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Long-time existence for signed solutions of the heat equation with a noise term

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Summary. Let $\mathbb{I}$ be the circle $[0, J]$ with the ends identified. We prove long-time existence for the following equation.

$$\begin{aligned} u_t &= u_{xx} + g(u)\dot{W} \quad , \qquad t > 0, \ \ x \in \mathbb{I} \\ u(0, x) &= u_0(x) \end{aligned}$$

Here, $\dot{W} = \dot{W}(t, x)$ is 2-parameter white noise, and we assume that $u_0(x)$ is a continuous function on $\mathbb{I}$. We show that if $g(u)$ grows no faster than $C_0(1 + |u|)^\gamma$ for some $\gamma < 3/2$, $C_0 > 0$, then this equation has a unique solution $u(t, x)$ valid for all times $t > 0$.

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1. Introduction

It is well known that stochastic differential equations with Lipschitz coefficients have unique solutions valid for all time. The same statement is usually also true for stochastic partial differential equations (SPDE). Indeed, sometimes one can go beyond Lipschitz coefficients.

In this paper, we will only deal with SPDE driven by a 2-parameter white noise $\dot{W} = \dot{W}(t, x)$. In this context, it was shown in [Mue91] that the following equation has a unique solution valid for all time.

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$$\begin{aligned} \bar{u}_t &= \bar{u}_{xx} + \bar{u}^{\gamma}\dot{W}, \qquad t > 0, \ x \in [0, J] \\ \bar{u}(t,0) &= \bar{u}(t,J) = 0 \\ \bar{u}(0,x) &= \bar{u}_0(x) \end{aligned} \tag{1.1}$$

Here, $1 \leq \gamma < 3/2$, and $\bar{u}_0(x)$ is a continuous nonnegative function on $[0, J]$, vanishing at the endpoints. One of the key steps of the proof was showing that $\bar{U}(t) = \int_0^J \bar{u}(t,x)\,dx$ is a continuous local supermartingale. Since $\bar{u}(t,x)$ can be shown to remain nonnegative, this implies that $\sup_{0 \leq t < \infty} \bar{U}(t) < \infty$ almost surely. Therefore, tall peaks of $\bar{u}(t,x)$ are very thin, and would soon be smoothed out by the action of the heat kernel. This heuristic idea is the basis of the proof of long time existence given in [Mue91]. In [Kry94], Krylov gives a second proof, for a more general class of equations. He derives certain Sobolev space estimates for the solution, and then uses the boundedness of $\sup_{0 \leq t < \infty} \bar{U}(t)$ to show that $\bar{u}(t,x)$ cannot blow up in finite time.

It was expected, at least by the author, that equations similar to (1.1), but with signed solutions, would not have solutions valid for all time. Indeed, if $u(t,x)$ is such a solution, the analogue of $\bar{U}(t)$ would be

$$U(t) = \int_{\mathbb{I}} |u(t,x)|\,dx \tag{1.2}$$

where $\mathbb{I}$ is the domain for x. However, there seems to be no reason why $U(t)$ should be bounded in t.

The goal of this article is to show that, on the contrary, long-time existence can hold for equations similar to (1.1), but with signed solutions. Let $\mathbb{I}$ be the circle $[0, J]$, with the endpoints identified, and let $\rho(x,y)$ be the distance from x to y along the circle $\mathbb{I}$. That is, let

$$\rho(x,y) = \min_{k \in \mathbf{Z}} |x - y + kJ| \ .$$

We consider the following equation.

$$\begin{aligned} u_t &= u_{xx} + g(u)\dot{W}, \qquad t > 0, \ x \in \mathbb{I} \\ u(0,x) &= u_0(x) \end{aligned} \tag{1.3}$$

For convenience, we suppose that $g(u)$ is a nonnegative function. Of course, this does not lead to a loss of generality. Let $C_0 > 0$ and

$$1 \leq \gamma < \tfrac{3}{2} \ . \tag{1.4}$$

We assume that $g(u)$ is locally Lipschitz and satisfies

$$|g(u)| \leq C_0(1 + |u|)^{\gamma} \ . \tag{1.5}$$

Then we have

Theorem 1. *Suppose that conditions* (1.4) *and* (1.5) *hold, and that* $u_0(x)$ *is a continuous function on* $\mathbb{I}$. *Then* (1.3) *has a unique solution valid for all* $t > 0$.

Our main contribution is to compare $u(t,x)$ with another random function $v(t,x)$ which is easier to control. We introduce a bounded drift into (1.3) which forces solutions to stay positive with high probability. Call the new solution (with the drift) $v(t,x)$, and suppose that $v(t,x)$ satisfies

$$\begin{aligned} v_t &= v_{xx} + f(v) + g(v)\dot{W} \qquad t > 0, \ x \in \mathbb{I} \\ v(t,x) &= v_0(x) \end{aligned} \tag{1.6}$$

where $f(v)$ will be specified later. Let

$$V(t) = \int_{\mathbb{I}} v(t,x)\, dx$$

We show that except for the drift, $V(t)$ is a continuous local martingale. Since the drift is bounded, it cannot push $V(t)$ up more than a finite amount, over a finite time. Thus, over a finite time, $V(t)$ remains bounded.

Then we revert back to the argument of [Mue91] and [Kry94] to show that with high probability $v(t,x)$ does not blow up in finite time. A comparison theorem shows that $u(t,x) \le v(t,x)$, so that with high probability, $u(t,x)$ does not blow up to $+\infty$ in finite time. The same argument applied to $-u(t,x)$ shows that with high probability, $u(t,x)$ does not blow up to $-\infty$ in finite time. To make our argument self-contained, we give what we believe is a simpler substitute for the continuity estimates of [Kry94], at least for the white noise case.

The following corollary of Lemma 2.7 may be of interest.

Corollary 1.1. *Assume that conditions* (1.4) *and* (1.5) *hold. Let* $v_0(x)$ *be continuous, positive, and bounded away from* 0. *If* $f(v) = v^{-\alpha}$ *with* $\alpha > 3$, *then* $v(t,x)$ *remains strictly positive for all time.*

We do not know if the inequality $\alpha > 3$ is sharp.

One might try to study $|u(t,x)|$, and thus give a direct analysis of $U(t)$. Several recent papers deal with SPDE with reflection, for example [DMP92]. This paper show that $|u(t,x)|$ satisfies an SPDE with singular drift occurring at the zeroes of $u(t,x)$. However, we could not see how to get enough information about the drift to control $U(t)$.

