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Large deviations and continuum limit in the 2D Ising model

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Summary. We study the 2D Ising model in a rectangular box Λ_L of linear size $O(L)$. We determine the exact asymptotic behaviour of the large deviations of the magnetization $\sum_{t\in\Lambda_L}\sigma(t)$ when $L\to\infty$ for values of the parameters of the model corresponding to the phase coexistence region, where the order parameter m^* is strictly positive. We study in particular boundary effects due to an arbitrary real-valued boundary magnetic field. Using the self-duality of the model a large part of the analysis consists in deriving properties of the covariance function $\langle\sigma(0)\sigma(t)\rangle$, as $|t|\to\infty$, at dual values of the parameters of the model. To do this analysis we establish new results about the high-temperature representation of the model. These results are valid for dimensions $D\geq 2$ and up to the critical temperature. They give a complete non-perturbative exposition of the high-temperature representation.

We then study the Gibbs measure conditioned by $\{|\sum_{t\in\Lambda_L}\sigma(t) - m|\Lambda_L||\leq|\Lambda_L|L^{-c}\}$, with $0<c<1/4$ and $-m^*<m<m^*$. We construct the continuum limit of the model and describe the limit by the solutions of a variational problem of isoperimetric type.

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1 Introduction

We analyze the large deviations of the magnetization of the two-dimensional Ising model in the phase coexistence region, paying attention to boundary conditions. Our new results lead to a new approach of the

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wetting phenomenon, an important surface phenomenon, which can be described in the Ising model [FP1], [FP2]. The theoretical physical aspects of the problem (wetting phenomenon) are discussed in a separate publication [PV2].

1.1 Historical remarks

When there is a unique Gibbs measure the rate function describing the large deviations of the magnetization is given by the specific free energy of the model. It is therefore sufficient to control the *bulk* thermodynamical properties of the model in order to compute large deviations bounds. The situation is different when two Gibbs states coexist (phase coexistence region) because the large deviations of the magnetization are now driven by *boundary* effects. Consequently we must control the surface tension and surface free energies in order to get sharp large deviations bounds.

In their famous papers [MS1] and [MS2] Minlos and Sinai started the analysis of the large deviations of the magnetization for the D-dimensional Ising model, $D \geq 2$, in the phase coexistence region. They showed that the phenomenon of phase segregation is at the origin of the large deviations behaviour of the magnetization. In the eighties the problem was considered again for $D = 2$. First Schonmann [S] (see also [CCS]) established lower and upper bounds for the large deviations with a completely different approach, which is non-perturbative as opposed to the work of Minlos and Sinai. Also, with different techniques, we have the works of Föllmer and Ort [FO] and [O]. A breakthrough was then made by Dobrushin, Kotecký and Shlosman in the late eighties [DKS]. They were able to get *exact* large deviations bounds for the magnetization and to get a detailed description of the typical configurations associated with large deviations in terms of the Wulff shape. Their results are valid at low temperature and for periodic boundary condition. After the announcement of these results Pfister [Pf2] obtained similar results valid at low temperature for + boundary condition. His method works as well for periodic boundary condition. Notice that the results depend on the choice of the boundary condition; see [Sh] for a study of some effects due to boundary conditions. More importantly, new tools are developed and several crucial estimates are done non-perturbatively. In particular sharp upper bounds for the probability of long contours are derived using moment inequalities (GKS-inequalities) and the self-duality of the model. These new techniques allow to considerably shorten some parts of analysis of [DKS]. Similar ideas appear independently in [ACC], where similar questions are studied in the percolation model. Substantial improvements have been obtained by Ioffe [I1], [I2], who derived exact lower and upper bounds for the large deviations of the magnetization *for all temperatures below the critical one*. Deuschel and Pisztora [DPi] and [Pi] studied large deviations for percolation, Ising and Potts models, $D \geq 3$.

1.2 Isoperimetric inequality and large deviations

Consider an Ising model in the finite box

$$\Lambda_L := \{t = (t(1), t(2)) \in \mathbb{Z}^2 : -r_1 L \le t(1) \le r_1 L ;\ 0 \le t(2) \le 2r_2 L\} \ , \quad (1.1)$$

where $r_1, r_2 \in \mathbb{N}$ are two fixed numbers. Let $\sigma(t) = \pm 1$, $t \in \Lambda_L$, and define

$$H_{\Lambda_L} = - \sum_{\substack{\langle t,t'\rangle : \\ t,t' \in \Lambda_L}} \sigma(t)\sigma(t') - \sum_{\substack{t \in \Lambda_L : \\ t(2)=0}} h\sigma(t) - \sum_{\substack{t \in \Lambda_L : t(2)=2r_2 L \\ \text{or } t(1)=\pm r_1 L}} \sigma(t) \ . \quad (1.2)$$

Here $\langle t, t'\rangle$ denotes a pair of nearest neighbours points of the lattice $\mathbb{Z}^2$. The last two sums prescribed the boundary condition; h is a real parameter, the boundary magnetic field. The Gibbs probability measure associated with the energy function H_{Λ_L} and inverse temperature β is

$$\mu_L^h := \Xi(\Lambda_L)^{-1} \exp(-\beta H_{\Lambda_L}) \ ; \quad (1.3)$$

$\Xi(\Lambda_L)$ is the normalization constant,

$$\Xi(\Lambda_L) := \sum_{\substack{\sigma(t)=\pm 1 : \\ t \in \Lambda_L}} \exp(-\beta H_{\Lambda_L}) \ . \quad (1.4)$$

Probability with respect to that measure is also denoted by $P_L^h[\,\cdot\,]$.

We study the asymptotic behaviour of $P_L^h[A(m;c)]$ when $A(m;c)$ is the event

$$A(m;c) := \left\{ \left| \sum_{t \in \Lambda_L} \sigma(t) - m|\Lambda_L| \right| \le |\Lambda_L| \cdot L^{-c} \right\} \ . \quad (1.5)$$

The solution to this problem is given in terms of a variational problem, which is the following isoperimetric problem with constraints defined in the rectangle

$$Q := \{x = (x(1), x(2)) : -r_1 \le x(1) \le r_1 ;\ 0 \le x(2) \le 2r_2\} \ . \quad (1.6)$$

The horizontal bottom part of the boundary of Q plays a special role; we set

$$w_Q := \{x \in Q : x(2) = 0\} \ . \quad (1.7)$$

Suppose that $\hat{\tau} : \mathbb{R}^2 \to \mathbb{R}$ is a positive convex function, which is positively homogeneous of degree one and such that $\hat{\tau}(x) = \hat{\tau}(-x)$; the function $\hat{\tau}$ depends on the parameter β, $\hat{\tau}(x) = \hat{\tau}(x;\beta)$. Suppose also that $\hat{\tau}_{\mathrm{bd}} = \hat{\tau}_{\mathrm{bd}}(\beta, h) \in \mathbb{R}$ satisfies the condition

$$|\hat{\tau}_{\mathrm{bd}}| \le \hat{\tau}((1,0)) \ . \quad (1.8)$$

On the space of rectifiable curves in Q we introduce the functional

$$\mathrm{W}(\mathscr{C}) := \int_0^r \hat{\tau}(\dot{u}(t), \dot{v}(t))\, dt + \Big[\hat{\tau}_{\mathrm{bd}} - \hat{\tau}((1,0))\Big] |\mathscr{C} \cap w_Q| \ , \quad (1.9)$$

where $(u(t), v(t))$, $t \in [0, r]$, is a parametrization of the curve $\mathscr{C}$; $|\mathscr{C} \cap w_Q|$ is the Lebesgue measure of the subset $\mathscr{C} \cap w_Q$.

We define in the standard way the interior and exterior of $\mathscr{C}$; $\mathrm{vol}\,\mathscr{C}$ is the area of the interior of $\mathscr{C}$.

Isoperimetric problem: *Find the minimum of functional* W *among all closed curves* $\mathscr{C} \subset Q$, *with* $\mathrm{vol}\,\mathscr{C}$ *fixed.*

This isoperimetric problem is similar, but not equivalent to the problem treated by Wulff [Wu] in his theory of crystal. The solution in our case depends on the choice of $\hat{\tau}_{\mathrm{bd}}$ and on the shape of the box Q, see [KP]. The problem considered by Wulff was solved by [D]; a detailed study is done in [DKS]; see also [DP] for a recent completely different proof. Ideas from [DP] are used in the last section.

It is convenient to introduce $m^* = m^*(\beta)$, the order parameter of the Ising model (spontaneous magnetization). Suppose that $m^*(\beta) > 0$ i.e. $\beta > \beta_c$, the critical inverse temperature, and write the volume of $\mathscr{C}$,

$$\mathrm{vol}\,\mathscr{C} := 4r_1r_2\frac{m^* - m}{2m^*}, \quad -m^* < m < m^* \ . \tag{1.10}$$

(The parameter m has the interpretation of a mean magnetization: inside $\mathscr{C}$ we have the phase with magnetization $-m^*$ and outside with magnetization m^*.) We set

$$\mathrm{W}^*(m) := \inf\Big\{\mathrm{W}(\mathscr{C}) : \mathscr{C} \subset Q\,,\ \mathrm{vol}\,\mathscr{C} = 4r_1r_2\frac{m^* - m}{2m^*}\Big\} \ . \tag{1.11}$$

An important property is that the infimum can be computed with $\mathscr{C}$ the boundary of a convex body (use Jensen's inequality and the convexity of $\hat{\tau}$).

Theorem 1.1 *Let* $h \in \mathbb{R}$, $\beta > \beta_c$, $-m^* < m < m^*$, $c = 1/4 - \delta$ *with* $0 < \delta < 1/4$. *There exists a function* $\hat{\tau} : \mathbb{R}^2 \to \mathbb{R}$, *which is positive, convex, positively homogeneous of degree one, such that* $\hat{\tau}(x;\beta) = \hat{\tau}(-x;\beta)$ *and a real number* $\hat{\tau}_{\mathrm{bd}} = \hat{\tau}_{\mathrm{bd}}(\beta, h) \in \mathbb{R}$ *verifying* (1.8), *with the following property. If* W *is defined by* (1.9) *and* $0 < \eta < \delta$, *then for* L *large enough*

$$\left|\frac{1}{L}\ln P_L^h[A(m;c)] + \mathrm{W}^*(m)\right| \le O(L^{\eta-\delta}) \ . \tag{1.12}$$

We prove even stronger results, similar to those of [Pf2] (see Theorems 11.1 and 11.2). This allows us to take the continuum limit in which we scale every lengths by $1/L$, so that all results are formulated in the fixed box Q. Let $\mathscr{D}(m)$ be the set of macroscopic droplets at equilibrium in Q,

$$\mathscr{D}(m) := \left\{\mathscr{V} \subset Q : |\mathscr{V}| = \frac{m^* - m}{2m^*}|Q|\,,\ \mathrm{W}(\partial\mathscr{V}) = \mathrm{W}^*(m)\right\} \ . \tag{1.13}$$

For each $\mathscr{V} \in \mathscr{D}(m)$ we have a magnetization profile,

$$\rho_{\mathscr{V}}(x) := \begin{cases} m^* & \text{if } x \in Q\backslash\mathscr{V}\,, \\ -m^* & \text{if } x \in \mathscr{V} \ . \end{cases} \tag{1.14}$$

Let f be a real-valued function on Q; we set

$$d_1(f, \mathscr{D}(m)) := \inf_{\mathscr{V} \in \mathscr{D}(m)} \int_Q dx \, |f(x) - \rho_{\mathscr{V}}(x)| \; . \tag{1.15}$$

For each ω we define a magnetization profile $\rho_L(x;\omega)$ on Q. We subdivide the box Λ_L by the cells of a grid of lattice spacing $2[L^a]$. In each cell C of the grid $m_C(\omega)$ is the empirical magnetization,

$$m_C(\omega) := \frac{1}{|C|} \sum_{t \in C} \sigma(t)(\omega) \; . \tag{1.16}$$

Then we set, for each $x \in Q$,

$$\rho_L(x;\omega) := m_C(\omega) \; \text{ if } Lx \in C \tag{1.17}$$

where Lx is the point $x \in Q$ scaled by L.

Theorem 1.2 *Let $\beta > \beta_c$, $h \in \mathbb{R}$, $-m^* < m < m^*$ and $c = 1/4 - \delta > 0$. Then there exist a positive function $\bar{\varepsilon}(L)$ such that $\lim_{L\to\infty} \bar{\varepsilon}(L) = 0$ and two real numbers $\kappa > 0$ (see (12.42)) and $1 > a > 0$ such that for L large enough*

$$P_L^h[\{ d_1(\rho_L(\cdot\,;\omega), \mathscr{D}(m)) \leq \bar{\varepsilon}(L) \} \, | \, A(m;c)] \geq 1 - \exp\{-O(L^\kappa)\} \; . \tag{1.18}$$

1.3 Outline of the paper

The proof of Theorems 1.1 and 1.2 are long. The basic strategy is taken from [Pf2]. To understand the large deviations in presence of two Gibbs measures we must study the phase boundaries, which in dimension two are random lines. A large part of the paper is devoted to that question. We use a special feature of the model, self-duality, to identify problems concerning the phase boundaries (at low temperatures) with problems concerning the two-point function (at high temperatures), which is defined as the covariance of the Gibbs random field with free boundary condition. We can therefore identify the functions $\hat{\tau}$, respectively $\hat{\tau}_{\mathrm{bd}}$, with the decay-rates of the two-point function, respectively the boundary two-point function. The first part of the paper gives a complete non-perturbative exposition of the high-temperature representation of the model, which is then used to study the two-point function through its high-temperature representation, which is close to its representation via the random-cluster model. This part of the paper is not restricted to $D = 2$; it has its own interest and is written in an independent way. In the second part we prove our main theorems.

We would like to stress here that we do not use stability properties of the solution of the variational problem, even not the existence of such a solution. The only property, which is important, is that $W^*(m)$ can be computed using convex bodies. We also do not use the sharp triangle inequality property of $\hat{\tau}$ [I1].

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2 Ising model, contours, duality and inequalities

We give a list of the main definitions. The notion of contour of subsection 2.2 is essential for the whole paper. Additional definitions are given in parts I and II, when they are more specifically related to these parts. Throughout the paper $O(x)$ denotes a non-negative function of $x \in \mathbb{R}^+$, such that there exists a constant C with $O(x) \leq Cx$. The function $O(x)$ may be different at different places.

2.1 Ising model

We use the following notation and terminology.

The lattice is $\mathbb{Z}^2 := \{t = (t(1), t(2)) : t(i) \in \mathbb{Z}, i = 1, 2\}$. Its elements are called **sites**. We set $\mathbb{L} := \{t \in \mathbb{Z}^2 : t(2) \geq 0\}$ and $\Sigma_0 := \{t \in \mathbb{Z}^2 : t(2) = 0\}$. An **edge**, $e = \langle t, t' \rangle$, is an unordered pair of elements $t, t' \in \mathbb{Z}^2$ such that $|t(1) - t'(1)| + |t(2) - t'(2)| = 1$. We sometimes identify the edge $e = \langle t, t' \rangle$ with the unit length segment in $\mathbb{R}^2$ with end-points t, t'. The set of all edges is $\mathscr{E}$. An edge e is **adjacent** to $t \in \mathbb{Z}^2$ if $e = \langle t, t' \rangle$. Let $B \subset \mathscr{E}$; the **index** of a site t in B is the number of edges of B, which are adjacent to t. A **configuration** ω is an element of the product space $\Omega := \{-1, 1\}^{\mathbb{Z}^2}$. The value of ω at $t \in \mathbb{Z}^2$ is $\omega(t)$; $\sigma(t)$ is the random variable $\sigma(t)(\omega) := \omega(t)$. Let $\Lambda \subset \mathbb{Z}^2$; $\mathscr{F}_\Lambda$ is the σ-algebra generated by $\sigma(t)$, $t \in \Lambda$. We set $\mathscr{F} := \mathscr{F}_{\mathbb{Z}^2}$. A function f is **Λ-local** if it is $\mathscr{F}_\Lambda$-measurable and Λ finite.

Let $\Lambda \subset \mathbb{Z}^2$ be a finite subset; a configuration ω **satisfies the Λ^+-boundary condition** if $\omega(t) = 1$, $t \notin \Lambda$. For each edge e we introduce a non-negative number $J(e)$, called **coupling constant**. The energy in Λ for the configuration ω is

$$H_\Lambda(\omega) := -\sum_{\substack{e=\langle t,t'\rangle: \\ e\cap\Lambda\neq\emptyset}} J(e)[\sigma(t)(\omega)\sigma(t')(\omega) - 1] \ . \tag{2.1}$$

The **Gibbs measure in Λ with $+$ boundary condition** is by definition the measure on $(\Omega, \mathscr{F})$ given by the formula

$$\mu_\Lambda^+(\omega) := \begin{cases} \Xi^+(\Lambda)^{-1} \exp(-H_\Lambda(\omega)) & \text{if } \omega \text{ satisfies the } \Lambda^+\text{-bd. cond.,} \\ 0 & \text{otherwise.} \end{cases} \tag{2.2}$$

$\Xi^+(\Lambda)$ is the normalization constant so that μ_Λ^+ is a probability measure. Expectation value with respect to μ_Λ^+ is denoted by $P_\Lambda^+[\,\cdot\,]$, $\langle \cdot \rangle_\Lambda^+$ or $\langle \cdot \rangle_\Lambda^{+,J}$. In a similar way we define the **Gibbs measure in Λ with $-$ boundary condition**. The

free boundary Gibbs measure in Λ is by definition the probability measure on $\{-1,1\}^{\Lambda}$ defined by

$$\mu_{\Lambda} := \Xi(\Lambda)^{-1} \prod_{e=\langle t,t'\rangle \subset \Lambda} \exp(J(e)\sigma(t)\sigma(t')) \ . \tag{2.3}$$

$\Xi(\Lambda)$ is the normalization constant, called **partition function**,

$$\Xi(\Lambda) := \sum_{\sigma(t),\, t\in\Lambda} \prod_{e=\langle t,t'\rangle \subset \Lambda} \exp(J(e)\sigma(t)\sigma(t')) \ . \tag{2.4}$$

Expectation value with respect to μ_{Λ} is denoted by $P_{\Lambda}[\,\cdot\,]$, $\langle\,\cdot\,\rangle_{\Lambda}$ or $\langle\,\cdot\,\rangle_{\Lambda}^{J}$.

