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A central limit theorem for stationary random fields

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Summary. We prove a central limit theorem for strictly stationary random fields under a projective assumption. Our criterion is similar to projective criteria for stationary sequences derived from Gordin's theorem about approximating martingales. However our approach is completely different, for we establish our result by adapting Lindeberg's method. The criterion that it provides is weaker than martingale-type conditions, and moreover we obtain as a straightforward consequence, central limit theorems for α-mixing or ϕ-mixing random fields.

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1 Introduction

Let $(X_i)_{i\in\mathbb{Z}}$ be a stationary sequence of random variables with mean zero and finite variance, and write $S_n = \sum_{k=1}^{n} X_k$. As far as we know, one of the best way to prove the asymptotic normality of $n^{-1/2}S_n$ is to approximate S_n by a naturally related martingale with stationary differences. More precisely, assume that the sequence is ergodic and that $n^{-1}E(S_n^2)$ converges to a strictly positive σ^2, then S_n behaves asymptotically like a sum of n martingale differences, each with variance σ^2. Therefore, under fairly weak additional condition, the central limit theorem can be deduced from the martingale case. This approach was

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first explored by Gordin (1969). Next, Hall and Heyde (1980), Dürr and Goldstein (1984) or more recently Volný (1993), used Gordin's approach to provide projective criteria for the central limit theorem. These criteria imply Ibragimov's central limit theorem for stationary and strongly mixing sequences (1962).

Unfortunately, we cannot follow this way to study stationary random fields, because the σ-algebras which naturally appear are no more nested. Nevertheless it is still natural to ask for projective criterions which imply the existence of central limit theorems for stationary random fields. This question has been partially answered over the past few years, first by considering martingale-type conditions (see Nahapetian and Petrosian (1992) and Nahapetian (1995)), and then by studying the case of conditionally centered random fields (see Jensen and Künsch (1994), Janžura and Lachout (1995), and Comets and Janžura (1995) in the non-stationary case). This notion has been first introduced by Guyon and Künsch (1992) in order to study the asymptotic behaviour of a certain estimator of the interaction for the Ising model at the critical temperature. In that case, the mixing coefficients have no good properties of decrease and one cannot used any mixing theorems, whereas conditional centering applies to certain fields subordinated to the Ising model. Conversely, it is easy to understand that martingale-type conditions as well as conditional centering may fail to hold for a large class of random fields with long range interaction: for instance, one cannot infer from any of these assumptions Bolthausen's central limit theorem for strongly mixing random fields (1982) (result recently improved by Perera (1996) in the unbounded case).

Many proofs of these theorems are based on a method introduced by Stein (1973). However, this method does not always lead to optimal assumptions, as Bolthausen notes in Remark 1 of his article. As a matter of fact, to control the terms which naturally appear by following Stein's approach, one needs to make strong assumptions about the moments of the random field, or to introduce some unnecessary mixing coefficients. Stein's method has been also used by Gordin (1993) who proves a central limit theorem for dynamical systems. We agree with the author when he writes in the concluding remarks of his paper, that a natural application of his approach could be the central limit theorem for random fields. However, until now, we are unable to compare the conditions that it might provide with ours.

Our aim in this paper is first to propose a projective criterion comparable to the $\mathbb{L}^1$ criterion stated by Gordin (1973) in the case of stationary and ergodic sequences, and second to present a self-normalized sequence whose limit, under this assumption, is a standard gaussian. To establish our results, we use Lindeberg's method introduced in 1922 to study independent sequences of random variables, adapted by Billingsley (1961) and Ibragimov (1963) to the case of stationary and ergodic martingale difference sequences, and by Rio (1995) to the case of strongly mixing sequences. In order to exhibit our criterion, we extend a decomposition proposed by Rio to our context. The tools that are needed are quite different from the strongly mixing case, because the remainder terms cannot be controlled with the help of covariance inequalities as in Bolthausen, Rio or Perera. Since our approach needs to be more precise, we obtain as a straightforward consequence the α-mixing condition expected by Bolthausen (see again Remark 1 of his paper). Another interest of this approach is that it does not require any assumption about the ergodicity of the random field. Consequently, the normalized partial sum sequence converges in distribution to a mixture of gaussian law. More precisely, let $\mathscr{I}$ be the invariant σ-algebra, the limit is a product of an $\mathscr{I}$-measurable variable by an independent standard gaussian.

This paper is organized as follows: Section 2 sets up the notations and the preliminary results which will be useful in the sequel. In Section 3, our main results are stated. In Theorem 1 the normalized sequence converges in distribution to a mixture of gaussian law, where the variance term η is a positive $\mathscr{I}$-measurable random variable. Theorem 1 provides also a consistency estimator of η. In Corollary 1, we give a random normalization which ensures the asymptotic normality of the partial sum sequence. In Corollary 2, we are interested in the degeneracy of the variable η. Corollary 3 is devoted to mixing assumptions, and to be complete, Theorem 2 proposes the finite-dimensional version of Theorem 1. The results are proved in Sections 4, 5, 6 and 7.

2 Preliminaries

In order to develop our results, we need some preliminary notations.

2.1 Real random fields

Let us consider the space $\mathbb{R}$ with its borel σ-algebra $\mathscr{B}$. By a real random field we mean a probability space $(\mathbb{R}^{\mathbb{Z}^d}, \mathscr{B}^{\mathbb{Z}^d}, \mathbb{P})$. We denote by X the identical application from $\mathbb{R}^{\mathbb{Z}^d}$ to $\mathbb{R}^{\mathbb{Z}^d}$, and by X_i the projection from $\mathbb{R}^{\mathbb{Z}^d}$ to $\mathbb{R}$ defined by $X_i(\omega) = \omega_i$, for any ω in $\mathbb{R}^{\mathbb{Z}^d}$. From now on, the application X, or the field of all projections $(X_i)_{i\in\mathbb{Z}^d}$ will designate the whole random field $(\mathbb{R}^{\mathbb{Z}^d}, \mathscr{B}^{\mathbb{Z}^d}, \mathbb{P})$.

For k in $\mathbb{Z}^d$, T_k denotes the translation operator from $\mathbb{R}^{\mathbb{Z}^d}$ to $\mathbb{R}^{\mathbb{Z}^d}$ which is defined by: $[T_k(\omega)]_i = \omega_{i+k}$. An element A of $\mathscr{B}^{\mathbb{Z}^d}$ is said to be invariant if $T_k(A) = A$ for any k in $\mathbb{Z}^d$. We denote by $\mathscr{I}$ the σ-algebra of all invariant sets. A random field is said to be strictly stationary if $T_k \circ \mathbb{P} = \mathbb{P}$, for any k in $\mathbb{Z}^d$. Throughout, X is a strictly stationary random field with $\mathbb{E}(X_0) = 0$ and $\mathbb{E}(X_0^2) < +\infty$.

On $\mathbb{Z}^d$ we define the lexicographic order: If $i = (i_1, i_2, \ldots, i_d)$, and $j = (j_1, j_2, \ldots, j_d)$ are distinct elements of $\mathbb{Z}^d$, the notation $i <_{\text{lex}} j$ means that either $i_1 < j_1$ or for some p in $\{2, 3, \ldots, d\}$, $i_p < j_p$ and $i_q = j_q$ for $1 \le q < p$. Note that the lexicographic order provides a total ordering of $\mathbb{Z}^d$.

Let the sets $\{V_i^k : i \in \mathbb{Z}^d, k \in \mathbb{N}^*\}$ be defined as follows:

$$V_i^1 = \left\{ j \in \mathbb{Z}^d \ : \ j <_{\text{lex}} i \right\} ,$$

and for $k \ge 2$:

$$V_i^k = V_i^1 \cap \left\{ j \in \mathbb{Z}^d \ : \ |i-j| \ge k \right\} \quad \text{where } |i-j| = \max_{1\le k\le d} |i_k - j_k| \ .$$

For any subset Γ of $\mathbb{Z}^d$, let $\mathscr{F}_\Gamma$ be the σ-algebra defined by: $\mathscr{F}_\Gamma = \sigma(X_i : i \in \Gamma)$. To conclude this section, let us define the tail σ-algebra $\mathscr{F}_{-\infty} = \cap_{k\in\mathbb{N}^*} \mathscr{F}_{V_0^k}$. Then, using the same argument as in Georgii (1988) Proposition (14.9), the following result holds:

Proposition 1. *Let X be a stationary random field. Then $\mathscr{I}$ is included in the $\mathbb{P}$-completion of $\mathscr{F}_{-\infty}$.*

2.2 Mixing coefficients

Let $(\Omega, \mathscr{A}, \mathbb{P})$ be a probability space. Given two σ-algebras $\mathscr{U}$ and $\mathscr{V}$ of $\mathscr{A}$, different measures of their dependence have been considered in the literature. We are interested by two of them. The strong mixing coefficient of Rosenblatt (1956) is defined by:

$$\alpha(\mathscr{U}, \mathscr{V}) = \sup\{|\mathbb{P}(U)\mathbb{P}(V) - \mathbb{P}(U \cap V)|; U \in \mathscr{U}, V \in \mathscr{V}\} \ .$$

The ϕ-mixing coefficient has been introduced by Ibragimov (1962) and can be defined by:

$$\phi(\mathscr{U}, \mathscr{V}) = \sup\left\{ \|\mathbb{P}(V|\mathscr{U}) - \mathbb{P}(V)\|_\infty , V \in \mathscr{V} \right\} \ .$$

Between those two coefficients, the following relation holds:

$$2\alpha(\mathscr{U}, \mathscr{V}) \le \phi(\mathscr{U}, \mathscr{V}) \ . \tag{2.1}$$

Mixing coefficients for real random fields.

Let $(\mathbb{R}^{\mathbb{Z}^d}, \mathscr{B}^{\mathbb{Z}^d}, \mathbb{P})$ be a real random field. The mixing coefficients we will use in the sequel are defined by: if $n \in \mathbb{N}, k, l \in \mathbb{N} \cup \{\infty\}$,

$$\alpha_{k,l}(n) = \sup\{\alpha(\mathscr{F}_{\Gamma_1}, \mathscr{F}_{\Gamma_2}), |\Gamma_1| \le k, |\Gamma_2| \le l, d(\Gamma_1, \Gamma_2) \ge n\} \ ,$$
$$\phi_{k,l}(n) = \sup\{\phi(\mathscr{F}_{\Gamma_1}, \mathscr{F}_{\Gamma_2}), |\Gamma_1| \le k, |\Gamma_2| \le l, d(\Gamma_1, \Gamma_2) \ge n\} \ ,$$

where $d(\Gamma_1, \Gamma_2) = \min\{|j-i|, i \in \Gamma_1, j \in \Gamma_2\}$. For more about the mixing properties of random fields with respect to those coefficients, see Doukhan (1994).

2.3 Toward a new central limit theorem for stationary random fields

Let Γ be any subset of $\mathbb{Z}^d$. We denote by $|\Gamma|$ the cardinality of this set, and we introduce:

$$\partial\Gamma = \{i \in \Gamma \ : \ \exists j \notin \Gamma \quad \text{such that} \quad |i-j| = 1\} \ .$$

If Γ is a finite subset of $\mathbb{Z}^d$, S_Γ denotes the partial sum of the random field X over this set: $S_\Gamma = \sum_{i \in \Gamma} X_i$. Throughout $(\Gamma_n)_{n \in \mathbb{N}}$ is a sequence of finite subsets of $\mathbb{Z}^d$ satisfying:

$$\lim_{n \to +\infty} |\Gamma_n| = +\infty \quad \text{and} \quad \lim_{n \to +\infty} |\Gamma_n|^{-1} |\partial\Gamma_n| = 0 \ . \tag{2.2}$$

The $\mathbb{L}^2$-ergodic theorem (see Georgii 1988) ensures that $|\Gamma_n|^{-1} S_{\Gamma_n}$ converges to $\mathbb{E}(X_0|\mathscr{I})$ in $\mathbb{L}^2$. In order to prove a central limit theorem for $|\Gamma_n|^{-1/2} S_{\Gamma_n}$, it will be necessary to impose some conditions ensuring that $\mathbb{E}(X_0|\mathscr{I}) = 0$.