We do not expect (1.3) to have a long-time solution if $g(u)$ increases too rapidly with u. Indeed, in [MS95] it was shown that if $u_0(x)$ is not identically 0, then (1.1) has a solution $u(t,x)$ such that $\sup_{0 \le x \le J} u(t,x)$ blows up to $+\infty$ in finite time, with positive probability.

As mentioned above, our second contribution is to give a new proof that solutions to (1.1) do not blow up in finite time, for $\gamma < 3/2$. This proof was inspired by [Kry94], but is only loosely related. We show that the formation of tall, thin peaks is incompatible with certain estimates on the Hölder continuity of solutions.

Now we discuss the rigorous meaning of (1.3), following the formalism of Walsh [Wal86], chapter 3. Before giving details, we set up some notation. Let $G(t,x,y)$ be the fundamental solution of the heat equation on $\mathbb{I}$. If $G(t,x)$ is a function of 2 variables, we let $G(t,x)$ be the fundamental solution of the heat equation on $\mathbb{R}$. In other words

$$G(t,x) = \frac{1}{\sqrt{4\pi t}} \exp\left(-\frac{x^2}{4t}\right) .$$

It is well known that

$$G(t,x,y) = \sum_{k=-\infty}^{\infty} G(t, x - y + kJ) .$$

We regard (1.3) as shorthand for the following integral equation.

$$u(t,x) = \int_{\mathbb{I}} G(t,x,y)u_0(y)\,dy + \int_0^t \int_{\mathbb{I}} G(t-s,x,y)g(u(s,y))W(dy\,ds) \tag{1.7}$$

where the final term in (1.7) is a white noise integral in the sense of [Wal86], Chapter 2. Because $g(u)$ is locally Lipschitz, standard arguments show that (1.3) has a unique solution $u(t,x)$ valid up to the time τ_L at which $|u(t,x)|$ first reaches the level L for some $x \in \mathbb{I}$. Letting $L \to \infty$, we find that (1.3) has a unique solution for $t < \tau$, where $\tau = \lim_{L\to\infty} \tau_L$. If $\tau < \infty$, one has

$$\lim_{t\uparrow\tau} \sup_{x\in\mathbb{I}} |u(t,x)| = \infty .$$

Our goal is to show that $\tau = \infty$ with probability 1.

More generally, we regard

$$\begin{aligned} v_t &= v_{xx} + f(v) + g(v)\dot{W}, \qquad t > 0, \ x \in \mathbb{I} \\ v(0,x) &= v_0(x) \end{aligned}$$

as a shorthand for the following integral equation.

$$\begin{aligned} v(t,x) = &\int_{\mathbb{I}} G(t,x,y)v_0(y)\,dy + \int_0^t \int_{\mathbb{I}} G(t-s,x,y)f(v(s,y))\,dy\,ds \\ &+ \int_0^t \int_{\mathbb{I}} G(t-s,x,y)g(v(s,y))W(dy\,ds) \end{aligned} \tag{1.8}$$

Lastly, we will always work with the σ-fields $\mathscr{F}_t$ generated by the white noise up to time t. That is, $\mathscr{F}_t$ is the σ-field generated by the random variables $\int_0^t \int_{\mathbb{I}} \phi(s,x)W(dx\,ds)$, where ϕ varies over all continuous functions on $[0,t] \times \mathbb{I}$.

2. Some lemmas

In this section, we give some lemmas which we will need for the proof of Theorem 1. One of our most important goals is to show that we can modify our solution $u(t,x)$ so that it remains positive. This result is given in the last lemma in this section.

Our first goal is to show that a nonnegative function with a given modulus of continuity and a large supremum must also have a large $\mathbb{L}^1$ norm. Later we will show that our solution $u(t,\cdot)$ has a certain modulus of continuity, and has bounded $\mathbb{L}^1$ norm. This argument will show that $u(t,\cdot)$ remains bounded.

Lemma 2.1. *Let* $0 < \eta < 1$, *and let* $f(x)$ *be a nonnegative function whose domain is* $\mathbb{I}$. *Suppose that for all* $x, y \in \mathbb{I}$, *we have*

$$|f(x) - f(y)| \leq A\rho^{\eta}(x, y) \tag{2.1}$$

and that

$$\int_{\mathbb{I}} f(x)\,dx \leq K \ . \tag{2.2}$$

Then,

$$\sup_{x \in \mathbb{I}} f(x) \leq \max\left[(2A)^{1/(\eta+1)} K^{\eta/(\eta+1)}, 2K/J\right]$$

Proof. Let $M(f) = \sup_{x \in \mathbb{I}} f(x)$ and $I(f) = \int_{\mathbb{I}} f(x)\,dx$. Suppose that, on the contrary,

$$M(f) > \max\left[(2A)^{1/(\eta+1)} K^{\eta/(\eta+1)}, 2K/J\right] \ . \tag{2.3}$$

Since f is continuous, there exists $x_0 \in \mathbb{I}$ such that $M(f) = f(x_0)$. Note that (2.3) implies that $K/M(f) < J/2$. We claim that there exists $x_1 \in \mathbb{I}$ such that $f(x_1) \leq M(f)/2$ and $\rho(x_0, x_1) \leq K/M(f)$. If not, we would have $f(x) > M(f)/2$ for all $x \in \mathbb{I}$ satisfying $\rho(x, x_0) \leq K/M(f) \leq J/2$, that is, on an interval of length $2K/M(f)$. Then we would have $I(f) > [2K/M(f)] \cdot [M(f)/2] = K$, which contradicts assumption (2.2). Therefore, x_1 exists and has the properties claimed.

But then, using assumption (2.1), we would have

$$\begin{aligned}\frac{M(f)}{2} &\leq |f(x_0)| - |f(x_1)| \\ &\leq |f(x_0) - f(x_1)| \\ &\leq A\rho^{\eta}(x_0, x_1) \\ &\leq A\left(\frac{K}{M(f)}\right)^{\eta} \ .\end{aligned}$$

Solving for $M(f)$, we find

$$M(f) \leq (2A)^{1/(\eta+1)} K^{\eta/(\eta+1)}$$

which contradicts (2.3). This proves Lemma 2.1. □

Secondly, we specialize Theorem 2.5 of Kotelenez [Kot92] to show that adding a nonnegative drift to (1.3) can never lead to a smaller solution. Note that equation (2.11) below is really a shorthand for an integral equation similar to (1.7).