Let $\Lambda'_L := \{t \in \mathbb{Z}^2 : -L \le t(i) \le L,\ i = 1,2\}$. There exists a limiting measure μ^+ on $(\Omega, \mathscr{F})$, $\mu^+ := \lim_{L\to\infty} \mu^+_{\Lambda'_L}$. Expectation value with respect to μ^+ is denoted by $P^+[\,\cdot\,]$, $\langle\,\cdot\,\rangle^+$ or $\langle\,\cdot\,\rangle^{+,J}$. The same construction is possible with Λ^--boundary condition instead of Λ^+-boundary condition. The limiting measure is μ^-. Similarly, there exists a limiting measure μ on $(\Omega, \mathscr{F})$, $\mu := \lim_{L\to\infty} \mu_{\Lambda'_L}$. Expectation value with respect to μ is denoted by $P[\,\cdot\,]$, $\langle\,\cdot\,\rangle$ or $\langle\,\cdot\,\rangle^J$. Let $J(e) = \beta$ for every edge e. Then all measures defined above are translation-invariant. There exists $\beta_c := 1/2\log(1+\sqrt{2})$, called **critical coupling**, which is characterized by the following properties (see subsection 2.3): the measures $\langle\,\cdot\,\rangle^{+,\beta}$ and $\langle\,\cdot\,\rangle^{-,\beta}$ are equal if and only if $\beta \le \beta_c$; the **spontaneous magnetization** $m^* = m^*(\beta) = \langle\sigma(t)\rangle^{+,\beta}$ is strictly positive if and only if $\beta > \beta_c$. The **two-point function** $\langle\sigma(t_1)\sigma(t_2)\rangle$ is:

$$\langle\sigma(t_1)\sigma(t_2)\rangle := \lim_{L\to\infty} \langle\sigma(t_1)\sigma(t_2)\rangle_{\Lambda'_L} \ . \tag{2.5}$$

It is translation-invariant, $\langle\sigma(t_1)\sigma(t_2)\rangle = \langle\sigma(t_1+t)\sigma(t_2+t)\rangle$, $t \in \mathbb{Z}^2$. It is also invariant under axial symmetries with horizontal, vertical and diagonal axis. It is a non-trivial fact that

$$\langle\sigma(t_1)\sigma(t_2)\rangle = \langle\sigma(t_1)\sigma(t_2)\rangle^+ = \langle\sigma(t_1)\sigma(t_2)\rangle^- \ . \tag{2.6}$$

2.2 Contours

A **path** is an ordered sequence of sites and edges, $t_0, e_0, t_1, e_1, \ldots, t_n$, where $t_i \in \mathbb{Z}^2$ for all $i = 0, \ldots, n$, and $e_j = \langle t_j, t_{j+1}\rangle \in \mathscr{E}$, $j = 0, \ldots, n-1$. By definition all edges of a path are different, but not necessarily all sites of the path. The initial point of the path is t_0 and the final point is t_n. The initial edge of the path is e_0 and the final edge is e_{n-1}. A path is **closed** if its final point coincides with its initial point; otherwise it is **open**. We say that a path is in a subset $A \subset \mathbb{Z}^2$ if $t_i \in A$, $\forall\, i = 0, \ldots, n$; we say that it is in a subset $B \subset \mathscr{E}$ if $e_i \in B$, $\forall\, i = 0, \ldots, n-1$. A subset $A \subset \mathbb{Z}^2$ is **connected** if for any pair of elements $t, t' \in A$ there is a path in A with initial point $t_0 = t$ and final point $t_n = t'$. A subset $B \subset \mathscr{E}$ is **connected** if for any pair of elements $e, e' \in B$ there is a path in B with initial edge $e_0 = e$ and final edge $e_{n-1} = e'$. Let $t \in \mathbb{Z}^2$; the **plaquette** $p(t)$ of center t is the subset of $\mathbb{R}^2$,

$$p(t) := \{s = (s(1), s(2)) \in \mathbb{R}^2 : |s(i) - t(i)| \le 1/2\,, i = 1,2\} \ . \tag{2.7}$$

A subset $A \subset \mathbb{Z}^2$ is **simply connected** if the subset of $\mathbb{R}^2$, $\cup_{t \in A} p(t)$ is simply connected in $\mathbb{R}^2$. The **boundary** of $B \subset \mathscr{E}$ is the subset of $\mathbb{Z}^2$

$$\delta B := \left\{ t \in \mathbb{Z}^2 : \text{ index of } t \text{ in } B \text{ is odd} \right\} . \tag{2.8}$$

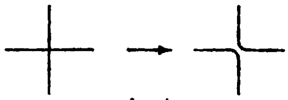
Let $B \subset \mathscr{E}$ be a finite non-empty subset. We decompose uniquely (up to orientation) B into a finite number of paths, such that they are pairwise disjoint, when considered as subsets of $\mathscr{E}$. (On the other hand sites may belong to two different paths.)

1. If $\delta B = \emptyset$, then choose an edge $e = \langle t, t' \rangle$ in B and set $t_0 := t$, $e_0 := e$ and $t_1 = t'$. The path is uniquely continued using rule A specified in the picture below and by requiring that it is maximal and that its final point is t_0. We have thus defined a closed path. Repeat this construction until all edges of B belong to some (closed) path.
2. If $\delta B \neq \emptyset$, then choose first $t \in \delta B$, and set $t_0 := t$. Then choose e_0 among the adjacent edges to t_0 according to rules A' specified in the picture below. Initial points are marked by dots in the picture specifying the rules A'. The path is uniquely continued using rules A and A' and by requiring that it is maximal and its final point $t_n \in \delta B$. We have thus defined an open path, since $t_0 \neq t_n$. Repeat this construction starting with a new point of δB until all points of δB are initial or final points of open paths; if there are still edges of B which do not belong to some paths, then do the construction 1. above.

The unoriented paths, which are defined by the above procedure, are called **contours**; a contour is **closed** or **open**, if the corresponding unoriented path is closed or open. The set of all contours of a configuration is denoted by $\underline{\gamma} = \{\gamma_1, \ldots, \gamma_n\}$. The **diameter** of a contour γ is

$$d(\gamma) := \max\{|t(1) - t'(1)| + |t(2) - t'(2)| : t, t' \in \gamma\} . \tag{2.9}$$

The **length** of a contour, $|\lambda|$, is the number of edges of λ. The length $|\underline{\gamma}|$ of a family of contours $\underline{\gamma}$ is the sum of the lengths of the contours of the family.



rule A

rules A'

the dots denote initial points of open paths

Let $\{\gamma_1, \ldots, \gamma_n\}$ be a family of contours. Let $\mathscr{E}(\gamma_1, \ldots, \gamma_n)$ be the set of all edges of these contours. We say that $\{\gamma_1, \ldots, \gamma_n\}$ is **compatible** if either $\mathscr{E}(\gamma_1, \ldots, \gamma_n) = \emptyset$, or $\{\gamma_1, \ldots, \gamma_n\}$ corresponds to the decomposition of the set

$\mathscr{E}(\gamma_1, \ldots, \gamma_n)$ into contours. If we want to add the condition that for a compatible family $\{\gamma_1, \ldots, \gamma_n\}$ all edges of $\mathscr{E}(\gamma_1, \ldots, \gamma_n)$ are pairs of points of $\Lambda \subset \mathbb{Z}^2$, then we say that the family is **Λ-compatible**. A contour is an unoriented path; it is however useful to choose sometimes an orientation and to consider a contour as a unit-speed parametrized curve in $\mathbb{R}^2$,

$$[0, |\lambda|] \ni s \mapsto \lambda(s) \in \mathbb{R}^2 \quad , \tag{2.10}$$

with initial point $\lambda(0) = t_1$ and final point t_2. The contour is closed iff $t_1 = t_2$; it is open if $\delta\lambda = \{t_1, t_2\}$.

2.3 Duality

An important concept is the duality transformation [W]. It relates the properties of the Ising model for the couplings $J(e) = \beta < \beta_c$ to the properties of the dual model for the couplings $J^*(e^*) = \beta^* > \beta_c$. When the dimension is two the model is self-dual. The proper framework to study the duality transformation is the theory of cell-complexes. However we need only elementary facts, so that we define the duality transformation as follows. It consists of a geometric and an analytic part.

1. *Geometric part*. The **dual lattice** $(\mathbb{Z}^2)^*$ is

$$(\mathbb{Z}^2)^* := \{t = (t(1), t(2)) : t(i) + 1/2 \in \mathbb{Z}\, , \, i = 1, 2\} \quad . \tag{2.11}$$

   To each edge e of $\mathbb{Z}^2$ we associate a **dual edge** e^* of $(\mathbb{Z}^2)^*$: it is the edge which crosses e in the middle, when both edges, e and e^*, are considered as unit length segments in $\mathbb{R}^2$.
2. *Analytic part*. The ***-transformation** is the transformation $x \mapsto x^*$ defined on $\{x : 0 \leq x \leq \infty\}$ into itself, given by the identity

$$\exp\{-2x\} = \tanh x^* \quad . \tag{2.12}$$

   The $*$-transformation is such that $(x^*)^* = x$; it has a unique fixed-point $x_c := 1/2 \log(1 + \sqrt{2})$.

The critical coupling β_c of the 2-dimensional Ising model has been identified to x_c in [KW], using the duality transformation. Let $J(e)$ be a non-negative coupling constant. The **dual coupling constant** for the dual edge e^*, $J^*(e^*)$, is defined by the $*$-transformation, $\exp\{-2\, J(e)\} = \tanh J^*(e^*)$.

2.4 Correlation inequalities

The main tools for analyzing the Ising model are correlations inequalities, also called moment inequalities. The Gibbs measures on Λ with Λ^+-boundary condition, Λ^--boundary condition or free boundary condition are special cases of the probability measure

$$\nu_\Lambda := \frac{\exp\Big\{\sum_{t,t'\in\Lambda} J(t,t')\sigma(t)\sigma(t') + \sum_{t\in\Lambda} k(t)\sigma(t)\Big\}}{\text{normalization}} \quad . \tag{2.13}$$

Let $A \subset \Lambda$ and set

$$\sigma_A := \prod_{t\in A} \sigma(t) \quad . \tag{2.14}$$

A function f is **increasing** if

$$\omega(t) \le \omega'(t) \quad \forall t \quad \Longrightarrow \quad f(\omega) \le f(\omega') \quad . \tag{2.15}$$

Proposition 2.1 *Let Λ be finite and $J(t,t') \ge 0$ for all $t,t' \in \Lambda$.*

1. *If $k(t) \ge 0$ for all t, then GKS-inequalities hold* [Gr]

$$\langle \sigma_A \rangle_{\nu_\Lambda} \ge 0 \quad , \tag{2.16}$$

$$\frac{d\,\langle \sigma_A \rangle_{\nu_\Lambda}}{dJ(t,t')} \ge 0 \quad . \tag{2.17}$$

2. *If $k(t) \in \mathbb{R}$ and f and g are two increasing functions, then FKG-inequality hold* [FKG]

$$\langle f\, g \rangle_{\nu_\Lambda} \ge \langle f \rangle_{\nu_\Lambda} \langle g \rangle_{\nu_\Lambda} \quad . \tag{2.18}$$

3. *If $k(t) \ge 0$ for all t, then* GHS-*inequalities hold* [GHS]

$$\frac{d^2\,\langle \sigma(t) \rangle_{\nu_\Lambda}}{dk(t')dk(t'')} \le 0 \quad . \tag{2.19}$$

Let $t_1, t_2 \in \mathbb{Z}^2$; a subset $B \subset \mathbb{Z}^2$ **separates** t_1 from t_2 if and only if $t_1 \notin B$, $t_2 \notin B$ and any path from t_1 to t_2 contains an element of B.

Proposition 2.2 *Let $J(e)$ be non-negative for all edges e and $t_1, t_2 \in \mathbb{Z}^2$.*

1. *If $B \subset \mathbb{Z}^2$ is a finite subset which separates t_1 from t_2, then*

$$\langle \sigma(t_1)\sigma(t_2) \rangle \le \sum_{t\in B} \langle \sigma(t_1)\sigma(t) \rangle \langle \sigma(t)\sigma(t_2) \rangle \quad . \tag{2.20}$$

2. *Let $J(e) = \beta$, $\beta > 0$, for all edges e and $t = (t(1), t(2)) \in \mathbb{Z}^2$, such that $0 \le t(1) \le t(2)$. Then*

$$\langle \sigma(0)\sigma(t') \rangle < \langle \sigma(0)\sigma(t) \rangle \quad , \tag{2.21}$$

if either $t'(1) = t(1) + 1$ and $t'(2) = t(2)$, or $t'(1) = t(1)$ and $t'(2) = t(2) + 1$, or $t'(1) = t(1) - 1$ and $t'(2) = t(2) + 1$.

3. *The two-point function $\langle \sigma(0)\sigma(t) \rangle$ is invariant under symmetries with horizontal, vertical and diagonal axis.*

Proposition 2.2.1. is proven in [Sim] and 2. in [MM]. (To prove the strict inequality follow the proof of [MM] and apply the inequalities of section 3.5 in [FP1].)

Part I: Ising model at high temperature

We present here results concerning the Ising model on $\mathbb{Z}^2$ for $\beta < \beta_c$ (i.e. above the critical temperature). They are essential tools for the study of the large deviations estimates at low temperature and are of interest independently of the large deviations analysis. Our method is non-perturbative and the validity of these results is for all $\beta < \beta_c$. Some results are not restricted to $D = 2$.

3 High-temperature representation

We recall the high-temperature representation of the model. The goal is to derive formula (3.13) which gives a representation of the two-point function in terms of random lines. The correct point of view here is to consider the free boundary Gibbs measure on a graph. All graphs considered in this paper are subgraphs of the graph $(\mathbb{Z}^2, \mathscr{E})$. We use the following conventions. If a subgraph of $(\mathbb{Z}^2, \mathscr{E})$ is specified by its set of vertices $\Lambda \subset \mathbb{Z}^2$, then by definition the set $\mathscr{E}(\Lambda)$ of its edges is the set of all edges $e = \langle t, t' \rangle$ with $t, t' \in \Lambda$. If it is specified by a set B of edges, then by definition the set of vertices is the set of all sites t of $\mathbb{Z}^2$ which are boundary points of edges of B.

The partition function $\Xi(\Lambda)$ can be written as

$$\sum_{\sigma(t),\, t\in\Lambda} \prod_{e=\langle t,t'\rangle \in \mathscr{E}(\Lambda)} \cosh J(e)(1 + \sigma(t)\sigma(t') \tanh J(e)) \ . \tag{3.1}$$

We expand the product in (3.1). Each term of the expansion is labelled by a set of edges $\langle t, t' \rangle$: we specify the edges corresponding to factors $\tanh J(e)$. Then we sum over $\sigma(t)$, $t \in \Lambda$; after summation only terms labelled by sets of edges with empty boundary (see (2.8)) give a non-zero contribution. Any term of the expansion of (3.1), which gives a non-zero contribution, can be uniquely labelled by a Λ-compatible family $\underline{\gamma}$ of closed contours. Let e be an edge, γ a contour and $\underline{\gamma}'$ a compatible family of contours; we set

$$w(e) := \tanh J(e), \quad w(\gamma) := \prod_{e\in\gamma} w(e), \quad w(\underline{\gamma}') := \prod_{\gamma\in\underline{\gamma}'} w(\gamma) \ . \tag{3.2}$$

If $\underline{\gamma}' = \emptyset$, then $w(\underline{\gamma}') := 1$; $w(\gamma)$ is the **weight** of γ. The partition function is

$$\Xi(\Lambda) = 2^{|\Lambda|} \prod_{e\in\mathscr{E}(\Lambda)} \cosh J(e) \sum_{\substack{\underline{\gamma}:\, \delta\underline{\gamma}=\emptyset \\ \Lambda\text{-comp.}}} w(\underline{\gamma}) \ . \tag{3.3}$$

It is natural to introduce the **normalized partition function**

$$Z(\Lambda) := \sum_{\substack{\underline{\gamma}:\, \delta\underline{\gamma}=\emptyset \\ \Lambda\text{-comp.}}} w(\underline{\gamma}) \ . \tag{3.4}$$

More generally, given any Λ-compatible family $\underline{\gamma}'$ we set

$$Z(\Lambda|\underline{\gamma}') := \sum_{\substack{\underline{\gamma}:\, \delta\underline{\gamma}=\emptyset \\ \underline{\gamma}\cup\underline{\gamma}'\ \Lambda-\text{comp.}}} w(\underline{\gamma}) \ . \tag{3.5}$$

In particular

$$Z(\Lambda) = Z(\Lambda|\emptyset) \ , \tag{3.6}$$

and

$$\Xi(\Lambda) = Z(\Lambda)2^{|\Lambda|} \prod_{e\in\mathscr{E}(\Lambda)} \cosh J(e) \ . \tag{3.7}$$

Remark: For normalized partition functions we may have

$$Z(\Lambda_1) = Z(\Lambda_2) \tag{3.8}$$

with $\Lambda_1 \neq \Lambda_2$. Indeed, the condition for equality is that the graphs $(\Lambda_1, \mathscr{E}(\Lambda_1))$ and $(\Lambda_2, \mathscr{E}(\Lambda_2))$ have the same set of closed contours.

