) > \log|B_{t_n}| + c\log\log|B_{t_n}|\right)\\&\leq |B_{t_n}|\mathrm{Prob}(\eta(0) > \log|B_{t_n}| + c\log\log|B_{t_n}|)\\&= \frac{1}{(\log|B_{t_n}|)^c} \sim \frac{1}{(n\log\theta)^c}\end{aligned}$$

and

$$\begin{aligned}&\mathrm{Prob}\left(\max_{x\in B_{t_n}}\eta(x) < \log|B_{t_n}| - c\log\log|B_{t_n}|\right)\\&= \left(1 - \frac{(\log|B_{t_n}|)^c}{|B_{t_n}|}\right)^{|B_{t_n}|} \leq \exp\{-(\log|B_{t_n}|)^c\}\\&= \exp\{-(n\log\theta)^c(1+o(1))\}\ .\end{aligned}$$

Hence, the probabilities on the left of both inequalities are summable over n for $c > 1$, and our assertion follows by an application of the Borel-Cantelli lemma. □

Taking into account Remark 2.1 b), we obtain the corresponding result for the transformed field $\xi(\cdot)$.

Corollary 2.7. *Let Assumption (F) be satisfied. Then almost surely*

$$\max_{x\in B_t}\xi(x) = \psi(\log|B_t|) + o(1) \quad \text{as } t\to\infty\ .$$

Remark 2.8. Since $|B_t|$ behaves like $(2t)^d$, we may replace in Lemma 2.6, and therefore also in Corollary 2.7, $\log|B_t|$ by $d\log t$. Corollary 2.7 therefore explains the appearance of the term $\psi(d\log t)$ in our considerations.

Given $\gamma > 0$ and $t > 0$, consider the point process of high exceedances

$$\tilde{E}_t^\gamma := \{x\in B_t: \eta(x) > \mathrm{e}^{-\gamma}\log|B_t|\}\ .$$

We next want to show that almost surely for large t the set $\tilde{E}_t^\gamma$ consists of islands the size of which does not exceed e^γ. After that this result will be reformulated in terms of the high exceedances of $\xi(\cdot)$.

Lemma 2.9. *For each $\gamma > 0$ and each natural number R, the following is true almost surely. There exists a random time $t_0 = t_0(\gamma, R, \eta(\cdot)) > 0$ such that for $t > t_0$ each R-connected component of $\tilde{E}_t^\gamma$ consists of at most e^γ elements.*

Proof. Fix $\gamma > 0$, $R\in\mathbb{N}$, and $\theta > 1$ arbitrarily. Consider an increasing sequence (t_n) such that $|B_{t_n}| \sim \theta^n$ as $n\to\infty$. Then

$$\mathrm{e}^{-\gamma} \log |B_{t_{n+1}}| = \mathrm{e}^{-\gamma+o(1)} \log |B_{t_n}|, \quad n \to \infty \ .$$

Because of this it will suffice to prove our lemma for the sequence (t_n) instead of t.

Fix a natural number $m > \mathrm{e}^{\gamma}$ arbitrarily and denote by $A_t^{\gamma,m}$ the event that $\tilde{E}_t^{\gamma}$ contains an R-connected subset of m elements. By the Borel-Cantelli lemma it will be enough to check that

$$\sum_n \mathrm{Prob}\left(A_{t_n}^{\gamma,m}\right) < \infty \ . \tag{2.9}$$

There are at most $C_{m,R}|B_t|$ R-connected subsets of B_t consisting of m elements, where $C_{m,R}$ is a positive constant which depends on m and R only. For each of these sets the probability to be contained in $\tilde{E}_t^{\gamma}$ equals

$$\exp\{-m\mathrm{e}^{-\gamma} \log |B_t|\} = |B_t|^{-m\mathrm{e}^{-\gamma}} \ .$$

Therefore,

$$\mathrm{Prob}\left(A_{t_n}^{\gamma,m}\right) \leq C_{m,R}|B_{t_n}|^{1-m\mathrm{e}^{-\gamma}} \sim C_{m,R}\theta^{-(m\mathrm{e}^{-\gamma}-1)n} \ .$$

Since $m\mathrm{e}^{-\gamma} > 1$, we arrive at (2.9). □

Given $\gamma > 0$ and $t > 0$, consider now the point process

$$E_t^{\gamma} := \left\{x \in B_t \colon \xi(x) > \max_{B_t} \xi - \gamma\right\} \ .$$

Corollary 2.10. *Let Assumption (F) be satisfied for some* $\varrho \in (0, \infty]$. *Then for each* $\gamma > 0$ *and each natural number* R, *the following is true almost surely. There exists a random time* $t_0 = t_0(\gamma, R, \xi(\cdot)) > 0$ *such that for* $t > t_0$ *each* R*-connected component of* E_t^{γ} *consists of at most* $\mathrm{e}^{\gamma/\varrho}$ *elements.*

In other words, for $0 < \varrho \leq \infty$, the high exceedances of the potential $\xi(\cdot)$ form islands of asymptotically bounded size which are located far from each other. For $\varrho = \infty$, these islands shrink to single lattice sites as $t \to \infty$.

Proof of Corollary 2.10. Suppose first that $0 < \varrho < \infty$. Then in Lemma 2.9 the point process $\tilde{E}_t^{\gamma/\varrho}$ coincides in law with

$$\left\{x \in B_t \colon \xi(x) > \psi\left(\mathrm{e}^{-\gamma/\varrho} \log |B_t|\right)\right\}$$

and, by Assumption (F),

$$\psi\left(\mathrm{e}^{-\gamma/\varrho} \log |B_t|\right) = \psi(\log |B_t|) - \gamma + o(1) \ .$$

Combining this with Corollary 2.7, we arrive at the desired result. The case $\varrho = \infty$ may be treated similarly. □

We now turn to the investigation of the typical shapes of high peaks of the field $\eta(\cdot)$ (resp. $\xi(\cdot)$) in a large ball around 0.

Lemma 2.11. *For each $R \in \mathbb{N}$ and almost all realizations of the random field $\eta(\cdot)$,*

$$\limsup_{t\to\infty} \max_{x\in B_t} \frac{\sum_{y\in B_R(x)} \eta(y)}{\log |B_t|} \le 1 \ . \tag{2.10}$$

Proof. Fix $\theta > 1$ arbitrarily and select an increasing sequence (t_n) so that $|B_{t_n}| \sim \theta^n$ as $n \to \infty$. It will be enough to prove (2.10) for the sequence (t_n) instead of t. Fix further $\gamma > 1$ arbitrarily. Then, applying Chebyshev's exponential inequality, we obtain

$$\begin{aligned} &\mathrm{Prob}\left(\max_{x\in B_{t_n}} \sum_{y\in B_R(x)} \eta(y) > \gamma^2 \log |B_{t_n}| \right) \\ &\le |B_{t_n}| \mathrm{Prob}\left(\gamma^{-1} \sum_{y\in B_R} \eta(y) > \gamma \log |B_{t_n}| \right) \\ &\le |B_{t_n}| \exp\{-\gamma \log |B_{t_n}|\} \left\langle \exp\left\{ \gamma^{-1} \sum_{y\in B_R} \eta(y) \right\} \right\rangle \\ &= \left(\frac{\gamma}{\gamma-1} \right)^{|B_R|} |B_{t_n}|^{1-\gamma} \sim \left(\frac{\gamma}{\gamma-1} \right)^{|B_R|} \theta^{-(\gamma-1)n} \ . \end{aligned}$$