Lemma 2.2. *For* $i = 1, 2$, *let* $f_i(v)$ *be a Lipschitz function on* $\mathbb{R}$, *and suppose that* $v^{(i)}(t, x)$ *is a solution to the following equation.*

$$
\begin{aligned}
v_t^{(i)} &= v_{xx}^{(i)} + f_i(v^{(i)}) + g(v^{(i)})\dot{W} \qquad t > 0,\ \ x \in \mathbb{I} \qquad (2.4)\\
v^{(i)}(t,x) &= v_0^{(i)}(x)
\end{aligned}
$$

For $i = 1,2$, *assume that* $v_0^{(i)}(x)$ *is a continuous function on* $\mathbb{I}$, *such that* $v_0^{(1)}(x) \le v_0^{(2)}(x)$ *for all* $x \in \mathbb{I}$. *Also assume that* $f_1(v) \le f_2(v)$ *for all* $v \in \mathbb{R}$. *Let* T *be the minimum of the blow-up times for* $v^{(1)}(t,x)$ *and* $v^{(2)}(t,x)$. *Then, with probability* 1,

$$
v^{(1)}(t,x) \le v^{(2)}(t,x)
$$

for all $(t,x) \in [0,T) \times \mathbb{I}$.

Proof. Lemma 2.2 would follow immediately from Theorem 2.5 of [Kot92], if $g(v)$ were a Lipschitz function. Let T_n be the first time t that $\max_{i=1,2} \sup_{x\in\mathbb{I}} |v^{(i)}(t,x)| \ge n$. Thus, for $t \le T_n$, replacing $g(v)$ by $g_n(v) = g(-n\vee (n \wedge v))$ does not affect $v^{(i)}(t,x)$, $i = 1,2$. Since $g_n(v)$ is Lipschitz, Theorem 2.5 of [Kot92] implies that with probability 1, $v^{(1)}(t,x) \le v^{(2)}(t,x)$ for all $(t,x) \in [0,T_n] \times \mathbb{I}$. Taking $n \to \infty$, we get the conclusion of Lemma 2.2. □

Thirdly, we estimate the modulus of continuity of our solution $u(t,\cdot)$. More precisely, we estimate the modulus of a random function $N_H(t,x)$ related to the final term in the integral equation (1.7). Suppose that $H(s,y)$ is a nonanticipating random function such that $|H(s,y)| \le L$ for all s,y almost surely. The following estimate is similar to those in [Mue91], among other places. Let

$$
N_H(t,x) = \int_0^t \int_{\mathbb{I}} G(t-s,x,y)H(s,y)W(dy\,ds)
$$

and note that $N_H(t,x)$ is similar to the final term in (1.7), except for the L. We have:

Lemma 2.3. *For* $T > 0$ *and* $0 < \kappa < 1/4$ *there exist constants* $c_0, C_1 > 0$ *depending on* T, κ *and* J *such that for all* $\Delta > 0$,

$$
P\left(\sup_{0\le t\le T} \sup_{x,y\in\mathbb{I}, x\ne y} \frac{|N_H(t,x) - N_H(t,y)|}{\rho^{2\kappa}(x,y)} > \Delta \right) \le \exp\left(-\frac{c_0\Delta^2}{L^2} \right) . \qquad (2.5)
$$

and

$$
P\left(\sup_{0\le s<t\le T} \sup_{x\in\mathbb{I}} \frac{|N_H(t,x) - N_H(s,x)|}{(t-s)^{\kappa}} > \Delta \right) \le \exp\left(-\frac{c_0\Delta^2}{L^2} \right) \qquad (2.6)
$$

provided that

$$
\exp\left(-\frac{c_0\Delta^2}{L^2} \right) \le C_1 \ .
$$

Note that we need only prove Lemma 2.3 for s, t, x, y dyadic rationals (numbers of the form $k2^{-n}$) since then, standard arguments show that N_H has a continuous version.

First, (2.6) is a special case of Proposition A.2 of Sowers [Sow92]. The proof of (2.5) rests on the following estimates, proved in [Sow92], Proposition A.1 and equation (A.4).

Lemma 2.4. *Let*

$$D_1(t,x,y) = \int_0^t \int_{\mathbb{I}} [G(t-r,x,z) - G(t-r,y,z)]^2 \, dz \, dr$$
$$D_2(s,t,x) = \int_0^s \int_{\mathbb{I}} [G(t-r,x,z) - G(s-r,x,z)]^2 \, dz \, dr$$
$$D_3(s,t,x) = \int_s^t \int_{\mathbb{I}} [G(t-r,y)]^2 \, dy \, dr$$

For $\kappa \in (0, 1/4)$ *and* $T > 0$, *there exists a constant* c *depending on* T, κ *and on the length of* $\mathbb{I}$, *such that for* $0 \le s < t \le T$ *and* $x \in \mathbb{I}$, *we have*

$$D_1(t,x,y) \le c\rho^{4\kappa}(x,y)$$
$$D_2(s,t,x) \le c|t-s|^{2\kappa}$$
$$D_3(s,t,x) \le c|t-s|^{2\kappa}$$

Sowers only states that $D_1(t,x,y) \le c\rho^{2\kappa}(x,y)$. However, his proof works with no changes if 2κ is replaced by 4κ.

Next, we use Lemma 2.4 to get some probability estimates on the differences $|N_H(t,x) - N_H(t,y)|$ and $|N_H(t,x) - N_H(s,x)|$.

Lemma 2.5. *For* $\kappa \in (0, 1/4)$ *and* $T > 0$, *there exist constants* $c, C > 0$ *depending on* T, κ *and on* J, *such that for* $0 \le s < t \le T$ *and* $x, y \in \mathbb{I}$, $x \ne y$, *we have*

$$P(|N_H(t,x) - N_H(t,y)| > \Delta) \le \exp\left(-\frac{c\Delta^2}{L^2\rho^{4\kappa}(x,y)}\right) \tag{2.7}$$

$$P(|N_H(t,x) - N_H(s,x)| > \Delta) \le \exp\left(-\frac{c\Delta^2}{L^2(t-s)^{2\kappa}}\right) \tag{2.8}$$

provided that the right hand sides in the above 2 *equations are both less than or equal to* C.