On the set of all families of Λ-compatible *closed* contours we define a probability measure

$$P_\Lambda[\underline{\gamma}] := \frac{w(\underline{\gamma})}{Z(\Lambda)} \ . \tag{3.9}$$

Let $\underline{\gamma}'$ be a Λ-compatible family of contours, not necessarily closed. We set

$$q_\Lambda(\underline{\gamma}') := \begin{cases} w(\underline{\gamma}')\dfrac{Z(\Lambda|\underline{\gamma}')}{Z(\Lambda)} & \text{if } \underline{\gamma}' \ \Lambda\text{-compatible,} \\ 0 & \text{otherwise.} \end{cases} \tag{3.10}$$

If $\underline{\gamma}'$ is a Λ-compatible family of *closed* contours, then

$$q_\Lambda(\underline{\gamma}') = P_\Lambda[\{\underline{\gamma} : \underline{\gamma}' \subset \underline{\gamma}\}] \ . \tag{3.11}$$

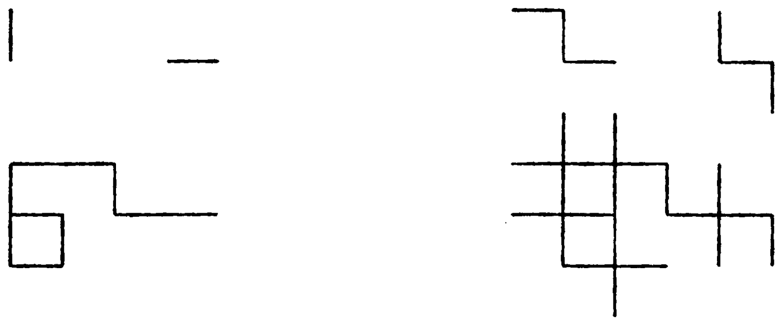
Let us consider the numerator of the two-point function $\langle\sigma(t_1)\sigma(t_2)\rangle_\Lambda$,

$$\sum_{\sigma(t),\, t\in\Lambda} \prod_{e=\langle t,t'\rangle\in\mathscr{E}(\Lambda)} \cosh J(e)(1+\sigma(t)\sigma(t')\tanh J(e))\sigma(t_1)\sigma(t_2) \ . \tag{3.12}$$

We expand the product as above. The presence of the variables $\sigma(t_1)$ and $\sigma(t_2)$ implies that the only terms in the expansion of the numerator of $\langle\sigma(t_1)\sigma(t_2)\rangle_\Lambda$, which give non-zero contributions, are those labelled by compatible families of contours containing one open contour λ such that $\delta\lambda = \{t_1, t_2\}$. The two-point function has the simple expression

$$\langle\sigma(t_1)\sigma(t_2)\rangle_\Lambda = Z(\Lambda)^{-1} \sum_{\substack{\lambda:\, \Lambda\text{-comp.} \\ \delta\lambda=\{t_1,t_2\}}} Z(\Lambda|\lambda)w(\lambda) = \sum_{\substack{\lambda:\, \Lambda\text{-comp.} \\ \delta\lambda=\{t_1,t_2\}}} q_\Lambda(\lambda) \ . \tag{3.13}$$

Definition 3.1 *Let e be an edge and $B(e)$ the set formed by e and all edges adjacent to e. The* **edge-boundary** *of e is the contour $\Delta(e) \ni e$ of the decomposition of $B(e)$ into contours. Let $A \subset \mathscr{E}$; the* **edge-boundary** *$\Delta(A)$ of A is $\Delta(A) := \cup_{e\in A}\Delta(e)$.*



Two edges e, e' and a contour γ with their edge-boundaries $\Delta(e)$, $\Delta(e')$, $\Delta(\gamma)$

The notions of compatibility and edge-boundary are related.

Lemma 3.1 *Let $\underline{\gamma}'$ be a family of compatible contours (closed or open). Then a non-empty compatible family of n closed contours $\underline{\gamma} = \{\gamma_1, \dots, \gamma_n\}$ is compatible with $\underline{\gamma}'$, that is $\underline{\gamma} \cup \underline{\gamma}'$ is compatible, if and only if no edge of γ_i is an edge of $\Delta(\underline{\gamma}')$, $\forall i = 1, \dots, n$.*

Proof. Suppose that $\underline{\gamma} \cup \underline{\gamma}'$ is compatible and $e_i \in \mathscr{E}(\gamma_i)$. Then $e_i \notin \mathscr{E}(\underline{\gamma}')$ since compatibility implies $\mathscr{E}(\underline{\gamma}') \cap \mathscr{E}(\underline{\gamma}) = \emptyset$. We show that $e_i \notin \Delta(\underline{\gamma}')$. Suppose that $e_i \in \Delta(\underline{\gamma}')$ and $\mathscr{E}(\gamma_i) \cap \mathscr{E}(\underline{\gamma}') = \emptyset$. Then one end-point t of e_i is of index at least three in $\Delta(\underline{\gamma}') \cup \mathscr{E}(\gamma_i)$: t has index 2 in γ_i since γ_i is closed, and at least one in $\mathscr{E}(\underline{\gamma}')$. This implies that the decomposition of $\underline{\gamma}' \cup \mathscr{E}(\gamma_i)$ into contours is not $(\gamma_i, \underline{\gamma}')$, hence γ_i and $\underline{\gamma}'$ are not compatible.

Suppose that e_1, e_2 is a pair of edges adjacent to a site t of $\underline{\gamma}'$ such that $\{e_1, e_2\} \cap \Delta(\underline{\gamma}') = \emptyset$. Then the decomposition of $\mathscr{E}(\underline{\gamma}') \cup \{e_1, e_2\}$ into contours is $(\underline{\gamma}', \{e_1, e_2\})$. Therefore, if $\mathscr{E}(\gamma_i) \cap \Delta(\underline{\gamma}') = \emptyset$, then γ_i is compatible with $\underline{\gamma}'$. □

Let $B \subset \mathscr{E}$ be a finite set of edges. Let $\mathscr{G}(B)$ be the graph defined by B. On $\mathscr{G}(B)$ we consider the Ising model defined by formula (2.4), taking the product over the edges of the graph. Its normalized partition function is $Z(\mathscr{G}(B))$.

Lemma 3.2 *Let Λ, $B \subset \mathscr{E}(\Lambda)$ and $\underline{\gamma}'$, a family of Λ-compatible contours, be given. If the graph $\mathscr{G}(B)$ has the same set of closed contours as the graph $\mathscr{G}(\mathscr{E}(\Lambda) \backslash \Delta(\underline{\gamma}'))$, then*

$$Z(\mathscr{G}(B)) = Z\left(\Lambda | \underline{\gamma}'\right) . \tag{3.14}$$

Proof. By hypothesis (see (3.8))

$$Z(\mathscr{G}(B)) = Z\left(\mathscr{G}\left(\mathscr{E}(\Lambda) \backslash \Delta\left(\underline{\gamma}'\right)\right)\right) . \tag{3.15}$$

The conclusion follows now from Lemma 3.1. □

4 Exponential decay-rate

4.1 Two-point function

In this subsection we suppose that $J(e) = \beta \geq 0$ for all edges e of $\mathbb{Z}^2$.

Definition 4.1 *Let $t_1, t_2 \in \mathbb{Z}^2$ and $n \in \mathbb{N}$. The* **decay-rate** *of the two-point function is defined on $\mathbb{Z}^2$ by*

$$\tau(t_2 - t_1; \beta) := \lim_{n\to\infty} -\frac{1}{n} \ln \langle \sigma(nt_1)\sigma(nt_2)\rangle^{\beta} \ . \tag{4.1}$$

Proposition 4.1 *Let $J(e) = \beta \geq 0$ for all edges e of $\mathbb{Z}^2$. The decay-rate has the following properties.*

1. *The decay-rate is non-negative and is a decreasing function of β.*
2. *If $t_1, t_2 \in \mathbb{Z}^2$, then $\langle \sigma(t_1)\sigma(t_2)\rangle^{\beta} \leq \exp\{-\tau(t_2 - t_1; \beta)\}$.*
3. *If $\beta \geq \beta_c$, then $\tau(t; \beta) = 0$ for all t.*
4. *If $\beta < \beta_c$, then $\tau(t; \beta) > 0$ for all $t \neq 0$.*
5. *Let $|\cdot|$ be the Euclidean norm; the function $\tau(t;\beta)/|t|$, $t \neq 0$, can be extended by continuity to any $x \in \mathbb{R}^2$ with $|x| = 1$; it is defined on $\mathbb{R}^2$ by $\tau(x;\beta) := |x| \cdot \tau(x/|x|;\beta)$. It is invariant under the axial symmetries with horizontal, vertical and diagonal axis. There exists a constant K such that for any x and y, $|x| = 1$ and $|y| = 1$,*
$$|\tau(x;\beta) - \tau(y;\beta)| \leq K|x - y| \ . \tag{4.2}$$

Proof. Points 3. and 4. are consequences of the duality transformation and of [LP] (see also remark below). The continuity statement (4.2) is proved in [Pf2] section 6 (Lemmas 6.4 and 6.5). For the sake of completeness we prove the existence of the limit (4.1) and point 2. of the proposition. By GKS-inequalities and translation invariance we have with $t = t_2 - t_1$ and $n = n_1 + n_2$,

$$\langle \sigma(0)\sigma(nt)\rangle \geq \langle \sigma(0)\sigma(n_1 t)\rangle\langle \sigma(n_1 t)\sigma(nt)\rangle \geq \langle \sigma(0)\sigma(n_1 t)\rangle\langle \sigma(0)\sigma(n_2 t)\rangle \ . \tag{4.3}$$

Hence the standard subadditivity argument gives

$$\lim_{n\to\infty} -\frac{1}{n}\ln\langle \sigma(0)\sigma(nt)\rangle = \inf_n -\frac{1}{n}\ln\langle \sigma(0)\sigma(nt)\rangle \ . \tag{4.4}$$

□

Remark: The decay-rate is known explicitly, see [MW] chapters XI and XII. In particular $x = (1, 0)$ is a minimum of $\tau(x;\beta)$ on the unit sphere.

4.2 Boundary two-point function

In this subsection we consider the model on the semi-infinite lattice $\mathbb{L}$. We choose the coupling constants

$$J(e) := \begin{cases} \beta \geq 0 & \text{if } e = \langle t,t'\rangle \text{ with } t \notin \Sigma_0 \text{ or } t' \notin \Sigma_0\,, \\ h\beta \geq 0 & \text{if } e = \langle t,t'\rangle \text{ with } t \in \Sigma_0 \text{ and } t' \in \Sigma_0 \end{cases} . \tag{4.5}$$

The **boundary two-point function** is defined for $t_1, t_2 \in \Sigma_0$ by

$$\langle \sigma(t_1)\sigma(t_2)\rangle_{\mathbb{L}} := \lim_{L\to\infty} \langle \sigma(t_1)\sigma(t_2)\rangle_{\Lambda'_L \cap \mathbb{L}} \ . \tag{4.6}$$

It is invariant under translations $t \in \Sigma_0$,

$$\langle \sigma(t_1 + t)\sigma(t_2 + t)\rangle_{\mathbb{L}} = \langle \sigma(t_1)\sigma(t_2)\rangle_{\mathbb{L}} \ . \tag{4.7}$$

Definition 4.2 *Let $t_1, t_2 \in \Sigma_0$ and $n \in \mathbb{N}$. The* **boundary decay-rate** *of the boundary two-point function is defined on Σ_0 by*

$$\tau_{\text{bd}}(t_2 - t_1; \beta, h) := \lim_{n\to\infty} -\frac{1}{n} \ln \langle \sigma(nt_1)\sigma(nt_2)\rangle_{\mathbb{L}}^{\beta,h} \ . \tag{4.8}$$

Proposition 4.2 *Let $\beta \geq 0$ and $h \geq 0$. The boundary decay-rate has the following properties.*

1. *The boundary decay-rate is non-negative and decreasing in β and h.*
2. *If $t_1, t_2 \in \Sigma_0$, then $\langle \sigma(t_1)\sigma(t_2)\rangle_{\mathbb{L}}^{\beta,h} \leq \exp\{-\tau_{\text{bd}}(t_2 - t_1; \beta, h)\}$.*
3. *For any $t \in \Sigma_0$, $\tau_{\text{bd}}(t; \beta, h) \leq \tau(t; \beta)$.*
4. *If $\beta \geq \beta_c$, then $\tau_{\text{bd}}(t; \beta, h) = 0$ for all $t \in \Sigma_0$.*
5. *If $\beta < \beta_c$, then $\tau_{\text{bd}}(t; \beta, h) > 0$ for all $t \neq 0$. Moreover there exists a positive $h_c(\beta) > 1$ so that for $t \in \Sigma_0$, $t \neq 0$,*

$$\begin{aligned} \tau(t;\beta) &= \tau_{\text{bd}}(t;\beta,h) \quad \textit{if} \quad h \leq h_c(\beta), \\ \tau(t;\beta) &> \tau_{\text{bd}}(t;\beta,h) \quad \textit{if} \quad h > h_c(\beta) \ . \end{aligned} \tag{4.9}$$

Remark: The proof of the first part of Proposition 4.2 is the same as that of Proposition 4.1. Points 3. to 5. are proven using the duality transformation and results of [FP2]. In particular, in [FP2] the following inequalities are proven for $\beta < \beta_c$,

$$1 \geq \exp\{2\beta(1 - h_c(\beta))\} \geq \frac{1 - \exp\{-2(\beta^* - \beta)\}}{1 - \exp\{-2(\beta + \beta^*)\}} \ . \tag{4.10}$$

Abraham computed $h_c(\beta)$ explicitly [Ab]; the boundary two-point function can also be computed explicitly [P]. Let $\hat{\beta}$ be defined by

$$\exp\left\{-2\hat{\beta}\right\} := \tanh \beta \ , \tag{4.11}$$

and $h_w(\hat{\beta})$ by the equation

$$\exp\left\{2\hat{\beta}\right\}\left\{\cosh 2\hat{\beta} - \cosh 2\hat{\beta} h_w\left(\hat{\beta}\right)\right\} = \sinh 2\hat{\beta} \ . \tag{4.12}$$

Then $h_c(\beta)$ is defined by

$$\exp\left\{-2\hat{\beta} h_w(\hat{\beta})\right\} = \tanh \beta h_c(\beta) \ . \tag{4.13}$$

5 Basic estimates

We prove basic estimates for the high-temperature representation. These estimates are non-perturbative, valid for all $\beta < \beta_c$[1] and *not restricted* to dimension two. The main ideas are from [Pf2] section 6. The basic quantity in the high-temperature representation is $q_\Lambda(\underline{\gamma})$, see (3.10); it is a function of the coupling constants J. The dependence of $q_\Lambda(\underline{\gamma})$ on J is studied in Lemmas 5.2 and 5.3. Lemmas 5.4 and 5.5 are essential.

All results are established for a finite graph Λ, which is always a subgraph of $\mathbb{Z}^2$ or $\mathbb{L}$. Implicitly all contours are defined on the graph Λ. We do not always write explicitly the parameters β and h to simplify the notations. However, in cases we want to emphasize the coupling constants J in the partition functions we write for example $Z(\Lambda; J)$ instead of $Z(\Lambda)$.

Lemma 5.1 *Let $\beta < \beta_c$.*

1. *If $J(e) = \beta$ for all edges e, then for any $t_1, t_2 \in \Lambda$,*

$$\sum_{\substack{\lambda: \\ \delta\lambda = \{t_1, t_2\}}} q_\Lambda(\lambda) \leq \langle \sigma(t_2)\sigma(t_1)\rangle^\beta \leq \exp\{-\tau(t_2 - t_1; \beta)\} \ . \tag{5.1}$$

2. *If $\Lambda \subset \mathbb{L}$, $h \geq 0$ and $J(e)$ is defined by* (4.5), *then for any $t_1, t_2 \in \Sigma_0 \cap \Lambda$,*

$$\sum_{\substack{\lambda: \\ \delta\lambda = \{t_1, t_2\}}} q_\Lambda(\lambda) \leq \langle \sigma(t_2)\sigma(t_1)\rangle^J_{\mathbb{L}} \leq \exp\{-\tau_{\mathrm{bd}}(t_2 - t_1; \beta, h)\} \ . \tag{5.2}$$

3. *If $\Lambda \subset \mathbb{L}$, $h \geq 0$ and $J(e)$ is defined by* (4.5), *then for any $t_1, t_2 \in \Lambda$,*

$$\sum_{\substack{\lambda: \mathscr{E}(\lambda) \cap \mathscr{E}(\Sigma_0) = \emptyset \\ \delta\lambda = \{t_1, t_2\}}} q_\Lambda(\lambda) \leq \exp\{-\tau(t_2 - t_1; \beta)\} \ . \tag{5.3}$$

Proof. 1. follows from formula (3.13), GKS inequalities and Proposition 4.1.
2. is proved in the same manner.
3. The case $h \leq 1$ is easy. Indeed,

$$\sum_{\substack{\lambda: \mathscr{E}(\lambda) \cap \mathscr{E}(\Sigma_0) = \emptyset \\ \delta\lambda = \{t_1, t_2\}}} q_\Lambda(\lambda) \leq \sum_{\substack{\lambda: \\ \delta\lambda = \{t_1, t_2\}}} q_\Lambda(\lambda) = \langle \sigma(t_1)\sigma(t_2)\rangle^{\beta,h}_\Lambda \ . \tag{5.4}$$

$\langle \sigma(t_1)\sigma(t_2)\rangle^{\beta,h}_\Lambda$ is increasing in h and in Λ. Therefore $\langle \sigma(t_1)\sigma(t_2)\rangle^{\beta,h}_\Lambda \leq \langle \sigma(t_1)\sigma(t_2)\rangle^\beta$. The result follows from 1.

[1] It is natural in this context to define β_c as the smallest β such that $\tau(t; \beta)$ is equal to zero (see Def. 4.1). Due to results of Aizenman, Barsky and Fernandez [ABF] this β_c coincides with the previous definition in terms of the spontaneous magnetization. For $D = 2$ this follows from [LP], see Proposition 4.1.