Hence, the above probabilities are summable over n, and our assertion again follows from the Borel-Cantelli lemma. □

Corollary 2.12. *Let Assumption (F) be satisfied for some $\varrho \in (0, \infty)$. Then for each $R \in \mathbb{N}$ and almost all realizations of the random field $\xi(\cdot)$,*

$$\limsup_{t\to\infty} \max_{x\in B_t} \sum_{y\in B_R(x)} \exp\{[\xi(y) - \psi(\log |B_t|)]/\varrho\} \le 1 \ .$$

Remark 2.13. The corresponding assertion for $\varrho = \infty$ is obvious from Corollary 2.7 and Corollary 2.10. In this case, given $\gamma < 0 < \delta$ and $R \in \mathbb{N}$, the following holds true a.s. for sufficiently large t. In each ball $B_R(x)$, $x \in B_t$, the 'vertically' shifted potential $\xi(\cdot) - \psi(\log |B_t|)$ exceeds γ at not more than one lattice site and does not exceed δ at all.

Proof of Corollary 2.12. Since $\eta(\cdot) = \varphi(\xi(\cdot))$, the assertion of Lemma 2.11 may be rewritten in the form

$$\limsup_{t\to\infty} \max_{x\in B_t} \sum_{y\in B_R(x)} \frac{\varphi(\xi(y))}{\log|B_t|} \le 1 \quad \text{a.s.} \tag{2.11}$$

Assume that $0 < \varrho < \infty$. It then follows from Remark 2.1 c) that

$$\lim_{t\to\infty} \frac{\varphi(\psi(\log|B_t|) + \beta)}{\log|B_t|} = \mathrm{e}^{\beta/\varrho} \quad \text{uniformly in } \beta \le \beta_0 \tag{2.12}$$

for each β_0. Because of Corollary 2.7, a.s. the field $\xi(\cdot) - \psi(\log|B_t|)$ is bounded from above on B_{t+R} uniformly for large t. Taking this into account, we conclude from (2.12) that a.s.

$$\frac{\varphi(\xi(y))}{\log|B_t|} = \exp\{[\xi(y) - \psi(\log|B_t|)]/\varrho\} + o(1)$$

uniformly in $y \in B_{t+R}$ as $t \to \infty$. Substituting this in (2.11), we arrive at the desired result. □

We are now going to derive bounds on the profiles of high peaks opposite to that given in Lemma 2.11 and Corollary 2.12. To this end we will need to consider percolation clusters. Recall that, for each $R \in \mathbb{N}$, we fixed a level $\alpha = \alpha_R$ as in Lemma 2.4 and denoted by $A^+(R) = A^+_\alpha(R)$ and $W^+(R)$ the associated level set on the sublattice $(2R+1)\mathbb{Z}^d$ and its infinite $(2R+1)$-connected component, respectively. We will assume without loss of generality that the random field $\xi(\cdot)$ admits a representation of the form

$$\xi(x) = (1 - \zeta(x))\xi_-(x) + \zeta(x)\xi_+(x), \quad x \in \mathbb{Z}^d \ ,$$

where the random variables $\zeta(x)$, $\xi_-(x)$, $\xi_+(x)$ are mutually independent, $\zeta(x)$ attains the values 0 and 1 with probability $\mathrm{Prob}(\xi(x) \le \alpha)$ and $\mathrm{Prob}(\xi(x) > \alpha)$, respectively, $\xi_-(x) \le \alpha < \xi_+(x)$, and the distributions of $\xi_-(x)$ and $\xi_+(x)$ coincide with the conditional laws of $\xi(x)$ given $\xi(x) \le \alpha$ and $\xi(x) > \alpha$, respectively. Note that $\zeta(x) = 1$ if and only if $\xi(x)$ exceeds the level α. Accordingly, the field $\eta(\cdot) = \varphi(\xi(\cdot))$ admits the decomposition

$$\eta(x) = (1 - \zeta(x))\eta_-(x) + \zeta(x)\eta_+(x), \quad x \in \mathbb{Z}^d \ , \tag{2.13}$$

where $\eta_\pm(x) := \varphi(\xi_\pm(x))$. In particular, we have $\eta_-(x) \le \varphi(\alpha) \le \eta_+(x)$ and $\mathrm{Prob}(\eta_+(x) > s) = \exp\{\varphi(\alpha) - s\}$ for $s > \varphi(\alpha)$.

Lemma 2.14. a) *Suppose that* $d \ge 2$. *Given a natural number* R *and a function* $h\colon B_R \to \mathbb{R}_+$ *with*

$$\sum_{x\in B_R} h(x) < 1 \ , \tag{2.14}$$

the following holds true a.s. There exists a positive (random) time t_0 such that for all $t > t_0$ one finds a (random) site $z_0 \in W^+(R)$ such that $B_R(z_0) \subseteq B_t$ and

$$\eta(z_0 + \cdot) > h(\cdot) \log |B_t| \quad \text{on } B_R \ . \tag{2.15}$$

b) *With $W^+(R)$ replaced by $A^+(R)$, the above assertion is also true in dimension $d = 1$.*

Proof. a) Fix $R \in \mathbb{N}$ and $h: B_R \to \mathbb{R}_+$ satisfying (2.14) arbitrarily. Suppose without loss of generality that h is strictly positive. Since $\log |B_{n+1}| \sim \log |B_n|$, we may restrict ourselves to natural values of t. For $t \geq R$, define

$$W_t^+(R) := W^+(R) \cap B_{t-R} \ .$$

The balls $B_R(z)$, $z \in W_t^+(R)$, are pairwise disjoint and contained in B_t. Recall that $\text{Prob}(0 \in W^+(R)) > 0$ (Lemma (2.4 a)). Hence, we conclude from Birkhoff's ergodic theorem that there exists a positive constant C_R such that a.s.

$$|W_t^+(R)| \geq C_R |B_t| \quad \text{for sufficiently large } t \ . \tag{2.16}$$

Consider the events

$$E_{t,z} := \{\eta(z + \cdot) > h(\cdot) \log |B_t| \text{ on } B_R\} \ ,$$

$t \in \mathbb{N}, z \in (2R + 1)\mathbb{Z}^d$. Using the decomposition (2.13) and taking into account that $\zeta(x) = 1$ for all $x \in B_R(z)$ if $z \in W_t^+(R)$, the events $E_{t,z}$ coincide with

$$E_{t,z}^+ := \left\{\eta_+(z + \cdot) > h(\cdot) \log |B_t| \text{ on } B_R\right\}$$

for $z \in W_t^+(R)$. Therefore an application of the Borel-Cantelli lemma with respect to the conditional law given $\zeta(\cdot)$ reduces the proof of assertion a) to the verification of

$$\sum_{t=R}^{\infty} \text{Prob}\Big(\bigcap\nolimits_{z \in W_t^+(R)} (E_{t,z}^+)^c \Big| \zeta(\cdot)\Big) < \infty \quad \text{a.s.} \tag{2.17}$$

