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Probability Theory and Related Fields

Waves on fractal-like manifolds and effective energy propagation

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Summary. We prove that the average speed of effective energy propagation of the waves satisfying long wave length condition in some globally fractal-like manifolds is asymptotically zero.

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1 Introduction

Let M be a complete Riemannian manifold. Let $u \in C^{\infty}(R \times M; R)$ be a solution to the wave equation on M, i.e.

$$\frac{\partial^2}{\partial t^2} u(t,x) = \Delta u(t,x), \quad (t,x) \in R \times M \ .$$

Let us define the energy, $\Im(u)$, of u by

$$\Im(u) = \frac{1}{2}\int_M \left\{ \left| \left(\frac{\partial}{\partial t} u\right)(0,x) \right|^2 + |\nabla u(0,x)|^2 \right\} \mu(dx) \ ,$$

and the energy, $\Im(u; t, A)$, of u in a Borel set A $(A \subset M)$ at time t by

$$\Im(u; t, A) = \frac{1}{2}\int_A \left\{ \left| \left(\frac{\partial}{\partial t} u\right)(t,x) \right|^2 + |\nabla u(t,x)|^2 \right\} \mu(dx) \ ,$$

where $\mu(dx)$ denotes the Riemannian volume on the manifold M.

Let us assume that $0 < \Im(u) < \infty$. Let

$$A_0 = (\text{supp } u(0,\cdot)) \cup \left(\text{supp}\left(\frac{\partial}{\partial t}u\right)(0,\cdot)\right) ,$$

and

$$A(r) = \{x \in M; \text{dis}(x, A_0) > r\}, \quad r > 0 .$$

Then it is well-known that the propagation speed of wave is one (c.f. Chavel [3] p. 198), i.e. $\Im(u, t, A(|t|)) = 0, \ t \in \mathbb{R}$.

For each $\epsilon \in (0,1)$ and $t > 0$, let

$$l(t; \epsilon, u) = \inf\{r > 0; \Im(u; t, A(r)) < \epsilon \cdot \Im(u)\} .$$

We call $t^{-1}l(t;\epsilon,u)$ the average speed of effective energy propagation.

The purpose of the present paper is to show that the average speed of effective energy propagation of the waves satisfying long wave length condition in some globally fractal-like manifolds is asymptotically zero.

2 Main result

We say that $(\Omega, \sim)$ is a connected countable graph, if the following conditions are satisfied.

(1) Ω is a countable set.
(2) $\sim$ is a relation in Ω satisfying
 (i) $\omega \sim \omega$ for any $\omega \in \Omega$,
 (ii) if $\omega \sim \omega'$, then $\omega' \sim \omega$,

and

 (iii) for any $\omega, \omega' \in \Omega$, there are $n \geq 1, \omega_i, i = 1, \ldots, n-1$, such that $\omega_{i-1} \sim \omega_i$, $i = 1, \ldots, n$, where $\omega_0 = \omega$ and $\omega_n = \omega'$.

For a conneced countable graph $(\Omega, \sim)$, we define a metric function $d_\Omega\colon \Omega \times \Omega \to [0,\infty)$ on Ω by

$$d_\Omega(\omega,\omega') = \min\{n \geq 0; \omega_i \in \Omega, i = 0, \ldots, n, \\ \omega_0 = \omega, \omega_n = \omega', \omega_i \sim \omega_{i-1}, i = 1, \ldots, n\} .$$

for any $\omega, \omega' \in \Omega$.

Definition 2.1 *An $(N-1)$-dimensional Sierpinski Gasket graph, $N \geq 2$, is a connected countable graph $(\Omega, \sim)$ given by the following.*

(1) $\Omega = \{1, \ldots, N\}^{\mathbb{Z}_-}$, *where $\mathbb{Z}_-$ is the set of all nonpositive integers.*
(2) $\omega \sim \omega'$ *if there is an $n \leq 0$ such that $\omega_k = \omega'_k$, $k \leq n-1$, $\omega_k = \omega'_n$, $k \geq n+1$, and $\omega'_k = \omega_n$, $k \geq n+1$. Here $\omega = \{\omega_k\}_{k \leq 0}$ and $\omega' = \{\omega'_k\}_{k\leq 0}$.*

Definition 2.2 *Let* $(\Omega, \sim)$ *be a conneced countable graph. Let* M *be a complete Riemannian manifold,* dis_M *be its Riemannian distance, and* μ *be its Riemannian volume. We say that a complete Riemannian manifold* M *is a globally* $(\Omega, \sim)$*-like manifold, if there are a constant* $C_0 > 0$*, connected open sets* U_ω *in* M*, probability measures* ρ_ω *on* M *and* $\phi_\omega \in C_0^\infty(M; \mathbb{R})$*,* $\omega \in \Omega$*, such that*

(1) $\cup_{\omega\in\Omega} U_\omega = M$, diameter $(U_\omega) \leq C_0$, $\mu(U_\omega) \leq C_0$, $\omega \in \Omega$,

$$\mathrm{dis}_M(U_\omega, U_{\omega'}) \leq C_0 \cdot d_\Omega(\omega, \omega'), \quad \omega, \omega' \in \Omega \ ,$$

and

$$d_\Omega(\omega, \omega') \leq C_0(\mathrm{dis}_M(U_\omega, U_{\omega'}) + 1), \quad \omega, \omega' \in \Omega \ ,$$

(2) supp $\rho_\omega \subset U_\omega$, $\omega \in \Omega$, *and*

$$C_0^{-1} \cdot \mu \leq \sum_{\omega\in\Omega} \rho_\omega \leq C_0 \cdot \mu \ .$$

(3) supp $\phi_\omega \subset U_\omega$, $\omega \in \Omega$, $\phi_\omega \geq 0$, $\int_M \phi_\omega d\mu \geq C_0^{-1}$ *and* $|\nabla \phi_\omega| \leq C_0$, $\omega \in \Omega$, *and*

$$\sum_{\omega\in\Omega} \phi_\omega = 1 \ ,$$

(4) *for any* $\omega, \omega' \in \Omega$ *with* $\omega \sim \omega'$ *and* $u \in C^\infty(M, \mathbb{R})$,

$$\left| \int_M u \, d\rho_\omega - \int_M u \, d\rho_{\omega'} \right|^2 \leq C_0 \cdot \int_{U_\omega \cup U_{\omega'}} |\nabla u|^2 \, d\mu \ .$$

Definition 2.3 *A globally* $(N-1)$*-dim Sierpinski gasket like manifold is a globally* $(\Omega, \sim)$*-like complete Riemannian manifold, where* $(\Omega, \sim)$ *is a* $(N-1)$*-dimensinal Sierpinski gasket graph.*

Definition 2.4 *We say that a complete Riemannian manifold* M *satisfies uniform local Harnack inequality, if the following is satisfied.*

(ULH) *For any* $R_1 > R_0 > 0$*, there is an* $\varepsilon = \varepsilon(R_0, R_1) > 0$ *such that the following assertion holds. If* $x \in M$*,* $u \in C^\infty(B(x; R_1); (0, \infty))$ *and* $\Delta u(y) \leq 0$*,* $y \in B(x; R_1)$*, then* $\min\{u(y); y \in B(x; R_0)\} \geq \varepsilon \cdot \max\{u(y); y \in B(x; R_0)\}$.

Our main result is the following.

Theorem 2.5 *Let* M *be an globally* $(N-1)$*-dim Sierpinski gasket like Riemannian manifold satisfying uniform local Harnack inequality. Here we assume that* $N \geq 2$*. Suppose that* $\{u_n\}_{n=0}^\infty \subset C^\infty(\mathbb{R} \times M; \mathbb{R})$ *satisfies the following three conditions.*

(1) $u_n(t,\cdot) \in \mathrm{Dom}(\Delta),\ t \in \mathbb{R},\ t \to u_n(t,\cdot) \in L^2(M,dx)$ *is a* C^2*-function in* t, *and*

$$\frac{\partial^2}{\partial t^2} u_n(t,x) = \Delta u_n(t,x), \quad (t,x) \in \mathbb{R} \times M, \quad n = 1,2,\ldots,$$

(2) $\Im(u_n) = 1, \quad n = 1,2,\ldots,$

(3) (*long wave condition*) $\lim_{n\to\infty} 2^{-\varepsilon n} \log(1 + \|u_n(0)\|_{L^2}) = 0$ *for any* $\varepsilon > 0$, *and there is a* $C > 0$ *such that*

$$\Im\left(\frac{\partial}{\partial t} u_n\right) \le C \cdot 2^{-2n}, \quad n = 1,2,\ldots$$

Then for any $a \in (1, 2 - (2 - 2/d_w)^{-1}(1 - 2/d_w))$,

$$\lim_{n\to\infty} 2^{-n} \sup\{l(t;\epsilon,u_n); t \in [0, 2^{an}]\} = 0, \quad \forall \epsilon > 0 \ .$$

where $d_w = (\log(N+2)/\log 2) > 2$. *In particular*,

$$\limsup_{n\to\infty} 2^{-n} l(2^n;\varepsilon,u_n) = 0, \quad \varepsilon > 0 \ .$$

We give the proof of this theorem in Section 5.