Let $h \geq 1$. The condition $\mathscr{E}(\lambda) \cap \mathscr{E}(\Sigma_0) = \emptyset$ implies that $w(\lambda)$ is independent of h; assume that

$$\frac{Z(\Lambda|\lambda;h)}{Z(\Lambda;h)} \leq \frac{Z(\Lambda|\lambda;1)}{Z(\Lambda;1)} \quad \text{if} \quad h \geq 1 \ , \tag{5.5}$$

which is proven in the next lemma. Then

$$\sum_{\substack{\lambda:\mathscr{E}(\lambda)\cap\mathscr{E}(\Sigma_0)=\emptyset \\ \delta\lambda=\{t_1,t_2\}}} q_\Lambda(\lambda;h) \leq \sum_{\substack{\lambda:\mathscr{E}(\lambda)\cap\mathscr{E}(\Sigma_0)=\emptyset \\ \delta\lambda=\{t_1,t_2\}}} q_\Lambda(\lambda;1) \tag{5.6}$$

and we conclude using the preceding case. □

Lemma 5.2 *Let $\underline{\gamma}'$ be a Λ-compatible family of contours. Then*

$$\frac{Z\left(\Lambda|\underline{\gamma}';J\right)}{Z(\Lambda;J)} \tag{5.7}$$

is decreasing in $J(e)$ for any e.

Proof. Let

$$B := \mathscr{E}(\Lambda)\backslash\Delta\left(\underline{\gamma}'\right) \ , \tag{5.8}$$

and $\mathscr{G}(B)$ the graph defined by this set of edges. Let $\Lambda(B)$ be the set of vertices of $\mathscr{G}(B)$. By Lemma 3.2 we have

$$Z\left(\Lambda|\underline{\gamma}';J\right) = Z(\mathscr{G}(B)) \ . \tag{5.9}$$

Therefore

$$\ln\frac{Z\left(\Lambda|\underline{\gamma}';J\right)}{Z(\Lambda;J)} = \ln\frac{\Xi(\Lambda(B))}{\Xi(\Lambda)} + \ln\prod_{e\in\mathscr{E}(\Lambda)\backslash B}\cosh J(e) + (|\Lambda| - |\Lambda(B)|)\ln 2 \ . \tag{5.10}$$

If $e = \langle t,t'\rangle \in B$, then

$$\frac{\partial}{\partial J(e)}\ln\frac{Z\left(\Lambda|\underline{\gamma}';J\right)}{Z(\Lambda;J)} = \langle\sigma(t)\sigma(t')\rangle_{\Lambda(B)} - \langle\sigma(t)\sigma(t')\rangle_\Lambda \leq 0 \ , \tag{5.11}$$

by GKS-inequalities, since $\Lambda(B) \subset \Lambda$. If $e = \langle t,t'\rangle \in \mathscr{E}(\Lambda)\backslash B$, then

$$\frac{\partial}{\partial J(e)}\ln\frac{Z\left(\Lambda|\underline{\gamma}';J\right)}{Z(\Lambda;J)} = -\langle\sigma(t)\sigma(t')\rangle_\Lambda + \tanh J(e) \leq 0 \ , \tag{5.12}$$

since by GKS-inequalities

$$\langle\sigma(t)\sigma(t')\rangle_\Lambda \geq \langle\sigma(t)\sigma(t')\rangle_{\{t,t'\}} = \tanh J(e) \ . \tag{5.13}$$

□

Lemma 5.3 *Let $J(e) \geq 0$. Let $\underline{\gamma}'$ be a Λ-compatible family of contours.*

1. *$q_\Lambda(\underline{\gamma}';J)$ is decreasing in $J(e)$ for all edges $e \in \mathscr{E}(\Lambda)\backslash\mathscr{E}(\underline{\gamma}')$. In particular $q_\Lambda(\underline{\gamma}';J)$ is decreasing in Λ.*
2. *Let $\Lambda'_L := \{t \in \mathbb{Z}^2 : |t(i)| \leq L\}$. Then the following limits exist,*
$$q\left(\underline{\gamma}'\right) := \lim_{L\to\infty} q_{\Lambda'_L}\left(\underline{\gamma}'\right), \quad q_{\mathbb{L}}\left(\underline{\gamma}'\right) := \lim_{L\to\infty} q_{\Lambda'_L\cap\mathbb{L}}\left(\underline{\gamma}'\right) . \tag{5.14}$$
Moreover, $q(\underline{\gamma}') \leq q_\Lambda(\underline{\gamma}')$ and if $\Lambda \subset \mathbb{L}$, then $q_{\mathbb{L}}(\underline{\gamma}') \leq q_\Lambda(\underline{\gamma}')$.
3. *Let J be given by (4.5) and $\Lambda \subset \mathbb{L}$. If $0 \leq h \leq 1$, then $q_\Lambda(\underline{\gamma}';\beta,h) \geq q(\underline{\gamma}';\beta)$. If $1 \leq h$, then $q_\Lambda(\underline{\gamma}';\beta,h) \geq q_{\mathbb{L}}(\underline{\gamma}';\beta,h)$.*
4. *Let J be given by (4.5) and $\Lambda \subset \mathbb{L}$. Assume that no edge of $\Delta(\underline{\gamma}')$ is adjacent to a site of Σ_0. Then*
$$q_\Lambda\left(\underline{\gamma}';J\right) \geq q\left(\underline{\gamma}';\beta\right)$$
$$\cdot \exp\left\{-\beta^2 \sum_{e=\langle t,t'\rangle\in\Delta\left(\underline{\gamma}'\right)} \sum_{\substack{t'':\\ t''(2)=1}} \left(e^{-\tau(t-t'';\beta)} + e^{-\tau(t'-t'';\beta)}\right)\right\} .$$

Proof. Lemma 5.2 implies that $q_\Lambda(\underline{\gamma}';J)$ is decreasing in J and therefore also in Λ. This proves 1., 2. and 3..

We prove 4.. We have $q_{\mathbb{L}\backslash\Sigma_0}(\underline{\gamma}';J) \geq q(\underline{\gamma}';\beta)$; hence
$$q_\Lambda\left(\underline{\gamma}';J\right) \geq q_{\mathbb{L}}\left(\underline{\gamma}';J\right) = q_{\mathbb{L}\backslash\Sigma_0}\left(\underline{\gamma}';J\right)\frac{q_{\mathbb{L}}\left(\underline{\gamma}';J\right)}{q_{\mathbb{L}\backslash\Sigma_0}\left(\underline{\gamma}';J\right)} \geq q\left(\underline{\gamma}';\beta\right)\frac{q_{\mathbb{L}}\left(\underline{\gamma}';J\right)}{q_{\mathbb{L}\backslash\Sigma_0}\left(\underline{\gamma}';J\right)} . \tag{5.16}$$

We estimate the last quotient. Let
$$J_s(e) := \begin{cases} J(e) & \text{if} \quad e \notin \Delta\left(\underline{\gamma}'\right), \\ sJ(e) & \text{if} \quad e \in \Delta\left(\underline{\gamma}'\right) . \end{cases} \tag{5.17}$$

Then $Z(\Lambda;J_0) = Z(\Lambda|\underline{\gamma}';J)$, since only family of closed contours $\underline{\gamma}$, such that $\underline{\gamma} \cap \Delta_\Lambda(\underline{\gamma}') = \emptyset$, give a nonzero contribution to $Z(\Lambda;J_0)$. On the other hand we have $Z(\Lambda;J_1) = Z(\Lambda;J)$. Therefore
$$\begin{aligned}\ln\frac{Z\left(\Lambda|\underline{\gamma}';J\right)}{Z(\Lambda;J)} &= \ln\frac{\Xi(\Lambda;J_0)}{\Xi(\Lambda;J_1)} + \ln\prod_{e\in\Delta(\underline{\gamma}')}\cosh J(e) \\ &= -\int_0^1 ds\,\frac{d}{ds}\ln\Xi(\Lambda;J_s) + \ln\prod_{e\in\Delta(\underline{\gamma}')}\cosh J(e) \\ &= -\sum_{e=\langle t,t'\rangle\in\Delta(\underline{\gamma}')} J(e)\int_0^1 ds\,\langle\sigma(t)\sigma(t')\rangle_\Lambda^{J_s} + \ln\prod_{e\in\Delta(\underline{\gamma}')}\cosh J(e) .\end{aligned} \tag{5.18}$$

Therefore

$$\frac{q_{\mathbb{L}}\left(\underline{\gamma}';J\right)}{q_{\mathbb{L}\backslash\Sigma_0}\left(\underline{\gamma}';J\right)} = \exp\left\{ - \sum_{e=\langle t,t'\rangle\in\Delta(\underline{\gamma}')} \beta \int_0^1 ds \left(\langle\sigma(t)\sigma(t')\rangle_{\mathbb{L}}^{J_s} - \langle\sigma(t)\sigma(t')\rangle_{\mathbb{L}\backslash\Sigma_0}^{J_s} \right) \right\} . \tag{5.19}$$

GKS-inequalities give

$$\langle\sigma(t)\sigma(t')\rangle_{\mathbb{L}}^{J_s} - \langle\sigma(t)\sigma(t')\rangle_{\mathbb{L}\backslash\Sigma_0}^{J_s} \leq \langle\sigma(t)\sigma(t')\rangle_{\mathbb{L}\backslash\Sigma_0}^{+,J_s} - \langle\sigma(t)\sigma(t')\rangle_{\mathbb{L}\backslash\Sigma_0}^{J_s} . \tag{5.20}$$

The $(\mathbb{L}\backslash\Sigma_0)^+$-boundary condition in (5.20) is obtained by introducing an external field on Σ_0 and then letting this field go to ∞. Notice that $-\sigma(t)\sigma(t') + \sigma(t) + \sigma(t')$ is an increasing function, so that by FKG-inequalities

$$\begin{aligned}&\langle\sigma(t)\sigma(t')\rangle_{\mathbb{L}\backslash\Sigma_0}^{+,J_s} - \langle\sigma(t)\sigma(t')\rangle_{\mathbb{L}\backslash\Sigma_0}^{J_s} \\ &\quad \leq \langle\sigma(t)\rangle_{\mathbb{L}\backslash\Sigma_0}^{+,J_s} - \langle\sigma(t)\rangle_{\mathbb{L}\backslash\Sigma_0}^{J_s} + \langle\sigma(t')\rangle_{\mathbb{L}\backslash\Sigma_0}^{+,J_s} - \langle\sigma(t')\rangle_{\mathbb{L}\backslash\Sigma_0}^{J_s} .\end{aligned} \tag{5.21}$$

Define new coupling constants $J''(e)$,

$$J''(e) := \begin{cases} J_s(e) & \text{if } e \text{ not adjacent to } \Sigma_0, \\ aJ_s(e) & \text{otherwise.} \end{cases} \tag{5.22}$$

$\langle\sigma(t)\rangle_{\mathbb{L}\backslash\Sigma_0}^{+,J_s} = \langle\sigma(t)\rangle_{\mathbb{L}\backslash\Sigma_0}^{+,J''}$ with $a = 1$ and $\langle\sigma(t)\rangle_{\mathbb{L}\backslash\Sigma_0}^{J_s} = \langle\sigma(t)\rangle_{\mathbb{L}\backslash\Sigma_0}^{+,J''}$ with $a = 0$. Hence

$$\begin{aligned}&\langle\sigma(t)\sigma(t')\rangle_{\mathbb{L}\backslash\Sigma_0}^{+,J_s} - \langle\sigma(t)\sigma(t')\rangle_{\mathbb{L}\backslash\Sigma_0}^{J_s} \\ &\quad \leq \sum_{\substack{t'':\\ t''(2)=1}} \beta \int_0^1 da \left(\langle\sigma(t)\,;\,\sigma(t'')\rangle_{\mathbb{L}\backslash\Sigma_0}^{+,J''} + \langle\sigma(t')\,;\,\sigma(t'')\rangle_{\mathbb{L}\backslash\Sigma_0}^{+,J''} \right) ,\end{aligned} \tag{5.23}$$

where

$$\langle\sigma(t)\,;\,\sigma(t'')\rangle_{\mathbb{L}\backslash\Sigma_0}^{+,J''} := \langle\sigma(t)\,\sigma(t'')\rangle_{\mathbb{L}\backslash\Sigma_0}^{+,J''} - \langle\sigma(t)\rangle_{\mathbb{L}\backslash\Sigma_0}^{+,J''} \cdot \langle\sigma(t'')\rangle_{\mathbb{L}\backslash\Sigma_0}^{+,J''} . \tag{5.24}$$

GHS-inequalities imply that $\langle\sigma(t)\,;\,\sigma(t'')\rangle_{\mathbb{L}\backslash\Sigma_0}^{+,J''}$ is decreasing in a; putting $a = 0$ we get

$$\langle\sigma(t)\,;\,\sigma(t'')\rangle_{\mathbb{L}\backslash\Sigma_0}^{+,J''} \leq \langle\sigma(t)\,;\,\sigma(t'')\rangle_{\mathbb{L}\backslash\Sigma_0}^{J''} = \langle\sigma(t)\,\sigma(t'')\rangle_{\mathbb{L}\backslash\Sigma_0}^{J_s} , \tag{5.25}$$

because the last expectation value is with respect to the Gibbs measure on $\mathbb{L}\backslash\Sigma_0$ with free boundary condition and consequently by symmetry

$$\langle\sigma(t)\rangle_{\mathbb{L}\backslash\Sigma_0}^{J_s} = 0 . \tag{5.26}$$

GKS-inequalities imply now

$$\langle\sigma(t)\sigma(t'')\rangle_{\mathbb{L}\backslash\Sigma_0}^{J_s} \leq \langle\sigma(t)\sigma(t'')\rangle^{\beta} \leq \exp\{-\tau(t - t'';\beta)\} . \tag{5.27}$$

Summarizing, we have

$$\begin{aligned} 0 &\le \langle\sigma(t)\sigma(t')\rangle^{+,J_s}_{\mathbb{L}\backslash\Sigma_0} - \langle\sigma(t)\sigma(t')\rangle^{J_s}_{\mathbb{L}\backslash\Sigma_0} \\ &\le \beta \sum_{\substack{t'':\\ t''(2)=1}} \left(\exp\{-\tau(t-t'';\beta)\} + \exp\{-\tau(t'-t'';\beta)\}\right) \ . \end{aligned} \tag{5.28}$$

We conclude using (5.19) and (5.28). □

Lemma 5.4 *Let $J(e) \ge 0$ and $t, t_1, t_2 \in \Lambda$. Let λ_1 be an open contour with $\delta\lambda_1 = \{t_1, t\}$ and λ_2 be an open contour with $\delta\lambda_2 = \{t, t_2\}$, so that $\lambda = \lambda_1 \cup \lambda_2$ is an open contour with $\delta\lambda = \{t_1, t_2\}$. Then*

$$\begin{aligned} \sum_{\substack{\lambda:\delta\lambda=\{t_1,t_2\}\\ t\in\lambda,\,\Lambda\text{-comp}}} q_\Lambda(\lambda) &\le \sum_{\substack{\lambda_1:\delta\lambda_1=\{t_1,t\}\\ \Lambda\text{-comp}}} q_\Lambda(\lambda_1) \sum_{\substack{\lambda_2:\delta\lambda_2=\{t,t_2\}\\ \Lambda\text{-comp}}} q_\Lambda(\lambda_2) \\ &= \langle\sigma(t_1)\sigma(t)\rangle_\Lambda \langle\sigma(t)\sigma(t_2)\rangle_\Lambda ; \end{aligned} \tag{5.29}$$

$$\sum_{\substack{\lambda_1:\delta\lambda_1=\{t_1,t\}\\ \Lambda\text{-comp}}} q_\Lambda(\lambda_1 \cup \lambda_2) \le 2\, q_\Lambda(\lambda_2) \sum_{\substack{\lambda_1:\delta\lambda_1=\{t_1,t\}\\ \Lambda\text{-comp}}} q_\Lambda(\lambda_1) \ ; \tag{5.30}$$

$$q_\Lambda(\lambda) \ge q_\Lambda(\lambda_1) q_\Lambda(\lambda_2) \ . \tag{5.31}$$

Proof. We prove (5.29).