Proof. First note that

$$N_H(t,x) - N_H(t,y) = \int_0^t \int_{\mathbb{I}} (G(t-s,x,z) - G(t-s,y,z))H(s,z)W(dz\,ds) \tag{2.9}$$

We use the fact, given in [Wal86], Chapter 2, that for a predictable term $R(t,x)$,

$$M(t) = \int_0^t \int_{\mathbb{I}} R(s,x)W(dx\,ds)$$

is a continuous local martingale, and hence is a time-changed Brownian motion $B(\sigma(t))$. Walsh also shows in [Wal86], chapter 2, that

$$\langle M\rangle_t = \int_0^t \int_{\mathbb{I}} R^2(s,x)\, dx\, ds \tag{2.10}$$

Standard martingale theory implies that the time scale $\sigma(t) = \langle M\rangle_t$. Now we apply these facts to $M_H = N_H(t,x) - N_H(t,y)$. Let $\sigma_H(t)$ denote the corresponding time-change. Note that by (2.9), (2.10), and Lemma 2.4 we have

$$\sigma_H(t) \le cL^2\rho^{2\kappa}(x,y) \ . \tag{2.11}$$

Using (2.17), the reflection principle for Brownian motion, and standard estimates for the normal density, we find

$$\begin{aligned} P\big(|N_H(t,x) - N_H(t,y)| > \Delta\big) &\le P\left(\sup_{0\le t\le cL^2\rho^{2\kappa}(x,y)} |B(t)| > \Delta\right) \\ &\le 4P\big(B(cL^2\rho^{4\kappa}(x,y)) > \Delta\big) \\ &\le \exp\left(-\frac{c\Delta^2}{L^2\rho^{4\kappa}(x,y)}\right) \end{aligned}$$

if the final term is small enough. Here, the constant c may vary from line to line. This proves (2.7). The proof of (2.8) is similar, and we leave it to the reader. □

Now we use Lemma 2.5 to prove (2.5) in Lemma 2.3. For simplicity of notation, we restrict ourselves to the case in which the length of $\mathbb{I}$ is 1, although our argument would carry over to the general case. In [Wal86], chapter 3, Walsh shows that solutions $u(t,x)$ are continuous with probability 1. Therefore, if $\mathscr{R} \subset \mathbb{R}$ is the set of dyadic rationals, (numbers of the form $k2^{-n}$), it suffices to show (2.5) with t, x, y restricted to $\mathscr{R}$. Since $0 < \kappa < 1/4$ we may choose $\bar\kappa$ such that $\kappa < \bar\kappa < 1/4$. Let $\mathscr{A}_1(i,\ j,\ m,\ n,\ \Delta,\ \beta)$ be the event that

$$|N_H(i2^{-m},(j+1)2^{-n}) - N_H(i2^{-m},j2^{-n})| \le \beta\Delta 2^{-2n\bar\kappa} \ .$$

Let $\mathscr{A}_2(i,j,m,n,\Delta,\beta)$ be the event that

$$|N_H((i+1)2^{-m},\ j2^{-n}) - N_H(i2^{-m},j2^{-n})| \le \beta\Delta 2^{-m\bar\kappa}$$

and setting $m = 2n$, let

$$\mathscr{A}(\beta) = \bigcap_{n=1}^{\infty}\bigcap_{i=1}^{2^{2n}}\bigcap_{j=1}^{2^n}\big[\mathscr{A}_1(i,j,2n,n,\Delta,\beta)\cap\mathscr{A}_2(i,j,2n,n,\Delta,\beta)\big]$$

Using Lemma 2.5, we have

$$\begin{aligned} P(\mathscr{A}^c) &\le \sum_{n=1}^{\infty} 6\cdot 2^{2n}\exp\left(-\frac{c\beta^2\Delta^2 2^{4n(\bar\kappa-\kappa)}}{L^2}\right) \\ &\le \exp\left(-\frac{c'\beta^2\Delta^2}{L^{2\gamma}}\right) \end{aligned} \tag{2.12}$$

Lemma 2.3 follows from (2.12) once we have shown that

Lemma 2.6. *If $\beta > 0$ is small enough, and if the event $\mathscr{A}(\beta)$ occurs, then for all dyadic rationals $t \in [0,1]$ and $x, y \in \mathbb{I}$, we have*

$$|N_H(t,x) - N_H(t,y)| \leq \Delta\rho^{2\kappa}(x,y) \ .$$

Proof. To simplify notation, we may assume without loss that x, y are positioned such that $\rho(x,y) = |x-y|$. To further simplify the explanation, we only deal with the case in which $T \leq 1$. Let $t = .t_1t_2\ldots t_{n(t)}$ be the base 4 expansion of t, and let $x = .x_1x_2\ldots x_{n(x)}$, $y = .y_1y_2\ldots y_{n(y)}$ be the base 2 expansions of x and y, respectively. Let $k(x,y)$ be the smallest index k such that $x_{k+1} \neq y_{k+1}$. For any index $i > 0$, let $\bar{t}_i = .t_1\ldots t_i$. In other words, $\bar{t}_i$ is t truncated at the i^{th} digit. Define $\bar{x}_i$ and $\bar{y}_i$ in the same way, and note that $\bar{x}_{k(x,y)} = \bar{y}_{k(x,y)}$. It is easy to check that there is a constant $\bar{c}$ not depending on t, x, y such that

$$\begin{aligned}|t - \bar{t}_{k(x,y)}| &\leq \bar{c}|x-y|^2\\ |x - \bar{x}_{k(x,y)}| &\leq \bar{c}|x-y|\\ |y - \bar{x}_{k(x,y)}| &\leq \bar{c}|x-y|\end{aligned}$$

and

$$|x-y| \leq \bar{c}2^{-k(x,y)} \ .$$

Let $p_0 = (\bar{t}_{k(x,y)}, \bar{x}_{k(x,y)})$. We claim that there exists a path from p to (t,x) with a finite number of steps, with each step having the form $(k2^{-2n}, \ell 2^{-n}) \to (k2^{-2n}, (\ell+1)2^{-n})$ or $(k2^{-2n}, \ell 2^{-n}) \to ((k+1)2^{-2n}, \ell 2^{-n})$. Furthermore, there are at most 4 such steps for each value of n. Such a path can be constructed by adding back the missing digits in $\bar{t}_{k(x,y)}$ and in $\bar{x}_{k(x,y)}$ one by one. In other words, each step has the form $(.t_1\ldots t_m,.\ x_1\ldots x_m) \to (.t_1\ldots t_m, .x_1\ldots x_{m+1})$, or $(.t_1\ldots t_{m-1}j, .x_1\ldots x_m) \to (.t_1\ldots t_{m-1}(j+1), .x_1 \ldots x_{m+1})$ where $j = 0,1,2$. Call $p_0, \ldots p_m$ the points along this path.