Remark 2.6 (Waves in Euclidean space). Let us think of waves on an Euclidean space $\mathbb{R}^d$. Let $B(x,r)$ denote the set $\{y \in \mathbb{R}^d; |x-y| \le r\}, x \in \mathbb{R}^d$, $r > 0$. Let $f \in C_0^\infty(\mathbb{R}^d;\mathbb{R})$ be such that

$$f \ge 0, \quad \mathrm{supp}\, f = B(0,1) \quad \textit{and} \quad \int_{\mathbb{R}^d} |f(x)|^2\, dx = 1 \ .$$

Then there is a unique solution $v \in C^\infty(\mathbb{R}\times\mathbb{R}^d;\mathbb{R})$ such that

$$\frac{\partial^2}{\partial t^2} v(t,x) = \Delta v(t,x), \quad (t,x) \in \mathbb{R}\times\mathbb{R}^d \ ,$$
$$v(0,\cdot) = 0, \quad \frac{\partial}{\partial t} v(0,\cdot)) = f$$

(c.f. Petrovskii [7] Chapter 2). We have if $d = 1$

$$v(t,x) = \frac{1}{2}\int_{-t}^{t} f(s+x)\, ds, \quad (t,x) \in \mathbb{R}\times\mathbb{R} \ ,$$

and if $d \ge 2$

$$v(t,x) = \frac{1}{(d-2)!}\frac{\partial^{d-2}}{\partial t^{d-2}} \int_0^t I(x,r) r (t^2 - r^2)^{(d-3)/2}\, dr, \quad (t,x) \in \mathbb{R}\times\mathbb{R}^d$$

$$\text{where} \qquad I(x,r) = \frac{\Gamma(d/2)}{2\pi^{d/2}} \int_{|x-y|=r} f(y) r^{1-n}\, dS_y$$

(c.f. John [4] p. 33). Then we can easily see that

$$\left(\bigcup_{2<t<2+\delta} \operatorname{supp} v(t,\cdot)\right) \cap (\mathbb{R}^d \setminus B(0,3)) \neq \emptyset \quad \forall \delta > 0 \ .$$

So again by the uniqueness of a solution we have

$$(\operatorname{supp} v(2,\cdot)) \cap (\mathbb{R}^d \setminus B(0,2)) \neq \emptyset \ .$$

Also, we have $\Im(v) = 1$. Now let

$$u_n(t,x) = 2^{-((d/2)-1)n} v(2^{-n}t, 2^{-n}x), \quad (t,x) \in \mathbb{R} \times \mathbb{R}^d, \ \ n = 1,2,\ldots$$

Then we have

$$\operatorname{supp} u_n(0,\cdot) \cup \operatorname{supp}\left(\frac{\partial}{\partial t}u_n\right)(0,\cdot) \subset B(0,2^n),$$
$$\|u_n(0,\cdot)\|_{L^2}^2 = 2^{2n}\|v\|_{L^2}^2,$$
$$\Im(u_n) = \Im(v) = 1,$$
$$\Im\left(\frac{\partial}{\partial t}u_n\right) = 2^{-2n}\Im\left(\frac{\partial}{\partial t}v\right) \ ,$$

and

$$\Im(u_n; 2\cdot 2^n, \mathbb{R}^d \setminus B(2\cdot 2^n, 0)) = \Im(v; 2, \mathbb{R}^d \setminus B(2,0)) > 0 \ .$$

So we see that $l(2\cdot 2^n; \varepsilon, u_n) \geq 2^n$ if we choose $\varepsilon = \Im(v; \mathbb{R}^d \setminus B(2,0))/2$. In particular we have

$$\liminf_{n\to\infty} (2\cdot 2^n)^{-1} l(2\cdot 2^n; \varepsilon, u_n) \geq 1/2 \ . \tag{2.1}$$

So in the case of Euclidean space, even if we assume the long wave condition, the average speed of effective energy propagation does not converges to zero in general.

The frequency of $u_n(\cdot,x)$ is 2^{-n} times that of $v(\cdot,x)$. So the wave length of $u_n(\cdot,x)$ is 2^n times that of $v(\cdot,x)$. This is the reason why we call the condition (3) in Theorem 2.5 the long wave condition.

3 Basic results

Let $(\Omega,\sim)$ be an $(N-1)$-dim Sierpinski gasket graph. Let $\alpha = 2$ and $I = \{1,\ldots,N\}$. For each $n \geq 1$, let $\partial I^n = \{(i,\ldots,i) \in I^n;\ i = 1,\ldots,N\}$. Then it is obvious that $\partial I^n \neq I^n, n \geq 2$. Let $q\colon \Omega \times \Omega \to \{0,1\}$ be given by

$$q(\omega,\omega') = \begin{cases} 1, & \text{if } \omega \sim \omega', \\ 0, & \text{otherwise} \ . \end{cases}$$

Let $\mathscr{E}_0 \colon \mathrm{Dom}(\mathscr{E}_0) \times \mathrm{Dom}(\mathscr{E}_0) \to R$ be a bilinear form given by

$$\mathrm{Dom}(\mathscr{E}_0) = \left\{ f \in C(\Omega, R); \sum_{\omega \in \Omega} f(\omega)^2 < \infty, \right.$$
$$\left. \sum_{\omega,\omega' \in \Omega} q(\omega, \omega')(f(\omega) - f(\omega'))^2 < \infty \right\}$$

and

$$\mathscr{E}_0(f, g) = \sum_{\omega,\omega' \in \Omega} q(\omega, \omega')(f(\omega) - f(\omega'))(g(\omega) - g(\omega')),$$
$$f, g \in \mathrm{Dom}(\mathscr{E}_0) \ .$$

For any finite subset A of Ω, let $\mathscr{E}_{0,A}$: $\mathrm{Dom}(\mathscr{E}_0) \times \mathrm{Dom}(\mathscr{E}_0) \to R$ be a bilinear form given by

$$\mathscr{E}_{0,A}(f, g) = \sum_{\omega,\omega' \in A} q(\omega, \omega')(f(\omega) - f(\omega'))(g(\omega) - g(\omega')),$$
$$f, g \in \mathrm{Dom}(\mathscr{E}_0) \ .$$

For each $n \geq 1$ and $\omega \in \Omega$, let $B_{\omega,n} = \{\omega' \in \Omega; \omega'(k) = \omega(k), k \leq -n\}$ and $\mathscr{B}_n = \{B_{\omega,n}; \omega \in \Omega\}$, $n \geq 1$. For each $A, B \in \Omega$, we denote $A \sim B$ if there are $\omega \in A$ and $\omega' \in B$ with $\omega \sim \omega'$. For each $n \geq 1$ and $B \in \mathscr{B}_n$, we denote by $\partial_n B$ the set $\{\omega \in B; (\omega(-n), \omega(-(n-1)), \ldots, \omega(-1)) \in \partial I^n\}$. For each $A \subset \Omega$ and $n \geq 1$, let $E_n(A) = \cup\{B \in \mathscr{B}_n; A \sim B\}$. Let $\langle u \rangle_A$ denote $\sharp(A)^{-1} \cdot \sum_{x \in A} u(x)$ for any $u \in C(A; R)$ and subset A of Ω.

For each $n \geq 1$, let

$$\lambda_n = \sup\left\{ \sum_{x \in B} (u(x) - \langle u \rangle_B)^2; u \in C(B; R), \mathscr{E}_{0,B}(u, u) = 1, B \in \mathscr{B}_n \right\} \ .$$

For any $B, B' \in \mathscr{B}_n$ with $B \sim B'$ and $B \neq B'$, let $\sigma_n(B, B')$ be given by

$$\sigma_n(B, B') = \sup\{N^n(\langle u \rangle_B - \langle u \rangle_{B'})^2; .$$
$$u \in C(B \cup B'; R), \mathscr{E}_{0,B \cup B'}(u, u) = 1\} \ .$$

Furthermore, let σ_n, $n \geq 0$, be given by

$$\sigma_n = \sup\{\sigma(B, B'); B, B' \in \mathscr{B}_n, B \sim B', B \neq B'\} \ ,$$

and let $\lambda_n^{(D)}, n \geq 1$, be given by

$$\lambda_n^{(D)} = \sup\{N^n \cdot \langle u \rangle_B^2; u \in C(B; R), B \in \mathscr{B}_n, u|_{\partial_n B} = 0, \mathscr{E}_{0,B}(u, u) = 1\} \ .$$

For any subsets A and B of I^n with $A \cap B = \emptyset$, let

$$R(A,B) = \min\{\mathscr{E}_0(u,u); u \in \mathrm{Dom}(\mathscr{E}_0), u|_A = 1, u|_B = 0\}^{-1} \ .$$

Here we used the convention that $\min \emptyset = \infty$. For any $n \geq 1$, let

$$R_n = \inf\{R(B, \Omega \setminus E_n(B)), B \in \mathscr{B}_n\} \ .$$

Then we have the following (c.f.[6]).