Proposition 2. *Let* $\Lambda_n = [-n, n]^d \cap \mathbb{Z}^d$. *Assumptions* (a) *and* (b) *are equivalent*:

$$(a) \ \lim_{n \to +\infty} |\Lambda_n|^{-1} \sum_{k \in \Lambda_n} \mathrm{Cov}(X_0, X_k) = 0; \qquad (b) \ \mathbb{E}(X_0|\mathscr{I}) = 0 \ a.s.$$

The condition (a) is very weak, and is automatically realized as soon as we make some assumption concerning the dependency of the variables. For example, if we define, for all positive integers k and all i in $\mathbb{Z}^d$, $\mathbb{E}_k(X_i) = \mathbb{E}(X_i|\mathscr{F}_{V_i^k})$, (a) holds if we suppose that the martingale-type condition $\mathbb{E}_1(X_0) = 0$ is realized. However, in that case, the classical central limit theorem may fail, for this kind of condition does not imply the ergodicity of the field. More precisely, if $d = 1$, Eagleson (1975) has shown that the sequence $n^{-1/2} S_n$ converges weakly to a mixture of gaussian law $\varepsilon \mathbb{E}^{1/2}(X_0^2|\mathscr{I})$, where $\varepsilon \sim \mathscr{N}(0,1)$ and ε is independent of $\mathscr{I}$. The fact that a single variable X_0 appears through the conditional expectation with respect to $\mathscr{I}$ can be easily understood. As a matter of fact, the martingale-type condition ensures that: $|\Lambda_n|^{-1}\mathbb{E}(S^2_{|\Lambda_n|}|\mathscr{I}) = \mathbb{E}(X_0^2|\mathscr{I})$ a.s.

In view of the martingale case, it is natural to think that the convergence of $|\Lambda_n|^{-1}\mathbb{E}(S^2_{|\Lambda_n|}|\mathscr{I})$ may be important to obtain a central limit theorem. This leads us to consider the condition:

$$\sum_{k\in V_0^1} \left|X_k \mathbb{E}_{|k|}(X_0)\right| \in \mathbb{L}^1 \quad , \tag{2.3}$$

which implies the convergence of $|\Lambda_n|^{-1}\mathbb{E}(S^2_{|\Lambda_n|}|\mathscr{I})$, as shown in the proposition below.

Proposition 3. *If X satisfies* (2.3), *then* $\mathbb{E}(X_0|\mathscr{F}_{-\infty}) = 0$ *almost surely. Moreover*:

$$\sum_{k\in\mathbb{Z}^d} |\mathbb{E}(X_0X_k|\mathscr{I})| \in \mathbb{L}^1 \quad \textit{and} \quad \lim_{n\to+\infty} |\Lambda_n|^{-1}\mathbb{E}\big(S^2_{|\Lambda_n|}|\mathscr{I}\big) = \sum_{k\in\mathbb{Z}^d} \mathbb{E}(X_0X_k|\mathscr{I}) \text{ a.s.}$$

3 Central limit theorems

Throughout this section, $(X_i)_{i\in\mathbb{Z}^d}$ is a strictly stationary random field, with $\mathbb{E}(X_0) = 0$ and $\mathbb{E}(X_0^2) < +\infty$. $(\Gamma_n)_{n\in\mathbb{N}^*}$ is a sequence of finite subsets of $\mathbb{Z}^d$ satisfying (2.2).

Now let us introduce the concept of stability (Rényi 1963), which enables us to interchange norming in the central limit theorem.

Definition 1. *Let $(Y_n)_{n\in\mathbb{N}}$ be a sequence of real random variables, and let Y be defined on some extension of the underlying probability space $(\Omega, \mathscr{A}, \mathbb{P})$. Let $\mathscr{U}$ be a σ-algebra of $\mathscr{A}$. Then $(Y_n)_{n\in\mathbb{N}}$ is said to converge $\mathscr{U}$-stably to Y if for any continuous bounded function φ and any bounded and $\mathscr{U}$-measurable variable Z, $\lim_{n\to+\infty}\mathbb{E}(\varphi(Y_n)Z) = \mathbb{E}(\varphi(Y)Z)$.*

Theorem 1. *Assume that condition* (2.3) *is satisfied, and set* $\eta = \sum_{k\in\mathbb{Z}^d}\mathbb{E}(X_0X_k|\mathscr{I})$. *The following results hold*:

(*a*) *The random variable $|\Gamma_n|^{-1/2}S_{\Gamma_n}$ converges $\mathscr{I}$-stably to $\varepsilon\eta^{1/2}$, where $\varepsilon \sim \mathscr{N}(0,1)$ and ε is independent of η.*
(*b*) *For any N in $\mathbb{N}^*$, set $G_N = \{(i,j)\in\Gamma_n\times\Gamma_n \ : \ |i-j|\le N\}$. Let ρ_n be a sequence of positive integers satisfying*:

$$\lim_{n\to+\infty}\rho_n = +\infty \quad \textit{and} \quad \lim_{n\to+\infty}\rho_n^{3d}\mathbb{E}\big(X_0^2\big(1\wedge|\Gamma_n|^{-1}X_0^2\big)\big) = 0 \quad . \tag{3.1}$$

Then:

$$A_n = |\Gamma_n|^{-1} \max\left(1, \sum_{(i,j)\in G_{\rho_n}} X_i X_j\right) \xrightarrow{\mathbb{P}} \eta \ .$$

As a direct consequence, we obtain the following corollary:

Corollary 1. *Assume that condition* (2.3) *is satisfied. Then, with the same notations as in Theorem* 1, $(|\Gamma_n|^{-1/2} S_{\Gamma_n}, A_n)$ *converges in distribution to* $(\varepsilon\eta^{1/2}, \eta)$. *Assume moreover that* $\mathbb{P}(\eta > 0) = 1$. *Then* $(A_n|\Gamma_n|)^{-1/2} S_{\Gamma_n}$ *converges in distribution to* ε.

Remark 1. Let us describe an important class of random fields which satisfies condition (2.3): let X and Y be two stationary centered random fields. As in Jensen and Künsch (1994), we say that X is conditionally centered with respect to Y if $\mathbb{E}(X_0|Y_i, i \neq 0) = 0$ and X_0 is $\sigma(Y_i, |i| \leq K)$-measurable for some integer K. Since $\sigma(X_i, i \in V_0^k)$ is contained in $\sigma(Y_i, i \neq 0)$ for $k > K$, it follows immediately that condition (2.3) is satisfied. This kind of random fields has also been studied by Comets and Janžura (1995) in the non-stationary case. They obtain a central limit theorem, assuming that the random variables have uniformly bounded fourth moments. It is rather interesting to compare conditional centering as it is defined here with the notion of martingale-difference random fields considered by Nahapetian (1992, 1995).

Corollary 1 gives an example of sequence whose limit is a Gaussian law, by choosing a random norming. Situations like this one under which we can obtain the asymptotic normality are of a special interest. Applying Proposition 2, the next corollary gives a condition which ensures the degeneracy of η.

Corollary 2. *Let* N *be a positive number, and set*: $X_i^N = (X_i \wedge N) \vee (-N)$. *Assume that condition* (2.3) *is fulfilled. Assume moreover that for any* k *in* $\mathbb{Z}^d$, *and any positive integer* N:

$$\lim_{n\to+\infty} |\Lambda_n|^{-1} \sum_{i\in\Lambda_n} \mathrm{Cov}\left(X_0 X_k^N, X_i X_{i+k}^N\right) = 0 \qquad (3.2)$$

Then Theorem 1 *holds with*: $\eta = \sigma^2 = \sum_{k\in\mathbb{Z}^d} \mathbb{E}(X_0 X_k)$.

Remark 2. Assume that the random variables X_i have finite fourth moments, then we do not need any truncation. In view of Proposition 2, the condition which ensures the degeneracy of η can be replaced by:

$$\lim_{n\to+\infty} |\Lambda_n|^{-1} \sum_{i\in\Lambda_n} \mathrm{Cov}(X_0 X_k, X_i X_{i+k}) = 0 \quad \text{for any } k \text{ in } \mathbb{Z}^d \ .$$

As a consequence of Theorem 1, we obtain central limit theorems under α-mixing or ϕ-mixing assumptions.

Corollary 3. *Let us consider the two following assumptions*:

$$\sum_{k\in\mathbb{Z}^d}\int_0^{\alpha_{1,\infty}(|k|)} Q_{X_0}^2(u)\,du < \infty \ , \tag{3.3}$$

where Q_{X_0} *denotes the cadlag inverse of the function* H_{X_0}: $t \to \mathbb{P}(|X_0| > t)$, *and*

$$\sum_{k\in\mathbb{Z}^d}\phi_{\infty,1}(|k|) < \infty \ . \tag{3.4}$$

The following results hold:
(*a*) (3.3) *implies* (2.3), *and hence also Theorem* 1(*a*)(*b*).
(*b*) *Under condition* (3.4) *Theorem* 1(*a*) *holds.*
(*c*) *Assume that* (3.3) *or* (3.4) *is realized, and moreover that*: $\lim_{k\to+\infty}\alpha_{2,2}(k) = 0$. *Then*, *with the same notations as in Theorem* 1, $\eta = \sigma^2 = \sum_{k\in\mathbb{Z}^d}\mathbb{E}(X_0X_k)$ *a.s.*

Remark 3. Bolthausen (1982) proves a central limit theorem for stationary and α-mixing random fields (see Guyon (1993) for a non-stationary version of this theorem), but he fails to make assumptions on $\alpha_{1,\infty}$ only (see Remark 1 of his paper). To compare our result with Bolthausen's, let us note that if $\mathbb{E}(|X_0|^{2+\delta}) < \infty$ for some $\delta > 0$, then the condition $\sum_{m=1}^{\infty} m^{d-1}\alpha_{1,\infty}^{\delta/2+\delta}(m) < \infty$ is more restrictive than condition (3.3).

We remark that in Bolthausen's article, the conditional expectation with respect to the σ-algebra $\mathscr{I}$ does not appear. Indeed $\alpha_{2,2}(n)$ is required to be asymptoticaly negligible, and this implies the degeneracy of η. In fact, one can see that this condition on $\alpha_{2,2}(n)$ is stronger than assumption of Corollary 2, since it implies that $\sigma(X_0, X_k)$ is independent of $\mathscr{I}$ for any k in $\mathbb{Z}^d$.

To conclude this section, let us state a multivariate version of Theorem 1.

Theorem 2. *Let* $(\Gamma_{i,n})_{i\in[1..q]}$ *be a sequence of disjoint subsets of* $\mathbb{Z}^d$. *Assume that condition* (2.3) *is fulfilled. Then*:

$$\begin{pmatrix} \frac{S_{\Gamma_{1,n}}}{|\Gamma_{1,n}|^{1/2}} \\ \cdot \\ \cdot \\ \cdot \\ \frac{S_{\Gamma_{q,n}}}{|\Gamma_{q,n}|^{1/2}} \end{pmatrix} \xrightarrow{\mathscr{D}} \begin{pmatrix} \varepsilon_1\eta^{1/2} \\ \cdot \\ \cdot \\ \cdot \\ \varepsilon_q\eta^{1/2} \end{pmatrix}$$

where $(\varepsilon_i)_{i\in[1..q]} \sim \mathcal{N}(0, Id)$ *and* $(\varepsilon_i)_{i\in[1..q]}$ *is independent of* η.

4 Proofs of propositions and corollaries

Proof of Proposition 2. Since $\mathbb{E}(X_0) = 0$ the condition (a) of Proposition 2 can be expressed as follows:

$$\lim_{n\to+\infty} |\Lambda_n|^{-1}\mathbb{E}(X_0 S_{\Lambda_n}) = 0 \ .$$

By the $\mathbb{L}^2$-ergodic theorem, we infer that condition (a) is equivalent to $\mathbb{E}(X_0\mathbb{E}(X_0|\mathscr{I})) = 0$, and the result easily follows.