Now assume that event $\mathscr{A}(\beta)$ holds. Using Lemma 2.6, we find that

$$\begin{aligned}|N_H(t,x) - N_H(p_0)| &\leq \sum_{i=1}^{m} |N_H(p_i) - N_H(p_{i-1})|\\ &\leq 4\sum_{k=k(x,y)}^{\infty} \beta\Delta 2^{-k/2}\\ &\leq \bar{c}\beta 2^{-k(x,y)/2}\\ &\leq \tfrac{1}{2}|x-y|^{1/2} \ .\end{aligned}$$

if $0 < \beta < 1/\bar{c}$. By a similar argument, we find that if $0 < \beta < 1/\bar{c}$ and if $\mathscr{A}(\beta)$ holds, then

$$|N_H(t,y) - N_H(p_0)| \leq \tfrac{1}{2}|x-y|^{1/2} \ .$$

Therefore,

$$|N_H(t,x) - N_H(t,y)| \leq |x-y|^{1/2} \ .$$

This completes the proof of Lemma 2.6, and hence also of Lemma 2.3. □

Fourthly, we show that by adding the appropriate drift to (1.3), we can force solutions to remain positive.

Lemma 2.7. *Suppose that $T > 0$ and $\alpha > 3$. For each $\varepsilon > 0$, there exists a $\delta_0 > 0$ such that if $0 < \delta < \delta_0$ then the following holds. Let $f(v) = (|v| \vee (\delta/2))^{-\alpha}$, and let c_0 be the constant appearing in Lemma 2.3. Suppose that $v(t,x)$ satisfies*

$$\begin{aligned} v_t &= v_{xx} + f(v) + g(v)\dot{W} \qquad t > 0, \ x \in \mathbb{I} \\ v(t,x) &= v_0(x) \end{aligned} \tag{2.13}$$

and assume that $v_0(x)$ is a continuous function on $\mathbb{I}$ satisfying $v_0(x) \geq \delta$ for all $x \in \mathbb{I}$. Let $v(t,x) = \infty$ if t is greater or equal to the blow-up time for $v(t,x)$. Then

$$P\left(\inf_{0 \leq t \leq T} \inf_{x \in \mathbb{I}} v(t,x) < \delta/2\right) < \varepsilon \ .$$

Proof. First, since $\alpha > 3$, it follows that

$$\frac{1}{\alpha+1} < 1/4 \ .$$

Choose κ such that $1/(\alpha+1) < \kappa < 1/4$. Note that if $\delta/2 \leq u \leq 2\delta$, then

$$(2\delta)^{-\alpha} \leq f(u) \leq (\delta/2)^{-\alpha}. \tag{2.14}$$

Let

$$\beta = 2^{-\alpha-2}\delta^{1+\alpha} \ .$$

For simplicity, assume that $M \equiv T/\beta$ is an integer; if not, increase T a bit. We wish to compare $v(t,x)$ with a random function $w(t,x)$ such that

$$P\left(\inf_{0 \leq t \leq T} \inf_{x \in \mathbb{I}} (v(t,x) - w(t,x)) \geq 0, \ \inf_{0 \leq t \leq T} \inf_{x \in \mathbb{I}} w(t,x) \geq \delta/2\right) \geq 1 - \varepsilon \tag{2.15}$$

Note that (2.15) implies Lemma 2.7. We define $w(t,x)$ as follows. For $k = 0, \ldots, M-1$, and for $k\beta \leq t < (k+1)\beta$ and $x \in \mathbb{I}$, let $w(t,x)$ satisfy

$$w_t(t,x) = w_{xx}(t,x) + f(w(t,x)) + g(w(t,x))\dot{W}(t,x) \tag{2.16}$$
$$w(k\beta, x) = \delta \ .$$

Let τ_w be the blow-up time for $|w|$. Let $w(t,x) = \infty$ if $t \geq \tau_w$. Let $L = \sup_{\delta/2 \leq u \leq 2\delta} g(u)$, and let

$$D_k(t,x) = \int_{k\beta}^{t} \int_{x \in \mathbb{I}} G(t-s,x,y) f(w(s,y))\, dy\, ds$$
$$N_{k,L}(t,x) = \int_{k\beta}^{t} \int_{x \in \mathbb{I}} G(t-s,x,y)[g(w(s,y)) \wedge L] W(dy\, ds) \ .$$

If $t \geq \tau_w$, let $D_k(t,x) = N_{k,L}(t,x) = \infty$. Then for $k\beta < t < (k+1)\beta$ and $x \in \mathbb{I}$, and assuming that $\delta/2 \leq w(s,x) \leq 2\delta$ for $k\beta < s \leq t$ and $x \in \mathbb{I}$, we have

$$w(t,x) = \delta + D_k(t,x) + N_{k,L}(t,x). \tag{2.17}$$

For $k = 0, \ldots, M-1$, let $\mathscr{W}_k$ be the event that for $k\beta < t < (k+1)\beta$ and $x \in \mathbb{I}$,

$$\delta/2 \leq w(t,x) \leq 2\delta$$

and that for $x \in \mathbb{I}$,

$$w((k+1)\beta, x) \geq \delta \ .$$

Let $\mathscr{N}_k$ be the event that for $k\beta < t \leq (k+1)\beta$ and $x \in \mathbb{I}$,

$$\left| \frac{N_{k,L}(t,x)}{(t-k\beta)^{\kappa}} \right| \leq \delta^{1-\kappa(1+\alpha)} 2^{-2-2\alpha+\kappa(\alpha+2)} \ .$$