Proposition 3.1 *Let* $\rho = N + 1$. *Then there are* $c_0, c_1 > 0$ *such that*

(1) $c_0 \cdot \rho^n \leq \sigma_n \leq c_1 \cdot \rho^n, \quad n \geq 2,$
(2) $c_0 \cdot \rho\rho^n \leq \lambda_n^{(D)} \leq c_1 \cdot \rho^n, \quad n \geq 2,$
(3) $c_0 \cdot \rho^n \leq \lambda_n \leq c_1 \cdot \rho^n, \quad n \geq 2,$
(4) $c_0 \cdot \rho^n N^{-n} \leq R_n \leq c_1 \cdot \rho^n N^{-n}, \quad n \geq 2.$

By [6] Theorem 7.2, we have the following.

Proposition 3.2 *There is a constant* $c_2 \in (0, \infty)$ *such that*

$$|u(\omega) - u(\omega')|^2 \leq c_2(N^{-n}\rho^n)\mathscr{E}_{0,B}(u,u), \quad \omega, \omega' \in B$$

for any $u \in C(B; R)$, $n \geq 1$, *and* $B \in \mathscr{B}_n$.

The following result is easy.

Proposition 3.3 *There are constants* $c_3, c_4, c_5 \in (0, \infty)$ *such that*

1. $d_\Omega(B, \Omega \setminus E_n(B)) \geq c_3 \alpha^n, \quad \forall n \geq 1, B \in \mathscr{B}_n,$
2. $\max\{d_\Omega(\omega, \omega'); \omega, \omega' \in B\} \leq c_4 \alpha^n, \quad \forall n \geq 1, B \in \mathscr{B}_n,$
3. $c_5 \alpha^n \leq \min\{d_\Omega(\omega, \omega'); \omega \in B, \omega' \in \Omega \setminus E_n(B)\}, \quad \forall n \geq 1, \ B \in \mathscr{B}_n.$

Now let M be a globally $(N-1)$-dim Sierpinski gasket like Riemannian manifold satisfying uniform local Harnack inequality. Let us denote by Δ the self-adjoint extension of the Laplacian operator, too. Let $\mathrm{Dom}(\mathscr{E}) = \mathrm{Dom}(\sqrt{-\Delta})$ and $\mathscr{E}$: $\mathrm{Dom}(\mathscr{E}) \times \mathrm{Dom}(\mathscr{E}) \to R$ be given by

$$\mathscr{E}(u,v) = \int_M \nabla u \cdot \nabla v \, d\mu, \quad u, v \in \mathrm{Dom}(\mathscr{E}) \ .$$

Let T: $C_0^\infty(M) \to C_0(\Omega)$ and S: $C_0(\Omega) \to C_0^\infty(M)$ be linear operators given by

$$(Tu)(\omega) = \int_M u \, d\rho_\omega, \quad \omega \in \Omega, \quad u \in C_0^\infty(M) \ ,$$

and

$$(Sv)(x) = \sum_{\omega \in \Omega} v(\omega)\phi_\omega(x), \quad x \in M, \quad v \in C_0(\Omega) \ .$$

Let us denote $U(A) = \cup_{\omega \in A} U_\omega$ for any subset A of Ω.

By Definition 2.2 we have the following result.

Proposition 3.4 *There is a constant* $C_1 \in (0, \infty)$ *such that*

1. $\mathscr{E}(Sv, Sv) \leq C_1 \cdot \mathscr{E}_0(v, v), \quad v \in C_0(\Omega),$
2. $\mathscr{E}_{0,B}(Tu, Tu) \leq C_1 \cdot \int_{U(B)} |\nabla u|^2, \quad u \in C_0^\infty(M),$ *for any* $n \geq 1$ *and* $B \in \mathscr{B}_n$.

By Proposition 3.3 and Definition 2.2 (1), we have the following result.

Proposition 3.5 *There is an* $n_0 \geq 1$ *such that*

$$\mathrm{dis}_M(U(\Omega \setminus E_n(B), U(B)) > 2C_0, \quad \forall B \in \mathscr{B}_n, \quad n \geq n_0 \ .$$

The following global Harnack inequality is the main result in this section.

Theorem 3.6 *There is an* $\epsilon_1 > 0$ *such that the following assertion holds. If* $n \geq n_0$, $B \in \mathscr{B}_n$, $u \in C^\infty(U(E_n^3(B)); (0, \infty))$ *and* $\Delta u(y) \leq 0$, $y \in U(E_n^3(B))$, *then* $\min\{u(x);\ x \in U(B)\} \geq \epsilon_1 \cdot \max\{u(x); x \in U(B)\}$, *where* $E_n^1(B) = E_n(B)$, *and* $E_n^k(B)$ *is defined iteratively by*:

$$E_n^k(B) = E_n(E_n^{k-1}(B)), \quad k \geq 2 \ .$$

Proof. By Proposition 3.1 (4), we see that for any $n \geq 1$ and $B \in \mathscr{B}_n$, there is a $\xi_{n,B} \in C_0(\Omega)$ such that $\mathscr{E}_0(\xi_{n,B}, \xi_{n,B}) \leq M_0^2 c_0^{-1} \rho^{-n} N^n$, $\xi_{n,B}(\omega) = 1, \omega \in E_n\ (B)$, and $\xi_{n,B}(\omega) = 0, \omega \in \Omega \setminus E_n^2(B)$. Let $\eta_{n,B} = S(\xi_{n,B}) \in C_0^\infty(M; R)$. If $n \geq n_0$, then by Proposition 3.5 and Definition 2.2 (3), we see that $\eta_{n,B}(x) = 1, x \in U(B)$, and $\eta_{n,B}(x) = 0$, $x \in U(\Omega \setminus E_n^3(B))$. By Proposition 3.4 we can see that

$$\mathscr{E}(\eta_{n,B}, \eta_{n,B}) \leq C_1 M_0^2 c_0^{-1} \rho^{-n} N^n \ .$$

Now let us assume $u \in (U(E_n^3(B); (0, \infty))$ with $\Delta u \leq 0$. Note that $U(\Omega \setminus E_n^3(B)) \cup U(E_n^3(B)) = M$. Then,

$$\begin{aligned}
\int_M \eta_{n,B}^2 |\nabla(\log u)|^2 \, d\mu &= -\int_M \eta_{n,B}^2 \nabla u \cdot \nabla\left(\frac{1}{u}\right) d\mu \\
&= \int_M \eta_{n,B}^2 \frac{1}{u} \Delta u \, d\mu + \int_M (\eta_{n,B} \nabla(\log u) \cdot \nabla \eta_{n,B}) \, d\mu \\
&\leq \left(\int_M \eta_{n,B}^2 |\nabla(\log u)|^2 \, d\mu\right)^{1/2} \mathscr{E}(\eta_{n,B}, \eta_{n,B})^{1/2} \ ,
\end{aligned}$$

which implies that

$$\int_{U(B)} |\nabla(\log u)|^2 \, d\mu \leq \int_M \eta_{n,B}^2 |\nabla(\log u)|^2 \, d\mu \leq C_1 M_0^2 c_0^{-1} \rho^{-n} N^n \ .$$

By Proposition 3.4 we have

$$\mathscr{E}_{0,B}(T(\log u), T(\log u)) \leq C_1^2 M_0^2 c_0^{-1} \rho^{-n} N^n \ .$$

Thus, by Proposition 3.2 we know that

$$|T(\log u)(\omega) - T(\log u)(\omega')| \leq c_3 C_1 M_0 c_0^{-1/2}, \quad \omega, \omega' \in B \ .$$

By the assumption (ULH) and Proposition 3.5 we see that

$$\min\{u(x); x \in U(\{\omega\})\} \geq \mathscr{E}(C_0, 2C_0) \max\{u(x); x \in U(\{\omega\})\}, \ \omega \in \Omega \ .$$

Therefore,

$$|\log u(x) - \log u(y)| \leq c_3 C_1 M_0 c_0^{-1/2} - 2 \log \mathscr{E}(C_0, 2C_0), \quad x, y \in U(B) \ .$$

This implies the desired result. □

Now let $\{P_x; x \in M\}$ be the family of probability measures on $C([0,\infty); M)$ induced by the Brownian motion on M starting at the point $x \in M$. For any measurable set A in M, let $\sigma_A : C([0,\infty); M) \to [0,\infty]$ be given by

$$\sigma_A(w) = \inf\{t > 0; w(t) \in A, w \in C([0,\infty); M)\} \ .$$

We have the following result.