Let λ be an open contour with $\delta\lambda = \{t_1, t_2\}$, considered as a unit-speed parametrized curve. Let s^* be the largest $s \in [0, |\lambda|]$ so that $\lambda(s^*) = t$. We decompose λ into $\{\lambda_1, \lambda_2\}$ by cutting λ at t and we set

$$\lambda_1 = \{\lambda(s) : s \in [0, s^*]\} \quad \text{and} \quad \lambda_2 = \{\lambda(s) : s \in [s^*, |\lambda|]\} \ . \tag{5.32}$$

Notice that by definition $\lambda_2(s) \ne t$ for any $s > s^*$, that is, λ_2 has exactly one adjacent edge to t. (The way we cut λ depends on the choice of the orientation of λ.) Define the graph $\mathscr{G}^\#(\lambda_2)$ by its set of bonds,

$$\{e^\#\} \cup \mathscr{E}(\Lambda)\backslash\Delta(\lambda_2) \ ; \tag{5.33}$$

$e^\#$ is the edge of $\Delta(\lambda_2)$, which is adjacent to t, but does not belong to λ_2 [2]. We claim that

$$Z(\Lambda|\lambda_1 \cup \lambda_2) = Z\big(\mathscr{G}^\#(\lambda_2)|\lambda_1\big) \ . \tag{5.34}$$

First, by definition

$$\Delta(\lambda_1 \cup \lambda_2) = \Delta(\lambda_1) \cup \Delta(\lambda_2) \ . \tag{5.35}$$

Let $\underline{\gamma}$, $\delta\underline{\gamma} = \emptyset$, be Λ-compatible with $\lambda_1 \cup \lambda_2$. By Lemma 3.1

$$\mathscr{E}(\underline{\gamma}) \cap \Delta(\lambda_1 \cup \lambda_2) = \emptyset \ . \tag{5.36}$$

Therefore by (5.35)

[2] We want that λ_1 be a contour of the graph $\mathscr{G}^\#(\lambda_2)$, so that $e^\#$ must be an edge of $\mathscr{G}^\#(\lambda_2)$.

$$\mathscr{E}(\underline{\gamma}) \subset \mathscr{E}\big(\mathscr{G}^{\#}(\lambda_2)\big) \ , \tag{5.37}$$

and

$$\mathscr{E}(\underline{\gamma}) \cap \Delta(\lambda_1) = \emptyset \ . \tag{5.38}$$

This implies that $\underline{\gamma}$ is $\Lambda^{\#}$-compatible with λ_1. Conversely, if $\underline{\gamma}$, $\delta\underline{\gamma} = \emptyset$, is $\mathscr{G}^{\#}(\lambda_2)$-compatible with λ_1, then

$$\mathscr{E}(\underline{\gamma}) \cap \Delta(\lambda_1) = \emptyset \ . \tag{5.39}$$

Suppose that e_1, e_2 are two edges of $\mathscr{E}(\underline{\gamma})$, which are adjacent to t. This is possible only if the edge e^* of λ_2, which is adjacent to t, belongs to $\Delta(\lambda_1)$. But this means that $\underline{\gamma}$ is Λ-compatible with $\lambda_1 \cup \lambda_2$. Using this result we get

$$\begin{aligned}\sum_{\substack{\lambda:\delta\lambda=\{t_1,t_2\}\\ t\in\lambda}} \frac{Z(\Lambda|\lambda)}{Z(\Lambda)} w(\lambda) &= \sum_{\{\lambda_1,\lambda_2\}=\lambda} \frac{Z(\Lambda|\lambda)}{Z(\Lambda)} w(\lambda_1) w(\lambda_2) \\ &= \sum_{\{\lambda_1,\lambda_2\}=\lambda} w(\lambda_1) \frac{Z(\Lambda|\lambda_1 \cup \lambda_2)}{Z\big(\mathscr{G}^{\#}(\lambda_2)\big)} w(\lambda_2) \frac{Z\big(\mathscr{G}^{\#}(\lambda_2)\big)}{Z(\Lambda)} \ . \end{aligned} \tag{5.40}$$

If we sum in (5.40) over all λ_1, given λ_2, and use (5.34), then we obtain the two-point function of the Ising model on the graph $\mathscr{G}^{\#}(\lambda_2)$:

$$\sum_{\lambda_1} w(\lambda_1) \frac{Z\big(\mathscr{G}^{\#}(\lambda_2)|\lambda_1\big)}{Z(\mathscr{G}(\lambda_2))} = \langle \sigma(t_1)\sigma(t) \rangle_{\mathscr{G}^{\#}(\lambda_2)} \leq \langle \sigma(t_1)\sigma(t) \rangle_{\Lambda} = \sum_{\lambda_1:\delta\lambda_1=\{t_1,t\}} q_{\Lambda}(\lambda_1) \ . \tag{5.41}$$

We can interpret in a similar way the remaining sum,

$$\sum_{\lambda_2} w(\lambda_2) \frac{Z\big(\mathscr{G}^{\#}(\lambda_2)\big)}{Z(\Lambda)} \ . \tag{5.42}$$

We have

$$\begin{aligned} Z\big(\mathscr{G}^{\#}(\lambda_2)\big) &= \sum_{\substack{\underline{\gamma}:\delta\underline{\gamma}=\emptyset\\ \mathscr{G}^{\#}(\lambda_2)\text{-comp}}} w(\underline{\gamma}) \\ &= \sum_{\substack{\underline{\gamma}:\delta\underline{\gamma}=\emptyset,\, e^{\#}\notin\underline{\gamma}\\ \mathscr{G}^{\#}(\lambda_2)\text{-comp}}} w(\underline{\gamma}) + \sum_{\substack{\underline{\gamma}:\delta\underline{\gamma}=\emptyset,\, e^{\#}\in\underline{\gamma}\\ \mathscr{G}^{\#}(\lambda_2)\text{-comp}}} w(\underline{\gamma}) \ . \end{aligned} \tag{5.43}$$

By construction, all open contours λ_2 have only one edge adjacent to t. In the first sum all closed contours of $\underline{\gamma}$ are compatible with λ_2, while in the second sum there is one closed contour containing $e^{\#}$; we glue this contour together with λ_2 to form a new open contour of index 3 at t. Therefore

$$\sum_{\lambda_2} w(\lambda_2) \frac{Z\big(\mathscr{G}^{\#}(\lambda_2)\big)}{Z(\Lambda)} = \sum_{\substack{\lambda:\delta\lambda=\{t,t_2\}\\ \Lambda\text{-comp}}} w(\lambda) \frac{Z(\Lambda|\lambda)}{Z(\Lambda)} = \sum_{\substack{\lambda:\delta\lambda=\{t,t_2\}\\ \Lambda\text{-comp}}} q_{\Lambda}(\lambda) = \langle \sigma(t)\sigma(t_2) \rangle_{\Lambda} \ . \tag{5.44}$$

We prove (5.30). The first part of the proof is the same up to (5.40) and (5.41), so that we get

$$\sum_{\substack{\lambda_1:\delta\lambda_1=\{t_1,t\}\\ \Lambda\text{-comp}}} q_\Lambda(\lambda_1\cup\lambda_2) \le w(\lambda_2)\frac{Z\big(\mathscr{G}^{\#}(\lambda_2)\big)}{Z(\Lambda)} \sum_{\substack{\lambda_1:\delta\lambda_1=\{t_1,t\}\\ \Lambda\text{-comp}}} q_\Lambda(\lambda_1) \ . \tag{5.45}$$

Let $\mathscr{G}(\lambda_2)$ be defined by its set of bonds, which is $\mathscr{E}(\Lambda)\backslash\Delta(\lambda_2)$. Then we write

$$w(\lambda_2)\frac{Z\big(\mathscr{G}^{\#}(\lambda_2)\big)}{Z(\Lambda)} = w(\lambda_2)\frac{Z(\mathscr{G}(\lambda_2))}{Z(\Lambda)}\frac{Z\big(\mathscr{G}^{\#}(\lambda_2)\big)}{Z(\mathscr{G}(\lambda_2))} \ . \tag{5.46}$$

Using (3.7) we can bound the last quotient by 2. We conclude using Lemma 3.2,

$$w(\lambda_2)\frac{Z(\mathscr{G}(\lambda_2))}{Z(\Lambda)} = w(\lambda_2)\frac{Z(\Lambda|\lambda_2)}{Z(\Lambda)} = q_\Lambda(\lambda_2) \ . \tag{5.47}$$

We prove (5.31). Since $Z(\mathscr{G}^{\#}(\lambda_2)) \ge Z(\mathscr{G}(\lambda_2))$,

$$\begin{aligned} q_\Lambda(\lambda) &= w(\lambda_1)w(\lambda_2)\frac{Z(\Lambda|\lambda_1\cup\lambda_2)}{Z(\Lambda)} \\ &= w(\lambda_1)\frac{Z(\Lambda|\lambda_1\cup\lambda_2)}{Z(\mathscr{G}(\lambda_2))}\cdot w(\lambda_2)\frac{Z(\mathscr{G}(\lambda_2))}{Z(\Lambda)} \\ &\ge w(\lambda_1)\frac{Z(\Lambda|\lambda_1\cup\lambda_2)}{Z\big(\mathscr{G}^{\#}(\lambda_2)\big)}\cdot w(\lambda_2)\frac{Z(\mathscr{G}(\lambda_2))}{Z(\Lambda)} \\ &= q_{\mathscr{G}^{\#}(\lambda_2)}(\lambda_1)\cdot q_\Lambda(\lambda_2) \\ &\ge q_\Lambda(\lambda_1)\cdot q_\Lambda(\lambda_2) \end{aligned} \tag{5.48}$$

by Lemma 5.2. □

Lemma 5.5 *Let* $\beta < \beta_c$.

1. *If* $J(e) = \beta$ *for all edges* e, *then for any* $t_0,\dots,t_n \in \Lambda$, *with* $t_{n+1}\equiv t_0$,

$$\sum_{\substack{\lambda:\delta\lambda=\emptyset\\ t_0,\dots,t_n\in\lambda}} q_\Lambda(\lambda) \le \exp\left\{-\sum_{i=0}^{n}\tau(t_{i+1}-t_i)\right\} \ . \tag{5.49}$$

2. *If* $h\ge 0$ *and* $J(e)$ *is defined by* (4.5), *then for* $t,t_1,t_2\in\Lambda\subset\mathbb{L}$,

$$\sum_{\substack{\lambda:\delta\lambda=\{t_1,t_2\},t\in\lambda\\ \mathscr{E}(\lambda)\cap\mathscr{E}(\Sigma_0)=\emptyset}} q_\Lambda(\lambda) \le \exp\{-\tau(t-t_1)-\tau(t_2-t)\} \ . \tag{5.50}$$

3. *If* $h\ge 0$ *and* $J(e)$ *is defined by* (4.5), *and each* λ_i, $i=1,\dots,k$, *is a closed contour, with* $t_{i0},\dots,t_{in_i}\in\lambda_i$, *then* $(t_{i(n_i+1)}\equiv t_{i0})$

$$\sum_{i=1}^{k}\sum_{\substack{\lambda_i:t_{i0},\dots,t_{in_i}\in\lambda_i\\ \mathscr{E}(\lambda_i)\cap\mathscr{E}(\Sigma_0)=\emptyset}} q_\Lambda(\lambda_1,\dots,\lambda_k) \le \prod_{i=1}^{k}\exp\left\{-\sum_{j=0}^{n_j}\tau\big(t_{i(j+1)}-t_{ij}\big)\right\} \ . \tag{5.51}$$

Proof. 1. follows from Lemma 5.4, GKS inequalities and Proposition 4.1. The proof of 2. is a consequence of Lemmas 5.1 and the equivalent of Lemma 5.4. The only modification in the proof of that lemma comes from the constraint $\mathscr{E}(\lambda) \cap \mathscr{E}(\Sigma_0) = \emptyset$. Before interpreting (5.42) we first take $h = 0$ in (5.42). The reason for doing this is to prevent that the contour, which we get by gluing an open contour and a closed contour, gives a contribution to the sum when it intersects $\mathscr{E}(\Sigma_0)$. □

Lemma 5.6 *Let $\beta < \beta_c$, $h \geq 0$ and $J(e)$ be defined by* (4.5); *let $\Lambda \subset \mathbb{L}$. The diameter of γ is $d(\gamma)$. There exist a positive constant $\alpha = \alpha(J)$ and a constant $C(\alpha)$ such that for $l \geq C(\alpha)$*

$$P_\Lambda[\,\exists\,\gamma\,,\, d(\gamma) \geq l\,] \leq |\Lambda| \cdot O(l^2) \exp\{-\alpha \cdot l\}\ . \tag{5.52}$$

$C(\alpha) = O(\frac{1}{\alpha} \ln \frac{1}{\alpha})$ *for small* α.

Proof. We give to all closed contours an origin by choosing a total order on the lattice:

$$t < t' \Longleftrightarrow t(2) < t'(2) \quad \text{or} \quad t(2) = t'(2) \quad \text{and} \quad t(1) < t'(1)\ . \tag{5.53}$$

The origin is also the initial point of the contour, viewed as a parametrized curve, which is counterclockwise oriented. To each γ with diameter $d(\gamma) > l$ we associate a sequence of points on the lattice as follows:

1. t'_0 is the origin of γ. If $t'_0(2) = 0$, then s_0 is the last time such that $\gamma(s_0)(2) = 0$; we set $t_0 := \gamma(s_0)$. (t_0 is the largest point of γ such that $t_0(2) = 0$.) Otherwise $t_0 := t'_0$.
2. Let s_1 be the first time such that γ leaves the square of center t_0 and side $l/2$; we set $t_1 := \gamma(s_1)$.
3. Let s_2 be the first time greater than s_1 such that γ leaves the square of center t_1 and side $l/2$; we set $t_2 := \gamma(s_2)$.
4. The procedure is iterated until it stops.

Thus for all closed γ we have a well-defined ordered sequence of points $(t'_0, t_0, t_1, \ldots, t_n)$.

For $x = (x(1), x(2)) \in \mathbb{R}^2$ let $|x|_1 := |x(1)| + |x(2)|$. Since $\beta < \beta_c$, $\tau(x) > 0$; we define α as the largest positive constant such that $\tau(x) \geq 2\alpha|x|_1$, $\forall x \in \mathbb{R}^2$. Clearly

$$P_\Lambda[\,\exists\gamma\ d(\gamma) \geq l\,] \leq \sum_{t \in \Lambda} \sum_{\substack{d(\gamma) \geq l \\ t'_0(\gamma) = t}} q_\Lambda(\gamma)\ , \tag{5.54}$$

since $P_\Lambda[\,\exists\,\gamma\,] = q_\Lambda(\gamma)$. Suppose that the points $t'_0, t_0, t_1, \ldots, t_n$ are fixed. Then

$$\begin{aligned} \sum_{\substack{\gamma:\, \delta\gamma = \emptyset\,, d(\gamma) \geq l \\ t'_0, \ldots, t_{n+1} \in \gamma}} q_\Lambda(\gamma) &\leq \exp\left\{ -\tau_{\text{bd}}(t'_0 - t_0) - \sum_{i=0}^{n} \tau(t_{i+1} - t_i) \right\} \\ &\leq \exp\{-\tau_{\text{bd}}(t'_0 - t_0)\} \exp\{-(\alpha/2) n l\}\ , \end{aligned} \tag{5.55}$$

with $t_{n+1} \equiv t'_0$. Therefore

$$\sum_{\gamma:\, t'_0(\gamma)=t,\, d(\gamma)\geq l} q_\Lambda(\gamma) \leq \sum_{t_0} \exp\{-\tau_{\mathrm{bd}}(t'_0 - t_0)\} \cdot \sum_{n\geq 2} (2[l+2])^n \exp\{-(\alpha/2)nl\} \ . \tag{5.56}$$

We can choose $C(\alpha)$ so that for $l \geq C(\alpha)$,

$$\sum_{\gamma:\, t'_0(\gamma)=t,\, d(\gamma)\geq l} q_\Lambda(\gamma) \leq O(l^2) \exp\{-\alpha \cdot l\} \ . \tag{5.57}$$

Remark: In the second part of the paper, we will have to consider the case $h = \infty$, which corresponds to $\tau_{\mathrm{bd}} = 0$. In this case the statement of the lemma is modified as follows

$$P_\Lambda[\exists\, \gamma\, ,\, d(\gamma) \geq l\,] \leq |\Lambda|^2 \cdot O(l^2) \exp\{-\alpha \cdot l\} \ . \tag{5.58}$$

6 Random-line representation of the two-point function

6.1 Two-point function

Let $\beta < \beta_c$ and suppose that $J(e) = \beta$ for all edges e. We study here the covariance of the Ising model at the thermodynamic limit, through its random-line representation. The main goal is to obtain a subset of the random-lines, which gives the main contribution to the covariance[3]

Let $\mathscr{L}$ be the set

$$\mathscr{L} := \left\{\lambda :\ \lambda = \emptyset \text{ or } \lambda \text{ is an open contour, } \delta\lambda = \{0, t\}\, ,\, 0 \neq t \in \mathbb{Z}^2\right\} \ . \tag{6.1}$$

Let $q(\lambda)$ be the quantity of formula (5.14) when λ is an open contour; set $q(\lambda) = 1$ when $\lambda = \emptyset$. We have

$$\begin{aligned}\chi := &\sum_{\lambda\in\mathscr{L}} q(\lambda) \\ &= 1 + \sum_{0\neq t\in\mathbb{Z}^2} \sum_{\lambda:\, \delta\lambda=\{0,t\}} q(\lambda) \\ &= \sum_{t\in\mathbb{Z}^2} \langle\sigma(0)\sigma(t)\rangle \ .\end{aligned} \tag{6.2}$$

The quantity χ is the **susceptibility** of the model. It is finite since $\beta < \beta_c$, see e.g. Lemma 5.1. On $\mathscr{L}$ we define a measure $\mathbb{M}$ with finite mass χ, by setting

[3] In this paper we do this by using monotonicity properties of the covariance. It is possible to improve these results [PV1], if we make use of the sharp triangle inequality of the decay-rate τ, see [I1].

$$\mathbb{M}(\lambda) := q(\lambda) \ . \tag{6.3}$$

Let $\{0 \to t\}$ denote the event $\{\lambda \in \mathscr{L} : \delta\lambda = \{0, t\}\}$. Then the two-point function is equal to

$$\langle \sigma(0)\sigma(t) \rangle = \mathbb{M}[\{0 \to t\}] \ . \tag{6.4}$$

The next proposition gives one of the main estimates of the paper. Let $t \in \mathbb{Z}^2$; if $|t(2)| \le t(1)$, then set

$$B_t := \left\{ s \in \mathbb{Z}^2 : 0 \le s(1) \le t(1) \, ; \frac{t(2) - t(1)}{2} \le s(2) \le \frac{t(1) + t(2)}{2} \right\} \ ; \tag{6.5}$$

if $|t(1)| \le t(2)$, then set

$$B_t := \left\{ s \in \mathbb{Z}^2 : \frac{t(1) - t(2)}{2} \le s(1) \le \frac{t(2) + t(1)}{2} \ ; 0 \le s(2) \le t(2) \right\} \ . \tag{6.6}$$

The boundary of B_t is defined as

$$\partial B_t := \left\{ t' \in B_t : \exists \, s \notin B_t \, , \, |s - t'| = 1 \right\} \ . \tag{6.7}$$

Proposition 6.1 *Let* $\beta < \beta_c$, $J(e) = \beta$ *for all edges* e, $t = (t(1), t(2)) \in \mathbb{Z}^2$, *with* $0 \le t(2) \le t(1)$ *and* $a \in \mathbb{N}$ *with* $2a < t(1)$. *Let* B_t *be the square box* (6.5) *and* ∂B_t *be its boundary* (6.7). *Then*

$$\sum_{\substack{\lambda : \delta\lambda = \{0,t\} \\ \lambda \text{ inside } B_t}} q(\lambda) \ge \langle \sigma(0)\sigma(t) \rangle \Big[1 - O(|t(1)| \exp\{-O(a)\}) \Big] \exp\{-O(a)\} \ . \tag{6.8}$$

λ inside B_t *means that* $\lambda \subset B_t$ *and that no edge of* λ, *except the first and the last one*, *is adjacent to a site of* ∂B_t.

Proof. The proof is divided into two parts. The first part is inspired by a similar result of [I1]; the second part follows a similar result proved in [Pf2].