Note that for $k = 0, \ldots, M-1$, we may use the Markov property, proved in [Wal86], chapter 3, and Lemma 2.3 from this paper to conclude that since $1 - \kappa(1+\alpha) < 0$, if δ is small enough then

$$P\left(\mathscr{N}_k^c\right) \leq \exp\left(-c_0 \delta^{2-2\kappa(1+\alpha)} L^{-2} 2^{-4-4\alpha 2\kappa(\alpha+2)}\right)$$
$$< \frac{\varepsilon}{T} 2^{-\alpha-2} \delta^{1+\alpha} \leq \frac{\varepsilon\beta}{T} = \frac{\varepsilon}{M} \ . \tag{2.18}$$

Now we show that on $\mathscr{N}_k$, $w(t,x)$ remains bounded between $\delta/2$ and 2δ for $k\beta \leq t \leq (k+1)\beta$ and $x \in \mathbb{I}$, and $w((k+1)\beta, x) \geq \delta$ for $x \in \mathbb{I}$, and hence $\mathscr{N}_k \subset \mathscr{W}_k$. Suppose that $\mathscr{N}_k$ occurs. Let t^* be the first time $t \in [k\beta, (k+1)\beta]$ such that for some $x \in \mathbb{I}$, $w(t^*, x) = \delta/2$ or 2δ. If there is no such time, let $t^* = (k+1)\beta$.

Our first goal is to show that on $\mathscr{N}_k$, $t^* = (k+1)\beta$. Since $t^* - k\beta \leq \beta$, and recalling the definition of β, it follows that for $x \in \mathbb{I}$,

$$0 \leq D_k(t^*, x) \leq \beta \sup_{\delta/2 \leq u \leq 2\delta} f(u)$$
$$\leq \beta \left(\frac{\delta}{2}\right)^{-\alpha}$$

$$= 2^{-\alpha-2}\delta^{1+\alpha}\left(\frac{\delta}{2}\right)^{-\alpha} = \frac{\delta}{4} \quad .$$

Also, since $\mathcal{N}_k$ occurs, if $x \in \mathbb{I}$ and if $t^* < (k+1)\beta$, then by the definition of β,

$$\begin{aligned}
\left|N_{k,L}(t^*,x)\right| &\le (t^* - k\beta)^{\kappa}\delta^{1-\kappa(1+\alpha)}2^{-2-2\alpha+\kappa(\alpha+2)} \\
&< \beta^{\kappa}\delta^{1-\kappa(1+\alpha)}2^{-2-2\alpha+\kappa(\alpha+2)} \\
&= \left(2^{-\alpha-2}\delta^{1+\alpha}\right)\delta^{1-\kappa(1+\alpha)}2^{-2-2\alpha+\kappa(\alpha+2)} \\
&= \delta 2^{-2-2\alpha} \quad .
\end{aligned}$$

Therefore, if $x \in \mathbb{I}$ and if $t^* < (k+1)\beta$, then

$$\begin{aligned}
|w(t^*,x) - \delta| &\le |D_k(t^*,x)| + \left|N_{k,L}(t^*,x)\right| \\
&< \frac{\delta}{2}
\end{aligned}$$

and therefore $t^* = (k+1)\beta$.

Our next goal is to show that on $\mathcal{N}_k$, if $x \in \mathbb{I}$, then $w((k+1)\beta,x) \ge \delta$. But

$$w((k+1)\beta,x) = \delta + D_k((k+1)\beta,x) + N_{k,L}((k+1)\beta,x)$$

so it suffices to show that

$$\left|N_{k,L}((k+1)\beta,x)\right| \le D_k((k+1)\beta,x) \tag{2.19}$$

However, since $t^* = (k+1)\beta$,

$$\begin{aligned}
D_k((k+1)\beta,x) &\ge \beta \inf_{\delta/2 \le u \le 2\delta} f(u) \qquad (2.20) \\
&= \beta(2\delta)^{-\alpha} \\
&= 2^{-\alpha-2}\delta^{1+\alpha}(2\delta)^{-\alpha} \\
&= \delta 2^{-2\alpha-2}
\end{aligned}$$

Since $\mathcal{N}_k$ occurs,

$$\begin{aligned}
\left|N_{k,L}((k+1)\beta,x)\right| &\le \beta^{\kappa}\delta^{1-\kappa(1+\alpha)}2^{-2-2\alpha+\kappa(\alpha+2)} \qquad (2.21) \\
&= \delta 2^{-2\alpha-2} \quad .
\end{aligned}$$

Together, (2.20) and (2.21) imply (2.19), and therefore imply that if $x \in \mathbb{I}$, then $w((k+1)\beta,x) \ge \delta$. This completes the proof that $\mathcal{N}_k \subset \mathcal{W}_k$.

Let

$$\mathcal{N} = \bigcap_{k=0}^{M-1} \mathcal{N}_k \quad .$$

Using induction, we now prove that if $\mathcal{N}$ occurs then $\delta/2 \le w(t,x) \le v(t,x)$ for $0 \le t \le T$ and $x \in \mathbb{I}$. Recall our definition of $\mathcal{N}_k$. It follows that if the event $\mathcal{N}_k$ occurs, then $\delta/2 \le w(t,x)$ for $0 \le t \le T$ and $x \in \mathbb{I}$, and that

$w(t,x) \leq v(t,x)$ for $0 \leq t \leq \beta$ and $x \in \mathbb{I}$. Suppose that we have shown that $w(t,x) \leq v(t,x)$ for $0 \leq t < k\beta$ and $x \in \mathbb{I}$. Since $\mathcal{N}_{k-1}$ occurs, we have

$$v(k\beta,x) = \lim_{t \uparrow k\beta} v(t,x) \geq \lim_{t \uparrow k\beta} w(t,x) \geq \delta = w(k\beta,x) \ .$$

Therefore, using Lemma 2.2, we conclude that $w(t,x) \leq v(t,x)$ for $0 \leq t < (k+1)\beta$. So the induction is complete, and we conclude that if $\mathcal{N}$ occurs then $v(t,x) \geq \delta/2$ for $0 \leq t \leq T$ and $x \in \mathbb{I}$.