Proposition 3.7 *There are constants $C_2, C_3 \in (0,\infty)$ such that for any $n \geq n_0$ and $B \in \mathscr{B}_n$,*

$$C_2 \rho^n \leq E^{P_x}\left[\sigma_{M \setminus U(E_n^4(B))}\right] \leq C_3 \rho^n, \quad x \in U(B) \ .$$

Proof. For any bounded open set G, let $\phi_G(x) = E^{P_x}[\sigma_{M\setminus G}], x \in M$. Then we see that $\sigma_G \leq \sigma_{G'}$, if $G \subset G'$. Moreover, if a bounded open set G in M has a smooth boundary, then σ_G is the unique solution of the PDE:

$$\begin{cases} \Delta \phi_G(x) = -1, & x \in G, \\ \phi_G(y) = 0, & y \in M \setminus G \ . \end{cases}$$

Hence,

$$\int_M \phi_G \, d\mu = \sup\left\{\left(\int_M u \, d\mu\right)^2; u \in C_0^\infty(M; R), u|_{M\setminus G} = 0, \mathscr{E}(u,u) = 1\right\} \ .$$

Using this we can show that

$$\sup\left\{\left(\int_M u \, d\mu\right)^2; u \in C_0^\infty(M; R), u|_{M\setminus U(E_n(B))} = 0, \mathscr{E}(u,u) = 1\right\} \leq \int_{U(E_n(B))} \phi_{M\setminus U(E_n^4(B))} \, d\mu \ , \tag{3.1}$$

and

$$\int_{U(E_n^9(B))} \phi_{M\setminus U(E_n^4(B))}\, d\mu$$

$$\leq \sup\left\{ \left(\int_M u\, d\mu\right)^2 ; u \in C_0^\infty(M;R), u|_{M\setminus U(E_n^5(B))} = 0, \mathscr{E}(u,u) = 1 \right\} . \tag{3.2}$$

By Proposition 3.1 (2), we see that there is a function $\tilde{f} \in C(B;R)$ such that $\tilde{f}|_{\partial_n B} = 0, \mathscr{E}_{0,B}(\tilde{f},\tilde{f}) = 1,$ and $N^n\langle f\rangle_B^2 \geq (c_0/2)\rho^n$. Let $f \in C_0(\Omega, R)$ be given by $f|_B = \tilde{f}$ and $f|_{\Omega\setminus B} = 0$. By Proposition 3.4 and Proposition 3.5 we see that $Sf|_{M\setminus U(E_n(B))} = 0$ and $\mathscr{E}(Sf,Sf) \leq C_1$. By Definition 2.2 (3), we also have

$$\left(\int_M Sf\ d\mu\right)^2 \geq C_0^{-2} \cdot N^{2n}\langle \tilde{f}\rangle_B^2 \geq (2C_0^2)^{-1}c_0 \cdot N^n\rho^n\ .$$

Thus,

$$(2C_0^2C_1)^{-1}c_0 \cdot N^n\rho^n \leq \int_{U(E_n(B))} \phi_{M\setminus U(E_n^4(B))}\, d\mu\ . \tag{3.3}$$

Let $u \in C_0^\infty(M;R)$ satisfy: $u|_{M\setminus U(E_n^5(B))} = 0$ and $\mathscr{E}(u,u) = 1$. Then we have $Tu|_{\Omega\setminus E_n^6(B)} = 0$ and $\mathscr{E}(Tu,Tu) \leq C_1$. By Definition 2.2 (2), we see that $\int_M u\ d\mu \leq C_0 \cdot \sum_{\omega\in\Omega}(Su)(\omega)$. However, by Proposition 3.1 (1) we see that

$$\left|\sum_{\omega\in B'}(Su)(\omega) - \sum_{\omega\in B''}(Su)(\omega)\right| \leq (c_1C_1^{-1})^{1/2}(N^n\rho^n)^{1/2} \tag{3.4}$$

or any $B', B'' \in \mathscr{B}_n$. Therefore, there is a constant $C_4 \in (0,\infty)$, independent on $n \geq n_0$, $B \in \mathscr{B}_n$ and u, such that

$$\left|\sum_{\omega\in\Omega}(Su)(\omega)\right| \leq C_4(N^n\rho^n)^{1/2}\ .$$

This implies that

$$\int_{U(E_n(B))} \phi_{M\setminus U(E_n^4(B))}\, d\mu \leq (C_0C_4)^2N^n\rho^n\ . \tag{3.5}$$

It is obvious that

$$\begin{aligned}
&\mu(E_n(B)) \cdot \min_{x\in U(E_n(B))} \phi_{M\setminus U(E_n^4(B))} \\
&\quad \le \int_{U(E_n(B))} \phi_{M\setminus U(E_n^4(B))}\, d\mu \\
&\quad \le \mu(E_n(B)) \cdot \max_{x\in U(E_n(B))} \phi_{M\setminus U(E_n^4(B))} \quad .
\end{aligned} \tag{3.6}$$

Thus we get the desired result from (3.3), (3.5), the assumpion (M-1) (1) (3) and Theorem 3.6. This completes the proof. □

Proposition 3.8 *There is a constant* $C_5 \in (0,\infty)$ *such that*

$$E^{P_x}[\sigma_{M\setminus U(E_n^4(B))}] \le C_5 \cdot \rho^n, \quad x \in U(E_n^4(B))$$

for any $n \ge n_0$ *and* $B \in \mathscr{B}_n$.

Proof. As in the proof of Proposition 3.7, we can show that there is a constant $C_6 \in (0,\infty)$ such that

$$E^{P_x}[\sigma_{M\setminus U(E_n^9(B'))}] \le C_6\rho^n, \quad x \in U(B')$$

for any $n \ge n_0$ and $B' \in \mathscr{B}_n$. Note that $E_n^4(B) \subset E_n^9(B')$ for any $B' \subset E_n^4(B)$. Thus we have the desired assertion. □

We will use an idea due to Barlow-Bass (see the proofs of [1] Proposition 4.4 and [2] Proposition 3.3) to prove a nice estimate for hitting time.

The following lemma is due to Barlow-Bass [1] Lemma 1.1.

Lemma 3.9 *Let* $X, Y_1, \ldots, Y_n$ *be non-negative random variables satisfying*

$$X \ge \sum_{i=1}^{n} Y_i,$$

$$P(Y_i \le s | Y_j, j = 1, \ldots, i-1) \le p + bs, \quad i = 1,2,\ldots,n, \;\; s \ge 0 \;\; ,$$

where $0 < p < 1$ *and* $b > 0$. *Then*

$$P(X \le s) \le \exp(2(bns/p)^{1/2} - n \cdot \log 1/p), \quad s \ge 0 \;\; .$$

Proposition 3.10 *There are constants* $p \in (0,1)$ *and* $b > 0$ *such that*

$$P_x(\sigma_{M\setminus U(E_n^4(B))} \le \rho^n s) \le ps + b, \quad s \ge 0$$

for any $x \in U(B), B \in \mathscr{B}_n$ *and* $n \ge n_0$.

Proof. Note that

$$\begin{aligned}
C_3 \cdot \rho^n &\le E^{P_x}\left[\sigma_{M\setminus U(E_n^4(B))}\right] \\
&\le \rho^n s + E^{P_x}\left[E^{P_{\omega(\rho^n s)}}\left[\sigma_{M\setminus U(E_n^4(B))}\right], \sigma_{M\setminus U(E_n^4(B))} > \rho^n s\right] \\
&\le \rho^n s + C_5 \cdot \rho^n \left(1 - P\left(\sigma_{M\setminus U(E_n^4(B))} \le \rho^n s\right)\right) \;\; .
\end{aligned}$$

By letting $p = 1 - C_5^{-1}C_3$ and $b = C_5^{-1}$, we have the desired result. $\square$

Combining Lemma 3.9 with Proposition 3.10 we can get the following result.

Corollary 3.11 *There are constants* $\rho \in (0,1)$ *and* $b > 0$ *such that*

$$P_x\big(\sigma_{M\setminus U(E_k^{5m}(B))} \le \rho^k s\big) \le \exp\big(2(bms/p)^{1/2} - m\cdot \log 1/p\big), \quad s \ge 0$$

for any $m \ge 1, x \in U(B), B \in \mathscr{B}_k$ *and* $k \ge n_0$.

Now let $d_w = \log\rho \backslash \log\alpha$, where d_w is sometimes called the random walk dimension.

Proposition 3.12 *For any* $v \in (1, d_w)$, *there are* $c(v) > 0$ *and* $n(v) \in N$ *such that*

$$P_x\big(\sigma_{M\setminus U(E_n(B))} \le \alpha^{vn}\big) \le \exp\big(-c(v)\cdot \alpha^{(d_w-1)^{-1}(d_w-v)n}\big)$$

for any $x \in U(B), B \in \mathscr{B}_n$ *and* $n \ge n(v)$.