Since the random cluster $W_t^+(R)$ depends on $\zeta(\cdot)$ only and the events $E_{t,z}^+$ are mutually independent and independent of $\zeta(\cdot)$, we obtain a.s.

$$\begin{aligned}
\text{Prob}\Big(\bigcap\nolimits_{z \in W_t^+(R)} (E_{t,z}^+)^c \Big| \zeta(\cdot)\Big) &= \Big(1 - \text{Prob}(E_{t,0}^+)\Big)^{|W_t^+(R)|} \\
&\leq \exp\Big\{-|W_t^+(R)| \text{Prob}(E_{t,0}^+)\Big\} \\
&= \exp\Big\{-|W_t^+(R)| e^{|B_R|\varphi(\alpha)} |B_t|^{-\sum_{x \in B_R} h(x)}\Big\} \\
&\leq \exp\Big\{-\tilde{C}_R |B_t|^{1-\sum_{x \in B_R} h(x)}\Big\}
\end{aligned}$$

for sufficiently large t, where $\tilde{C}_R$ denotes a positive constant. On the bottom line we have used the bound (2.16). Because of assumption (2.14), this proves (2.17).

b) With several simplifications, the proof of part b) goes along the same lines as that of part a). □

Corollary 2.15. *Suppose that* $d \geq 2$. *Let Assumption* (*F*) *be satisfied for some* $\varrho \in [0, \infty)$. *Fix* $R \in \mathbb{N}$ *arbitrarily. Then the following is valid for almost all realizations of the random field* $\xi(\cdot)$.

a) *If* $\varrho = 0$, *then for each* $\delta > 0$ *there exists a positive* (*random*) *time* t_0 *such that for every* $t > t_0$ *one finds a* (*random*) *site* $z_0 \in W^+(R)$ *such that* $B_R(z_0) \subseteq B_t$ *and*

$$\xi(z_0 + \cdot) > \psi(\log|B_t|) - \delta \quad \text{on } B_R \ .$$

b) *If* $0 < \varrho < \infty$, *then for each function* $h: B_R \to \mathbb{R}$ *with*

$$\sum_{x \in B_R} \mathrm{e}^{h(x)/\varrho} < 1 \tag{2.18}$$

there exists a positive (*random*) *time* t_0 *such that for every* $t > t_0$ *one finds a* (*random*) *site* $z_0 \in W^+(R)$ *such that* $B_R(z_0) \subseteq B_t$ *and*

$$\xi(z_0 + \cdot) > \psi(\log|B_t|) + h(\cdot) \quad \text{on } B_R \ .$$

c) *With* $W^+(R)$ *replaced by* $A^+(R)$, *the above assertions are also true in dimension* $d = 1$.

For $\varrho = 0$, assertion a) tells us that the size of the islands of high exceedances of the potential $\xi(\cdot)$ grows unboundedly as $t \to \infty$.

Proof of Corollary 2.15. Since $\xi(\cdot) = \psi(\eta(\cdot))$, assertion (2.15) implies that

$$\xi(z_0 + \cdot) > \psi(h(\cdot)\log|B_t|) \quad \text{on } B_R \ . \tag{2.19}$$

But, if $\varrho = 0$, then

$$\psi(h(\cdot)\log|B_t|) = \psi(\log|B_t|) + o(1)$$

independent of the specific choice of $h: B_R \to \mathbb{R}_+$ provided that h is strictly positive. This yields assertion a).

To prove b) we remark that assumption (2.18) is the same as (2.14) with $h(\cdot)$ replaced by $\mathrm{e}^{h(\cdot)/\varrho}$. Hence, instead of (2.19) we obtain

$$\xi(z_0 + \cdot) > \psi\left(\mathrm{e}^{h(\cdot)/\varrho}\log|B_t|\right) \quad \text{on } B_R \ .$$

But, since $0 < \varrho < \infty$, Assumption (F) yields

$$\psi\left(\mathrm{e}^{h(\cdot)/\varrho}\log|B_t|\right) = \psi(\log|B_t|) + h(\cdot) + o(1) \quad \text{on } B_R \ ,$$

and we are done.

To prove c), one has to use assertion b) of Lemma 2.14 instead of a). □

2.4. Related spectral problems

Given $R > 0$ and $h: B_R \to \mathbb{R}$, let us denote by $\lambda_R(h(\cdot))$ the principal eigenvalue of the operator $\kappa\Delta + h(\cdot)$ in $l^2(B_R)$ with Dirichlet boundary condition. In particular, $\lambda_t(\xi(\cdot))$ is the principal eigenvalue of the Anderson Hamiltonian

$$\mathscr{H} = \kappa\Delta + \xi(\cdot)$$

in $l^2(B_t)$ with zero boundary condition. The aim of this subsection is to prove the following theorem on the almost sure asymptotics of $\lambda_t(\xi(\cdot))$.

Theorem 2.16. *Let Assumption (F) be satisfied for some* $\varrho \in [0, \infty]$. *Then almost surely*

$$\lambda_t(\xi(\cdot)) = \psi(\log|B_t|) - 2d\kappa\chi\left(\frac{\varrho}{\kappa}\right) + o(1) \quad \text{as } t \to \infty \ . \tag{2.20}$$

Before carrying out the details, let us briefly explain the origin of formula (2.20) in the case when $0 < \varrho < \infty$. We have seen in Section 2.3 that almost surely the high peaks of $\xi(\cdot)$ in B_t are of the form

$$\psi(\log|B_t|) + h(\cdot) \ ,$$

where $h: B_R \to \mathbb{R}$ runs through the class of functions satisfying

$$\sum_{x \in B_R} \mathrm{e}^{h(x)/\varrho} < 1 \tag{2.21}$$

and R is arbitrarily large. Since the islands of these peaks are located far from each other, the upper part of the spectrum of $\mathscr{H}$ in $l^2(B_t)$ is expected to split into the union of the spectra on the single islands. Hence, as $t \to \infty$, $\lambda_t(\xi(\cdot))$ will be close to the upper boundary of

$$\lambda_R(\psi(\log|B_t|) + h(\cdot)) = \psi(\log|B_t|) + \lambda_R(h(\cdot))$$

taken over all profiles $h: B_R \to \mathbb{R}$ satisfying (2.21) for arbitrarily large R. The next lemma shows that this variational expression equals $\psi(\log|B_t|) - 2d\kappa\chi(\varrho/\kappa)$. It therefore makes plausible formula (2.20).

Let S_R^0 denote the Donsker-Varadhan functional on $\mathscr{P}(B_R)$ with Dirichlet boundary condition, and let I_R be the corresponding entropy functional. These functionals are defined in the same way as the functionals $S_d^{R,0}$ and I_d^R considered in the Sections 1.2 and 1.3 with the only difference that they are now given on $\mathscr{P}(B_R)$ instead of $\mathscr{P}(\mathbb{T}_R^d)$.