Proof of Proposition 3. We start by proving that $\mathbb{E}(X_0|\mathscr{F}_{-\infty}) = 0$ a.s. We denote by $\mathbb{E}_\infty$ the conditional expectation with respect to $\mathscr{F}_{-\infty}$, and by $\mathbb{E}_{\mathscr{I}}$ the conditional expectation with respect to $\mathscr{I}$. By the backward martingale convergence theorem, we know that $\lim_{n\to+\infty}\|\mathbb{E}_n(X_0) - \mathbb{E}_\infty(X_0)\|_2 = 0$. Now, for any k in V_0^1:

$$\mathbb{E}(|X_k\mathbb{E}_\infty(X_0)|) \le \mathbb{E}\big(|X_k\mathbb{E}_{|k|}(X_0)|\big) + \|X_0\|_2\|\mathbb{E}_{|k|}(X_0) - \mathbb{E}_\infty(X_0)\|_2 \ ,$$

hence $\mathbb{E}|X_k\mathbb{E}_\infty(X_0)|$ converges to 0 as $|k| \to +\infty$. Let us introduce the set $\Lambda_n^1 = \Lambda_n \cap V_0^1$. Applying the $\mathbb{L}^2$-ergodic theorem to the random variables $|X_k|$, and the Cesaro mean convergence theorem, we infer that:

$$\mathbb{E}(\mathbb{E}_{\mathscr{I}}(|X_0|)|\mathbb{E}_\infty(X_0)|) = \lim_{n\to+\infty}|\Lambda_n^1|^{-1}\sum_{i\in\Lambda_n^1}\mathbb{E}|X_i\mathbb{E}_\infty(X_0)| = 0 \ .$$

By Proposition 1 and Jensen's inequality, $\mathbb{E}_{\mathscr{I}}(|X_0|) \ge \mathbb{E}_{\mathscr{I}}(\mathbb{E}_\infty(|X_0|)) \ge \mathbb{E}_{\mathscr{I}}(|\mathbb{E}_\infty(X_0)|)$ a.s. Hence we have that $\mathbb{E}(\mathbb{E}_{\mathscr{I}}(|X_0|)|\mathbb{E}_\infty(X_0)|) \ge \mathbb{E}(|\mathbb{E}_\infty(X_0)|\mathbb{E}_{\mathscr{I}}\ (|\mathbb{E}_\infty\ (X_0)|))$, which ensures that $\mathbb{E}_{\mathscr{I}}(|\mathbb{E}_\infty(X_0)|) = 0$ a.s., and finally $\mathbb{E}_\infty(X_0) = 0$ a.s.

The second point is to prove that $\sum_{k\in\mathbb{Z}^d}\mathbb{E}(|\mathbb{E}(X_0X_k|\mathscr{I})|) < +\infty$. By Proposition 1 and the fact that $\mathscr{F}_{-\infty} \subset \mathscr{F}_{V_0^k}$, we have, for all k in V_0^1:

$$\mathbb{E}(|\mathbb{E}(X_0X_k|\mathscr{I})|) \le \mathbb{E}(|\mathbb{E}(X_0X_k|\mathscr{F}_{-\infty})|) \le \mathbb{E}\big(|X_k\mathbb{E}_{|k|}(X_0)|\big) \ .$$

Since $\mathbb{E}(X_0X_k|\mathscr{I}) = \mathbb{E}(X_0X_{-k}|\mathscr{I})$, we infer that:

$$\begin{aligned}\sum_{k\in\mathbb{Z}^d}\mathbb{E}(|\mathbb{E}(X_0X_k|\mathscr{I})|) &\le \mathbb{E}(X_0^2) + 2\sum_{k\in V_0^1}\mathbb{E}(|\mathbb{E}(X_0X_k|\mathscr{I})|)\\ &\le \mathbb{E}(X_0^2) + 2\sum_{k\in V_0^1}\mathbb{E}\big(|X_k\mathbb{E}_{|k|}(X_0)|\big) < +\infty \ .\end{aligned}$$

The last point is to prove that $\lim_{n\to+\infty}|\Lambda_n|^{-1}\mathbb{E}(S_{\Lambda_n}^2|\mathscr{I})=\sum_{k\in\mathbb{Z}^d}\mathbb{E}(X_0X_k|\mathscr{I})$. For any subset Γ of $\mathbb{Z}^d$ and any k in $\mathbb{Z}^d$, let $\Gamma-k=\{i-k,\ i\in\Gamma\}$. By stationarity of the random field:

$$|\Lambda_n|^{-1}\mathbb{E}\left(S_{|\Lambda_n|}^2|\mathscr{I}\right)=\sum_{k\in\Lambda_{2n}}|\Lambda_n|^{-1}|\Lambda_n\cap(\Lambda_n-k)|\mathbb{E}(X_0X_k|\mathscr{I})\ .$$

Now $|\Lambda_n|^{-1}|\Lambda_n\cap(\Lambda_n-k)||\mathbb{E}(X_0X_k|\mathscr{I})|\le|\mathbb{E}(X_0X_k|\mathscr{I})|$, and also $\sum_{k\in\mathbb{Z}^d}|\mathbb{E}(X_0X_k|\mathscr{I})|<+\infty$ a.s. Since $\lim_{n\to+\infty}|\Lambda_n|^{-1}|\Lambda_n\cap(\Lambda_n-k)|=1$, we may apply the dominated convergence theorem, yielding:

$$\lim_{n\to+\infty}|\Lambda_n|^{-1}\mathbb{E}\big(S_{|\Lambda_n|}^2|\mathscr{I}\big)=\sum_{k\in\mathbb{Z}^d}\mathbb{E}(X_0X_k|\mathscr{I})\ \text{a.s.}$$

Hence the result follows.

Proof of Corollary 1. Corollary 1 is an immediate consequence of the following lemma:

Lemma 1. *Let X_n and Y_n be two sequences of real random variables defined on $(\Omega,\mathscr{A},\mathbb{P})$. Let $\mathscr{U}$ be a σ-algebra of $\mathscr{A}$. Assume that X_n converges $\mathscr{U}$-stably to X and that Y_n converges in probability to some $\mathscr{U}$-measurable random variable Y. Then (X_n,Y_n) converges in distribution to (X,Y).*

Proof. Let f and g be two continuous bounded functions, and assume moreover that g is 1-Lipchitz. Clearly:

$$\begin{aligned}|\mathbb{E}(f(X_n)g(Y_n)-f(X)g(Y))|\le{}&\|f\|_\infty\mathbb{E}|g(Y_n)-g(Y)|\\&+|\mathbb{E}(g(Y)[f(X_n)-f(X)])|\ .\end{aligned}$$

The stability of the convergence of X_n to X ensures that the second term of the right hand inequality is asymptotically negligible, and the convergence in probability of Y_n to Y together with the fact that g is 1-Lipchitz imply that $\lim_{n\to+\infty}\mathbb{E}|g(Y_n)-g(Y)|=0$. Hence $\mathbb{E}(f(X_n)g(Y_n))$ converges to $\mathbb{E}(f(X)g(Y))$ and the result follows.

Proof of Corollary 2. By Proposition 2 and assumption (3.2), $\mathbb{E}(X_0X_k^N|\mathscr{I})=\mathbb{E}(X_0X_k^N)$ a.s. Now, applying the dominated convergence theorem, we get that:

$$\lim_{N\to+\infty}\mathbb{E}\big(X_0X_k^N|\mathscr{I}\big)=\mathbb{E}(X_0X_k|\mathscr{I})\quad\text{a.s.}\quad\text{and}$$
$$\lim_{N\to+\infty}\mathbb{E}(X_0X_k^N)=\mathbb{E}(X_0X_k)\ .$$

Finally for all k in $\mathbb{Z}^d$: $\mathbb{E}(X_0X_k|\mathscr{I})=\mathbb{E}(X_0X_k)$ almost surely. Since (2.3) is realized, we infer that:

$$\eta = \sigma^2 = \sum_{k\in\mathbb{Z}^d} \mathbb{E}(X_0X_k) \quad \text{and} \quad |\Gamma_n|^{-1/2}S_{\Gamma_n} \xrightarrow{\mathscr{D}} \mathscr{N}(0,\sigma^2) \ .$$

Proof of Corollary 3. First, we note that (c) follows immediately from Corollary 2.

To prove (a), let us remark that:

$$\mathbb{E}|X_k\mathbb{E}_{|k|}(X_0)| = \mathrm{Cov}\big(|X_k|\big(\mathbb{1}_{\mathbb{E}_{|k|}(X_0)\geq 0} - \mathbb{1}_{\mathbb{E}_{|k|}(X_0)<0}\big), X_0\big) \ .$$

By Theorem 1.1 in Rio (1993), it follows that

$$\mathbb{E}|X_k\mathbb{E}_{|k|}(X_0)| \leq 4\int_0^{\alpha_{1,\infty}(|k|)} Q^2_{X_0}(u)\,du \ ,$$

which proves (a).

To prove (b), we need a conditional version of Peligrad's inequality (1983). A complete proof of this inequality will be done in Appendix.

Proposition 4. *Let* $(\Omega,\mathscr{A},\mathbb{P})$ *be a probability space, and* $\mathscr{U},\mathscr{V},\mathscr{F}$ *three* σ*-algebras of* $\mathscr{A}$ *such that* $\mathscr{U}$ *and* $\mathscr{V}$ *are independent of* $\mathscr{F}$. *Let X and Y be two random variables from* $(\Omega,\mathscr{A},\mathbb{P})$ *to* $\mathbb{R}$ *such that X is* $\mathscr{U}$*-measurable in* $\mathbb{L}^p(\mathbb{P})$, *and Y is* $\mathscr{V}$*-measurable in* $\mathbb{L}^q(\mathbb{P})$, *where p and q are two positive numbers with* $p^{-1}+q^{-1}=1$. *We define* $\mathrm{Cov}(X,Y|\mathscr{F}) = \mathbb{E}(XY|\mathscr{F}) - \mathbb{E}(X)\mathbb{E}(Y)$. *Then*:

$$|\mathrm{Cov}(X,Y|\mathscr{F})| \leq 2\phi^{1/p}(\mathscr{F}\vee\mathscr{U},\mathscr{V})\phi^{1/q}(\mathscr{F}\vee\mathscr{V},\mathscr{U})\|X\|_p\|Y\|_q \quad a.s.$$

Now, to prove the asymptotic normality, we apply the truncation technique as in Ibragimov and Linnik (1971). Using the same notation as in Corollary 2, let $X_k^N = (X_k\wedge N)\vee(-N)$, and $\widetilde{X_k^N} = X_k - X_k^N$. We denote by $S^N_{\Gamma_n}$ the sum of the new centered field $X^N - \mathbb{E}(X^N)$ over the set Γ_n and we set $\widetilde{S^N_{\Gamma_n}} = S_{\Gamma_n} - S^N_{\Gamma_n}$. By assumption the equation (3.4): $\sum_{k\in\mathbb{Z}^d}\phi_{\infty,1}(|k|) < \infty$ is satisfied. Applying (2.1), $2\alpha_{1,\infty}(|k|) \leq \phi_{\infty,1}(|k|)$ and (3.4) implies that: $\sum_{k\in\mathbb{Z}^d}\alpha_{1,\infty}(|k|) < \infty$. Now, we can apply Corollary 3(a) to the random field X^N. As a matter of fact, the definition of $Q_{X_0^N-\mathbb{E}(X_0^N)}$ as the inverse cadlag of the tail function $H_{X_0^N-\mathbb{E}(X_0^N)} : t \to \mathbb{P}(|X_0^N - \mathbb{E}(X_0^N)| > t)$, ensures that $Q_{X_0^N-\mathbb{E}(X_0^N)} \leq 2N$. Therefore:

$$\sum_{k\in\mathbb{Z}^d}\int_0^{\alpha_{1,\infty}(|k|)} Q^2_{X_0^N-\mathbb{E}(X_0^N)}(u)\,du \ \leq\ 4N^2\sum_{k\in\mathbb{Z}^d}\alpha_{1,\infty}(|k|) < \infty$$

This means that (3.3) is realized, and Corollary 3(a) ensures that the random field $X^N - \mathbb{E}(X^N)$ satisfies condition (2.3). Consequently, the series $\sum_{k\in\mathbb{Z}^d}\mathrm{Cov}(X_0^N, X_k^N|\mathscr{I})$ converges in $\mathbb{L}^1$. Now, set $\eta_N = \sum_{k\in\mathbb{Z}^d}$ $\mathrm{Cov}(X_0^N, X_k^N|\mathscr{I})$. Let Z be any bounded $\mathscr{I}$-measurable random variable, and φ be a bounded 1-Lipschitz function. To obtain the theorem, we have to prove that, under (3.4),

$$\lim_{n\to+\infty} \mathbb{E}\big(Z\big[\varphi\big(|\Gamma_n|^{-1/2}S_{\Gamma_n}\big) - \varphi\big(\varepsilon\eta^{1/2}\big)\big]\big) = 0 \ .$$