Proof. By Proposition 3.5 we see that there is a $C_7 \in (0,\infty)$ such that $E_k^{5m}(B) \subset E_n(B)$ for any $B \in \mathscr{B}_n, n \ge k \ge n_0$ and $m \le C_7\alpha^{n-k}$. By Corollary 3.11 we have

$$\begin{aligned} &P_x\big(\sigma_{M\setminus U(E_n(B))} \le \alpha^{vn}\big) \\ &\quad\le \exp\big(2\big(bC_7\alpha^{n-k+vn-kd_w}/p\big)^{1/2} - \alpha^{n-k}\cdot \log 1/p\big) \\ &\quad= \exp\big(-\alpha^{(n-k)/2}\{\alpha^{(n-k)/2}\log 1/p - 2(bC_7/p)^{1/2}\alpha^{(vn-kd_w)/2}\}\big)\ . \end{aligned}$$

Let $k = (d_w - 1)^{-1}(v-1)n + r$. If $\frac{1}{2}(\log\frac{1}{p})\alpha^{-r/2} \ge 2(bC_7/p)^{1/2}\alpha^{-rd_w/2}$, we have

$$P_x\big(\sigma_{M\setminus U(E_n(B))} \le \alpha^{vn}\big) \le \exp\big(-((\log 1/p)\alpha^{-r}/2)\alpha^{(d_w-v)/(d_w-1)}\big)\ .$$

This then completes the proof. $\square$

4 Inverse of Laplace transform

In this section, we think of inverse of Laplace transform. The idea here is due to Bernshtein's proof on Weierstrass' polynomial approximation theorem. Let $Q_x, x \in [0,1]$, denote the probability measure on $\tilde{\Omega} = \{0,1\}^N$ such that $Q_x[\tilde{\omega}(k) = 1] = x, k \in N$, and $\tilde{\omega}(k), k = 1,2,\ldots$, are independent under $Q_x(d\tilde{\omega})$. Let $p_n\colon [0,1] \to [0,1], n \in N$, be given by

$$p_n(x) = Q_x\left[\frac{1}{n}\sum_{k=1}^{n}\tilde{\omega}(k) > \frac{1}{2}\right], \quad x \in [0,1]\ . \tag{4.1}$$

Then we have the following result.

Proposition 4.1 (1) $p_n(x) \le p_n(y)$ *for any* $x, y \in [0,1]$ *with* $x < y$.

(2) $p_n(x) = \sum_{l=[n/2]+1}^{n} \binom{n}{l} x^l (1-x)^{n-l}$.

(3) $p_n(x) = \sum_{k=[n/2]+1}^{n} x^k \binom{n}{k}) \{\sum_{j=0}^{k-([n/2]+1)} \binom{k}{j} (-1)^j\}, x \in [0,1], n \in N$. *Moreover,*

$$\sum_{k=[n/2]+1}^{n} \binom{n}{k} \left| \sum_{j=0}^{k-([n/2]+1)} \binom{k}{j} (-1)^j \right| \le 3^n, \quad n \in N \ .$$

(4) $p_n(x) + p_n(1-x) = 1, x \in [0,1]$, *if* n *is odd, and* $p_n(x) + p_n(1-x) = 1 - \binom{n}{n/2} \cdot 2^{-n}, \quad x \in [0,1]$, if n *is even.*

Proof. (1) Note that for $x, y \in [0,1]$, the law of $\{\tilde{\omega}(k)\tilde{\omega}'(k)\}_{k=1}^{\infty}$ under $Q_x(d\tilde{\omega}) \otimes Q_y(d\tilde{\omega}')$ is the same as Q_{xy}. Thus

$$p_n(xy) = Q_x \otimes Q_y \left[\frac{1}{n} \sum_{k=1}^{n} \tilde{\omega}(k)\tilde{\omega}'(k) > \frac{1}{2} \right] \le p_n(x) \ .$$

This proves the assertion (1).

The assertion (2) is obvious. The assertion (3) and (4) follow from the assertion (2). This then completes the proof. □

Proposition 4.2 (1) $p_n(1/2 + n^{-1/2}x) \to g(x)$ as $n \to \infty$ *for any* $x \in R$, *where* $g(x) = \int_{-\infty}^{x} (2/\pi)^{1/2} \exp(-2y^2)\, dy, x \in R$.

(2) *There are* $\epsilon > 0$ *and* $C > 0$ *such that*
$$p_n\left(\tfrac{1}{2} - n^{-1/2}x\right) \le C \cdot \exp(-\epsilon \cdot x^2)$$
for any $n \in N$ *and* $x \in [0, n^{1/2}/2]$.

(3) $\sup_n n^{1/2} \cdot \int_0^1 |x - \frac{1}{2}| \cdot p_n'(x)\, dx < \infty$.

(4) $\sup_n n^{1/2} \{\int_0^{1/2} x^{-1} p_n(x)\, dx + \int_{1/2}^{1} x^{-1}(1 - p_n(x))\, dx\} < \infty$.

Proof. (1) Let $\phi_n(\xi; x) = E^{Q_x}[\exp(\xi \cdot \sum_{k=1}^{n} (\tilde{\omega}(k) - x))], \xi \in R$. Then

$$\begin{aligned}\phi_n(\xi; x) &= \{x \cdot \exp(\xi(1-x)) + (1-x) \cdot \exp(-\xi x)\}^n \\ &= \{1 + x \cdot \psi(\xi(1-x)) + (1-x) \cdot \psi(-\xi x)\}^n \ ,\end{aligned}$$

where $\psi(\xi) = e^{\xi} - 1 - \xi, \xi \in R$. In particular,

$$\phi_n\left(n^{-1/2}\xi; \tfrac{1}{2} + n^{-1/2}x\right) \to \exp(\xi^2/8) \ ,$$

as $n \to \infty$. We remark that

$$\int_R e^{-\xi x} g(x)\, dx = e^{\xi^2/8}$$

and

$$E^{Q_{1/2+n^{-1/2}x}} \exp\left(n^{-1/2}\xi\left(\sum_{k=1}^{n}\tilde{\omega}(k) - \left(\frac{1}{2}+n^{-1/2}x\right)\right)\right)$$

$$= \phi_n\left(n^{-1/2}\xi, \tfrac{1}{2}+n^{-1/2}x\right) \ .$$

Hence by the definition of $p_n(x)$, we then know that

$$p_n\left(\tfrac{1}{2}+n^{-1/2}x\right) \to g(x)$$

as $n \to \infty$ for any $x \in R$. This proves the assertion (1).

(2) Since there is a $\delta \in (0,1)$ such that $\psi(\xi) \leq \xi^2, |\xi| < \delta$, we see that

$$\phi_n\left(n^{-1/2}\xi; x\right) \leq \left(1+\xi^2/n\right)^n \leq \exp\left(\xi^2\right), \quad |\xi| \leq \delta n^{1/2}, \ 0 \leq x \leq 1 \ .$$

Hence,

$$p_n\left(\tfrac{1}{2}-n^{-1/2}x\right) = Q_{1/2-n^{-1/2}x}\left[n^{-1/2}\left\{\sum_{k=1}^{n}\left(\tilde{\omega}(k) - \left(\frac{1}{2}-n^{-1/2}x\right)\right)\right\}x\right]$$

$$\leq \exp(-\xi x)\cdot\phi_n\left(-n^{-1/2}\xi; \tfrac{1}{2}-n^{-1/2}x\right)$$

$$\leq \exp\left(-\xi\cdot x+\xi^2\right)$$

for $0 \leq \frac{1}{2}-n^{-1/2}x \leq 1$ and $|\xi| \in [0, \delta n^{1/2}]$. Therefore, by letting $\xi = \frac{x}{2}$, we get

$$p_n\left(\tfrac{1}{2}-n^{-1/2}x\right) \leq \exp(-x^2/4)$$

if $x \in [0, \delta n^{1/2}/2]$.

By Cramer's large deviation theorey (c.f. Stroock [8]), we have

$$\limsup_{n\to\infty} n^{-1}\log p_n\left(\tfrac{1}{2}-\epsilon/2\right) < 0, \quad \forall \epsilon \in [\delta, 1] \ .$$

Thus there are $C > 0$ and $k > 0$ such that

$$p_n\left(\tfrac{1}{2}-\epsilon/2\right) \leq C\cdot\exp(-kn), \quad n \in N, \ \epsilon \in [\delta, 1] \ .$$

From this we get that for $x \in [\delta n^{1/2}/2, n^{1/2}/2]$,

$$p_n\left(\tfrac{1}{2}-n^{-1/2}x\right) \leq C\cdot\exp(-kn) \leq C\cdot\exp(-kx^2) \ ,$$

which proves the assertion (2).

(3) Note that

$$\begin{aligned} n^{1/2} \cdot \int_0^1 \left| x - \frac{1}{2} \right| \cdot p_n'(x)\, dx \\ &= 2n^{1/2} \cdot \int_0^{1/2} \left| x - \frac{1}{2} \right| \cdot p_n'(x)\, dx \\ &= -2 \cdot \int_0^{n^{1/2}/2} x \cdot \frac{d}{dx} p_n\left(\frac{1}{2} - n^{-1/2}x\right) dx \\ &= 2 \cdot \int_0^{n^{1/2}/2} p_n\left(\frac{1}{2} - n^{-1/2}x\right) dx \ . \end{aligned}$$

Thus the assertion (3) comes from the assertion (2).