Lemma 2.17. *If* $0 < \varrho < \infty$, *then*

$$\sup_{\sum_{x \in B_R} e^{h(x)/\varrho} < 1} \lambda_R(h(\cdot)) = - \min_{\mathscr{P}(B_R)} \left[\kappa S_R^0 + \varrho I_R\right]$$

for each $R \in \mathbb{N}$. *Moreover,*

$$\lim_{R \to \infty} \min_{\mathscr{P}(B_R)} \left[\kappa S_R^0 + \varrho I_R\right] = 2d\kappa\chi\left(\frac{\varrho}{\kappa}\right) \ .$$

Proof. Let us first note that

$$\sup_{\sum_{x \in B_R} e^{h(x)/\varrho} < 1} \lambda_R(h(\cdot)) = \sup_{\sum_{x \in B_R} e^{h(x)/\varrho} = 1} \lambda_R(h(\cdot)) \ . \tag{2.22}$$

This follows e.g. from the observation that $\lambda_R(h(\cdot) + c) = \lambda_R(h(\cdot)) + c$ for each constant c. According to the variational principle for the largest eigenvalue,

$$\lambda_R(h(\cdot)) = \sup_{\|v\|_2 = 1} \sum_{x \in B_R} (\kappa \Delta v(x) + h(x)v(x))v(x) \ ,$$

where, by convention, $v(x) = 0$ for $x \notin B_R$. Since it is enough to take the supremum over positive v, we may use the substitution $v^2 =: p$ to rewrite it in the form

$$\lambda_R(h(\cdot)) = \sup_{p \in \mathscr{P}(B_R)} \left[\sum_{x \in B_R} h(x)p(x) - \kappa S_R^0(p)\right] \ .$$

In other words, λ_R is the Legendre transform of κS_R^0. Using this, we find that the supremum on the right of (2.22) equals

$$\sup_{p \in \mathscr{P}(B_R)} \left\{ \sup_{\sum_{x \in B_R} e^{h(x)/\varrho} = 1} \left[\sum_{x \in B_R} h(x)p(x) - \varrho \log \sum_{x \in B_R} e^{h(x)/\varrho}\right] - \kappa S_R^0(p) \right\} \ .$$

Now observe that the expression in the square brackets does not change by adding a constant to h. Therefore the inner supremum may be taken over all $h\colon B_R \to \mathbb{R}$, and a straightforward computation shows that it coincides with $-\varrho I_R(p)$. In this way we arrived at the first assertion of our lemma. Since each ball B_R may be embedded in between two tori $\mathbb{T}_{R'}^d$ and $\mathbb{T}_{R''}^d$, the second assertion is a straightforward consequence of Lemma 1.10. □

It may be seen from the above proof that the maximum of $\lambda_R(h(\cdot))$ over all h with

$$\sum_{x \in B_R} e^{h(x)/\varrho} = 1 \tag{2.23}$$

is attained at h if and only if the square of the normalized positive eigenfunction of $\kappa\Delta + h(\cdot)$ in $l^2(B_R)$ minimizes the functional $\kappa S_R^0 + \varrho I_R$, i.e. if

$$h = -\frac{\kappa\Delta\sqrt{p}}{\sqrt{p}} + \text{const} \ ,$$

where $p \in \mathscr{P}(\mathbb{T}_R^d)$ is a minimizer of $\kappa S_R^0 + \varrho I_R$ and the constant adjusts h to fulfill (2.23). Now let $R \to \infty$ and take into account Remark 1.3 b) and Lemma 1.10. Then one finds that the relevant shapes of the potential should be of the form

$$h = 2\varrho \log\bigl(v_{\varrho/\kappa} \otimes \ldots \otimes v_{\varrho/\kappa}\bigr) - 2d\kappa\chi(\varrho/\kappa) \ .$$

This is in accordance with our claims after Theorem 2.2.

We are now going to prove that the principal eigenvalue $\lambda_t(\xi(\cdot))$ indeed may be approached by the maximum of the principal eigenvalues on the islands of high peaks provided that these islands are located far from each other. Since primarily this does not have to do anything with randomness, we will formulate the result in a nonrandom setting, although the proof will heavily rely on probabilistic arguments.

Let B be a finite connected subset of $\mathbb{Z}^d$. Fix $\kappa > 0$ and a potential $V : B \to \mathbb{R}$ arbitrarily. We want to estimate the principal eigenvalue $\lambda^{\mathscr{G}}$ of the Hamiltonian

$$\mathscr{G} = \kappa\Delta + V \quad \text{in } l^2(B)$$

with Dirichlet boundary condition by comparing it with the maximum of the principal eigenvalues of $\mathscr{G}$ on the 'islands of high peaks' of V. To this end, let G_i, $i = 1, \ldots, m$, denote connected subsets of B such that $\text{dist}(G_i, G_j) > 1$ for $i \neq j$, where $\text{dist}(\cdot,\cdot)$ denotes the lattice distance between subsets of $\mathbb{Z}^d$. For $i = 1, \ldots, m$, let g_i be a (not necessarily connected) non-empty subset of G_i. Think of the g_i's as the sites of high exceedances of V on some islands G_i in a surrounding ocean B. Let λ_i denote the principal eigenvalue of $\mathscr{G}$ in $l^2(G_i)$ with Dirichlet boundary condition, and set

$$\lambda_{\max} := \max_i \lambda_i, \ g := \bigcup_i g_i, \ G := \bigcup_i G_i \ .$$

Lemma 2.18. (*Cluster expansion*) *Suppose that*

$$\max_{B\setminus g} V \leq \lambda_{\max} \ . \tag{2.24}$$

Then the principal eigenvalue $\lambda^{\mathscr{G}}$ *of* $\mathscr{G}$ *in* $l^2(B)$ *satisfies*

$$\lambda_{\max} < \lambda^{\mathscr{G}} < \gamma$$

for all $\gamma > \lambda_{\max}$ *for which*

$$\frac{\gamma - \lambda_{\max}}{2d\kappa}\left[\left(1 + \frac{\gamma - \lambda_{\max}}{2d\kappa}\right)^{\operatorname{dist}(B\setminus G, g)} - 1\right] > \max_i |G_i| \ . \tag{2.25}$$

Proof. Since the principal eigenvalue $\lambda^{\mathscr{G}}$ depends on the potential V monotonically, the lower bound $\lambda^{\mathscr{G}} > \lambda_{\max}$ is obvious from replacing V by $-\infty$ outside of G. To prove the upper bound, we will apply some sort of cluster expansion of the resolvent $\mathscr{R}_\gamma$ associated with $\mathscr{G}$. We will show that, under (2.24) and (2.25), γ belongs to the resolvent set of $\mathscr{G}$. Using the probabilistic representation of the resolvent and taking into account that B is finite, it will be enough to check that

$$\mathscr{R}_\gamma \mathbb{1}(x) = \mathbb{E}_x \int_0^\eta dt \exp\left\{\int_0^t ds[V(x(s)) - \gamma]\right\} < \infty \tag{2.26}$$

for all $x \in B$. Here, as before, $(x(t), \mathbb{P}_x)$ denotes symmetric random walk on $\mathbb{Z}^d$ with generator $\kappa\Delta$, and η is the first exit time from B:

$$\eta := \inf\{t \geq 0 : x(t) \notin B\} \ .$$

We next introduce stopping times $0 \leq \sigma_0 < \tau_0 < \sigma_1 < \tau_1 \ldots$ of successive visits of the sets g and G^c by our random walk:

$$\begin{aligned} \sigma_0 &:= \inf\{t \geq 0 : x(t) \in g\}, \\ \tau_i &:= \inf\{t \geq \sigma_i : x(t) \notin G\}, \\ \sigma_{i+1} &:= \inf\{t \geq \tau_i : x(t) \in g\}, \quad i = 0, 1, 2, \ldots \end{aligned}$$