Finally, we conclude that

$$\begin{aligned} P\left(\inf_{0 \leq t \leq T} \inf_{x \in \mathbb{I}} v(t,x) < \delta/2 \right) &\leq P(\mathcal{N}^c) \\ &\leq \sum_{k=0}^{M-1} P\left(\mathcal{N}_k^c\right) \\ &< \varepsilon. \end{aligned}$$

This proves Lemma 2.7. □

3. Proof of Theorem 1

Now we use Lemmas 2.1 and 2.7 to prove Theorem 1. Fix $T, \varepsilon > 0$. We wish to show that with probability at least $1 - \varepsilon$, $u(t,x)$ does not blow up to $+\infty$ before time T. That is, we will show

Lemma 3.1. *For each* $\varepsilon > 0$,

$$P\left(\sup_{0 \leq t < T \wedge \tau} \sup_{x \in \mathbb{I}} u(t,x) = +\infty \right) \leq \varepsilon \ . \tag{3.1}$$

Theorem 1 would follow from Lemma 3.1 if we could show that with probability at least $1 - \varepsilon$, $u(t,x)$ does not blow up to $-\infty$ before time T. But this follows by applying Lemma 3.1 to $-u(t,x)$. Now we prove Lemma 3.1.

Proof. Let $\alpha > 3$. Let $v(t,x)$ satisfy (1.8), with $f(v) = (v \vee (\delta/2))^{-\alpha}$ and $v_0(x) = v_0 = \max(1, \sup_{x \in \mathbb{I}} u_0(x))$. Using Lemma 2.7, let $\delta > 0$ be chosen such that

$$P\left(\inf_{0 \leq t \leq T} \inf_{x \in \mathbb{I}} v(t,x) \leq \delta/2 \right) < \frac{\varepsilon}{4} \ .$$

By Lemma 2.2, for $0 \leq t \leq T$, $x \in \mathbb{I}$, we have $u(t,x) \leq v(t,x)$. Thus, it suffices to show that

$$P\left(\sup_{0 \leq t < T} \sup_{x \in \mathbb{I}} v(t,x) = +\infty \right) \leq \varepsilon \ .$$

Note that the maximum drift $f(v)$ for v is

$$D_0 \equiv \sup_{u \geq \delta/2} f(u) = (\delta/2)^{-\alpha} \ .$$

Actually, we will work with v_L rather than v. Recall that v_L and v agree up to the first time that $g(v)$ reaches the level L. Using the integral equation (1.8), and the fact that $\int_{\mathbb{I}} G(t,x,y)v_0 dy = v_0$, we have

$$\begin{aligned} v_L(t,x) &= v_0 + \int_0^t \int_{\mathbb{I}} G(t-s,x,y) f(v_L(s,y))\, dy\, ds + N_{v,L}(t,x) \\ &\le v_0 + D_0 t + N_{v,L}(t,x) \end{aligned}$$

Now, using the Markov property of solutions, we have

$$\begin{aligned} v_L(s+t,x) &= \int_{\mathbb{I}} G(t,x,y) v_L(s,y) dy \\ &\quad + \int_0^t \int_{\mathbb{I}} G(s+t-r,x,y) f(v_L(s+r,y))\, dy\, dr \\ &\quad + N_{v,L}(t,x)(\theta_s \omega) \\ &\le \int_{\mathbb{I}} G(t,x,y) v_L(s,y) dy + D_0 t + N_{v,L}(t,x) \end{aligned}$$

where $N_{v,L}(t,x)(\theta_s\omega)$ denotes the standard time-shift of $N_{v,L}(t,x) = N_{v,L}(t,x)(\omega)$.

Now let

$$M(t) = D_0 J \cdot (T-t) + \int_{\mathbb{I}} v_L(t,x)\, dx \quad .$$

We claim that $M(t)$ is a continuous supermartingale for $0 \le t \le T$. The continuity of $M(t)$ follows from the continuity of $v_L(t,x)$ in (t,x). Next, using the integral equation, the fact that $EN_{v,L}(t,x)(\omega) = 0$, and the standard fact that

$$\int_{\mathbb{I}} \int_{\mathbb{I}} G(t,x,y) v_L(s,y)\, dy\, dx = \int_{\mathbb{I}} v_L(s,y)\, dy$$

we have that for $s+t \le T$ and $s,t \ge 0$,

$$\begin{aligned} E[M(s+t)|\mathscr{F}_s] &\le D_0 J \cdot (T-t) + \int_{\mathbb{I}} v_L(s,y) dy + D_0 J t \\ &= M(s) \end{aligned}$$

Since $M(t)$ is a supermartingale, there is a nondecreasing process $A(t)$ with $A(0) = 0$ such that $M(t) + A(t)$ is a martingale. Next, let τ_M be the first time $t \ge 0$ that $M(t) = 0$, and let $\Lambda(t) = M(t \wedge \tau_M) + A(t \wedge \tau_M)$. Note that $M(0) = D_0 JT + v_0 J > 0$, so $\tau_M > 0$ almost surely. Then $\Lambda(t)$ is a continuous, nonnegative martingale. Hence, it is a time-changed Brownian motion which remains nonnegative, and thus stochastically bounded. That is, there exists a random variable $K = K(\omega)$ such that $\sup_{t\ge 0} \Lambda(t) \le K$. Note that on the event $v(t,x) \ge 0$ for $0 \le t \le T$, $x \in \mathbb{I}$, we have that $M(t) \le \Lambda(t) \le K$ for $0 \le t \le T$.

Let $\sigma = \sigma_L$ be the first time $t \in [0,T]$ that $\inf_{x\in\mathbb{I}} v(t,x) \le \delta/2$. If there is no such time, let $\sigma = T$. Let $\rho = \rho_L$ be the first time $t \in [0,T]$ that $\sup_{x\in\mathbb{I}} g(v(t,x)) \ge L$. If there is no such time, let $\rho = T$. Then, by Lemma 2.2, for $0 \le t \le \sigma \wedge \rho$, $x \in \mathbb{I}$, we have $u(t,x) \le v(t,x) = v_L(t,x)$. Therefore, to prove Lemma 3.1, it suffices to show that for some $L > 0$,

$$P(\sigma_L \wedge \rho_L < T) < \frac{2\varepsilon}{3} \ .$$

However, using Lemma 2.7, we see that to prove Lemma 3.1, it suffices to show that

$$P(\rho_L < T) < \frac{\varepsilon}{3} \ . \tag{3.2}$$

Now, let $\mathscr{K} = \mathscr{K}(K_0)$ be the event that

$$\sup_{0 \le t < \sigma} M(t) \le K_0 - D_0 TJ$$

and choose K_0 such that

$$P(\mathscr{K}^c(K_0)) < \frac{\varepsilon}{6} \ . \tag{3.3}$$

Now let

$$\begin{aligned}\bar{v}(t,x) &= v_0 + D_0 t + N_{v,L}(t,x) \\ &= v_L(t,x) + D_0 t - \int_0^t \int_{\mathbb{I}} G(t-s,x,y) f(v_L(s,y))\, dy\, ds\end{aligned}$$

and note that

$$v_L(t,x) \le \bar{v}(t,x) \ .$$

Furthermore, on the event $\mathscr{K}(K_0)$ and for $0 \le t \le T$, we have

$$\begin{aligned}\int_{\mathbb{I}} \bar{v}(t,x)\, dx &\le \int_{\mathbb{I}} v(t,x)\, dx + D_0 Jt \\ &\le M(t) + D_0 TJ \\ &\le K_0 \ .\end{aligned} \tag{3.4}$$