(4) Note that

$$\int_{1/2}^1 (1 - p_n(x))\, dx = \int_{1/2}^1 \left(x - \frac{1}{2}\right) p_n'(x)\, dx \ ,$$

and

$$\int_0^{1/2} p_n(x)\, dx = \int_0^{1/2} \left(\frac{1}{2} - x\right) p_n'(x)\, dx \ .$$

By Proposition 4.1 (2) we have

$$x^{-1} p_n(x) = \sum_{l=[n/2]+1}^{n} \binom{n}{l} x^{l-1} (1-x)^{n-l} \le 2^n x^{[n/2]} \ .$$

Thus

$$\begin{aligned} n^{1/2} \left\{ \int_0^{1/2} x^{-1} p_n(x)\, dx + \int_{1/2}^1 x^{-1} (1 - p_n(x))\, dx \right\} \\ \le n^{1/2} \cdot 2^n 8^{-[n/2]} + 8n^{1/2} \cdot \int_0^1 \left| x - \frac{1}{2} \right| \cdot p_n'(x)\, dx \ . \end{aligned}$$

The assertion (4) now follows from the assertion (3).

The proof is complete. □

Lemma 4.3 *There is a universal constant $C \in (0, \infty)$ such that the following conclusion holds. For any Banach space E, $n \in N, T > 0$ and C^1 function $u\colon [0, \infty) \to E$,*

$$\begin{aligned} \left\| T^{-1} \cdot \int_0^\infty u(t) \cdot p_n'\left(\mathrm{e}^{-t/T}\right) \mathrm{e}^{-t/T}\, dt - u(T \cdot \log 2) \right\|_E \\ \le 4 \cdot p_n(1/4) \cdot \sup_{t>0} \|u(t)\|_E + Cn^{-1/2} T \cdot \sup_{t>0} \|u'(t)\|_E \ , \quad (4.2) \end{aligned}$$

and

$$\left\| \int_0^\infty u(t) \cdot p_n(\mathrm{e}^{-t/T})\, dt - \int_0^{T\log 2} u(t)\, dt \right\|_E$$
$$\leq C \cdot n^{-1/2} T \cdot \sup_{t>0} \|u(t)\|_E \ . \tag{4.3}$$

Proof. Let $v(x) = u(-T\log x), x \in (0,1]$. Then,

$$\int_0^1 v(x)x^n\, dx = t^{-1} \cdot \int_0^\infty u(t)\mathrm{e}^{-(n+1)t/T}\, dt \ .$$

Moreover,

$$\left\| \int_0^1 v(x)p_n'(x)dx - v(1/2) \right\|_E$$
$$\leq \int_0^{1/4} \|v(x) - v(1/2)\|_E \cdot p_n'(x)\, dx + \int_{3/4}^1 \|v(x) - v(1/2)\|_E \cdot p_n'(x)\, dx$$
$$+ \int_{1/4}^{3/4} \|v(x) - v(1/2)\|_E \cdot p_n'(x)\, dx$$
$$\leq 4\Big(\sup_{x\in[0,1]} \|v(x)\|\Big) p_n(1/4)$$
$$+ n^{-1/2}\Big(\sup_{x\in[1/4,3/4]} \|v'(x)\|\Big)\Big(\sup_n n^{1/2} \int_0^1 \left|x - \frac{1}{2}\right| \cdot p_n'(x)\, dx\Big) \ .$$

Since $v'(x) = -T \cdot x^{-1}u'(-T\log x)$, we get the first assertion (*i.e.* (4.2)) from Proposition 4.2 (3).

Note that

$$T^{-1}\left\{ \int_0^\infty u(t)p_n\big(\mathrm{e}^{-t/T}\big)\, dt - \int_0^{T\log 2} u(t)\, dt \right\}$$
$$= \int_0^{1/2} v(x)x^{-1}p_n(x)\, dx - \int_{1/2}^1 v(x)x^{-1}(1 - p_n(x))\, dx \ .$$

We get the second assertion (4.3) from Proposition 4.2 (4).

The proof is now complete. □

Corollary 4.4 *Let* $q_n(x) = p_n'(x) + p_n''(x)x, x \in [0,1], n \in N$, *and* C *be the constant given in Lemma* 4.3. *Let* E *be a Banach space and* $u\colon [0,\infty) \to E$ *be a* C^2 *function. Then*

$$\left\| T^{-2} \cdot \int_0^\infty u(t) \cdot q_n \ (\mathrm{e}^{-t/T} \ )\mathrm{e}^{-t/T}\, dt - u'(T\log 2) \right\|_E$$
$$\leq 4p_n(1/4)\big(\sup_{t>0} \|u'(t)\|_E\big) + Cn^{-1/2}T\big(\sup_{t>0} \|u''(t)\|_E\big)$$

for any $T > 0$ and $n \geq 2$.

Proof. By Proposition 4.1 (2) we know $p_n'(0) = 0, n \geq 2$. If $n \geq 2$ and $\sup_{t>0}(\|u'(t)\|_E + \|u''(t)\|_E) < \infty$, we have

$$T^{-1}\int_0^\infty u'(t)p_n'\big(\mathrm{e}^{-\mathrm{t}/\mathrm{T}}\big)\mathrm{e}^{-\mathrm{t}/\mathrm{T}}\,\mathrm{dt} = \mathrm{T}^{-2}\int_0^\infty \mathrm{u(t)q_n}\big(\mathrm{e}^{-\mathrm{t}/\mathrm{T}}\big)\mathrm{e}^{-\mathrm{t}/\mathrm{T}}\,\mathrm{dt}\ .$$

Thus we get the desired result from (4.2). □

Proposition 4.5 *There is a universal constant $C \in (0, \infty)$ such that the following conclusion holds. For any $f \in L^2(M; dx)$ with $f \in$ Dom(Δ) and open set A in M,*

$$\left\| \int_0^{T\log 2} \chi_A \cdot \cos\Big(t\sqrt{-\Delta}\Big)f\ dt \right\|_{L^2}$$
$$\leq 3^\eta \sup\left\{ \int_0^\infty \mathrm{e}^{-\lambda t}\chi_A \cdot \cos\big(t\sqrt{-\Delta}\,\big)f\ dt\|_{L^2}; \lambda \geq \eta T^{-1}/4 \right\}$$
$$\times\, C \cdot \eta^{-1/2}T\|f\|_{L^2}\ , \tag{4.4}$$

$$\left\| \chi_A \cdot \cos\big(T\log 2\sqrt{-\Delta}\big)f \right\|_{L^2}$$
$$\leq \eta 3^\eta \sup\left\{ \left\| \int_0^\infty \mathrm{e}^{-\lambda t}\chi_A \cdot \cos\big(t\sqrt{-\Delta}\big)f\ dt \right\|_{L^2}; \lambda \geq \eta T^{-1}/4 \right\}$$
$$+ 4(3/4)^{\eta/2}\|f\|_{L^2} + C\eta^{-1/2}T \cdot \ \|\sqrt{-\Delta}f\|_{L^2}\ , \tag{4.5}$$

and

$$\left\| \chi_A \cdot \sin(T\log 2\sqrt{-\Delta})\sqrt{-\Delta}f \right\|_{L^2}$$
$$\leq 2\eta^2 3^\eta \sup\left\{ \left\| \int_0^\infty \mathrm{e}^{-\lambda t}\chi_A \cdot \cos\big(t\sqrt{-\Delta}\big)f\ dt \right\|_{L^2}; \lambda \geq \eta T^{-1}/4 \right\}$$
$$+ 4(3/4)^{\eta/2}\|\sqrt{-\Delta}f\|_{L^2} + C\eta^{-1/2}T\|\Delta f\|_{L^2}\ , \tag{4.6}$$

for $\eta \geq 4, T \geq 1$.

Proof. Let $n = [\eta]$ and $u(t) = \chi_A \cdot \cos(t\sqrt{-\Delta})f, t > 0$. Since $p_n(1/4) = 4^{-n}\sum_{l=[n/2]+1}^n \binom{n}{l}3^{n-l} \leq (3/4)^{n/2}$, our assertion follows from Proposition 4.1 (2), Lemma 4.3 and Corollary 4.4. □

5 Proof of main theorem

Let $P_t = \exp(t\Delta), t \geq 0$, and $K_\lambda = \int_0^\infty \exp(-\lambda^2 s/2) P_s\, ds, \lambda > 0$.

Proposition 5.1 $\int_0^\infty \cos(t\sqrt{-\Delta}) \mathrm{e}^{-\lambda t}\, dt = (\lambda/2) K_\lambda, \lambda > 0$.