We will use these stopping time cycles to estimate the resolvent from above by a geometric series. First, we may rewrite (2.26) in the form

$$\begin{aligned} \mathscr{R}_\gamma \mathbb{1}(x) = \mathbb{E}_x \int_0^{\sigma_0 \wedge \eta} dt \exp\left\{\int_0^t ds[V(x(s)) - \gamma]\right\} \\ + \sum_{i=0}^\infty \mathbb{E}_x \int_{\sigma_i \wedge \eta}^{\sigma_{i+1} \wedge \eta} dt \exp\left\{\int_0^t ds[V(x(s)) - \gamma]\right\} \ . \end{aligned} \tag{2.27}$$

Since $x(s) \in B \setminus g$ for $0 < s < \sigma_0 \wedge \eta$ and because of (2.24), we have

$$\begin{aligned} \mathbb{E}_x \int_0^{\sigma_0 \wedge \eta} dt \exp\left\{\int_0^t ds[V(x(s)) - \gamma]\right\} \\ \leq \int_0^\infty dt \ \mathrm{e}^{(\lambda_{\max} - \gamma)t} = \frac{1}{\gamma - \lambda_{\max}} < \infty \end{aligned} \tag{2.28}$$

for all $x \in B$. Using the strong Markov property together with (2.24) and $\lambda_{\max} < \gamma$, we find for $i = 0, 1, 2, \ldots$ that

$$\begin{aligned}
&\mathbb{E}_x \int_{\sigma_i\wedge\eta}^{\sigma_{i+1}\wedge\eta} dt \exp\left\{\int_0^t ds[V(x(s))-\gamma]\right\} \\
&= \mathbb{E}_x \exp\left\{\int_0^{\sigma_0} ds[V(x(s))-\gamma]\right\}\mathbb{1}(\sigma_0<\eta) \\
&\quad \times \mathbb{E}_{x(\sigma_0)} \exp\left\{\int_0^{\sigma_i} ds[V(x(s))-\gamma]\right\}\mathbb{1}(\sigma_i<\eta) \\
&\quad \times \mathbb{E}_{x(\sigma_i)} \int_0^{\sigma_1\wedge\eta} dt \exp\left\{\int_0^t ds[V(x(s))-\gamma]\right\} \\
&\le \left(\max_{y\in g} \mathbb{E}_y \exp\left\{\int_0^{\sigma_1} ds[V(x(s))-\gamma]\right\}\mathbb{1}(\sigma_1<\eta)\right)^i \\
&\quad \times \max_{y\in g} \mathbb{E}_y \int_0^{\sigma_1\wedge\eta} dt \exp\left\{\int_0^t ds[V(x(s))-\gamma]\right\} . \qquad (2.29)
\end{aligned}$$

Combining (2.27) with (2.28) and (2.29), we see that it will be enough to show that, under the assumptions (2.24) and (2.25),

$$\mathbb{E}_x \exp\left\{\int_0^{\sigma_1} ds[V(x(s))-\gamma]\right\}\mathbb{1}(\sigma_1<\eta) < 1 \qquad (2.30)$$

and

$$\mathbb{E}_x \int_0^{\sigma_1\wedge\eta} dt \exp\left\{\int_0^t ds[V(x(s))-\gamma]\right\} < \infty \qquad (2.31)$$

for all $x \in g$.

Let us first prove assertion (2.30). Applying the strong Markov property, we obtain for $x \in g$:

$$\begin{aligned}
&\mathbb{E}_x \exp\left\{\int_0^{\sigma_1} ds[V(x(s))-\gamma]\right\}\mathbb{1}(\sigma_1<\eta) \\
&= \mathbb{E}_x \exp\left\{\int_0^{\tau_0} ds[V(x(s))-\gamma]\right\}\mathbb{1}(\tau_0<\eta) \\
&\quad \times \mathbb{E}_{x(\tau_0)} \exp\left\{\int_0^{\sigma_0} ds[V(x(s))-\gamma]\right\}\mathbb{1}(\sigma_0<\eta) . \qquad (2.32)
\end{aligned}$$

To derive an appropriate bound for the last expectation, note that $x(\tau_0) \in B\backslash G$ and $x(s) \in B\backslash g$ for $0 \le s < \sigma_0$. But, by assumption (2.24), $V - \gamma \le \lambda_{\max} - \gamma < 0$ outside of g. Moreover, σ_0 may be estimated from below by the sum of $\mathrm{dist}(B\backslash G, g)$ independent exponentially distributed random variables with mean $(2d\kappa)^{-1}$. Hence, $\mathbb{P}_x$-a.s.

$$\begin{aligned}
&\mathbb{E}_{x(\tau_0)} \exp\left\{\int_0^{\sigma_0} ds[V(x(s))-\gamma]\right\}\mathbb{1}(\sigma_0<\eta) \le \mathbb{E}_{x(\tau_0)} \mathrm{e}^{-(\gamma-\lambda_{\max})\sigma_0} \\
&\le \left(\frac{2d\kappa}{2d\kappa+\gamma-\lambda_{\max}}\right)^{\mathrm{dist}(B\backslash G, g)} . \qquad (2.33)
\end{aligned}$$

Note that, $\mathbb{P}_x$-a.s. for $x \in g_i$, τ_0 coincides with the first exit time from G_i. Thus, for $i \in \{1, \dots, m\}$ and $x \in g_i$,

$$u(x) := \mathbb{E}_x \exp\left\{ \int_0^{\tau_0} ds[V(x(s)) - \gamma] \right\} \tag{2.34}$$

coincides with the solution to the boundary value problem

$$\begin{aligned} (\kappa\Delta + V - \gamma)u = 0 \quad & \text{in } G_i, \\ u = 1 \quad & \text{on } G_i^c \ . \end{aligned}$$

With the substitution $u =: 1 + v$, this turns into

$$\begin{aligned} (\kappa\Delta + V - \gamma)v = \gamma - V \quad & \text{in } G_i, \\ v = 0 \qquad & \text{on } G_i^c \ . \end{aligned}$$

For $\gamma > \lambda_i$, the solution exists and is given by

$$v = \mathscr{R}_\gamma^{(i)}(V - \gamma) \ ,$$

where $\mathscr{R}_\gamma^{(i)}$ denotes the resolvent of $\mathscr{G}$ in $l^2(G_i)$ with Dirichlet boundary condition. Since $V \le \lambda_i + 2d\kappa \le \gamma + 2d\kappa$ on G_i, and because of the positivity of the resolvent, we obtain

$$v(x) \le 2d\kappa \mathscr{R}_\gamma^{(i)} \mathbb{1}(x) \le 2d\kappa \left(\mathscr{R}_\gamma^{(i)} \mathbb{1}, \mathbb{1} \right)_{G_i}, \quad x \in G_i \ ,$$

where $(\cdot, \cdot)_{G_i}$ is the inner product in $l^2(G_i)$. Using the spectral representation of the resolvent (i.e. its Fourier expansion with respect to the orthonormal basis of eigenfunctions of $\mathscr{G}$ in $l^2(G_i)$), we find that

$$\left(\mathscr{R}_\gamma^{(i)} \mathbb{1}, \mathbb{1} \right)_{G_i} \le \frac{|G_i|}{\gamma - \lambda_i} \ .$$

This means that

$$u(x) \le 1 + \frac{2d\kappa}{\gamma - \lambda_i} |G_i|, \quad x \in G_i \ . \tag{2.35}$$

Combining (2.32) with (2.33), (2.34), and (2.35), we arrive at

$$\begin{aligned} & \mathbb{E}_x \exp\left\{ \int_0^{\sigma_1} ds[V(x(s)) - \gamma] \right\} \mathbb{1}(\sigma_1 < \eta) \\ & \quad \le \left(1 + \frac{2d\kappa}{\gamma - \lambda_{\max}} \max_i |G_i| \right) \left(1 + \frac{\gamma - \lambda_{\max}}{2d\kappa} \right)^{-\mathrm{dist}(\mathrm{B} \setminus \mathrm{G}, \mathrm{g})} \end{aligned}$$

for $x \in g$. But the expression on the right is less than 1 if and only if (2.25) is fulfilled. This proves (2.30).