First part. We choose two points $u_a \in \mathbb{Z}^2$ and $v_a \in \mathbb{Z}^2$ such that

1. u_a is the point on the vertical line $\{t' : t'(1) = a\}$ with $u_a(2)$ minimal and $u_a(2) \ge a \cdot (t(2)/t(1))$.
2. v_a is the point on the vertical line $\{t' : t'(1) = t(1) - a\}$ with $v_a(2)$ maximal and $v_a(2) \le t(2) - a \cdot (t(2)/t(1))$.

Then we choose two open contours λ_1 and λ_2 inside B_t with $\delta\lambda_1 = \{0, u_a\}$ and $\delta\lambda_2 = \{v_a, t\}$, such that λ_1 and λ_2 have minimal lengths.

Let λ' be an open contour with $\delta\lambda' = \{u_a, v_a\}$ which is inside B_t. We assume that u_a is the initial point. Let $s_1 \in [0, |\lambda'|]$ be the integer time defined by the condition that $t_1 := \lambda'(s_1) \in \lambda_1$ so that $t_1(1)$ is minimal; similarly let $s_2 \in [0, |\lambda'|]$ be the integer time defined by the condition that $t_2 := \lambda'(s_2) \in \lambda_2$ so that $t_2(1)$ is maximal. This gives a partition of λ' into three open contours;

we sum over the first and last ones using (5.30) of Lemma 5.4. We get the upper bound

$$\sum_{\substack{\lambda':\delta\lambda'=\{u_a,v_a\}\\ \lambda' \text{ inside } B_t}} q(\lambda') \leq \sum_{t_1,t_2} \sum_{\substack{\lambda:\delta\lambda=\{t_1,t_2\}\\ \lambda \text{ inside } B_t}} 4\,q(\lambda)\langle\sigma(u_a)\sigma(t_1)\rangle\langle\sigma(t_2)\sigma(v_a)\rangle \ . \tag{6.9}$$

Let λ be an open contour of the sum (6.9); we extend λ, using the contours λ_1 and λ_2, to an open contour $\overline{\lambda}$, so that $\overline{\lambda}$ is inside B_t and $\delta\overline{\lambda} = \{0, t\}$: $\overline{\lambda}$ is the union of λ_1', λ and λ_2', with λ_1' the part of λ_1 from 0 to t_1 and λ_2' the part of λ_2 from t_2 to t. By Lemma 5.4 we have

$$q(\overline{\lambda}) \geq q(\lambda_1')q(\lambda)q(\lambda_2') \ . \tag{6.10}$$

Using Lemma 5.3 (replace $J(e)$ by $J'(e) = \infty$ for all $e \notin \Delta(\lambda_j')$), we can show that

$$q(\lambda_j'; J) \geq \exp\{-O(|\lambda_j'|)\}, \quad j = 1, 2 \ . \tag{6.11}$$

Thus

$$4\,q(\lambda) \leq q(\overline{\lambda})\exp\{O(a)\} \ , \tag{6.12}$$

since $|\lambda_j'| = O(a)$. Therefore, after summation over t_1 and t_2 in (6.9),

$$\sum_{\substack{\lambda':\delta\lambda'=\{u_a,v_a\}\\ \lambda' \text{ inside } B_t}} q(\lambda') \leq \exp\{O(a)\} \sum_{\substack{\overline{\lambda}:\delta\overline{\lambda}=\{0,t\}\\ \overline{\lambda} \text{ inside } B_t}} q(\overline{\lambda}) \ . \tag{6.13}$$

Second part. We prove a lower bound for

$$\begin{aligned}\sum_{\substack{\lambda:\delta\lambda=\{u_a,v_a\}\\ \lambda \text{ inside } B_t}} q(\lambda) &= \sum_{\lambda:\delta\lambda=\{u_a,v_a\}} q(\lambda) - \sum_{\substack{\lambda:\delta\lambda=\{u_a,v_a\}\\ \lambda\cap\partial B_t\neq\emptyset}} q(\lambda) \\ &= \langle\sigma(u_a)\sigma(v_a)\rangle - \sum_{\substack{\lambda:\delta\lambda=\{u_a,v_a\}\\ \lambda\cap\partial B_t\neq\emptyset}} q(\lambda) \ . \end{aligned} \tag{6.14}$$

Suppose that $\delta\lambda = \{u_a, v_a\}$ and $\lambda \cap \partial B_t \neq \emptyset$. We consider λ as a unit-speed parametrized curve from u_a to v_a. Let s be the first time λ touches ∂B_t; we set $r := \lambda(s)$. We have

$$\sum_{\substack{\lambda:\delta\lambda=\{u_a,v_a\}\\ \lambda\cap\partial B_t\neq\emptyset}} q(\lambda) \leq \sum_{r\in\partial B_t} \sum_{\substack{\lambda:\delta\lambda=\{u_a,v_a\}\\ \lambda\ni r}} q(\lambda) \leq \sum_{r\in\partial B_t} \langle\sigma(u_a)\sigma(r)\rangle\langle\sigma(r)\sigma(v_a)\rangle \ . \tag{6.15}$$

Suppose that $r \in \partial B_t$ belongs to the vertical left part, or to the horizontal bottom part of ∂B_t. For simplicity assume that $(t(2) - t(1))/2 \in \mathbb{Z}$. Let $\overline{u}_a$ be the point obtained by a reflection of u_a with axis $\{t'(1) = 0\}$, or $\{t'(2) = \frac{t(2)-t(1)}{2}\}$. Then by symmetry, GKS inequalities and translation invariance

$$\langle\sigma(u_a)\sigma(r)\rangle\langle\sigma(r)\sigma(v_a)\rangle = \langle\sigma(\overline{u}_a)\sigma(r)\rangle\langle\sigma(r)\sigma(v_a)\rangle \leq \langle\sigma(\overline{u}_a)\sigma(v_a)\rangle \ . \tag{6.16}$$

The set S,

$$S := \{s' : |s'(1) - \overline{u}_a(1)| + |s'(2) - \overline{u}_a(2)| = a\} \quad , \tag{6.17}$$

separates $\overline{u}_a$ and v_a. One checks that we can apply Proposition 2.2, so that for any $s' \in S$,

$$\langle \sigma(s')\sigma(v_a)\rangle \leq \langle \sigma(u_a)\sigma(v_a)\rangle \quad . \tag{6.18}$$

Therefore

$$\begin{aligned}\langle \sigma(\overline{u}_a)\sigma(v_a)\rangle &\leq \sum_{s'\in S}\langle \sigma(\overline{u}_a)\sigma(s')\rangle\langle \sigma(s')\sigma(v_a)\rangle \\ &\leq \langle \sigma(u_a)\sigma(v_a)\rangle \sum_{s'\in S}\langle \sigma(\overline{u}_a)\sigma(s')\rangle \\ &= \exp\{-O(a)\}\langle \sigma(u_a)\sigma(v_a)\rangle \quad .\end{aligned} \tag{6.19}$$

A similar argument holds for the remaining part of ∂B_t, exchanging the role of u_a and v_a. Hence

$$\begin{aligned}\sum_{\substack{\lambda:\delta\lambda=\{u_a,v_a\}\\ \lambda \text{ inside } B_t}} q(\lambda) &\geq \langle \sigma(u_a)\sigma(v_a)\rangle[1 - O(|t(1)|\exp\{-O(a)\})] \\ &\geq \langle \sigma(0)\sigma(t)\rangle[1 - O(|t(1)|\exp\{-O(a)\})] \quad .\end{aligned} \tag{6.20}$$

□

6.2 Boundary two-point function

There is a similar random-line representation for the boundary two-point function. The coupling constants are given by (4.5). Let

$$\mathscr{L}_{\mathbb{L}} := \{\lambda \subset \mathbb{L} : \lambda = \emptyset \text{ or } \lambda \text{ is an open contour, } \delta\lambda = \{0,t\}, 0 \neq t \in \mathbb{L}\} \quad . \tag{6.21}$$

We define a measure on $\mathscr{L}_{\mathbb{L}}$ by setting

$$\mathbb{M}_{\mathbb{L}}(\lambda) := q_{\mathbb{L}}(\lambda) \quad . \tag{6.22}$$

The total mass of $\mathbb{M}_{\mathbb{L}}$ is

$$\chi_{\mathbb{L}} := \sum_{\lambda\in\mathscr{L}_{\mathbb{L}}} q_{\mathbb{L}}(\lambda) = \sum_{t\in\mathbb{L}}\langle \sigma(0)\sigma(t)\rangle_{\mathbb{L}} \quad , \tag{6.23}$$

and by GKS-inequalities $\chi_{\mathbb{L}} \leq \chi$. We have, for $x \in \Sigma_0$,

$$\langle \sigma(0)\sigma(t)\rangle_{\mathbb{L}} = \mathbb{M}_{\mathbb{L}}[\{0 \to t\}] \quad . \tag{6.24}$$

Proposition 6.2 *Let $\beta < \beta_c$, $h \geq 0$ and $J(e)$ be defined by* (4.5). *Let $t \in \Sigma_0$ with $0 < t(1)$ and $a \in \mathbb{N}$ with $2a < t(1)$. Let B_t be the square box*

$$B_t := \{s \in \mathbb{L} : 0 \leq s(1) \leq t(1) \,;\, 0 \leq s(2) \leq t(1)\} \tag{6.25}$$

and ∂B_t *be its boundary*

$$\partial B_t := \{t' \in B_t : t'(1) = 0\,,\, t'(1) = t(1)\,,\, t'(2) = t(1)\} \ . \tag{6.26}$$

Then

$$\sum_{\substack{\lambda:\delta\lambda=\{0,t\}\\ \lambda \text{ inside } B_t}} q_{\mathbb{L}}(\lambda) \geq \langle\sigma(0)\sigma(t)\rangle_{\mathbb{L}}$$
$$\times \Big[1 - O\big(|t(1)|^{5/2}\exp\{-2a\tau_{\mathrm{bd}}((1,0))\}\big)\Big]\exp\{-O(a)\} \ . \tag{6.27}$$

λ inside B_t *means that* $\lambda \subset B_t$ *and no edge of* λ, *except the first and the last one, is adjacent to a site* $t' \in \partial B_t$.

Proof. We define $u_a := (a,0)$ and $v_a := (t(1)-a,0)$. The first part of the proof is identical with the one of Proposition 6.1. Thus we have

$$\sum_{\substack{\lambda:\delta\lambda=\{0,t\}\\ \lambda \text{ inside } B_t}} q_{\mathbb{L}}(\lambda) \geq \exp\{-O(a)\} \sum_{\substack{\lambda:\delta\lambda=\{u_a,v_a\}\\ \lambda \text{ inside } B_t}} q_{\mathbb{L}}(\lambda) \ . \tag{6.28}$$

Consider λ such that $\delta\lambda = \{u_a, v_a\}$ with initial point u_a. Assume that λ touches the boundary of the box B_t at t_*. Let $\lambda(s_*) := t_*$. There are two cases.

1. $t_*(2) = t(1)$. Then there is a last time s_1 such that $s_1 < s_*$ with $\lambda(s_1) \in \Sigma_0$ and a first time $s_2 > s_*$ such that $\lambda(s_2) \in \Sigma_0$. Let $\tau^* := \tau((1,0))$. Using symmetry and monotonicity properties of the decay-rate and Lemma 5.1 we get

$$\sum_{\substack{\lambda:\delta\lambda=\{u_a,v_a\}\\ t_*\in\lambda}} q_{\mathbb{L}}(\lambda) \leq O(\exp\{-2t(1)\tau^*\}) \ . \tag{6.29}$$

We write the right-hand side of (6.29) as

$$O(\exp\{-2t(1)\tau^*\}) = \frac{O(\exp\{-2t(1)\tau^*\})}{\langle\sigma(u_a)\sigma(v_a)\rangle_{\mathbb{L}}}\langle\sigma(u_a)\sigma(v_a)\rangle_{\mathbb{L}} \ . \tag{6.30}$$

The lower bound on the boundary two-point function of Section 7.2, Proposition 4.2 and $t(1)\tau^* = \tau(t)$, with $t = (t(1),0)$, imply that

$$\frac{O(\exp\{-2t(1)\tau^*\})}{\langle\sigma(u_a)\sigma(v_a)\rangle_{\mathbb{L}}} \leq O\Big(|t(1)|^{3/2}\exp\{-\tau(t)\}\Big) \ . \tag{6.31}$$

Summing over t^*, we get

$$\sum_{t_*:t_*(2)=t(1)} \sum_{\substack{\lambda:\delta\lambda=\{u_a,v_a\}\\ t_*\in\lambda}} q_{\mathbb{L}}(\lambda) \leq \langle\sigma(u_a)\sigma(v_a)\rangle_{\mathbb{L}} O\Big(|t(1)|^{5/2}\exp\{-\tau(t)\}\Big) \ . \tag{6.32}$$

2. $t_*(1) = 0$ or $t_*(1) = t(1)$. From Lemma 5.4 and GKS inequalities we obtain

$$\begin{aligned}\sum_{\substack{\lambda:\delta\lambda=\{u_a,v_a\}\\ t_*\in\lambda,\, t_*(1)=0}} q_{\mathbb{L}}(\lambda) &\leq \langle\sigma(u_a)\sigma(t_*)\rangle_{\mathbb{L}}\langle\sigma(v_a)\sigma(t_*)\rangle_{\mathbb{L}} \\ &= \langle\sigma(-u_a)\sigma(t_*)\rangle_{\mathbb{L}}\langle\sigma(v_a)\sigma(t_*)\rangle_{\mathbb{L}} \\ &\leq \langle\sigma(-u_a)\sigma(v_a)\rangle_{\mathbb{L}} \\ &= \frac{\langle\sigma(-u_a)\sigma(v_a)\rangle_{\mathbb{L}}}{\langle\sigma(u_a)\sigma(v_a)\rangle_{\mathbb{L}}}\langle\sigma(u_a)\sigma(v_a)\rangle_{\mathbb{L}} \\ &\leq \langle\sigma(u_a)\sigma(v_a)\rangle_{\mathbb{L}} O\Big(|t(1)|^{3/2}\exp\{-2\tau_{\mathrm{bd}}(u_a)\}\Big)\ . \end{aligned} \tag{6.33}$$

We conclude as in the proof of Proposition 6.1 ($\tau_{\mathrm{bd}}(u_a) = a\cdot\tau_{\mathrm{bd}}((1,0))$),

$$\begin{aligned}\sum_{\substack{\lambda:\delta\lambda=\{u_a,v_a\}\\ \lambda \text{ inside } B_t}} q_{\mathbb{L}}(\lambda) &\geq \langle\sigma(u_a)\sigma(v_a)\rangle_{\mathbb{L}}\Big[1 - O\big(|t(1)|^{5/2}\exp\{-2\tau_{\mathrm{bd}}(u_a)\}\big)\Big] \\ &\geq \langle\sigma(0)\sigma(t)\rangle_{\mathbb{L}}\Big[1 - O\big(|t(1)|^{5/2}\exp\{-2\tau_{\mathrm{bd}}(u_a)\}\big)\Big]\ . \end{aligned} \tag{6.34}$$

7 Correction to the exponential decay

We need lower bounds for the two-point function and the boundary two-point function in section 10, in order to get precise estimates for the remainder terms.

7.1 Lower bound for the two-point function

We need a bound of the following kind,

$$\langle\sigma(0)\sigma(t)\rangle \geq C\,\frac{\exp\{-\tau(t)\}}{|t|^k}\ , \tag{7.1}$$

for some positive k. Such a bound can be derived ([PL], [DKS]) with $k = 1/2$ for small β by perturbative methods; see also [G], [BLP2], and in particular [BF], where the connection with the Central Limit Theorem for the random lines λ is made explicit. In case of the Bernoulli percolation Alexander proves such bounds for the corresponding quantity in a non-perturbative way [A1] with $k \leq 420$ if $D = 2$ and $k \leq 2328$ if $D = 3$; see also [A2]. In this paper we use the bounds obtained from the work of McCoy and Wu [MW] chapters XI and XII.

Lemma 7.1 *Let $J(e) = \beta$, $\beta < \beta_c$, for all edges e of $\mathbb{Z}^2$. Then there exists a constant C such that for all $t \neq 0$,*

$$\langle\sigma(0)\sigma(t)\rangle \geq \frac{C}{\sqrt{|t|}}\exp\{-\tau(t)\}\ . \tag{7.2}$$

7.2 Lower bound for the boundary two-point function

Proposition 7.1 *Let* $J(e)$ *be given by* (4.5), $\beta < \beta_c$ *and* $t \in \Sigma_0$.

1. *Let* $h = 1$. *Then there exists a constant* C *such that for* $t \in \Sigma_0$,

$$\langle \sigma(0)\sigma(t)\rangle_{\mathbb{L}}^{\beta,1} \geq \frac{C}{|t|^{3/2}} \exp\{-\tau_{\rm bd}(t)\} \ . \tag{7.3}$$

2. *For all* $h \geq 0$

$$\langle \sigma(0)\sigma(t)\rangle_{\mathbb{L}}^{\beta,h} \geq (\tanh \beta)^2 \cdot \langle \sigma(0)\sigma(t)\rangle_{\mathbb{L}}^{\beta,1} \ . \tag{7.4}$$

3. *Let* $h > h_c(\beta)$ (*see Proposition* 4.2). *Then there exists a constant* $C = C(h,\beta)$ *such that*

$$\langle \sigma(0)\sigma(t)\rangle_{\mathbb{L}}^{\beta,h} \geq C \exp\{-\tau_{\rm bd}(t;\beta,h)\} \ . \tag{7.5}$$

Remark: Since $h_c(\beta) \geq 1$, we can write (7.3) as

$$\langle \sigma(0)\sigma(t)\rangle_{\mathbb{L}}^{\beta,1} \geq \frac{C}{|t|^{3/2}} \exp\{-\tau(t)\} \ . \tag{7.6}$$

Proof. 1. follows from [MW] chapter VII; see also [P].

By GKS inequalities the boundary two-point function decreases if we set $h = 0$ and $J(e) = 0$ for all edges e adjacent to a site $t' \in \Sigma_0$, except the vertical edges adjacent to $t' = 0$ and $t' = t$. It is now not difficult to sum over the variables $\sigma(0)$ and $\sigma(t)$ explicitly and to get (7.4). This proves 2.