For $A > 0$, let $\mathscr{M}(A)$ be the event that for all $0 \le t \le T$ and $x, y \in \mathbb{I}$,

$$|N_L(t,x) - N_L(t,y)| \le A\rho(x,y)^{\kappa}.$$

By the definition of $\bar{v}$, we have that if $\mathscr{M}(A)$ occurs then for all $0 \le t \le T$ and x, $y \in \mathbb{I}$,

$$|\bar{v}(t,x) - \bar{v}(t,y)| \le A\rho(x,y)^{\kappa} \ . \tag{3.5}$$

Note that by Lemma 2.1, if $\mathscr{K}(K_0)$ and $\mathscr{M}(A)$ occur, and if $0 \le t \le T$ and $x \in \mathbb{I}$, then

$$v(t,x) \le \bar{v}(t,x) \le \max\left[(2A)^{1/(\eta+1)} K_0^{\eta/(\eta+1)}, 2K_0/J\right] \tag{3.6}$$

Let

$$L_0 = \left(\frac{L}{C_0}\right)^{\frac{1}{\gamma}} - 1$$

and note that if $0 \le v \le L_0$ then by (1.5) we have $g(v) \le L$ and hence $g(v) \wedge L = g(v)$. Since $\gamma < 3/2$, we may choose γ_0, κ such that

$$\gamma < \gamma_0$$

and

$$\frac{\gamma_0}{2\kappa + 1} < 1 \ .$$

Then we may choose L so large that

$$\begin{aligned} \max\left[(2A)^{1/(2\kappa+1)} K_0^{2\kappa/(2\kappa+1)}, 2K_0/J\right] &= \max\left[(2L^{\frac{\gamma_0}{\gamma}})^{1/(2\kappa+1)} K_0^{2\kappa/(2\kappa+1)}, 2K_0/J\right] \\ &< \frac{L}{C_0} \end{aligned} \tag{3.7}$$

and

$$\exp\left(-\frac{c_0 A^2}{L^2}\right) < \frac{\varepsilon}{6} \tag{3.8}$$

where c_0 was the constant appearing in Lemma 2.5. Thus, using Lemma 2.5 and (3.8), we find that

$$P(\mathscr{M}(A)) < \frac{\varepsilon}{6} \ . \tag{3.9}$$

Also, Lemma 2.1 and (3.7) imply that if $\mathscr{K}(K_0)$ and $\mathscr{M}(A)$ occur, and if $0 \le t \le T$ and $x \in \mathbb{I}$, then

$$v_L(t,x) \le L_0 \ .$$

If this conclusion is true, then

$$g(v_L(t,x)) = g(v_L(t,x)) \ .$$

Therefore,

$$\begin{aligned} P(\rho_L < T) &\le P(\mathscr{K}(K_0)^c) + P(\mathscr{M}(A)^c) \\ &< \frac{\varepsilon}{6} \ . \end{aligned}$$

Thus we have shown (3.2), and as remarked earlier, this suffices to prove Lemma 3.1. This also completes the proof of Theorem 1. □

Now we briefly outline the proof of Corollary 1.1. Fix $T > 0$ and $\varepsilon > 0$, and assume that $v_0(x)$ is bounded away from 0. Let $v(t,x)$ satisfy (1.8) with $f(v) = (v \vee (\delta/2))^{-\alpha}$. We have already shown that with $\delta > 0$ sufficiently small and if $\mathscr{A}(v)$ is the event that

$$\inf_{0 \le t \le T} \inf_{x \in \mathbb{I}} v(t,x) \ge \frac{\delta}{2}$$

then

$$P(\mathscr{A}(v)^c) < \varepsilon \ .$$

However, on the event $\mathscr{A}$, $f(v(t,x)) = v(t,x)^{-\alpha}$ for $0 \le t \le T$ and $x \in \mathbb{I}$. Therefore, if $\bar{v}$ is the solution to (1.8) with $f(v) = v^{-\alpha}$, then

$$P(\mathscr{A}(v)^c) < \varepsilon \ .$$

Since T and ε were arbitrary, this proves Corollary 1.1.

We remark that the proof of our main theorem would carry through with very little change to the following situation. Consider $\mathbb{R}^d$-valued solutions $u(t,x)$ to the following equation.

$$\begin{aligned} u_t &= u_{xx} + g(u)\dot{W}, \qquad t > 0, \ x \in \mathbb{I} \\ u(0,x) &= u_0(x) \end{aligned} \tag{3.10}$$

Here, $\dot{W} = (\dot{W}_1(t,x), \ldots, \dot{W}_d(t,x))$ is a vector of independent 2-parameter white noises. We assume that $u_0(x)$ is a continuous function from $\mathbb{I}$ to $\mathbb{R}^d$. Funaki [Fun84] has considered such equations as a model for a random string. We usually use the wave equation to model a string, but the heat equation may be appropriate if the mass of the string is small and the string moves in a viscous medium.

Let $u = (u_1, \ldots, u_d)$ be the vector representation of $u(t,x)$. We could apply our argument to each of the components u_i. That is, for each i, we could add a positive drift in the i direction to keep u_i positive with probability 1. It would then follow, as before, that $U_i(t) = \int_{\mathbb{I}} u_i(t,x)dx$ is a local martingale plus a bounded drift, provided $0 \le t \le T$. We could again conclude that with high probability, each component $u_i(t,x)$ is bounded in x for $0 \le t \le T$. The conclusion would be that $u(t,x)$ does not blow up in finite time.

It should also be easy to replace $g(u)$ by a function $g(t,x,u)$ which is Lipschitz in (t,x), locally Lipschitz in u, and which grows no faster than $c(1 + |u|)^{(3/2)-\varepsilon}$.

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