It remains to verify (2.31). Given $i \in \{1, \dots, m\}$ and $x \in g_i$, an application of the strong Markov property and (2.28) yields

$$\begin{aligned}&\mathbb{E}_x \int_0^{\sigma_1\wedge\eta} dt \exp\left\{\int_0^t ds[V(x(s))-\gamma]\right\}\\&= \mathbb{E}_x\left(\int_0^{\tau_0}+\int_{\tau_0\wedge\eta}^{\sigma_1\wedge\eta}\right)dt\exp\left\{\int_0^t ds[V(x(s))-\gamma]\right\}\\&= \mathscr{R}_\gamma^{(i)}\mathbb{1}(x)+\mathbb{E}_x\exp\left\{\int_0^{\tau_0} ds[V(x(s))-\gamma]\right\}\mathbb{1}(\tau_0<\eta)\\&\quad\times \mathbb{E}_{x(\tau_0)}\int_0^{\sigma_0\wedge\eta} dt\exp\left\{\int_0^t ds[V(x(s))-\gamma]\right\}\\&\le \mathscr{R}_\gamma^{(i)}\mathbb{1}(x)+\frac{1}{\gamma-\lambda_{\max}}\mathbb{E}_x\exp\left\{\int_0^{\tau_0} ds[V(x(s))-\gamma]\right\}\ .\end{aligned}$$

Since $\gamma > \lambda_i$, $\mathscr{R}_\gamma^{(i)}\mathbb{1}(x)$ is finite. The finiteness of the expectation on the right of the last estimate was shown before, see (2.34) and (2.35). Hence, we arrived at (2.31). This completes the proof of our lemma. □

We have now collected all the auxiliary material for the proof of our theorem.

Proof of Theorem 2.16. a) *Lower bound.* Let us first assume that $0 < \varrho < \infty$. Fix $R \in \mathbb{N}$ and $h: B_R \to \mathbb{R}$ with

$$\sum_{x\in B_R} e^{h(x)/\varrho} < 1$$

arbitrarily. Corollary 2.15 b) (resp. c) in dimension one) tells us that a.s. for sufficiently large t there exists a site z_0 such that $B_R(z_0) \subset B_t$ and

$$\xi(z_0+\cdot) > \psi(\log|B_t|) + h(\cdot) \quad \text{on } B_R\ .$$

This implies that

$$\lambda_t(\xi(\cdot)) \ge \psi(\log|B_t|) + \lambda_R(h(\cdot))\ .$$

From this we conclude that

$$\liminf_{t\to\infty}\,[\lambda_t(\xi(\cdot)) - \psi(\log|B_t|)] \ge \sup_{\sum_{x\in B_R} e^{h(x)/\varrho}<1} \lambda_R(h(\cdot)) \quad \text{a.s.}$$

Together with Lemma 2.17, this yields the lower bound.

In the case $\varrho = 0$, using Corollary 2.15 a) (resp. c)), we obtain

$$\liminf_{t\to\infty}\,[\lambda_t(\xi(\cdot)) - \psi(\log|B_t|)] \ge \lambda_R(0)\ .$$

But $\lambda_R(0) \to 0$ as $R \to \infty$, and we are done.

For $\varrho = \infty$, the lower bound follows from Corollary 2.7 and the fact that

$$\lambda_t(\xi(\cdot)) \geq \max_{B_t} \xi - 2d\kappa \ .$$

The latter is obvious from the observation that, for each $x \in B_t$, $\lambda_t(\xi(\cdot))$ may be estimated from below by the principal eigenvalue of $\mathscr{H}$ on the set $\{x\}$ with zero boundary condition which equals $\xi(x) - 2d\kappa$.

b) *Upper bound.* We first treat the case $0 < \varrho < \infty$. Fix $\delta > 0$ arbitrarily and choose $R \in \mathbb{N}$ so large that

$$\frac{\delta}{2d\kappa}\left[\left(1+\frac{\delta}{2d\kappa}\right)^R - 1\right] > \mathrm{e}^{2d\kappa/\varrho}|B_R| \ . \tag{2.36}$$

We know from Corollary 2.10 that a.s. for sufficiently large t the level set

$$E_t := \left\{x \in B_t : \xi(x) > \max_{B_t} \xi - 2d\kappa\right\}$$

splits into $(2R+1)$-connected clusters of size not exceeding $\mathrm{e}^{2d\kappa/\varrho}$. Given $x \in E_t$, denote by g_x the $(2R+1)$-connected component of E_t which contains x, and let

$$G_x := \bigcup_{y \in g_x} B_R(y) \cap B_t$$

denote its R-neighborhood in B_t. By construction, the sets G_x are connected. Moreover, any two of these sets either coincide or have a distance larger than one. Let λ_x denote the principal eigenvalue of $\mathscr{H}$ in $l^2(G_x)$ with Dirichlet boundary condition. Since

$$\lambda_x \geq \max_{G_x} \xi - 2d\kappa, \quad x \in E_t \ ,$$

and the potential in B_t does not exceed $\max_{B_t} \xi - 2d\kappa$ outside of $E_t = \bigcup_x g_x$, we find that

$$\lambda_{\max} := \max_{x \in E_t} \lambda_x \geq \max_{B_t \setminus \bigcup_{x \in E_t} g_x} \xi \ .$$

Hence, we are in a situation where we may apply Lemma 2.18 to estimate $\lambda_t(\xi(\cdot))$ from above. In our case $g = \bigcup_{x \in E_t} g_x$, $G = \bigcup_{x \in E_t} G_x$,

$$\mathrm{dist}(B_t \setminus G, g) \geq R, \text{ and } \max_{x \in E_t} |G_x| \leq \mathrm{e}^{2d\kappa/\varrho}|B_R| \ .$$

Because of (2.36), this means that condition (2.25) is fulfilled for $\gamma = \lambda_{\max} + \delta$. Thus, we conclude from Lemma 2.18 that a.s. for large t,

$$\lambda_t(\xi(\cdot)) \leq \max_{x \in E_t} \lambda_x + \delta \ . \tag{2.37}$$