We prove 3., assuming Lemma 7.2. Given $x_1, x_2 \in \Sigma_0$, we define the interval $[x_1, x_2]$ as the set

$$[x_1, x_2] := \{t \in \mathbb{L} : x_1(1) \leq t(1) \leq x_2(1) \, , \, t(2) = 0\} \ . \tag{7.7}$$

Let $a \in \mathbb{N}$, $t(1) > a$ and $t = (t(1), 0)$. We set $t_k := kt$ for $k = 1, 2, \ldots, n$, and

$$I := [t_1 - (a,0), t_1] \ . \tag{7.8}$$

We have

$$\begin{aligned} \langle \sigma(0)\sigma(t_n)\rangle_{\mathbb{L}} &= \mathbb{M}_{\mathbb{L}}[\{0 \to t_n\}] \\ &= \mathbb{M}_{\mathbb{L}}\big[E_I^c \,|\, \{0 \to t_n\}\big]\mathbb{M}_{\mathbb{L}}[\{0 \to t_n\}] + \mathbb{M}_{\mathbb{L}}[E_I \cap \{0 \to t_n\}] \ , \end{aligned} \tag{7.9}$$

where E_I is the event $\{\lambda \cap I \neq \emptyset\}$ and E_I^c the complementary event. We choose a so that

$$\mathbb{M}\big[E_I^c \,|\, \{0 \to t_n\}\big] \leq 1/2 \ , \tag{7.10}$$

which is possible according to Lemma 7.2 if $t(1)$ is large enough. Thus we have

$$\begin{aligned}\langle\sigma(0)\sigma(t_n)\rangle_{\mathbb{L}} &\leq 2\mathbb{M}_{\mathbb{L}}[E_I\cap\{0\to t_n\}]\\ &\leq 2\sum_{u\in I}\sum_{\substack{\lambda:\delta\lambda=\{0,t_n\}\\ u\in\lambda}} q_{\mathbb{L}}(\lambda)\\ &\leq 2a\langle\sigma(0)\sigma(t_1-(a,0))\rangle_{\mathbb{L}}\langle\sigma(t_1)\sigma(t_n)\rangle_{\mathbb{L}}\ .\end{aligned}\tag{7.11}$$

We have used Lemma 5.1 and the monotonicity property of the boundary two-point function, which is proven in the same way as the corresponding property for the two-point function on $\mathbb{Z}^2$. By GKS inequalities and translation-invariance

$$\begin{aligned}\frac{\langle\sigma(0)\sigma(t_1-(a,0))\rangle_{\mathbb{L}}}{\langle\sigma(0)\sigma(t_1)\rangle_{\mathbb{L}}} &\leq \frac{\langle\sigma(0)\sigma(t_1-(a,0))\rangle_{\mathbb{L}}}{\langle\sigma(0)\sigma(t_1-(a,0))\rangle_{\mathbb{L}}\langle\sigma(t_1-(a,0))\sigma(t_1)\rangle_{\mathbb{L}}}\\ &=\frac{1}{\langle\sigma(0)\sigma((a,0))\rangle_{\mathbb{L}}}\ .\end{aligned}\tag{7.12}$$

If we set

$$C_* := \frac{\langle\sigma(0)\sigma((a,0))\rangle_{\mathbb{L}}}{2a}\ ,\tag{7.13}$$

then

$$\langle\sigma(0)\sigma(t_n)\rangle_{\mathbb{L}} \leq C_*^{-1}\langle\sigma(0)\sigma(t_1)\rangle_{\mathbb{L}}\langle\sigma(t_1)\sigma(t_n)\rangle_{\mathbb{L}}\ .\tag{7.14}$$

We can iterate this result,

$$\langle\sigma(0)\sigma(t_n)\rangle_{\mathbb{L}} \leq C_*^{-n}\Big(\langle\sigma(0)\sigma(t_1)\rangle_{\mathbb{L}}\Big)^n\ .\tag{7.15}$$

Therefore, if $t(1)$ is large enough, then

$$-\tau_{\mathrm{bd}}(t;\beta,h) = \lim_{n\to\infty}\frac{1}{n}\ln\langle\sigma(0)\sigma(t_n)\rangle_{\mathbb{L}} \leq -\ln C_* + \ln\langle\sigma(0)\sigma(t)\rangle_{\mathbb{L}}\ .\tag{7.16}$$

□

Lemma 7.2 *Let* $\beta<\beta_c$, $h>h_c(\beta)$, $x_1,x_2,t\in\Sigma_0$, *such that* $0<x_1(1)<x_2(1)<t(1)$, *and* $I:=[x_1,x_2]$ (*see* (7.7)). *Then there exist* ε *positive*, n_ε *and* C_1 *such that for all* x_1,x_2 *with* $|x_2-x_1|\geq n_\varepsilon$

$$\mathbb{M}_{\mathbb{L}}[\{\lambda\cap I=\emptyset\}\,|\,\{0\to t\}] \leq C_1\exp\{-\varepsilon|x_2-x_1|\}\ .\tag{7.17}$$

Proof. We have

$$\mathbb{M}_{\mathbb{L}}[\{0\to t\}] = \langle\sigma(0)\sigma(t)\rangle_{\mathbb{L}}\ .\tag{7.18}$$

Let λ be a random line such that $\delta\lambda=\{0,t\}$ and $\lambda\cap I=\emptyset$. Let s_1 be the last time when λ touches Σ_0 at the left hand side of I, and let s_2 be the first time that λ touches Σ_0 at the right hand side of I. We set $u:=\lambda(s_1)$ and $v:=\lambda(s_2)$. We necessarily have $u(1)<x_1(1)<x_2(1)<v(1)$. From Lemmas 5.4 and 5.1 we get

$$\mathbb{M}_{\mathbb{L}}[\{\lambda \cap I = \emptyset\} \cap \{0 \to t\}] \leq \sum_{u,v} \exp\{-\tau(v-u)\} \langle\sigma(0)\sigma(u)\rangle_{\mathbb{L}} \langle\sigma(v)\sigma(t)\rangle_{\mathbb{L}} \ . \tag{7.19}$$

By GKS inequalities

$$\langle\sigma(0)\sigma(t)\rangle_{\mathbb{L}} \geq \langle\sigma(0)\sigma(u)\rangle_{\mathbb{L}} \langle\sigma(u)\sigma(v)\rangle_{\mathbb{L}} \langle\sigma(v)\sigma(t)\rangle_{\mathbb{L}} \ , \tag{7.20}$$

so that

$$\mathbb{M}_{\mathbb{L}}[\{\lambda \cap I = \emptyset\} \mid \{0 \to t\}] \leq \sum_{u,v} \frac{\exp\{-\tau(v-u)\}}{\langle\sigma(u)\sigma(v)\rangle_{\mathbb{L}}} \ . \tag{7.21}$$

We know that

$$\lim_{n\to\infty} -\frac{1}{n} \ln\langle\sigma(0)\sigma(nt_1)\rangle_{\mathbb{L}} = \tau^*_{\mathrm{bd}} \ , \tag{7.22}$$

where

$$\tau^*_{\mathrm{bd}} = \tau_{\mathrm{bd}}(t_1), \quad t_1 = (1,0) \ . \tag{7.23}$$

Let $0 < 2\varepsilon < \tau^* - \tau^*_{\mathrm{bd}}$; $\tau^* = \tau(t_1)$. We can find n_ε so that for all $n \geq n_\varepsilon$,

$$\lim_{n} -\frac{1}{n} \ln\langle\sigma(0)\sigma(nt_1)\rangle_{\mathbb{L}} \leq \tau^*_{\mathrm{bd}} + \epsilon \ , \tag{7.24}$$

so that

$$\langle\sigma(0)\sigma(nt_1)\rangle_{\mathbb{L}} \geq \exp\{-n(\tau^*_{\mathrm{bd}} + \varepsilon)\} \ . \tag{7.25}$$

From this inequality and $\tau(u-v) = |u-v| \cdot \tau^*$

$$\mathbb{M}_{\mathbb{L}}[\{\lambda \cap I = \emptyset\} \mid \{0 \to t\}] \leq \sum_{u,v} \exp\{-\epsilon|u-v|\} \ . \tag{7.26}$$

Using $u < x_1 < x_2 < v$ the lemma follows. □

Remark: Using Proposition 7.1 we can improve Lemma 7.2. There exists a constant C such that for any interval $I = [x_1, x_2]$ we have

$$\mathbb{M}_{\mathbb{L}}[\{\lambda \cap I = \emptyset\} \mid \{0 \to t\}] \leq C \exp\{-(\tau^* - \tau^*_{\mathrm{bd}}) \cdot |x_2 - x_1|\} \ . \tag{7.27}$$

Part II: Ising model at low temperature

We study the large deviations of the magnetization of the Ising model for $\beta > \beta_c$ (i.e. below the critical temperature). We analyze in particular boundary effects. Some estimates of part I are essential. The results are valid only for the two-dimensional case.

We have written part I with coupling constants β and h in order to simplify the notations, and because this part has its own interest. However, the proper notations would be β^* and h^*, since these coupling constants are the dual coupling constants of β and h. In particular $h = 0$ in part II corresponds to $h^* = \infty$ in part I.

8 Low-temperature representation

There is a representation of the Gibbs measure in Λ with $+$ boundary condition in terms of contours, which is similar to the one of (2.4). To each configuration ω, which satisfies the Λ^+-boundary condition, we associate a family $\underline{\gamma} = \underline{\gamma}(\omega)$ of compatible contours on the dual lattice $(\mathbb{Z}^2)^*$: let $\mathscr{E}^*(\omega)$ be the subset of edges

$$\mathscr{E}^*(\omega) := \{e^* : [\sigma(t)(\omega)\sigma(t')(\omega) - 1] = -2, \ \langle t, t' \rangle = e\} \ . \tag{8.1}$$

We decompose the set $\mathscr{E}^*(\omega)$ into compatible contours $\underline{\gamma}$. Two important remarks: All contours γ of $\underline{\gamma}(\omega)$ are closed, i.e. $\delta\gamma = \emptyset$; We do *not* obtain all families of compatible contours as it is the case for the high-temperature representation. This motivates the

Definition 8.1 *A family of compatible contours* $\underline{\gamma}$ *in* Λ^* *is* Λ^+**-compatible** *if and only if there exists a configuration* ω *satisfying the* Λ^+*-boundary condition, such that* $\underline{\gamma}$ *is the family of contours of* ω.

Let $\Lambda \subset \mathbb{Z}^2$. The set $\Lambda^* \subset (\mathbb{Z}^2)^*$ is by definition the set

$$\Lambda^* := \left\{ t^* \in \left(\mathbb{Z}^2\right)^* : \ t^* \text{ is a corner of a plaquette } p(t), t \in \Lambda \right\} \ . \tag{8.2}$$

Any family of contours of a configuration ω satisfying the Λ^+-boundary condition is in Λ^*. Given a closed contour γ on the dual lattice $(\mathbb{Z}^2)^*$ there exists a unique configuration ω_γ having γ as single contour and such that $\omega(t) = 1$ for all t, except for a finite number. The **interior** of γ is

$$\operatorname{int}\gamma := \left\{ t \in \mathbb{Z}^2 : \ \omega_\gamma(t) = -1 \right\} \ . \tag{8.3}$$

The **exterior** of γ is $\operatorname{ext}\gamma := \mathbb{Z}^2 \backslash \operatorname{int}\gamma$. The **volume** of γ is

$$\operatorname{vol}\gamma := |\operatorname{int}\gamma| \ . \tag{8.4}$$

A contour γ of a configuration ω is **external** if there is no other contour γ' of the configuration such that $\operatorname{int}\gamma \subset \operatorname{int}\gamma'$. Let γ be a contour of a configuration ω satisfying the Λ^+-boundary condition. The **closure of the interior of** γ **in** Λ, $\overline{\operatorname{int}}\,\gamma$, is the union of $\operatorname{int}\gamma$ and the set of all $t \in \Lambda\backslash\operatorname{int}\gamma$, such that $\omega'(t) = 1$ for *any* configuration ω' with the properties: 1) ω' satisfies the Λ^+-boundary condition; 2) γ is an external contour in ω'. [4] The **closure of the exterior of** γ **in** Λ, $\overline{\operatorname{ext}}\,\gamma$, is the union of $\operatorname{ext}\gamma$ and the set of all $t \in \Lambda\backslash\operatorname{ext}\gamma$, such that $\omega'(t) = -1$ for *any* configuration ω' with the properties: 1) ω' satisfies the Λ^+-boundary condition; 2) γ is an external contour in ω'.

Let $J^*(e^*)$ be the dual coupling to $J(e)$. The $*$**-weight** of a contour is

$$w^*(\gamma) := \prod_{e^* \in \gamma} \tanh J^*(e^*) \equiv \prod_{e^* \in \gamma} \exp\{-2\,J(e)\} \ . \tag{8.5}$$

[4] $\overline{\operatorname{int}}\,\gamma$ depends on the rule A; if Λ is large enough it is independent of Λ.

The normalization constant $\Xi(\Lambda)^+$ appearing in the definition of μ_Λ^+ can be written as

$$\begin{aligned}\Xi(\Lambda)^+ &= \sum_{\omega} \prod_{\substack{e=\langle t,t'\rangle:\\ e\cap\Lambda\neq\emptyset}} \exp\{J(e)[\sigma(t)(\omega)\sigma(t')(\omega)-1]\} \\ &= \sum_{\substack{\underline{\gamma}:\delta\underline{\gamma}=\emptyset\\ \Lambda^+\text{-comp.}}} \prod_{\gamma\in\underline{\gamma}} \prod_{e^*\in\gamma} \exp\{-2\,J(e)\} \\ &= \sum_{\substack{\underline{\gamma}:\delta\underline{\gamma}=\emptyset\\ \Lambda^+\text{-comp.}}} \prod_{\gamma\in\underline{\gamma}} w^*(\gamma) \ . \end{aligned} \tag{8.6}$$

Let $\Lambda \subset \mathbb{Z}^2$ and $\underline{\gamma}'$ be a family of Λ^*-compatible closed contours. We set

$$Z^+\left(\Lambda|\underline{\gamma}';J\right) := \sum_{\substack{\underline{\gamma}:\delta\underline{\gamma}=\emptyset\\ \underline{\gamma}\cup\underline{\gamma}'\ \Lambda^+\text{-comp.}}} w^*(\underline{\gamma}) \ . \tag{8.7}$$

If $\underline{\gamma}' = \emptyset$, then

$$Z^+(\Lambda|\emptyset;J) = Z^+(\Lambda;J) \ . \tag{8.8}$$

Lemma 8.1 *Let Λ be a finite subset of $\mathbb{Z}^2$; $J(e)$ be non-negative coupling constants. Then*

$$\Xi(\Lambda;J)^+ = Z^+(\Lambda;J) \ . \tag{8.9}$$

Let $\underline{\gamma}'$ be a family of Λ^*-compatible closed contours. Then the probability $P_\Lambda^+[\underline{\gamma}']$, computed with respect to the measure μ_Λ^+, is given by

$$P_\Lambda^+\left[\underline{\gamma}'\right] = w^*\left(\underline{\gamma}'\right)\frac{Z^+\left(\Lambda|\underline{\gamma}';J\right)}{Z^+(\Lambda;J)} \ . \tag{8.10}$$

If Λ is simply connected, then any family of Λ^*-compatible contours is Λ^+-compatible; furthermore

$$Z^+(\Lambda;J) = Z(\Lambda^*;J^*) \tag{8.11}$$

and

$$P_\Lambda^+\left[\underline{\gamma}'\right] = w^*\left(\underline{\gamma}'\right)\frac{Z\left(\Lambda^*|\underline{\gamma}';J^*\right)}{Z(\Lambda^*;J^*)} = q_{\Lambda^*}\left(\underline{\gamma}'\right) \ . \tag{8.12}$$

Proof. (8.11) follows by comparing the high-temperature and low-temperature representations. If Λ is simply connected then we construct explicitly the configuration ω starting from the boundary for any family of Λ^+-compatible contours. □

9 Phase of small contours

A basic idea in [DKS] is the introduction of an intermediate length-scale in the analysis of the large deviations of the magnetization. One distinguishes between small and large contours. We study here the large deviations for the magnetization under the condition that all contours of a configuration ω are small. Our main result is Proposition 9.1. It is inspired by the appendix of Schonmann and Shlosman in [SS1]; see also Ioffe [I2] and Pisztora [Pi] for related and former results of this kind.

9.1 Definition of the phase of small contours

Let l be some positive integer; we set

$$B(0;l) := \left\{ t = (t(1), t(2)) \in \left(\mathbb{Z}^2\right)^* : \ -l \le t(i) < l \,,\, i = 1,2 \right\} \ . \tag{9.1}$$

Definition 9.1 *Let $s \in \mathbb{N}$. A contour γ is s-**small**, or **small**, if there is a translate of $B(0;s/2)$ which contains γ.*

Let Λ be a finite subset of $\mathbb{Z}^2$. The phase of small contours is described by the conditioned measure

$$P_\Lambda^{+,s}[\,\cdot\,] := P_\Lambda^+[\,\cdot\,|\{\text{all contours } s\text{-small}\}] \ . \tag{9.2}$$

The expectation value is denoted by $\langle\,\cdot\,\rangle_\Lambda^{+,s}$, or $P_\Lambda^{+,s}[\,\cdot\,]$. It is convenient to use

$$I_\Lambda^s(\omega) := \begin{cases} 1 & \text{if } \omega \text{ satisfies the } \Lambda^+\text{-boundary condition} \\ & \text{and each contour of } \omega \text{ is } s\text{-small,} \\ 0 & \text{otherwise} \ . \end{cases} \tag{9.3}$$

The function $\omega \mapsto I_\Lambda^s(\omega)$ is increasing. Furthermore, if Λ is the union of two disjoint connected components Λ_1 and Λ_2, then

$$I_\Lambda^s(\omega) = I_{\Lambda_1}^s(\omega) \cdot I_{\Lambda_2}^s(\omega) \ . \tag{9.4}$$

The main property of the phase of small contours is the decoupling property expressed in the next lemma.