Clearly:

$$\begin{aligned}\mathbb{E}\big(Z\big[\varphi\big(|\Gamma_n|^{-1/2}S_{\Gamma_n}\big)-\varphi\big(\varepsilon\eta^{1/2}\big)\big]\big) = \\ &\mathbb{E}\big(Z\big[\varphi\big(|\Gamma_n|^{-1/2}S_{\Gamma_n}\big) - \varphi(|\Gamma_n|^{-1/2}S^N_{\Gamma_n})\big]\big) \\ &+ \mathbb{E}\big(Z\big[\varphi\big(|\Gamma_n|^{-1/2}S^N_{\Gamma_n}\big) - \varphi(\varepsilon\eta_N^{1/2})\big]\big) \\ &+ \mathbb{E}\big(Z\big[\varphi\big(\varepsilon\eta_N^{1/2}\big) - \varphi(\varepsilon\eta^{1/2})\big]\big) \ .\end{aligned}$$

By Theorem 1(a), the second term of the right hand expression converges to 0 as $n \to +\infty$. Let us now study the first term of the right hand expression:

$$\big|\mathbb{E}\big(Z\big[\varphi\big(|\Gamma_n|^{-1/2}S_{\Gamma_n}\big) - \varphi\big(|\Gamma_n|^{-1/2}S^N_{\Gamma_n}\big)\big]\big)\big| \le \|Z\|_\infty |\Gamma_n|^{-1/2}\mathbb{E}^{1/2}\big(\big[\widetilde{S^N_{\Gamma_n}}\big]^2\big) \ .$$

Now, by Proposition 4:

$$|\Gamma_n|^{-1}\mathbb{E}\big(\big[\widetilde{S^N_{\Gamma_n}}\big]^2\big) \le \|\widetilde{X^N_0}\|_2^2 \sum_{k\in\mathbb{Z}^d} \phi_{\infty,1}(|k|) \ .$$

Since $\|\widetilde{X^N_0}\|_2$ converges to 0 as $N \to +\infty$, the first term of the right hand expression can be chosen as small as we wish. Now, to ensure that the third term of the right hand expression is asymptotically negligible, it is enough to prove that $\lim_{N\to+\infty}\eta_N = \eta$ almost surely. The dominated convergence theorem implies that:

$$\lim_{N\to+\infty} \mathrm{Cov}\big(X^N_0, X^N_k|\mathscr{I}\big) = \mathbb{E}(X_0X_k|\mathscr{I}) \text{ a.s.}$$

Let us remark that the convergence of $\phi_{\infty,1}(n)$ to zero implies that for all k in $\mathbb{Z}^d$, $\sigma(X_k)$ is independent of $\mathscr{F}_{-\infty}$. Therefore, applying Proposition 1 and Proposition 4:

$$\mathrm{Cov}\big(X^N_0, X^N_k|\mathscr{I}\big) \le \|X_0\|_2^2\phi_{\infty,1}(|k|) \text{ a.s.}$$

Since (3.4) is realized, we may apply once more the dominated convergence theorem yielding:

$$\lim_{N\to+\infty} \sum_{k\in\mathbb{Z}^d} \mathrm{Cov}\big(X^N_0, X^N_k|\mathscr{I}\big) = \sum_{k\in\mathbb{Z}^d} \mathbb{E}(X_0X_k|\mathscr{I}) = \eta \text{ a.s.}$$

This ends the proof of (b).

5 Proof of the main result

In this section we prove Theorem $1(a)$. The two main references concerning this part of Theorem 1 are Ibragimov (1963) and Rio (1995). From the first article, which deals with stationary and ergodic martingales differences sequences, we get the structure of our proof. From the second one we borrow a decomposition which can be adapted to our case although we do not use mixing assumptions.

Notations 1. Let f be a one to one map from $[1,N]\cap\mathbb{N}^*$ to $f([1,N]\cap\mathbb{N}^*)\subset\mathbb{Z}^d$, and $(\xi_i)_{i\in\mathbb{Z}^d}$ a real random field. For all integer k in $[1,N]$ we introduce:

$$S_{f(k)}(\xi)=\sum_{i=1}^{k}\xi_{f(i)}\quad\text{and}\quad S^c_{f(k)}(\xi)=\sum_{i=k}^{N}\xi_{f(i)}\ .$$

with the convention: $S_{f(0)}(\xi)=S^c_{f(N+1)}(\xi)=0$.

Let Γ be a bounded subset of $\mathbb{Z}^d$. To describe this set, we define the one to one map f_Γ from $[1,|\Gamma|]\cap\mathbb{N}^*$ to Γ by: f_Γ is the unique function such that for $1\le m<n\le|\Gamma|$, $f(m)<_{\text{lex}}f(n)$.

Let Γ_n be a sequence of finite subsets of $\mathbb{Z}^d$, satisfying (2.2). We introduce the sequence of one to one maps f_{Γ_n}. In the sequel, we will omit the index Γ_n.

Notations 2. From now on, we consider a strictly stationary random field $(X_i)_{i\in\mathbb{Z}^d}$ which satisfies the condition (2.3) and $(\varepsilon_i)_{i\in\mathbb{Z}^d}$ an i.i.d. random field independent of X, such that $\varepsilon_0\sim\mathcal{N}(0,1)$ (a classical argument ensures the existence of two such fields). We introduce the two sequences of fields: $Y_i^n=|\Gamma_n|^{-1/2}X_i$ and $\gamma_i^n=|\Gamma_n|^{-1/2}\varepsilon_i\eta^{1/2}$. In the sequel, we will omit the index n.

Notations 3. Let h be any Borel function from $\mathbb{R}$ to $\mathbb{R}$. For $0\le k<l\le|\Gamma_n|+1$, we introduce: $h_{k,l}(Y)=h(S_{f(k)}(Y)+S^c_{f(l)}(\gamma))$.

With the above convention: $h_{k,|\Gamma_n|+1}(Y)=h(S_{f(k)}(Y))$ and $h_{0,l}(Y)=h(S^c_{f(l)}(\gamma))$. For sake of brevity, we will often write $h_{k,l}$ instead of $h_{k,l}(Y)$.

We denote by $B_1^4(\mathbb{R})$ the unit ball of $C_b^4(\mathbb{R})$: h belongs to $B_1^4(\mathbb{R})$ if and only if it belongs to $C^4(\mathbb{R})$ and satisfies $\max_{0\le i\le 4}\|h^{(i)}\|_\infty\le 1$.

5.1 Lindeberg's method

Let Z be a $\mathcal{I}$-measurable random variable bounded by 1. We shall prove that, under the assumptions of Theorem 1, for all h in $B_1^4(\mathbb{R})$:

$$\lim_{n\to+\infty} \mathbb{E}\big(Zh\big(|\Gamma_n|^{-1/2}S_{\Gamma_n}\big)\big) = \mathbb{E}\big(Zh(\varepsilon\eta^{1/2})\big) \ . \tag{5.1}$$

We use Lindeberg's decomposition:

$$\begin{aligned}
&\mathbb{E}\big(Z\big[h\big(|\Gamma_n|^{-1/2}S_{\Gamma_n}\big)\big) - \mathbb{E}\big(h\big(\varepsilon\eta^{1/2}\big)\big]\big) \\
&= \mathbb{E}\big(Z\big[h_{|\Gamma_n|,|\Gamma_n|+1} - h_{0,1}\big]\big) \\
&= \sum_{k=1}^{|\Gamma_n|} \mathbb{E}\big(Z\big[h_{k,k+1} - h_{k-1,k}\big]\big) \ .
\end{aligned} \tag{5.2}$$

Now:

$$h_{k,k+1} - h_{k-1,k} = h_{k,k+1} - h_{k-1,k+1} + h_{k-1,k+1} - h_{k-1,k} \ .$$

Applying Taylor's formula we get that:

$$h_{k,k+1} - h_{k-1,k+1} = Y_{f(k)}h'_{k-1,k+1} + \frac{Y^2_{f(k)}}{2}h''_{k-1,k+1} + R_k \ ,$$

and

$$h_{k-1,k+1} - h_{k-1,k} = -\gamma_{f(k)}h'_{k-1,k+1} - \frac{\gamma^2_{f(k)}}{2}h''_{k-1,k+1} + r_k \ ,$$

where $|R_k| \le Y^2_{f(k)}(1 \wedge |Y_{f(k)}|)$ and $|r_k| \le \gamma^2_{f(k)}(1 \wedge |\gamma_{f(k)}|)$.

Since $(Y, (\varepsilon_i)_{i\neq f(k)})$ is independent of $\varepsilon_{f(k)}$, it follows that $\mathbb{E}(Z\gamma_{f(k)}h'_{k-1,k+1}) = 0$, and furthermore $\mathbb{E}(Z\gamma^2_{f(k)}h''_{k-1,k+1}) = \mathbb{E}(Z|\Gamma_n|^{-1} \times\eta h''_{k-1,k+1})$. We obtain:

$$\begin{aligned}
\mathbb{E}\big(Zh\big(|\Gamma_n|^{-1/2}S_{\Gamma_n}\big)\big) - \mathbb{E}\big(Zh\big(\varepsilon\eta^{1/2}\big)\big) &= \sum_{k=1}^{|\Gamma_n|} \mathbb{E}\big(Z\big(Y_{f(k)}h'_{k-1,k+1}\big)\big) \\
&+ \sum_{k=1}^{|\Gamma_n|} \mathbb{E}\left(Z\left(Y^2_{f(k)} - \frac{\eta}{|\Gamma_n|}\right)\frac{h''_{k-1,k+1}}{2}\right) \\
&+ \sum_{k=1}^{|\Gamma_n|} \mathbb{E}(R_k + r_k) \ .
\end{aligned} \tag{5.3}$$

Arguing as in Rio (1995), it is easily proven that $\lim_{n\to+\infty}\sum_{k=1}^{|\Gamma_n|}\mathbb{E}(|R_k| + |r_k|) = 0$.

On the other hand, if we define the random variable η_N by $\eta_N = \sum_{k\in\Lambda_{N-1}}\mathbb{E}(X_0X_k|\mathscr{I})$, the following upper bound $\mathbb{E}|\eta - \eta_N| \le 2\sum_{k\in V_0^N}\mathbb{E}|\mathbb{E}(X_0X_k|\mathscr{I})|$ holds for any positive integer N. Hence according to condition (2.3), $\lim_{n\to+\infty}\mathbb{E}|\eta - \eta_N| = 0$, and consequently Theorem $1(a)$ will be proved if we show that:

$$\lim_{N\to+\infty}\limsup_{n\to+\infty}\sum_{k=1}^{|\Gamma_n|}\mathbb{E}\big(Z\big(Y_{f(k)}h'_{k-1,k+1}\big)\big) + \mathbb{E}\left(Z\left(Y_{f(k)}^2 - \frac{\eta_N}{|\Gamma_n|}\right)\frac{h''_{k-1,k+1}}{2}\right) = 0 \ . \quad (5.4)$$

5.2 First reduction

In this section, we focus on $\sum_{k=1}^{|\Gamma_n|}\mathbb{E}(Z(Y_{f(k)}h'_{k-1,k+1}))$. Since Y does not satisfy a martingale type condition, this term has a non negligible contribution.

Notations 4. For all N in $\mathbb{N}^*$ and all integer k in $[1,|\Gamma_n|]$, we define:

$$E_k^N = f([1,k]\cap\mathbb{N}^*)\cap V_{f(k)}^N \quad \text{and} \quad S_{f(k)}^N(Y) = \sum_{i\in E_k^N} Y_i \ .$$

For any Borel function g from $\mathbb{R}$ to $\mathbb{R}$, we define: $g_{k-1,l}^N = g(S_{f(k)}^N(Y) + S_{f(l)}^c(\gamma))$. (Afterwards, we shall apply this notation to the successive derivatives of the function h.)