Now observe that each of the sets G_x is contained in a ball of radius $R' := \mathrm{e}^{2d\kappa/\varrho} R$. But, according to Corollary 2.12, we have

$$\max_{x\in B_t}\sum_{y\in B_{R'}(x)}\exp\{[\xi(y)-\psi(\log|B_t|)]\}<\mathrm{e}^{\delta/\varrho} \tag{2.38}$$

a.s. for large t. Since the principal eigenvalue depends on the potential monotonically, we conclude from (2.37) and (2.38) that

$$\begin{aligned}\lambda_t(\xi(\cdot))-\psi(\log|B_t|) &\le \max_{x\in B_t}\lambda_{R'}(\xi(x+\cdot)-\psi(\log|B_t|))+\delta\\ &\le \sup_{\sum_{y\in B_{R'}}\mathrm{e}^{h(y)/\varrho}<\mathrm{e}^{\delta/\varrho}}\lambda_{R'}(h)+\delta\\ &= \sup_{\sum_{y\in B_{R'}}\mathrm{e}^{h(y)/\varrho}<1}\lambda_{R'}(h)+2\delta\end{aligned}$$

a.s. for large t. Hence, for each $\delta>0$ and all sufficiently large R,

$$\limsup_{t\to\infty}[\lambda_t(\xi(\cdot))-\psi(\log|B_t|)]\le \sup_{\sum_{y\in B_{R'}}\mathrm{e}^{h(y)/\varrho}<1}\lambda_{R'}(h)+2\delta \quad \text{a.s.}$$

Combining this with Lemma 2.17, we arrive at the desired upper bound.

The proof for $\varrho=\infty$ is similar. Given $\delta>0$, one has to choose R so large that (2.36) holds with $\mathrm{e}^{2d\kappa/\varrho}$ replaced by 1. A.s. for sufficiently large t, the $(2R+1)$-connected components of the level set E_t consist of single lattice sites. In particular, $G_x=B_R(x)$, $x\in E_t$. Choose $\gamma<0$ arbitrarily. According to Remark 2.13, a.s. for large t, the 'vertically' shifted potential $\xi(\cdot)-\psi(\log|B_t|)$ does not exceed γ on $B_R(x)\setminus\{x\}$ and does not exceed δ at site x for each $x\in E_t$. Hence, applying Lemma 2.18, we find that

$$\limsup_{t\to\infty}[\lambda_t(\xi(\cdot))-\psi(\log|B_t|)]\le\lambda_R(h_{\delta,\gamma})+\delta \quad \text{a.s.}\ ,$$

where $h_{\delta,\gamma}(0)=\delta$ and $h_{\delta,\gamma}(x)=\gamma$ for $x\in B_R\setminus\{0\}$. But $\lambda_R(h_{\delta,\gamma})$ tends to $\delta-2d\kappa$ as $\gamma\to-\infty$. Since $\delta>0$ may be chosen arbitrarily small, this implies the desired bound.

The proof in the case $\varrho=0$ is a straightforward consequence of Corollary 2.7 and the observation that

$$\lambda_t(\xi(\cdot))\le\max_{B_t}\xi\ .$$

□

We close this subsection with a modification of Theorem 2.16 which takes into account the percolation effect explained in Section 2.2. Recall that $A^+(R)$ and $W^+(R)$ denote, respectively, the level set in the sublattice $(2R+1)\mathbb{Z}^d$ and its infinite $(2R+1)$-connected component considered in Lemma 2.4. For $d\ge 2$, define

$$\widehat{W}_t^+(R) := \bigcup_{z \in W^+(R) \cap B_{t-R}} B_R(z) \ .$$

This is the R-neighborhood of the part of the infinite percolation cluster $W^+(R)$ in the ball B_{t-R}. If $d = 1$, then we define $\widehat{W}_t^+(R)$ by the same formula but with $W^+(R)$ replaced by the level set $A^+(R)$. We denote by $\lambda_t^R(\xi(\cdot))$ the principal eigenvalue of our random Hamiltonian $\mathscr{H}$ in $l^2(\widehat{W}_t^+(R))$ with Dirichlet boundary condition. Note that $\lambda_t^R(\xi(\cdot)) \le \lambda_t(\xi(\cdot))$.

Corollary 2.19. *Let Assumption* (*F*) *be satisfied for some* $\varrho \in [0, \infty]$. *Then almost surely*

$$\liminf_{R\to\infty} \liminf_{t\to\infty} \left[\lambda_t^R(\xi(\cdot)) - \psi(\log|B_t|)\right] \ge -2d\kappa\chi\left(\frac{\varrho}{\kappa}\right) \ .$$

This means that the principal eigenvalue $\lambda_t(\xi(\cdot))$ is essentially 'generated' by those islands of high peaks of the potential $\xi(\cdot)$ which are located in the R-neighborhood of the cluster $W^+(R)$ for large R.

Proof of Corollary 2.19. For $0 < \varrho < \infty$ and also for $\varrho = 0$, this repeats part a) of the proof of Theorem 2.16. Namely, according to Corollary 2.15, in the proof the lattice site z_0 may be assumed to belong to $W^+(R)$. If $\varrho = \infty$, then one has to replace Corollary 2.7 by the asymptotic formula

$$\max_{x \in W^+(R) \cap B_{t-R}} \xi(x) = \psi(\log|B_t|) + o(1) \quad \text{a.s. as } t \to \infty$$

which holds in dimension $d \ge 2$ and the corresponding formula with $W^+(R)$ replaced by $A^+(R)$ in dimension $d = 1$. To understand how to treat such a restriction to the cluster $W^+(R)$, we refer to the proof of Lemma 2.14 where a similar problem had been considered. The details are left to the reader. □

2.5. *Completion of the proof*

We are now finally in a position to complete the proof of Theorem 2.2. Roughly speaking, we will show by an application of the Feynman-Kac formula and the spectral representation theorem that $u(t, 0)$ behaves like $e^{t\lambda_t(\xi(\cdot))}$ a.s. as $t \to \infty$. This combined with our asymptotic formula for the principal eigenvalue $\lambda_t(\xi(\cdot))$ will then yield the desired asymptotics of $u(t, 0)$.

To be precise, fix $\varepsilon > 0$ arbitrarily and set

$$\underline{r}(t) := \frac{t}{(\log t)^{1+\varepsilon}} \text{ and } \overline{r}(t) := t(\log t)^{1+\varepsilon} \ .$$

We want to show that under the assumptions of Theorem 2.2,

$$\exp\Big\{(t-1)\lambda^R_{\underline{r}(t)}(\xi(\cdot)) + o(t)\Big\} \le u(t,0) \le \exp\big\{t\lambda_{\overline{r}(t)}(\xi(\cdot)) + o(t)\big\} \tag{2.39}$$

a.s. as $t \to \infty$ for each $R \in \mathbb{N}$. We may then apply the asymptotic formulas for the principal eigenvalues $\lambda_{\overline{r}(t)}(\xi(\cdot))$ and $\lambda^R_{\underline{r}(t)}(\xi(\cdot))$ obtained in Theorem 2.16 and Corollary 2.19, respectively. Substituting them in (2.39) and taking into account that

$$\psi\big(\log|B_{\underline{r}(t)}|\big) = \psi(d\log t) + o(1),$$
$$\psi\big(\log|B_{\overline{r}(t)}|\big) = \psi(d\log t) + o(1) \ ,$$

and $\psi(d\log t) = o(t)$, we arrive at the desired asymptotics (2.4). The above properties of ψ are obvious from Remark 2.1 b).