Proof. This comes from the following fact:

$$\int_0^\infty \cos(tz)\mathrm{e}^{-\lambda t}\, dt = \mathrm{Re}\left(\int_0^\infty \mathrm{e}^{itz-\lambda t}\, dt\right) = \lambda\left(\lambda^2 + z^2\right)^{-1}$$
$$= (\lambda/2)\int_0^\infty \exp(-\lambda^2 s/2)\exp\left(-\lambda^2 s/2\right) ds, \quad \forall z \in R \ .$$

We also have the following result. □

Proposition 5.2 *For any open sets A, B in M, $T > 0, \lambda > 0$, and $p \in [1, \infty]$,*

$$\|\chi_A K_\lambda \chi_B\|_{L^p \to L_p} \leq 2\lambda^{-2}\left\{\mathrm{e}^{-\lambda^2 T/2} + \sup_{x\in A} P_x[\tau_B \leq T] + \sup_{x\in B} P_x[\tau_A \leq T]\right\} \ .$$

Proof. For any $f \in L^\infty(M; R, d\mu)$, we have

$$\begin{aligned}
&(\chi_A K_\lambda \chi_B f)(x) \\
&= \chi_A(x)\cdot E^{P_x}\left[\exp\left(-\lambda^2\sigma_B\right)\int_0^\infty \mathrm{e}^{-\lambda^2 s/2} f(\omega(\sigma_B + s))\, ds\right] \\
&\leq \sup_{x\in A} E^{P_x}\left[\exp\left(-\lambda^2 \tau_B\right)\right] \times (2\lambda^{-2})\|f\|_{L^\infty} \\
&\leq 2\lambda^{-2}\left\{\mathrm{e}^{-\lambda^2 T/2} + \sup_{x\in A} P_x[\tau_B \leq T]\right\}\|f\|_{L^\infty} \ .
\end{aligned}$$

Thus, the assertion is valid for $p = \infty$. By the duality we see that the assertion is valid for $p = 1$. Then our assertion follows from the interpolation theory. □

Corollary 5.3 *For any $\xi \in (0, d_w/2)$, there are constants $C, c \in (0, \infty)$ such that the following conclusion holds. For any $n \in N$ and open sets A, B in M with* $\mathrm{dis}(A, B) \geq \alpha^n$,

$$\sup\left\{\left\|\int_0^\infty (\chi_A \cos\left(t\sqrt{-\Delta}\right)\chi_B)\mathrm{e}^{-\lambda t}\, dt\right\|_{L^2 \to L^2}; \lambda \geq \alpha^{-\xi n}\right\}$$
$$\leq C\exp\left(-c\cdot\alpha^{(1-2\xi/d_w)n}\right) \ .$$

Proof. Let $v = 1 + 2\xi(1 - d_w^{-1})$. Then we see that $v \in (1, d_w), v - 2\xi = 1 - 2d_w^{-1}\xi$ and $(d_w - 1)^{-1}(d_w - v) = 1 - 2\xi/d_w$. Thus our assertion comes from Proposition 3.12, Proposition 5.1 and Proposition 5.2. □

Proposition 5.4 *Let $a \in (1, d_w/2)$ and $b \in (1/2, d_w/2)$, and assume that*

$$b < a < \frac{d_w + 2(d_w - 2)b}{2(d_w - 1)} \quad . \tag{5.1}$$

Then there are constants $C, c \in (0, \infty)$ such that the following conclusions hold for all $n \in N$, open sets A, B in M with $\mathrm{dis}(A, B) \geq \alpha^n$, *and* $f \in \mathrm{Dom}(\Delta)$ *with* $\mathrm{supp}\, f \subset B$,

$$\begin{aligned}
&\sup\big\{\big\| \int_0^T \chi_A \cdot \cos\big(s\sqrt{-\Delta}\big) f\, ds \big\|_{L^2}; T \in [0, \alpha^{an}]\big\} \\
&\quad \leq C\big\{\exp\big(-c \cdot \alpha^{(1-2(2b-a)/d_w)n}\big)\|f\|_{L^2} + \alpha^{bn}\|f\|_{L^2}\big\} \ , \\
&\sup\big\{\big\|\chi_A \cdot \cos\big(T\sqrt{-\Delta}\big) f\big\|_{L^2}; T \in \big[0, \alpha^{an}\big]\big\} \\
&\quad \leq C\big\{\exp\big(-c \cdot \alpha^{(1-2(2b-a)/d_w)n}\big)\|f\|_{L^2} \\
&\qquad + \exp\big(-c\alpha^{2(a-b)n}\big)\|f\|_{L^2} + \alpha^{bn}\big\|\sqrt{-\Delta} f\big\|_{L^2}\big\} \ ,
\end{aligned}$$

and

$$\begin{aligned}
&\sup\big\{\big\|\chi_A \cdot \sin\big(T\sqrt{-\Delta}\big)\sqrt{-\Delta} f\big\|_{L^2}; T \in \big[0, \alpha^{an}\big]\big\} \\
&\quad \leq C\big\{\exp(-c \cdot \alpha^{(1-2(2b-a)/d_w)n})\|f\|_{L^2} \\
&\qquad + \exp\big(-c\alpha^{2(a-b)n}\big)\big\|\sqrt{-\Delta} f\big\|_{L^2} + \alpha^{bn}\|\Delta f\|_{L^2}\big\} \ .
\end{aligned}$$

Proof. Note that

$$a < \frac{2d_w b + 2(d_w - 2)b}{2(d_w - 1)} = 2b \ .$$

Let $\eta = \alpha^{2(a-b)n}$ and $\xi = 2b - a$. Then $\xi \in (0, d_w/2)$, and

$$1 - 2\xi/d_w - 2(a - b) = d_w^{-1}\{d_w - 2(d_w - 1)a + 2(d_w - 2)b\} > 0 \ .$$

This shows that $\eta \ll \alpha^{(1-2\xi/d_w)n}$ as $n \to \infty$. If $T \in [0, \alpha^{an}]$, then $\eta T^{-1} \geq \alpha^{-\xi n}$ and $\eta^{-1/2} T \leq \alpha^{bn}$. Thus we get the desired result from Proposition 4.5 and Corollary 5.3. □

Now we are ready to give a proof of Theorem 2.5.

Proof of Theorem. Let $f_n = u_n(0)$ and $g_n = \frac{\partial}{\partial t} u_n(0)$. Then it is easy to see that

$$u_n(t) = \cos\big(t\sqrt{-\Delta}\big) f_n + \int_0^t \cos\big(s\sqrt{-\Delta}\big) g_n\, ds, \quad t > 0 \ . \tag{5.2}$$

This implies that

$$\frac{\partial}{\partial t}u_n(t) = \sin\left(t\sqrt{-\Delta}\right)\sqrt{-\Delta}f_n + \cos\left(t\sqrt{-\Delta}\right)g_n, \quad t > 0 \ , \tag{5.3}$$

$$\Im(u_n) = \left\|\sqrt{-\Delta}f_n\right\|_{L^2(d\mu)} + \|g_n\|_{L^2(d\mu)} \ , \tag{5.4}$$

and

$$\Im\Big(\frac{\partial}{\partial t}u_n\Big) = \left\|\Delta f_n\right\|^2_{L^2(d\mu)} + \left\|\sqrt{-\Delta}g_n\right\|^2_{L^2(d\mu)} \ . \tag{5.5}$$

Note that

$$2\kappa - (2 - 2/d_w)^{-1}(1 - 2\kappa/d_w) < d_w - (2 - 2/d_w)^{-1}(d_w - 1) = d_w/2 \ ,$$

and

$$2\kappa - (2 - 2/d_w)^{-1}(1 - 2\kappa/d_w) > 2\kappa - (2 - 2/d_w)^{-1}(2\kappa - 2\kappa/d_w) = \kappa \ .$$

Then, for any $a \in (\kappa, 2\kappa - (2 - 2/d_w)^{-1}(1 - 2\kappa/d_w))$ there are $a, b \in (1/2, \kappa)$ such that (5.1) holds. Let

$$A_n = \left\{x \in M; \mathrm{dis}_M\left(x, \left\{\left(\mathrm{supp}\, u_n(0,\cdot)\right) \cup \left(\mathrm{supp}\, \frac{\partial}{\partial t}u_n(0,\cdot)\right)\right\}\right) > 2\alpha^n\right\} \ ,$$

and

$$A'_n = \left\{x \in M; \mathrm{dis}_M(x, \mathrm{supp}\,(f_n) \cup (\mathrm{supp}\, g_n)) > \alpha^n\right\},$$

and let $\delta = \{2(a - b)\} \wedge \{1 - 2(2b - a)/d_w\} > 0$. Then, by Proposition 5.4 we know that there are $C_0, c_0 \in (0, \infty)$ such that

$$\begin{aligned}
&\sup\left\{\left(\int_{A'_n} |u_n(t)|^2 \, d\mu\right)^{1/2}; t \in [0, \alpha^{an}]\right\} \\
&\quad \le C_0\left[\exp\left(-c_0\alpha^{\delta n}\right)\left\{\|u_n(0)\|_{L^2} + \Im(u_n)^{1/2}\right\} + \alpha^{bn}\Im(u_n)^{1/2}\right] \\
&\quad = O(\alpha^{bn}) \ ,
\end{aligned} \tag{5.6}$$

as $n \to \infty$, and

$$\begin{aligned}
&\sup\left\{\left(\int_{A'_n} \left|\frac{\partial}{\partial t}u_n(t)\right|^2 d\mu\right)^{1/2}; t \in [0, \alpha^{an}]\right\} \\
&\quad \le C_0\left[\exp\left(-c_0\alpha^{\delta n}\right)\left\{\|u_n(0)\|_{L^2} + \Im(u_n)^{1/2} + \Im\left(\frac{\partial}{\partial t}u_n\right)^{1/2}\right\}\right. \\
&\qquad \left. + \alpha^{bn}\Im(u_n)^{1/2}\right] = O\left(\alpha^{-(\kappa - b)n}\right) \ ,
\end{aligned} \tag{5.7}$$

as $n \to \infty$.