Our aim in this section is to show that:

$$\lim_{N\to+\infty}\limsup_{n\to+\infty}\sum_{k=1}^{|\Gamma_n|}\mathbb{E}\big(Z\big(Y_{f(k)}h'_{k-1,k+1}\big) - Y_{f(k)}(S_{f(k-1)} - S_{f(k)}^N)h''_{k-1,k+1}\big) = 0 \ . \quad (5.5)$$

First we use the decomposition:

$$Y_{f(k)}h'_{k-1,k+1} = Y_{f(k)}h'^N_{k-1,k+1} + Y_{f(k)}(h'_{k-1,k+1} - h'^N_{k-1,k+1}) \ .$$

Let m be a one to one map from $[1,|E_k^N|]\cap\mathbb{N}^*$ to E_k^N such that $|m(i)-f(k)| \le |m(i-1)-f(k)|$. This choice of m ensures that $S_{m(i)}(Y)$ and $S_{m(i-1)}(Y)$ are $\mathscr{F}_{V_{f(k)}^{|m(i)-f(k)|}}$-measurable. On the other hand, the fact that γ is independant of Y together with the first result stated in Proposition 3 imply that $\mathbb{E}(ZY_{f(k)}h'(S_{f(k+1)}^c(\gamma))) = \mathbb{E}(h'(S_{f(k+1)}^c(\gamma)))$ $\times\mathbb{E}(Z\mathbb{E}(Y_{f(k)}|\mathscr{F}_{-\infty})) = 0$. Therefore:

$$\big|\mathbb{E}\big(ZY_{f(k)}h'^N_{k-1,k+1}\big)\big| = \left|\sum_{i=1}^{|E_k^N|}\mathbb{E}\big(ZY_{f(k)}\big[h'(S_{m(i)}(Y)+S_{f(k+1)}^c(\gamma)) - h'\big(S_{m(i-1)}(Y)+S_{f(k+1)}^c(\gamma)\big)\big]\big)\right| \ .$$

Since $S_{m(i)}(Y)$ and $S_{m(i-1)}(Y)$ are $\mathscr{F}_{V_{f(k)}^{|m(i)-f(k)|}}$-measurable, we can take the conditional expectation of $Y_{f(k)}$ with respect to $\mathscr{F}_{V_{f(k)}^{|m(i)-f(k)|}}$ in the right hand side of the above equation. On the other hand the function h' is 1-Lipschitz (see Notations 3), which implies that

$$\left|h'\left(S_{m(i)}(Y)+S^c_{f(k+1)}(\gamma)\right)-h'\left(S_{m(i-1)}(Y)+S^c_{f(k+1)}(\gamma)\right)\right|\le\left|Y_{m(i)}\right| \ .$$

Therefore

$$\begin{aligned}&\left|\mathbb{E}\left(ZY_{f(k)}\left[h'(S_{m(i)}(Y)+S^c_{f(k+1)}(\gamma))-h'\left(S_{m(i-1)}(Y)+S^c_{f(k+1)}(\gamma)\right)\right]\right)\right|\\&\quad\le\mathbb{E}\left|Y_{m(i)}\mathbb{E}_{|m(i)-f(k)|}(Y_{f(k)})\right| \ ,\end{aligned}$$

and

$$\left|\mathbb{E}\left(ZY_{f(k)}h'^{N}_{k-1,k+1}\right)\right|\le\sum_{i=1}^{|E_k^N|}\mathbb{E}\left|Y_{m(i)}\mathbb{E}_{|m(i)-f(k)|}(Y_{f(k)})\right| \ .$$

Consequently:

$$\begin{aligned}\left|\sum_{k=1}^{|\Gamma_n|}\mathbb{E}\left(ZY_{f(k)}h'^{N}_{k-1,k+1}\right)\right|&\le\sum_{k=1}^{|\Gamma_n|}\sum_{i=1}^{|E_k^N|}|\Gamma_n|^{-1}\mathbb{E}\left|X_{m(i)}\mathbb{E}_{|m(i)-f(k)|}(X_{f(k)})\right|\\&\le\sum_{k\in V_0^N}\mathbb{E}\left|X_k\mathbb{E}_{|k|}(X_0)\right| \ .\end{aligned}$$

Since (2.3) is realized, this last term is as small as we wish by choosing N large enough.

Applying again Taylor's formula, it remains to consider

$$Y_{f(k)}\left(h'_{k-1,k+1}-h'^{N}_{k-1,k+1}\right)=Y_{f(k)}\left(S_{f(k-1)}-S^N_{f(k)}\right)h''_{k-1,k+1}+R'_k \ ,$$

where $|R'_k|\le 2|Y_{f(k)}(S_{f(k-1)}-S^N_{f(k)})(1\wedge|S_{f(k-1)}-S^N_{f(k)}|)|$. It follows that

$$\begin{aligned}\sum_{k=1}^{|\Gamma_n|}\mathbb{E}|R'(k)|&\le 2\sum_{k=1}^{|\Gamma_n|}|\Gamma_n|^{-1}\mathbb{E}\left(|X_0|\left(\sum_{i\in\Lambda_N}|X_i|\right)\left(1\wedge|\Gamma_n|^{-1/2}\sum_{i\in\Lambda_N}|X_i|\right)\right)\\&\le 2\mathbb{E}\left(|X_0|\left(\sum_{i\in\Lambda_N}|X_i|\right)\left(1\wedge|\Gamma_n|^{-1/2}\sum_{i\in\Lambda_N}|X_i|\right)\right) \ .\end{aligned}$$

By the dominated convergence theorem, this last term converges to zero as $n\to+\infty$, and (5.5) follows.

5.3 The second order terms

It remains to control

$$W_1 = \mathbb{E}\left(Z\sum_{k=1}^{|\Gamma_n|} h''_{k-1,k+1}\left(\frac{Y_{f(k)}^2}{2} + Y_{f(k)}\left(S_{f(k-1)} - S_{f(k)}^N\right) - \frac{\eta_N}{2|\Gamma_n|}\right)\right) . \tag{5.6}$$

Notations 5. We introduce the two sets:

$$\Gamma_n^N = \{i \in \Gamma_n : d(\{i\}, \partial\Gamma_n) \geq N\} \quad \text{and}$$
$$I_n^N = \left\{1 \leq i \leq |\Gamma_n| : f(i) \in \Gamma_n^N\right\} ,$$

and the function g from $\mathbb{R}^{\mathbb{Z}^d}$ to $\mathbb{R}$ such that:

$$g(X) = X_0^2 + \sum_{i \in V_0^1 \cap \Lambda_{N-1}} 2X_0X_i .$$

For k in $[1, |\Gamma_n|]$, we set: $D_k^N = \eta_N - g \circ T_{f(k)}(X)$.

By definition of g and of the set I_n^N, we have, for any k in I_n^N:

$$g \circ T_{f(k)}(X) = X_{f(k)}^2 + 2X_{f(k)}\left(S_{f(k-1)}(X) - S_{f(k)}^N(X)\right) .$$

Therefore, for k in I_n^N:

$$|\Gamma_n|^{-1}D_k^N = |\Gamma_n|^{-1}\eta_N - Y_{f(k)}^2 - 2Y_{f(k)}\left(S_{f(k-1)}(Y) - S_{f(k)}^N(Y)\right) .$$

The assumption (2.2) ensures that $\lim_{n\to+\infty} |\Gamma_n|^{-1}|I_n^N| = 1$. Hence, it remains to prove that

$$\lim_{N\to+\infty} \limsup_{n\to+\infty} \mathbb{E}\left(Z\sum_{k=1}^{|\Gamma_n|} |\Gamma_n|^{-1}h''_{k-1,k+1}D_k^N\right) = 0 . \tag{5.7}$$

5.4 Conditional expectation with respect to the tail σ-algebra

Our aim in this section is to replace D_k^N by $\mathbb{E}(D_k^N|\mathscr{F}_{-\infty})$. We introduce the expression:

$$H_n^N = \sum_{k=1}^{|\Gamma_n|} \mathbb{E}\left(\frac{Z}{|\Gamma_n|}h''_{k-1,k+1}\left[g \circ T_{f(k)}(X) - \mathbb{E}\left(g \circ T_{f(k)}(X)|\mathscr{F}_{-\infty}\right)\right]\right) .$$

For sake of brevity, we have written $h''_{k-1,k+1}$ instead of $h''_{k-1,k+1}(Y)$. Using the stationarity of the field we get that

$$H_n^N = \sum_{k=1}^{|\Gamma_n|} \mathbb{E}\left(\frac{Z}{|\Gamma_n|}\left(h''_{k-1,k+1} \circ T_{-f(k)}\right)(Y)[g(X) - \mathbb{E}(g(X)|\mathscr{F}_{-\infty})]\right) .$$

For any positive integer p, we decompose H_n^N in two parts:

$$H_n^N = \sum_{k=1}^{|\Gamma_n|} J_k^1(p) + \sum_{k=1}^{|\Gamma_n|} J_k^2(p) \ ,$$

with

$$J_k^1(p) = \mathbb{E}\left(\frac{Z}{|\Gamma_n|}\left(h''^p_{k-1,k+1} \circ T_{-f(k)}\right)(Y)[g(X) - \mathbb{E}(g(X)|\mathscr{F}_{-\infty})]\right)$$

and

$$J_k^2(p) = \mathbb{E}\Big(\frac{Z}{|\Gamma_n|}\left[h''_{k-1,k+1} \circ T_{-f(k)} - h''^p_{k-1,k+1} \circ T_{-f(k)}\right](Y) \times [g(X) - \mathbb{E}(g(X)|\mathscr{F}_{-\infty})]\Big)$$

(cf. notations 4 for the definition of $h''^p_{k-1,k+1}$). The backward martingale theorem applied to the sequence $\mathbb{E}(g(X)|\mathscr{F}_{V_0^p})$ implies that $\lim_{p\to+\infty} \mathbb{E}|\mathbb{E}(g(X)|\mathscr{F}_{V_0^p}) - \mathbb{E}(g(X)\ |\mathscr{F}_{-\infty})| = 0$, and consequently,

$$\lim_{p\to+\infty} \limsup_{n\to+\infty} \left|\sum_{k=1}^{|\Gamma_n|} J_k^1(p)\right| = 0 \ .$$

Now,

$$\left|\sum_{k=1}^{|\Gamma_n|} J_2^k(p)\right| \leq \mathbb{E}\left[\left(2 \wedge \sum_{|i|<p} \frac{|X_i|}{|\Gamma_n|^{1/2}}\right)|g(X) - \mathbb{E}(g(X)|\mathscr{F}_{-\infty})|\right] \ .$$

Hence, applying the dominated convergence theorem, we conclude that $\lim_{n\to+\infty} H_n^N = 0$. It remains to consider:

$$W_2 = \mathbb{E}\left(Z\sum_{k=1}^{|\Gamma_n|} h''_{k-1,k+1}|\Gamma_n|^{-1}\mathbb{E}\left(D_k^N|\mathscr{F}_{-\infty}\right)\right) \ . \tag{5.8}$$

5.5 Truncation

Notations 6. For any integer k in $[1, |\Gamma_n|]$, and any M in $\mathbb{R}^+$ we introduce:

$$B_k^N(M) = \mathbb{E}\left(D_k^N|\mathscr{F}_{-\infty}\right)\mathbb{1}_{|\mathbb{E}(D_k^N|\mathscr{F}_{-\infty})|\leq M} \text{ and}$$
$$\bar{B}_k^N(M) = \mathbb{E}\left(D_k^N|\mathscr{F}_{-\infty}\right) - B_k^N(M) \ .$$

The stationarity of the field ensures that for all k in $[1, |\Gamma_n|]$, $\mathbb{E}|\bar{B}_k^N(M)| = \mathbb{E}|\bar{B}_1^N(M)|$. Now, applying the dominated convergence theorem, we have: $\lim_{M\to+\infty} \mathbb{E}|\bar{B}_1^N(M)| = 0$. It follows that

$$\lim_{M\to+\infty}\sum_{k=1}^{|\Gamma_n|}\mathbb{E}\big(h''_{k-1,k+1}|\Gamma_n|^{-1}\bar{B}_k^N(M)\big)=0\ .$$

Therefore, instead of W_2 it remains to consider:

$$W_3=\mathbb{E}\left(Z\sum_{k=1}^{|\Gamma_n|}h''_{k-1,k+1}|\Gamma_n|^{-1}B_k^N(M)\right)\ . \tag{5.9}$$

5.6 An ergodic lemma

The next result is the central point of our proof.