It now only remains to prove (2.39) by exploiting the Feynman-Kac representation (2.3) of $u(t,0)$. To derive the *lower bound* for $u(t,0)$, fix $R \in \mathbb{N}$ arbitrarily. Recall that $\widehat{W}^+_{\underline{r}(t)}(R)$ is the R-neighborhood of the part of the infinite percolation cluster $W^+(R)$ (resp. the level set $A^+(R)$ if $d=1$) which is contained in the ball $B_{\underline{r}(t)-R}$. Let e^R_t denote the normalized positive eigenfunction corresponding to the principal eigenvalue $\lambda^R_{\underline{r}(t)}(\xi(\cdot))$ of the random Hamiltonian $\mathscr{H}$ in $l^2(\widehat{W}^+_{\underline{r}(t)}(R))$ with Dirichlet boundary condition. Let $z_0 \in \widehat{W}^+_{\underline{r}(t)}(R)$ be a random site (depending on t and R) at which e^R_t attains its maximum. Then $|z_0| \le \underline{r}(t)$ and $(\mathrm{e}^R_t(z_0))^2 \ge |\widehat{W}^+_{\underline{r}(t)}(R)|^{-1} \ge |B_{\underline{r}(t)}|^{-1}$. Let σ^R_t denote the first exit time of the random walk $x(t)$ from $\widehat{W}^+_{\underline{r}(t)}(R)$. As before, let τ_{z_0} be the first hitting time of z_0. Repeatedly applying the strong Markov property to the Feynman-Kac representation of $u(t,0)$, we find that

$$\begin{aligned} u(t,0) \ge\ & \mathbb{E}_0 \exp\bigg\{\int_0^{\tau_{z_0}} \xi(x(u))du\bigg\} \mathbb{1}(\tau_{z_0} \le 1) \\ & \times \mathbb{E}_{z_0} \exp\bigg\{\int_0^{t-1} \xi(x(u))du\bigg\} \mathbb{1}\big(\sigma^R_t > t-1, x(t-1) = z_0\big) \\ & \times \inf_{0\le s\le 1} \mathbb{E}_{z_0} \exp\bigg\{\int_0^{s} \xi(x(u))du\bigg\} \ . \end{aligned} \tag{2.40}$$

In other words, we have forced the random walk to hit z_0 until time 1 and then to stay in $\widehat{W}^+_{\underline{r}(t)}(R)$ during a time period of length $t-1$ at the end of which it has to return to z_0. The remaining time the random walk is allowed to move freely. The expression on the last line of (2.40) is independent of t and strictly positive a.s. Since $|z_0| \le \underline{r}(t) = t/(\log t)^{1+\varepsilon}$, an application of Lemma 2.5 b) shows that a.s. the first expectation on the right is of order $\mathrm{e}^{o(t)}$ as $t \to \infty$. The main asymptotics is therefore hidden in the second expectation. But this is the probabilistic representation of the fundamental solution of $\mathscr{H}$ in $l^2(\widehat{W}^+_{\underline{r}(t)}(R))$ with zero boundary condition considered at time $t-1$ with starting point and end point equal to z_0. The spectral representation of the fundamental solution shows that the considered expectation may be estimated from below by

$$\mathrm{e}^{(t-1)\lambda^R_{\underline{r}(t)}(\xi(\cdot))}(\mathrm{e}^R_t(z_0))^2 \ge \mathrm{e}^{(t-1)\lambda^R_{\underline{r}(t)}(\xi(\cdot))}|B_{\underline{r}(t)}|^{-1} \quad .$$

In this way we arrive at the lower bound in (2.39).

To derive the *upper bound*, set $R_n(t) := n\overline{r}(t)$ for $n \in \mathbb{N}$ and $t > 0$. As before, let $\tau(R_n(t))$ denote the first exit time from the ball $B_{R_n(t)}$. Then, using the Feynman-Kac formula, we obtain

$$\begin{aligned} u(t,0) \le \mathbb{E}_0 \exp\left\{\int_0^t \xi(x(u))du\right\}\mathbb{1}(\tau(\overline{r}(t)) > t) \\ + \sum_{n=1}^{\infty} \mathbb{E}_0 \exp\left\{\int_0^t \xi(x(u))du\right\}\mathbb{1}(\tau(R_n(t)) \le t < \tau(R_{n+1}(t))) \quad . \end{aligned} \tag{2.41}$$

We will show that the first term on the right provides the correct asymptotics and the remaining sum tends to zero as $t \to \infty$ a.s. First note that

$$v(s,x) := \mathbb{E}_x \exp\left\{\int_0^s \xi(x(u))du\right\}\mathbb{1}(\tau(\overline{r}(t)) > s), \quad (s,x) \in \mathbb{R}_+ \times B_{\overline{r}(t)} \quad ,$$

is the solution of the initial-boundary value problem for the parabolic equation

$$\frac{\partial v}{\partial s} = \mathscr{H} v \quad \text{on } \mathbb{R}_+ \times B_{\overline{r}(t)}$$

with initial datum $v(0,x) \equiv 1$ and Dirichlet boundary condition. Using the spectral representation of $v(t,\cdot)$ (i.e. its Fourier expansion with respect to the eigenfunctions of $\mathscr{H}$ in $l^2(B_{\overline{r}(t)})$, we find that

$$v(t,0) \le \sum_{x \in B_{\overline{r}(t)}} v(t,x) \le \mathrm{e}^{t\lambda_{\overline{r}(t)}(\xi(\cdot))}|B_{\overline{r}(t)}| \quad .$$

Consequently,

$$\mathbb{E}_0 \exp\left\{\int_0^t \xi(x(u))du\right\}\mathbb{1}(\tau(\overline{r}(t)) > t) \le \exp\{t\lambda_{\overline{r}(t)}(\xi(\cdot)) + o(t)\}$$

a.s. as $t \to \infty$. This is the desired upper bound. It remains to check that the sum on the right of (2.41) tends to zero a.s. We obtain

$$\sum_{n=1}^{\infty} \mathbb{E}_0 \exp\left\{ \int_0^t \xi(x(u))du \right\} \mathbb{1}(\tau(R_n(t)) \le t < \tau(R_{n+1}(t)))$$

$$\le \sum_{n=1}^{\infty} \exp\left\{ t \max_{B_{R_{n+1}(t)}} \xi \right\} \mathbb{P}_0(\tau(R_n(t)) \le t) \ . \tag{2.42}$$

Using Corollary 2.7 and taking into account that $\psi(s) = o(s)$ by Remark 2.1 b), we find that a.s.

$$\max_{B_{R_{n+1}(t)}} \xi = \psi\left(\log |B_{R_{n+1}(t)}|\right) + o(1) = o(\log R_{n+1}(t))$$

as $t \to \infty$ uniformly in n. According to Lemma 2.5 a),

$$\mathbb{P}_0(\tau(R_n(t)) \le t) \le 2^{d+1} \exp\left\{ -R_n(t) \log \frac{R_n(t)}{d\kappa t} + R_n(t) \right\} \ .$$

Using these estimates and remembering that $R_n(t) = nt(\log t)^{1+\varepsilon}$, one easily checks that the sum on the right of (2.42) tends to zero a.s. as $t \to \infty$.

The proof of Theorem 2.2 is now complete.

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