Now let ϕ_n: $M \to R$ be given by

$$\phi_n(x) = \{(\alpha^{-n}\mathrm{dis}_M(x, (\mathrm{supp}\, f_n) \cup (\mathrm{supp}\, g_n)) - 1) \vee 0\}\wedge, \quad x \in M \ .$$

Then we have $|\nabla\phi_n| \le \alpha^{-n}$, a.e. $x \in M$. Note that

$$\begin{aligned}
&\int_{A_n} |\nabla u_n(t)|^2 d\mu \\
&\quad \le \int_M \phi_n |\nabla u_n(t)|^2 \, d\mu \\
&\quad = -\int_M \phi_n u_n(t) \Delta u_n(t) \, d\mu + \int_M u_n \nabla\phi_n \cdot \nabla u_n \, d\mu \\
&\quad \le \left\{ \int_{A'_n} |u_n(t)|^2 \, d\mu \right\}^{1/2} \left\{ \int_M \left| \frac{\partial^2}{\partial t^2} u_n(t) \right|^2 d\mu \right\}^{1/2} \\
&\qquad + \alpha^{-n} \left\{ \int_{A'_n} |u_n(t)|^2 \, d\mu \right\}^{1/2} \|\sqrt{-\Delta} u_n(t)\|_{L^2(d\mu)} \\
&\quad \le \left\{ \int_{A'_n} |u_n(t)|^2 \, d\mu \right\}^{1/2} \left\{ \Im\left(\frac{\partial}{\partial t} u_n\right)^{1/2} + \alpha^{-n} \Im(u_n)^{1/2} \right\} \ .
\end{aligned}$$

Thus, by (5.6) we have

$$\sup\left\{ \int_{A_n} |\nabla u_n(t)|^2 d\mu; t \in [0, \alpha^{an}] \right\} = O\left(\alpha^{-(\kappa - b)n}\right) \ , \qquad (5.8)$$

as $n \to \infty$. By (5.7) and (5.8) we then have

$$\limsup_{n\to\infty} (2\alpha^n)^{-1} \sup\{l(t; \epsilon, u_n); t \in [0, \alpha^{an}]\} \le 1, \quad \epsilon > 0$$

for any $a \in (1, 2\kappa - (2 - 2/d_w)^{-1}(1 - 2\kappa/d_w))$. This implies the desired result.

The proof of Theorem 2.5 is complete.

6 Remark on general recurrent fractals

So far we think of Sierpinski gasket graph. However, our method works on more general fractals.

Let $\alpha > 1$ and $I = \{1, \ldots, N\}$. Let $\{\psi_i; i \in I\}$ be a family of α-similitudes in $\mathbb{R}^D$, *i.e.* ψ_i's are maps on $\mathbb{R}^D$ satisfying $|\psi_i(x) - \psi_i(y)| = \alpha^{-1}|x - y|$ for any $x, y \in \mathbb{R}^D$. Then it is well-known that there is a unique non-void compact set E in $\mathbb{R}^D$ satisfying $E = \cup_{i\in I}\psi_i(E)$. Let us assume the following.

(A-1) (the open set condition) There exists a non-void open set V such that $\cup_{i\in I}\psi_i(V) \subset V$, and $\psi_i(V) \cap \psi_j(V) = \emptyset$ for any $i, j \in I$ with $i \neq j$.

Then ψ_i's, $i \in I$, have distinct fixed points, and the Hausdorff dimension of the set E is d_f, where $d_f = (\log\alpha)^{-1}(\log N)$.

Now let $Z_- = \{l \in Z; l \leq -1\}$ and let $\Omega = \{\omega\colon Z_- \to I; \omega(-k) = 1, k \geq n$ for some $n \geq 1\}$. We regard Ω as a topological space with discrete topology. It is easy to see that for each $\omega \in \Omega$ there is a map $\tilde{\psi}_\omega\colon\ \mathbb{R}^D \to \mathbb{R}^D$ given by

$$\tilde{\psi}_\omega(x) = \lim_{n\to\infty} \psi_1^{-n}\psi_{\omega(-n)}\psi_{\omega(-n+1)}\cdots\psi_{\omega(-1)}(x), \qquad x \in \mathbb{R}^D\ .$$

Let us take a $d_0 \in [0, d_f)$ and fix it throughout this section. We introduce a relation $\sim$ on Ω by the following.

$\omega \sim \omega'$ if the Hausdorff dimension of $\tilde{\psi}_\omega(E) \cap \tilde{\psi}_{\omega'}(E)$ is greater than or equal to d_0

For each $n \geq 1$, we define ∂I^n to be the set of $\xi \in I^n$ such that there are $\omega, \omega' \in \Omega$ for which $\tilde{\psi}_\omega(E) \cap \tilde{\psi}_{\omega'}(E) \neq \emptyset$, $\omega(k) \neq \omega'(k)$ for some $k \leq -n-1$ and that $\{\omega(-k)\}_{k=1}^n = \xi$.

We also assume the following.

(A-2) There is an $n \geq 1$ such that $\partial I^n \neq I^n$.

Let $q : \Omega \times \Omega \to \{0, 1\}$ be given by

$$q(\omega, \omega') = \begin{cases} 1, & \text{if } \omega \sim \omega', \\ 0, & \text{otherwise}\ . \end{cases}$$

Now we define λ_n, σ_n, $\lambda_n^{(D)}$, R_n, $n \geq 1$ as similar as in Section 3.

We now make the following strong assumptions.

(A-3) There are $\rho > N$ and $c_0, c_1 > 0$ such that

(1) $c_0 \cdot \rho^n \leq \sigma_n \leq c_1 \cdot \rho^n, \quad n \geq 1,$
(2) $c_0 \cdot \rho^n \leq \lambda_n^{(D)} \leq c_1 \cdot \rho^n, \quad n \geq 1,$
(3) $c_0 \cdot \rho^n \leq \lambda_n \leq c_1 \cdot \rho^n, \quad n \geq 1,$
(4) $c_0 \cdot \rho^n N^{-n} \leq R_n \leq c_1 \cdot \rho^n N^{-n}, \quad n \geq 1.$

Remark 6.1 (1) The assmption that $\rho > N$ is related to the recurrence property.

(2) By [6] Theorem 7.16 and the discussions given in [6] Section 7, we see that if the assumptions (R), (KM), (LS) and (B-1) in [6] are satisfied, then the assumption (A-3) is satisfied.

Finally, we assume the following.

(A-4) There are constants $c_4, c_5 \in (0, \infty)$ and $\gamma \in [1, \infty)$ such that

(1) $\max\{d_\Omega(\omega, \omega'); \omega, \omega' \in B\} \leq c_4\alpha^{\gamma n}, \quad \forall n \geq 1,\ B \in \mathscr{B}_n,$
(2) $c_5\alpha^{\gamma n} \leq \min\{d_\Omega(\omega, \omega'); \omega \in B, \omega' \in \Omega \setminus E_n(B)\}, \forall n \geq 1,\ B \in \mathscr{B}_n.$

Remark 6.2 Kumagai [5] proved that the assumption (A-6) is satisfied for any nested fractals. The assumption (A-6) is a geometrical assumption, and so is easy to check for each fractal.

Then we can regard $(\Omega, \sim)$ as a connected countable graph. We can prove the following similarly to Theorem 2.5.

Theorem 6.3 *Let M be an $(\Omega, \sim)$-like manifold satisfying uniform local Harnack inequality. Suppose that u_n: $\mathbb{R} \times M \to \mathbb{R}$, $n = 1, 2, \ldots$, satisfy the following conditions.*

(1) $u_n(t, \cdot) \in \mathrm{Dom}(\Delta), t \in \mathbb{R}$; $t \to u_n(t, \cdot) \in L^2(M, dx)$ *is a* C^2*-function in* t *and*

$$\frac{\partial^2}{\partial t^2} u_n(t,x) = \Delta_n(t,x), \quad (t,x) \in \mathbb{R} \times M,\ n = 1, 2, \ldots \ ,$$

(2) $\Im(u_n) = 1, \quad n = 1, 2, \ldots,$

(3) (long wave length condition) $\lim_{n\to\infty} \alpha^{-\epsilon n} \log(1 + \|u_n(0)\|_{L^2}) = 0$ *for any* $\epsilon > 0$, *and there are* $C \in (0, \infty)$ *and* $\kappa \in [1, \gamma(d_w/2))$ *such that*

$$\Im\left(\frac{\partial}{\partial t} u_n\right) \le C \cdot \alpha^{-2\kappa n}, \quad n = 1, 2, \ldots$$

Then for any $a \in (1, 2\kappa - (2 - 2/d_w)^{-1}(1 - 2\kappa/d_w))$,

$$\lim_{n\to\infty} \alpha^{-\gamma n} \sup\{l(t; \epsilon, u_n); t \in [0, \alpha^{an}]\} = 0, \quad \forall \epsilon > 0 \ .$$

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7 References

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