Lemma 2. *For all M in $\mathbb{R}_+$, we introduce*

$$\beta_N(M)=\mathbb{E}\big([\eta_N-\mathbb{E}(g(X)|\mathscr{F}_{-\infty})]\mathbb{1}_{|\eta_N-\mathbb{E}(g(X)|\mathscr{F}_{-\infty})|\le M}|\mathscr{I}\big)\ .$$

Then

$$\lim_{M\to+\infty}\beta_N(M)=0\ a.s.\quad and\quad \lim_{n\to+\infty}\mathbb{E}\left|\beta_N(M)-\frac{1}{|\Gamma_n|}\sum_{k=1}^{|\Gamma_n|}B_k^N(M)\right|=0\ .$$

Proof of Lemma 2. Let $u(X)=[\eta_N-\mathbb{E}(g(X)|\mathscr{F}_{-\infty})]\mathbb{1}_{|\eta_N-\mathbb{E}(g(X)|\mathscr{F}_{-\infty})|\le M}$. Using the function u, we write $\beta_N(M)=\mathbb{E}(u(X)|\mathscr{I})$. The fact that $\lim_{M\to+\infty}\beta_N(M)=0$ follows from the dominated convergence theorem. As a matter of fact, $\lim_{M\to+\infty}u(X)=\eta_N-\mathbb{E}(g(X)|\mathscr{F}_{-\infty})$, and $u(X)$ is bounded by $|\eta_N-\mathbb{E}(g(X)|\mathscr{F}_{-\infty})|$, which belongs to $\mathbb{L}^1$. This implies that:

$$\lim_{M\to+\infty}\beta_N(M)=\mathbb{E}(\eta_N-\mathbb{E}(g(X)|\mathscr{F}_{-\infty})|\mathscr{I})\ a.s.$$

Since $\mathscr{I}$ is included in the $\mathbb{P}$-completion of $\mathscr{F}_{-\infty}$ (see Proposition 1), and bearing in mind that η_N is $\mathscr{I}$-measurable, it follows that $\lim_{M\to+\infty}\beta_N(M)=\eta_N-\mathbb{E}(g(X)|\mathscr{I})$ *a.s.* By stationarity of the random field, $\mathbb{E}(X_0X_k|\mathscr{I})=\mathbb{E}(X_0X_{-k}|\mathscr{I})$, which implies that

$$\mathbb{E}(g(X)|\mathscr{I})=\sum_{k\in\Lambda_{N-1}}\mathbb{E}(X_0X_k|\mathscr{I})=\eta_N$$

and the result follows.

To prove the second point of Lemma 2, we apply the $\mathbb{L}^1$-ergodic theorem. First note that

$$\begin{aligned}B_k^N(M)&=\big[\eta_N-\mathbb{E}\big(g\circ T_{f(k)}(X)|\mathscr{F}_{-\infty}\big)\big]\mathbb{1}_{|\eta_N-\mathbb{E}(g\circ T_{f(k)}(X)|\mathscr{F}_{-\infty})|\le M}\\&=u\circ T_{f(k)}(X)\ .\end{aligned}$$

Consequently: $\sum_{k=1}^{|\Gamma_n|} B_k^N(M) = \sum_{i\in\Gamma_n} u \circ T_i(X)$, and the $\mathbb{L}^1$-ergodic theorem ensures that $|\Gamma_n|^{-1}\sum_{i\in\Gamma_n} u \circ T_i(X)$ converges in $\mathbb{L}^1$ to $\mathbb{E}(u(X)|\mathscr{I})$. This means exactly that

$$\lim_{n\to+\infty} \mathbb{E}\left|\beta_N(M) - \frac{1}{|\Gamma_n|}\sum_{k=1}^{|\Gamma_n|} B_k^N(M)\right| = 0 \ ,$$

and the proof of Lemma 2 is complete.

As a direct application of this lemma, we see that:

$$\left|\mathbb{E}\left(Z\sum_{k=1}^{|\Gamma_n|} h''_{k-1,k+1}\frac{\beta_N(M)}{|\Gamma_n|}\right)\right| \le \mathbb{E}|\beta_N(M)| \ ,$$

is as small as we wish, by choosing M large enough. So, instead of W_3, we consider:

$$W_4 = \mathbb{E}\left(Z\sum_{k=1}^{|\Gamma_n|} h''_{k-1,k+1}\frac{\left[B_k^N(M) - \beta_N(M)\right]}{|\Gamma_n|}\right) \ . \tag{5.10}$$

5.7 Abel transformation

$$W_4 = \mathbb{E}\left[\sum_{k=1}^{|\Gamma_n|}\left(\sum_{i=1}^{k}\frac{\left[B_i^N(M) - \beta_N(M)\right]}{|\Gamma_n|}\right)Z\left(h''_{k-1,k+1} - h''_{k,k+2}\right)\right]$$

$$+ \mathbb{E}\left(Zh''_{|\Gamma_n|,|\Gamma_n|+2}\sum_{k=1}^{|\Gamma_n|}\frac{\left[B_k^N(M) - \beta_N(M)\right]}{|\Gamma_n|}\right) \ .$$

Now

$$\left|\mathbb{E}\left(Zh''_{|\Gamma_n|,|\Gamma_n|+2}\sum_{k=1}^{|\Gamma_n|}\frac{\left[B_k^N(M) - \beta_N(M)\right]}{|\Gamma_n|}\right)\right|$$

$$\le \mathbb{E}\left|\beta_N(M) - \frac{1}{|\Gamma_n|}\sum_{k=1}^{|\Gamma_n|} B_k^N(M)\right| \ .$$

Then, applying Lemma 2, we get that:

$$\lim_{n\to+\infty}\left|\mathbb{E}\left(Zh''_{|\Gamma_n|,|\Gamma_n|+2}\sum_{k=1}^{|\Gamma_n|}\frac{\left[B_k^N(M) - \beta_N(M)\right]}{|\Gamma_n|}\right)\right| = 0 \ .$$

Therefore it remains to prove that, for any positive integer N and any positive real M,

$$\lim_{n\to+\infty}\mathbb{E}\left[\sum_{k=1}^{|\Gamma_n|}\left(\sum_{i=1}^{k}\frac{\left[B_i^N(M)-\beta_N(M)\right]}{|\Gamma_n|}\right)Z\left(h''_{k-1,k+1}-h''_{k,k+2}\right)\right]=0\ . \tag{5.11}$$

5.8 Last reductions

We use the same decomposition as in Section 5.1:

$$h''_{k,k+2}-h''_{k-1,k+1}=h''_{k,k+2}-h''_{k,k+1}+h''_{k,k+1}-h''_{k-1,k+1}\ .$$

Applying Taylor's formula:

$$h''_{k,k+2}-h''_{k,k+1}=-\gamma_{f(k+1)}h'''_{k,k+2}+s_k \quad \text{and}$$
$$h''_{k,k+1}-h''_{k-1,k+1}=Y_{f(k)}h'''_{k-1,k+1}+S_k\ ,$$

where $|s_k|\le\gamma^2_{f(k+1)}$ and $|S_k|\le Y^2_{f(k)}$. To examine the remainder terms, we consider:

$$\mathbb{E}\left(\sum_{k=1}^{|\Gamma_n|}\frac{1}{|\Gamma_n|}\left(\sum_{i=1}^{k}\frac{\left[B_i^N(M)-\beta_N(M)\right]}{|\Gamma_n|}\right)ZX^2_{f(k)}\right)\ .$$

The definition of $B_i^N(M)$ and of $\beta_N(M)$ enables us to write, for all integer k in $[1,|\Gamma_n|]$,

$$\sum_{i=1}^{k}|B_i^N(M)-\beta_N(M)|\le 2M|\Gamma_n|\ .$$

Therefore:

$$\begin{aligned}\mathbb{E}&\left|\sum_{k=1}^{|\Gamma_n|}\frac{1}{|\Gamma_n|}\left(\sum_{i=1}^{k}\frac{\left[B_i^N(M)-\beta_N(M)\right]}{|\Gamma_n|}\right)ZX^2_{f(k)}\mathbb{1}_{|X_{f(k)}|>K}\right|\\&\le 2M\mathbb{E}(X_0^2\mathbb{1}_{|X_0|>K})\ ,\end{aligned}$$

and, applying the dominated convergence theorem, this last term is as small as we wish by choosing K large enough. Now, for all K in $\mathbb{R}_+$, Lemma 2 ensures that:

$$\lim_{n\to+\infty}\mathbb{E}\left(\sum_{k=1}^{|\Gamma_n|}\frac{1}{|\Gamma_n|}\left(\sum_{i=1}^{k}\frac{\left[B_i^N(M)-\beta_N(M)\right]}{|\Gamma_n|}\right)ZX^2_{f(k)}\mathbb{1}_{|X_{f(k)}|\le K}\right)=0\ .$$

So, we have proved that

$$\lim_{n\to+\infty}\mathbb{E}\left(\sum_{k=1}^{|\Gamma_n|}\left(\sum_{i=1}^{k}\frac{\left[B_i^N(M)-\beta_N(M)\right]}{|\Gamma_n|}\right)ZS_k\right)=0\ .$$

In the same way, we obtain that

$$\lim_{n\to+\infty}\mathbb{E}\left(\sum_{k=1}^{|\Gamma_n|}\left(\sum_{i=1}^{k}\frac{\left[B_i^N(M)-\beta_N(M)\right]}{|\Gamma_n|}\right)Zs_k\right)=0\ .$$

Moreover, since $(X,(\varepsilon_i)_{i\neq f(k+1)})$ is independent of $\varepsilon_{f(k+1)}$ we have:

$$\mathbb{E}\left(\left(\sum_{i=1}^{k}\frac{\left[B_i^N(M)-\beta_N(M)\right]}{|\Gamma_n|}\right)\gamma_{f(k+1)}Zh'''_{k,k+2}\right)=0\ .$$

Finally, it remains to consider:

$$W_5=\mathbb{E}\left[\sum_{k=1}^{|\Gamma_n|}\left(\sum_{i=1}^{k}\frac{\left[B_i^N(M)-\beta_N(M)\right]}{|\Gamma_n|}\right)ZY_{f(k)}h'''_{k-1,k+1}\right]\ . \qquad (5.12)$$

Let p be a fixed positive integer. Since $|h'''_{k-1,k+1}-h'''^{p}_{k-1,k+1}|\leq|S_{f(k-1)}(Y)-S^p_{f(k)}(Y)|$, we can apply the same truncation argument as before: first we choose the level of our truncation by applying the dominated convergence theorem, and then we use Lemma 2. So, it follows that

$$\lim_{n\to+\infty}\mathbb{E}\left[\sum_{k=1}^{|\Gamma_n|}\left(\sum_{i=1}^{k}\frac{\left[B_i^N(M)-\beta_N(M)\right]}{|\Gamma_n|}\right)ZY_{f(k)}\left(h'''_{k-1,k+1}-h'''^{p}_{k-1,k+1}\right)\right]=0\ .$$

Therefore, to prove Theorem 1(a) it is enough to show that:

$$\lim_{p\to+\infty}\limsup_{n\to+\infty}\mathbb{E}\left[\sum_{k=1}^{|\Gamma_n|}\left(\sum_{i=1}^{k}\frac{\left[B_i^N(M)-\beta_N(M)\right]}{|\Gamma_n|}\right)ZY_{f(k)}h'''^{p}_{k-1,k+1}\right]=0\ . \qquad (5.13)$$

Let m be a one to one map from $[1,|E_k^p|]\cap\mathbb{N}^*$ to E_k^p, satisfying $|m(i)-f(k)|\leq|m(i-1)-f(k)|$. We use the same argument as in Section 5.2:

$$h'''^{p}_{k-1,k+1}-h'''\left(S^c_{f(k)}(\gamma)\right)=\sum_{i=1}^{|E_k^p|}h'''\left(S_{m(i)}(Y)+S^c_{f(k)}(\gamma)\right) - h'''\left(S_{m(i-1)}(Y)+S^c_{f(k)}(\gamma)\right)\ .$$

Since $B_i^N(M)-\beta_N(M)$ is $\mathscr{F}_{-\infty}$-measurable, Proposition 3 implies that

$$\mathbb{E}\left(\frac{\left[B_i^N(M)-\beta_N(M)\right]}{|\Gamma_n|}Z\frac{X_{f(k)}}{|\Gamma_n|^{1/2}}h'''\left(S^c_{f(k)}(\gamma)\right)\right)=0\ .$$

Therefore, using the conditional expectation, we find:

$$\mathbb{E}\left[\sum_{k=1}^{|\Gamma_n|}\left(\sum_{i=1}^{k}\frac{\left[B_i^N(M)-\beta_N(M)\right]}{|\Gamma_n|}\right)ZY_{f(k)}h_{k-1,k+1}^{\prime\prime\prime p}\right]$$
$$\leq 2M\sum_{k=1}^{|\Gamma_n|}\sum_{i=1}^{|E_k^p|}\mathbb{E}|\frac{1}{|\Gamma_n|}X_{m(i)}\mathbb{E}_{|m(i)-f(k)|}(X_{f(k)})|$$
$$\leq 2M\sum_{k\in V_0^p}\mathbb{E}|X_k\mathbb{E}_{|k|}(X_0)| \ .$$

Since (2.3) is realized the last term is as small as we wish, by choosing p large enough. Hence (5.11) holds, which ends up the control of W_4.

Finally we have proved (5.1), and the proof of Theorem 1(a) is complete.

6 Proof of Theorem 1(b)

Obviously, instead of A_n, we can consider:

$$A'_n = |\Gamma_n|^{-1}\sum_{(i,j)\in G_{\rho_n}} X_iX_j \ .$$

So we have to prove that under conditions (2.3) and (3.1), A'_n is a consistent estimator of η.

For any positive integer N, put:

$$A_n^N = \frac{1}{|\Gamma_n|}\sum_{(i,j)\in G_N} X_iX_j \quad \text{and} \quad \eta_{N+1} = \sum_{k\in\Lambda_N}\mathbb{E}(X_0X_k|\mathscr{I}) \ .$$

First we need to prove that A_n^N is a consistent estimator of η_{N+1}. Clearly:

$$A_n^N = \sum_{k\in\Lambda_N}\frac{1}{|\Gamma_n|}\left(\sum_{i\in\Gamma_n\cap(\Gamma_n-k)} X_iX_{i+k}\right) \ .$$

The assumption (2.2) implies that $\lim_{n\to+\infty}|\Gamma_n|^{-1}|\Gamma_n\cap(\Gamma_n-k)|=1$, and the $\mathbb{L}^1$-ergodic theorem enables us to conclude that A_n^N converges to η_{N+1} in $\mathbb{L}^1$.

The second step is to compare A'_n to A_n^N. Since $\mathbb{E}|\eta-\eta_{N+1}|$ is asymptotically negligible, the consistency of A'_n will be established if we prove that for any positive number δ:

$$\lim_{N\to+\infty}\limsup_{n\to+\infty}\mathbb{P}\left(|A'_n - A_n^N|>\delta\right)=0 \ . \qquad (6.1)$$

In order to prove (6.1) we shall adapt Lindeberg's method to our context.

Notation 7. Let the function φ be defined by $\varphi'(0) = \varphi(0) = 0$ and $\varphi''(t) = (1 - |t|)\mathbb{1}_{|t|<1}$.

To study $\mathbb{P}(|A'_n - A_n^N| > \delta)$ we use the function φ. Since φ is an even function, increasing from $\mathbb{R}_+$ to $\mathbb{R}_+$, (6.1) follows from the assertion

$$\lim_{N\to+\infty} \limsup_{n\to+\infty} \mathbb{E}\big(\varphi(A'_n - A_n^N)\big) = 0 \ . \tag{6.2}$$

Notations 8. For i in $\mathbb{Z}^d$ let us introduce the sets $B_i^n(N) = \{j \in V_i^1 \cap \Gamma_n \ : \ N < |j - i| \le \rho_n\}$. Bearing in mind notations 1, we consider the one to one maps $f = f_{\Gamma_n}$ and $g_k = f_{B^n_{f(k)}(N)}$.

For any integer j in $[1, |\Gamma_n|]$ and any integer l in $[0, |B^n_{f(j)}(N)|]$, we define:

$$\Delta_{j,l} = \left(\bigcup_{k=1}^{j-1} \{k\} \times \left([0, |B^n_{f(k)}(N)|] \cap \mathbb{N}^*\right)\right) \bigcup \left(\{j\} \times ([0, l] \cap \mathbb{N}^*)\right) \ ,$$

with the convention: $\Delta_{0,l} = \emptyset$.

Let Δ be any subset of $\Delta_{|\Gamma_n|,|B^n_{f(|\Gamma_n|)}(N)|}$. We set:

$$D_\Delta = 2 \sum_{(p,q)\in\Delta} \frac{X_{f(p)} X_{g_p(q)}}{|\Gamma_n|} \ .$$

Clearly, with the notations above, if $\rho_n > N$

$$A'_n - A_n^N = \frac{2}{|\Gamma_n|} \sum_{i\in\Gamma_n} \sum_{j\in B_i^n(N)} X_i X_j \ .$$

To prove (6.2) we introduce the decomposition below:

$$\mathbb{E}\big(\varphi(A'_n - A_n^N)\big) = \sum_{j=1}^{|\Gamma_n|} \sum_{l=1}^{|B^n_{f(j)}(N)|} \mathbb{E}\big(\varphi\big(D_{\Delta_{j,l}}\big)\big) - \mathbb{E}(\varphi(D_{\Delta_{j,l-1}})) \ .$$

The definition of φ'' an φ' ensures that: $\|\varphi''\|_\infty = 1$ and $\|\varphi'\|_\infty = 1/2$. Hence, applying Taylor's formula, $|\varphi(x+h) - \varphi(x) - h\varphi'(x)| \le |h|(1 \wedge |h|)$. Therefore:

$$\begin{aligned}\mathbb{E}\big(\varphi(A'_n - A_n^N)\big) \le& \sum_{j=1}^{|\Gamma_n|} \sum_{l=1}^{|B^n_{f(j)}(N)|} 2\left|\mathbb{E}\left(\varphi'\big(D_{\Delta_{j,l-1}}\big) \frac{X_{f(j)} X_{g_j(l)}}{|\Gamma_n|}\right)\right| \\ &+ \sum_{j=1}^{|\Gamma_n|} \sum_{l=1}^{|B^n_{f(j)}(N)|} \frac{2}{|\Gamma_n|} \mathbb{E}\left|X_{f(j)} X_{g_j(l)} \left(1 \wedge \frac{|X_{f(j)} X_{g_j(l)}|}{|\Gamma_n|}\right)\right| \ .\end{aligned} \tag{6.3}$$

Control of the main term.

Notations 9. For any integer j in $[1,|\Gamma_n|]$ and any integer l in $[1,|B^n_{f(j)}(N)|]$, we define:

$C_{j,l}=\{(p,q)\in\Delta_{j,l-1}:\min(|f(p)-f(j)|,|g_p(q)-f(j)|)<\rho_n\}$ and $C^c_{j,l}=\Delta_{j-1,l}\backslash C_{j,l}$.

With these notations,

$$\mathbb{E}\left(\varphi'(D_{\Delta_{j,l-1}})\frac{X_{f(j)}X_{g_j(l)}}{|\Gamma_n|}\right)=\mathbb{E}\left(\varphi'(D_{C^c_{j,l}})\frac{X_{f(j)}X_{g_j(l)}}{|\Gamma_n|}\right)+\mathbb{E}\left((\varphi'(D_{\Delta_{j,l-1}})-\varphi'(D_{C^c_{j,l}}))\frac{X_{f(j)}X_{g_j(l)}}{|\Gamma_n|}\right) .$$

Bearing in mind that $\|\varphi''\|_\infty\le 1$, it follows:

$$\left|\mathbb{E}\left(\varphi'(D_{\Delta_{j,l-1}})\frac{X_{f(j)}X_{g_j(l)}}{|\Gamma_n|}\right)\right|\le\left|\mathbb{E}\left(\varphi'(D_{C^c_{j,l}})\frac{X_{f(j)}X_{g_j(l)}}{|\Gamma_n|}\right)\right|+\mathbb{E}\left|\frac{X_{f(j)}X_{g_j(l)}}{|\Gamma_n|}(1\wedge|D_{C_{j,l}}|)\right| .$$

First of all, we focus on the first term of the right hand inequality. Since $\varphi'(D_{C^c_{j,l}})$ is $\mathscr{F}_{V_{f(j)}^{|g_j(l)-f(j)|}}$-measurable, the following inequality holds:

$$\left|\mathbb{E}\left(\varphi'(D_{C^c_{j,l}})\frac{X_{f(j)}X_{g_j(l)}}{|\Gamma_n|}\right)\right|\le\mathbb{E}\left|\frac{X_{g_j(l)}}{|\Gamma_n|}\mathbb{E}_{|g_i(l)-f(j)|}(X_{f(j)})\right|$$
$$\le\mathbb{E}\left|\frac{X_{g_j(l)-f(j)}}{|\Gamma_n|}\mathbb{E}_{|g_j(l)-f(j)|}(X_0)\right| .$$

Hence, summing in j,l, we get that:

$$\sum_{j=1}^{|\Gamma_n|}\sum_{l=1}^{|B^n_{f(j)}(N)|}\left|\mathbb{E}\left(\varphi'(D_{C^c_{j,l}})\frac{X_{f(j)}X_{g_j(l)}}{|\Gamma_n|}\right)\right|$$
$$\le\sum_{k\in\Lambda_{\rho_n}\cap V_0^N}\mathbb{E}|X_k\mathbb{E}_k(X_0)|\le\sum_{k\in V_0^N}\mathbb{E}|X_k\mathbb{E}_k(X_0)| .$$

This last term is as small as we wish, by choosing N large enough.

So, it remains to consider:

$$\mathbb{E}\left|\frac{X_{f(j)}X_{g_i(l)}}{|\Gamma_n|}(1\wedge(|D_{C_{j,l}}|)\right| .$$

If $(p,q)\in C_{j,l}$ then $|g_p(q)-f(p)|\le\rho_n$ and $|f(p)-f(j)|\le 2\rho_n$. This implies:

$$\mathbb{E}\left|\frac{X_{f(j)}X_{g_j(l)}}{|\Gamma_n|}\left(1\wedge|D_{C_{j,l}}|\right)\right|$$
$$\leq \sum_{r-f(j)\in\Lambda_{2\rho_n}}\sum_{s\in\Lambda_{\rho_n}}\mathbb{E}\left(\left(1\wedge\frac{2|X_rX_{r+s}|}{|\Gamma_n|}\right)\frac{|X_{f(j)}X_{g_j(l)}|}{|\Gamma_n|}\right) .$$

By the stationarity of the field, it follows:

$$\sum_{l=1}^{|B^n_{f(j)}(N)|}\mathbb{E}\left|\frac{X_{f(j)}X_{g_j(l)}}{|\Gamma_n|}\left(1\wedge|D_{C_{j,l}}|\right)\right|$$
$$\leq \sum_{k\in\Lambda_{\rho_n}}\sum_{r\in\Lambda_{2\rho_n}}\sum_{s\in\Lambda_{\rho_n}}\mathbb{E}\left(\left(1\wedge\frac{2|X_rX_{r+s}|}{|\Gamma_n|}\right)\frac{|X_0X_k|}{|\Gamma_n|}\right) . \tag{6.4}$$

To conclude this section, we need the following lemma which will be proved in Appendix:

Lemma 3. *Let* X_1, X_2, X_3, X_4 *be identically distributed real random variables. Then*:

$$\mathbb{E}(|X_1X_2|(1\wedge 2|X_3X_4|))\leq 2\mathbb{E}(X_1^2(1\wedge X_1^2)) .$$

By (6.4) and Lemma 3,

$$\sum_{j=1}^{|\Gamma_n|}\sum_{l=1}^{|B^n_{f(j)}(N)|}\mathbb{E}\left|\frac{X_{f(j)}X_{g_j(l)}}{|\Gamma_n|}\left(1\wedge|D_{C_{j,l}}|\right)\right|\leq 2|\Lambda_{2\rho_n}|^3\mathbb{E}\left(X_0^2\left(1\wedge\frac{X_0^2}{|\Gamma_n|}\right)\right) .$$

Now, $|\Lambda_{2\rho_n}|^3=(4\rho_n+1)^{3d}$ and condition (3.1) implies that $2\mathbb{E}(X_0^2(1\wedge X_0^2/|\Gamma_n|))|\Lambda_{2\rho(n)}|^3$ converges to 0 as $n\to+\infty$. This ensures the control of the main term.

To complete the proof, we need to control the second term of the right hand inequality (6.3). By Lemma 3 again,

$$\sum_{j=1}^{|\Gamma_n|}\sum_{l=1}^{|B^n_{f(j)}(N)|}\frac{1}{|\Gamma_n|}\mathbb{E}\left|X_{f(j)}X_{g_j(l)}\left(1\wedge\frac{2|X_{f(j)}X_{g_j(l)}|}{|\Gamma_n|}\right)\right|$$
$$\leq 2|\Lambda_{\rho_n}|\mathbb{E}\left(X_0^2\left(1\wedge\frac{X_0^2}{|\Gamma_n|}\right)\right) ,$$

and the choice of ρ_n implies the asymptotic negligibility of this term. Hence (6.2) holds, which implies the consistency of A_n.

7 Appendix

Proof of Proposition 4. Let X_1 and X_2 be two positive random variables with $\sigma(X_1)$ and $\sigma(X_2)$ independent of $\mathscr{F}$. Then almost surely,

$$\mathrm{Cov}(X_1, X_2|\mathscr{F}) = \int_0^{+\infty}\int_0^{+\infty} \mathbb{P}(X_1 > t, X_2 > s|\mathscr{F}) - \mathbb{P}(X_1 > t)\mathbb{P}(X_2 > s)\, ds\, dt \ .$$

Clearly $X = X_+ - X_-$ and $Y = Y_+ - Y_-$, where $X_+ = (X \wedge 0)$ and $X_- = -(X \vee 0)$. Hence,

$$|\mathrm{Cov}(X, Y|\mathscr{F}) \leq |\mathrm{Cov}(X_+, Y_+|\mathscr{F})| + |\mathrm{Cov}(X_-, Y_-|\mathscr{F})| + |\mathrm{Cov}(X_-, Y_+|\mathscr{F})| + |\mathrm{Cov}(X_+, Y_-|\mathscr{F})| \ .$$

To control $|\mathrm{Cov}(X_+, Y_+|\mathscr{F})|$, we note that:

$$\begin{aligned} &|\mathbb{P}(X_+ > t, Y_+ > s|\mathscr{F}) - \mathbb{P}(X_+ > t)\mathbb{P}(Y_+ > s)| \\ &= |\mathbb{E}(\mathbb{1}_{X_+ > t}(\mathbb{E}(\mathbb{1}_{Y_+ > s}|\mathscr{F} \vee \mathscr{U}) - \mathbb{P}(Y_+ > s))|\mathscr{F})| \\ &\leq \mathbb{P}(X_+ > t)\phi(\mathscr{F} \vee \mathscr{U}, \mathscr{V}) \quad \text{a.s.} \end{aligned}$$

In the same way:

$$\begin{aligned} &|\mathbb{P}(X_+ > t, Y_+ > s|\mathscr{F}) - \mathbb{P}(X_+ > t)\mathbb{P}(Y_+ > s)| \\ &\leq \mathbb{P}(Y_+ > t)\phi(\mathscr{F} \vee \mathscr{V}, \mathscr{U}) \quad \text{a.s.} \end{aligned}$$

Hence,

$$\begin{aligned} &|\mathbb{P}(X_+ > t, Y_+ > s|\mathscr{F}) - \mathbb{P}(X_+ > t)\mathbb{P}(Y_+ > s)| \\ &\leq \mathbb{P}(X_+ > t)\phi(\mathscr{F} \vee \mathscr{U}, \mathscr{V}) \wedge \mathbb{P}(Y_+ > t)\phi(\mathscr{F} \vee \mathscr{V}, \mathscr{U}) \ . \end{aligned}$$

The same inequalities hold for (X_-, Y_-), (X_-, Y_+) and (X_+, Y_-). Those inequalities together with the fact that $x_+ \wedge y_+ + x_+ \wedge y_- + x_- \wedge y_+ + x_- \wedge y_- \leq 2(x_+ + x_-) \wedge (y_+ + y_-)$ yield:

$$|\mathrm{Cov}(X, Y|\mathscr{F})| \leq 2\int_0^{+\infty}\int_0^{+\infty} \mathbb{P}(|X| > t)\phi(\mathscr{F} \vee \mathscr{U}, \mathscr{V}) \wedge \mathbb{P}(|Y| > t)\phi(\mathscr{F} \vee \mathscr{V}, \mathscr{U})\, ds\, dt$$

We set $a = \phi(\mathscr{F} \vee \mathscr{U}, \mathscr{V})$, $b = \phi(\mathscr{F} \vee \mathscr{V}, \mathscr{U})$ and $H_X(t) = \mathbb{P}(|X| > t)$. Bearing in mind the definition of Q_X as the inverse cadlag of H_X, it follows:

$$
\begin{aligned}
|\mathrm{Cov}(X, Y|\mathscr{F})| &\le 2\int_0^{+\infty}\int_0^{+\infty} aH_X(t)\wedge bH_Y(s)\,ds\,dt \\
&\le 2\int_0^{+\infty}\int_0^{+\infty}\int_0^{a\wedge b} \mathbb{1}_{u<aH_X(t)}\mathbb{1}_{u<bH_Y(s)}\,du\,ds\,dt \\
&\le 2\int_0^{+\infty}\int_0^{+\infty}\int_0^{a\wedge b} \mathbb{1}_{Q_X(\frac{u}{a})>t}\mathbb{1}_{Q_Y(\frac{u}{b})>s}\,du\,ds\,dt \\
&\le 2\int_0^{a\wedge b} Q_X\Big(\frac{u}{a}\Big)Q_Y\Big(\frac{u}{b}\Big)du \quad \text{a.s.}
\end{aligned}
$$

So, applying Hölder's inequality:

$$
\begin{aligned}
|\mathrm{Cov}(X, Y|\mathscr{F})| &\le 2\left(\int_0^a Q_X^p\Big(\frac{u}{a}\Big)du\right)^{1/p}\left(\int_0^b Q_Y^q\Big(\frac{u}{b}\Big)du\right)^{1/q} \\
&\le 2\left(\int_0^1 aQ_X^p(u)du\right)^{1/p}\left(\int_0^1 bQ_Y^q(u)du\right)^{1/q} \\
&\le 2\phi^{1/p}(\mathscr{F}\vee\mathscr{U}, \mathscr{V})\phi^{1/q}(\mathscr{F}\vee\mathscr{V}, \mathscr{U})\|X\|_p\|Y\|_q \quad \text{a.s.}
\end{aligned}
$$

Proof of Lemma 3. Since $2|ab| \le (a^2 + b^2)$ and $(1\wedge(a^2+b^2)) \le (1\wedge a^2) + (1\wedge b^2)$, we have:

$$
\begin{aligned}
2\mathbb{E}(|X_1X_2|(1\wedge 2|X_3X_4|)) &\le \mathbb{E}(X_1^2(1\wedge X_3^2)) + \mathbb{E}(X_1^2(1\wedge X_4^2)) \\
&\quad + \mathbb{E}(X_2^2(1\wedge X_3^2)) + \mathbb{E}(X_2^2(1\wedge X_4^2)) \ .
\end{aligned}
$$

Now let us recall a result due to Fréchet (1957): if Z and T are two positive random variables, then $\mathbb{E}(ZT) \le \int_0^1 Q_Z(u)Q_T(u)du$, where Q_Z is the inverse cadlag of the tail function H_Z: $t \to \mathbb{P}(Z > t)$. Therefore:

$$
\mathbb{E}(X_1^2(1\wedge X_3^2)) \le \int_0^1 Q_{X_1^2}(u)(1\wedge Q_{X_1^2}(u))du = \mathbb{E}(X_1^2(1\wedge X_1^2)) \ ,
$$

and the lemma easily follows.

References

[1] Billingsley, P.: The Lindeberg-Lévy theorem for martingales, Proc. Amer. Math. Soc. **12**, 788–792 (1961)

[2] Bolthausen, E.: On the central limit theorem for stationary mixing random fields, Ann. Probab. **10**, 1047–1050 (1982)

[3] Comets, F. and Janžura, M.: A central limit theorem for conditionally centered random fields with an application to Markov fields, Prépublication de l'université Paris 7 (1995)

[4] Doukhan, P.: Mixing: Properties and Examples. Lecture Notes in Statist. Vol. **85**, (Springer, Berlin, 1994)

[5] Dürr, D. and Goldstein, S.: Remarks on the CLT for weakly dependent random variables. Lecture Notes in Math. No. 1158 (Springer, Berlin, 1984)

[6] Eagleason, G.K.: Martingale convergence to mixture of infinitely divisible laws, Ann. Probab. **3**, 557–562 (1975)

[7] Fréchet, M.: Sur la distance de deux lois de probabilité, C.R Acad. Sci. Paris. 244, No **6**, 689–692 (1957)

[8] Georgii, H.O.: Gibbs Measures and Phase Transitions (De Gruyter, Berlin, 1988)

[9] Gordin, M.I.: The central limit theorem for stationary processes, Soviet Math. Dokl. **10**, 1174–1176 (1969)

[10] Gordin, M.I.: Abstracts of Communication, T.1:A-K, International Conference on Probability Theory, Vilnius (1973)

[11] Gordin, M.I.: Homoclinic approach to the central limit theorem for dynamical systems, Contempt. Math. **149**, 149–162 (American mathematical society, 1993)

[12] Guyon, X.: Champs aléatoires sur un réseau, (Masson, Paris, 1993)

[13] Guyon, X. and Künsch, H.R.: Asymptotic comparison of estimators in the Ising model, In Stochastic Models, Statistical Methods and Algorithms in Image Analysis. Lecture Notes in Statist. **74**, 177–198 (Springer, New York, 1992)

[14] Hall, P. and Heyde, C.C.: Martingale Limit Theory and its Applications (Academic Press, New York, 1980)

[15] Ibragimov, I.A.: Some limit theorems for stationary processes, Theory Probab. Appl. **7**, 349–382 (1962)

[16] Ibragimov,I.A.: A central limit theorem for a class of dependent random variables, Theory Probab. Appl. **8**, 83–89 (1963)

[17] Ibragimov, I.A. and Linnik, Yu.V.: Independent and Stationary Sequences of Random Variables (Wolters-Noordhoff, Groningen, 1971)

[18] Janžura, M. and Lachout, P: A central limit theorem for stationary random fields, Mathematical Method of Statistics **4**, 463–471 (1995)

[19] Jensen, J.L. and Künsch, H.R.: On asymptotic normality of pseudo likehood estimates for pairwise interaction processes, Ann. Inst. Statist. Math. **46**, 475–486 (1994)

[20] Lindeberg, J.W.: Eine neue Herleitung des Exponentialgezetzes in der Wahrscheinlichkeitsrechnung, Mathematische Zeitschrift **15**, 211–225 (1922)

[21] Nahapetian, B.: Billingsley-Ibragimov theorem for martingale-difference random fields and its applications to some models of classical statistical physics. C. R. Acad. Sci. Paris. **320**, Série I. 1539–1544 (1995)

[22] Nahapetian, B. and Petrosian, A.N.: Martingale-difference Gibbs random fields and central limit theorem. Ann. Acad. Sci. Fennicae, Series A-I. **17**, 105–110 (1992)

[23] Peligrad, M: A note on two measures of dependence and mixing sequences, Adv. Appl. Prob. **15**, 461–464 (1983)

[24] Perera, G: Geometry of Z^d and when does the Central Limit Theorem hold for weakly dependent random fields, Preprint, Centro de Matematica, Universidad de la Republica Uruguay (1996)

[25] Rényi, A: On stable sequences of events, Sankhyā Ser. A **25**, 189–206 (1963)

[26] Rio, E: Covariance inequalities for strongly mixing processes, Ann. Inst. Henri Poincaré Vol **29. 4**, 587–597 (1993)

[27] Rio, E: About the Lindeberg method for strongly mixing sequences, ESAIM. Vol. **1**, 35–61 (1995)

[28] Stein, C: A bound for the error in the normal approximation of a sum of dependent random variables, Proc. Sixth Berkeley Symp. Math. Statist. and Probab. **2**, 583–602 (1973)
[29] Volný, D: Approximating martingales and the central limit theorem for strictly stationary processes, Stoch. Processes Appl. **44**, 41–74 (1993)