Lemma 9.1 *Let $J(e) \ge 0$ for all edges e. Let $l, s \in \mathbb{N}$, and set $\Lambda_1 := B(0;l)$, $\Lambda_2 := B(0;l+s+1)$. Suppose that $\Lambda \supset \Lambda_2$.*

1. *Let f be Λ_1-local and g $\Lambda\backslash\Lambda_2$-local. Then*

$$\left|\langle f\,g\rangle_\Lambda^{+,s} - \langle f\rangle_\Lambda^{+,s} \cdot \langle g\rangle_\Lambda^{+,s}\right| \le \max_{\Lambda_1 \subset \Lambda' \subset \Lambda_2} \left|\langle f\rangle_\Lambda^{+,s} - \langle f\rangle_{\Lambda'}^{+,s}\right| \cdot \langle |g|\rangle_\Lambda^{+,s} \ . \tag{9.5}$$

2. *If furthermore f is increasing and g positive, then*

$$\langle f\,g\rangle_\Lambda^{+,s} \ge \langle f\rangle_{\Lambda_2}^{+} \cdot \langle g\rangle_\Lambda^{+,s} \ . \tag{9.6}$$

If furthermore f is decreasing and g positive, then

$$\langle f\,g\rangle_\Lambda^{+,s} \le \langle f\rangle_{\Lambda_1}^{+} \cdot \langle g\rangle_\Lambda^{+,s} \ . \tag{9.7}$$

Remark: In the proof of Lemma 9.1 we only use (9.4), the Markov property and FKG-inequalities. Lemma 9.1 is therefore also true if we replace the measure μ_Λ^+ by another measure, which has the Markov property, as long as FKG-inequalities remain valid; for example we may consider the Ising model with arbitrary external field.

Proof. By definition

$$\langle f\,g\rangle_\Lambda^{+,s} = \frac{\langle f\,g\,I_\Lambda^s\rangle_\Lambda^+}{\langle I_\Lambda^s\rangle_\Lambda^+} \quad . \tag{9.8}$$

Suppose that γ is an external contour in a configuration ω and that $I_\Lambda^s(\omega) = 1$. Then

$$\Lambda_2 \not\subset \operatorname{int}\gamma \quad . \tag{9.9}$$

Moreover, if

$$\operatorname{int}\gamma \cap (\Lambda\backslash\Lambda_2) \neq \emptyset \quad ; \tag{9.10}$$

then

$$\overline{\operatorname{int}}\,\gamma \cap \Lambda_1 = \emptyset \quad . \tag{9.11}$$

Let $\gamma_1(\omega), \ldots, \gamma_n(\omega)$ be all external contours of ω such that

$$\operatorname{int}\gamma_i(\omega) \cap (\Lambda\backslash\Lambda_2) \neq \emptyset, \quad i = 1, \ldots, n \quad ; \tag{9.12}$$

we define the random set

$$\Lambda(\omega) := (\Lambda\backslash\Lambda_2) \bigcup_{i=1,\ldots,n} \overline{\operatorname{int}}\,\gamma_i(\omega) \quad . \tag{9.13}$$

We have by Markov property and (9.4)

$$\begin{aligned}\langle f\,g\rangle_\Lambda^{+,s} &= \sum_{\Lambda''\subset\Lambda} \langle f\,g\,|\,\{\Lambda(\,\cdot\,) = \Lambda''\}\rangle_\Lambda^{+,s} \cdot P_\Lambda^{+,s}[\{\Lambda(\,\cdot\,) = \Lambda''\}] \\ &= \sum_{\Lambda''\subset\Lambda} \langle f\rangle_{\Lambda\backslash\Lambda''}^{+,s} \cdot \langle g\,|\,\{\Lambda(\,\cdot\,) = \Lambda''\}\rangle_\Lambda^{+,s} \cdot P_\Lambda^{+,s}[\{\Lambda(\,\cdot\,) = \Lambda''\}] \quad . \end{aligned} \tag{9.14}$$

If $P_\Lambda^{+,s}[\{\Lambda(\,\cdot\,) = \Lambda''\}] \neq 0$, then $\Lambda_1 \subset \Lambda\backslash\Lambda'' \subset \Lambda_2$. Hence, the result follows from

$$\begin{aligned}&\langle f\,g\rangle_\Lambda^{+,s} - \langle f\rangle_\Lambda^{+,s} \cdot \langle g\rangle_\Lambda^{+,s} \\ &\quad = \sum_{\Lambda''\subset\Lambda} \Big(\langle f\rangle_{\Lambda\backslash\Lambda''}^{+,s} - \langle f\rangle_\Lambda^{+,s}\Big) \langle g\,|\,\{\Lambda(\,\cdot\,) = \Lambda''\}\rangle_\Lambda^{+,s} \cdot P_\Lambda^{+,s}[\{\Lambda(\,\cdot\,) = \Lambda''\}] \quad . \end{aligned} \tag{9.15}$$

Suppose that f is increasing and g positive. FKG-inequalities and $\Lambda_1 \subset \Lambda\backslash\Lambda'' \subset \Lambda_2$ imply that

$$\langle f\rangle_{\Lambda\backslash\Lambda''}^{+,s} = \frac{\Big\langle f\,I_{\Lambda\backslash\Lambda''}^s\Big\rangle_{\Lambda\backslash\Lambda''}^+}{\Big\langle I_{\Lambda\backslash\Lambda''}^s\Big\rangle_{\Lambda\backslash\Lambda''}^+} \geq \langle f\rangle_{\Lambda\backslash\Lambda''}^+ \geq \langle f\rangle_{\Lambda_2}^+ \quad . \tag{9.16}$$

Hence (9.6) follows from (9.14) and (9.16). □

We derive some consequences of Lemma 9.1 for the model with the coupling constants

$$J(e) := \begin{cases} \beta \geq 0 & \text{if} \quad e = \langle t, t' \rangle,\ t(2) \geq 0 \text{ and } t'(2) \geq 0, \\ h\beta \geq 0 & \text{if} \quad e = \langle t, t' \rangle,\ t(2) \leq -1 \text{ or } t'(2) \leq -1 \quad . \end{cases} \tag{9.17}$$

We recall a result of [BLP2].

Lemma 9.2 *For any $\beta > \beta_c$ there exists $\overline{a}(\beta) > 0$ and K such that*

$$\left| \langle \sigma_A \rangle^+_{V_1} - \langle \sigma_A \rangle^+_{V_2} \right| \leq K \sum_{t \in A} \sum_{t' \in V_1 \Delta V_2} \exp\{-\overline{a}(\beta)|t - t'|\} \ , \tag{9.18}$$

where $A \subset V_1 \cap V_2$ and $V_1 \Delta V_2 = (V_1 \backslash V_2) \cup (V_2 \backslash V_1)$.

Lemma 9.3 *Let $J(e)$ be the coupling constants given by* (9.17) *with $\beta > \beta_c$. Let $s \in \mathbb{N}$ and $t \in \mathbb{L}$ with $t(2) > 2s + 1$. Let $\Lambda \subset \mathbb{L}$, such that Λ contains the square box*

$$\{u \in \mathbb{L} : \ |t(i) - u(i)| \leq 2s + 1\, , i = 1, 2\} \ . \tag{9.19}$$

Then there exists a positive constant $\kappa = \kappa(\beta)$ (see (9.31)) *such that*

$$\langle \sigma(t) \rangle^{+,\beta} \leq \langle \sigma(t) \rangle^{+,s,J}_{\Lambda} \leq \langle \sigma(t) \rangle^{+,\beta} + O(s^4) \exp\{-\kappa \cdot s\} \ . \tag{9.20}$$

Suppose furthermore that $t' \in \Lambda$ and

$$\min \{ |t'(i) - t(i)| : \ i = 1, 2\} > 2s + 1 \ . \tag{9.21}$$

Then

$$\left| \langle \sigma(t)\, \sigma(t') \rangle^{+,s}_{\Lambda} - \langle \sigma(t) \rangle^{+,s}_{\Lambda} \cdot \langle \sigma(t') \rangle^{+,s}_{\Lambda} \right| \leq O(s^4) \exp\{-\kappa \cdot s\} \cdot \langle \sigma(t') \rangle^{+,s}_{\Lambda} \ . \tag{9.22}$$

Proof. Let Λ_1 be a translate of the box $B(0; s/2)$ with t in its "center". Let Λ_2 be the translate (same translation) of the box Λ_2 of Lemma 9.1 with $l = s/2$. The first inequality follows from (9.6), with $g \equiv 1$, and from FKG-inequalities,

$$\langle \sigma(t) \rangle^{+,s,J}_{\Lambda} \geq \langle \sigma(t) \rangle^{+,J}_{\Lambda_2} \geq \langle \sigma(t) \rangle^{+,\beta} \ . \tag{9.23}$$

By (9.14) we have

$$\langle \sigma(t) \rangle^{+,s,J}_{\Lambda} = \sum_{\Lambda'' \subset \Lambda} \langle \sigma(t) \rangle^{+,s}_{\Lambda \backslash \Lambda''} \cdot P^{+,s}_{\Lambda}[\{\Lambda(\,\cdot\,) = \Lambda''\}] \ . \tag{9.24}$$

Only the terms with $\Lambda_1 \subset \Lambda \backslash \Lambda'' \subset \Lambda_2$ give a non-zero contribution. Therefore by FKG-inequalities

$$\langle\sigma(t)\rangle_{\Lambda}^{+,s,J} = \sum_{\Lambda''\subset\Lambda} \frac{\left\langle\sigma(t)I^s_{\Lambda\backslash\Lambda''}\right\rangle^{+,J}_{\Lambda\backslash\Lambda''}}{\left\langle I^s_{\Lambda\backslash\Lambda''}\right\rangle^{+,J}_{\Lambda\backslash\Lambda''}} \cdot P^{+,s}_{\Lambda}[\{\Lambda(\,\cdot\,)=\Lambda''\}]$$

$$\leq \sum_{\Lambda''\subset\Lambda} \frac{\left\langle\sigma(t)I^s_{\Lambda_1}\right\rangle^{+,J}_{\Lambda_1}}{\left\langle I^s_{\Lambda_1}\right\rangle^{+,J}_{\Lambda_1}} \cdot \frac{\left\langle I^s_{\Lambda_1}\right\rangle^{+,J}_{\Lambda_1}}{\langle I^s_{\Lambda}\rangle^{+,J}_{\Lambda\backslash\Lambda''}} \cdot P^{+,s}_{\Lambda}[\{\Lambda(\,\cdot\,)=\Lambda''\}]\ . \tag{9.25}$$

By FKG-inequalities and GKS-inequalities

$$\left\langle\sigma(t)I^s_{\Lambda_1}\right\rangle^{+,J}_{\Lambda_1} \geq \langle\sigma(t)\rangle^{+,J}_{\Lambda_1} \cdot \left\langle I^s_{\Lambda_1}\right\rangle^{+,J}_{\Lambda_1} \geq 0\ . \tag{9.26}$$

Since $\langle I^s_{\Lambda_1}\rangle^{+,J}_{\Lambda_1} \leq 1$ and $\langle I^s_{\Lambda}\rangle^{+,J}_{\Lambda\backslash\Lambda''} \geq \langle I^s_{\Lambda_2}\rangle^{+,J}_{\Lambda_2}$, we get

$$\langle\sigma(t)\rangle^{+,s,J}_{\Lambda} \leq \frac{1}{\left\langle I^s_{\Lambda_2}\right\rangle^{+}_{\Lambda_2}} \cdot \langle\sigma(t)\rangle^{+,s,J}_{\Lambda_1}\ . \tag{9.27}$$

In Λ_1 all contours are s-small, so that we have

$$\langle\sigma(t)\rangle^{+,s,J}_{\Lambda} \leq \frac{1}{\left\langle I^s_{\Lambda_2}\right\rangle^{+}_{\Lambda_2}} \cdot \langle\sigma(t)\rangle^{+,J}_{\Lambda_1}\ . \tag{9.28}$$

If the diameter $d(\gamma)$ of γ is smaller than s, then γ is s-small. Lemma 5.6 and Lemma 8.1 give

$$\frac{1}{\left\langle I^s_{\Lambda_2}\right\rangle^{+}_{\Lambda_2}} \leq 1 + O(s^4)\exp\{-s\alpha(\beta^*)\}\ , \tag{9.29}$$

with $\alpha(\beta^*)$ of Lemma 5.6. Lemma 9.2 gives

$$\left|\,\langle\sigma(t)\rangle^{+,J}_{\Lambda_1} - \langle\sigma(t)\rangle^{+,\beta}\,\right| \leq O(s)\exp\{-s\overline{a}(\beta)\}\ . \tag{9.30}$$

Define $\kappa(\beta)$ so that

$$\max\left[\exp\{-s\alpha(\beta^*)\}\,,\,\exp\{-s\overline{a}(\beta)\}\right] \leq \exp\{-\kappa(\beta)\cdot s\}\ . \tag{9.31}$$

The second affirmation is a consequence of (9.5) and (9.20). □

9.2 Large deviations in the phase of small contours

Proposition 9.1 *Let* $J(e) \geq 0$ *for all edges* e.
1. *Let*

$$\mathrm{var}^{+}_{\Lambda} := \frac{1}{|\Lambda|}\sum_{t,t'\in\Lambda}\Big(\langle\sigma(t')\sigma(t)\rangle^{+}_{\Lambda} - \langle\sigma(t')\rangle^{+}_{\Lambda}\,\langle\sigma(t)\rangle^{+}_{\Lambda}\Big)\ . \tag{9.32}$$

For any $x \geq 0$,

$$P_\Lambda^+\left[\left\{\sum_{t\in\Lambda}\left(\sigma(t)-\langle\sigma(t)\rangle_\Lambda^+\right)\geq x|\Lambda|\right\}\right]\leq \exp\left(-|\Lambda|\frac{x^2}{2\mathrm{var}_\Lambda^+}\right)\ . \tag{9.33}$$

2. *Let* $l,s\in\mathbb{N}$, $\Lambda_1=B(0;l)$, $\Lambda_2=B(0;l+s+1)$. *Suppose that* $\Lambda'\subset\mathbb{Z}^2$ *is the union of* n' *disjoint translates* $\mathscr{B}_i$ *of the box* Λ_2 *and that* Λ'' *is the union of* n'' *disjoint translates* $\mathscr{B}_j$ *of the box* Λ_2 *such that* $\Lambda'\cap\Lambda''=\emptyset$. *Let* $\Lambda:=\Lambda'\cup\Lambda''$, $N:=n'+n''$ *and* $P_\Lambda^s:=P_{\Lambda'}^{+,s}\otimes P_{\Lambda''}^{-,s}$.
Let

$$\Delta_{\Lambda'}^{+,s}:=\max_{\Lambda_1\subset\Lambda_3\subset\Lambda_2}\left|\frac{1}{|\Lambda'|}\sum_{t\in\Lambda'}\langle\sigma(t)\rangle_{\Lambda'}^{+,s}-\frac{1}{|\Lambda_1|}\sum_{t\in\Lambda_1}\langle\sigma(t)\rangle_{\Lambda_3}^{+,s}\right|\ , \tag{9.34}$$

$$\Delta_{\Lambda''}^{-,s}:=\max_{\Lambda_1\subset\Lambda_3\subset\Lambda_2}\left|\frac{1}{|\Lambda''|}\sum_{t\in\Lambda''}\langle\sigma(t)\rangle_{\Lambda''}^{-,s}-\frac{1}{|\Lambda_1|}\sum_{t\in\Lambda_1}\langle\sigma(t)\rangle_{\Lambda_3}^{-,s}\right|\ , \tag{9.35}$$

$$x:=y+2\frac{4(s+1)(2l+s+1)}{|\Lambda_2|}+\frac{n'\Delta_{\Lambda'}^{+,s}+n''\Delta_{\Lambda''}^{-,s}}{N}\ . \tag{9.36}$$

If $y\geq 0$, *then*

$$P_\Lambda^s\left[\left\{\left|\sum_{t\in\Lambda}\left(\sigma(t)-\langle\sigma(t)\rangle_\Lambda^s\right)\right|\geq x|\Lambda|\right\}\right]\leq\exp\left(-N\frac{y^2}{2}\right)\ . \tag{9.37}$$

Remarks: 1. A variant of 1. is: for any $x\geq 0$,

$$P_\Lambda^{+,s}\left[\left\{\sum_{t\in\Lambda}\left(\sigma(t)-\langle\sigma(t)\rangle_\Lambda^+\right)\geq x|\Lambda|\right\}\right]$$
$$\leq\left(1-P_\Lambda^+[\{\exists\,\gamma\ \text{ not small}\}]\right)^{-1}\exp\left(-|\Lambda|\frac{x^2}{2\mathrm{var}_\Lambda^+}\right)\ . \tag{9.38}$$

2. We have a similar proposition if we consider Λ^--boundary condition. In particular 1. becomes in this case: for any $x\geq 0$,

$$P_\Lambda^-\left[\left\{\sum_{t\in\Lambda}\left(\sigma(t)-\langle\sigma(t)\rangle_\Lambda^-\right)\leq -x|\Lambda|\right\}\right]\leq\exp\left(-|\Lambda|\frac{x^2}{2\mathrm{var}_\Lambda^-}\right)\ . \tag{9.39}$$

Notice that by symmetry

$$\mathrm{var}_\Lambda^-=\mathrm{var}_\Lambda^+\ . \tag{9.40}$$

3. In applications, we usually have $\Lambda=\Lambda'\cup\Lambda''\cup\delta\Lambda$, with $\delta\Lambda\neq\emptyset$. The conclusion of Proposition 9.1 still applies, provided y is defined by

$$x:=y+2\frac{4(s+1)(2l+s+1)}{|\Lambda_2|}+\frac{n'\Delta_{\Lambda'}^{+,s}+n''\Delta_{\Lambda''}^{-,s}}{N}+\frac{|\delta\Lambda|}{|\Lambda|}\ . \tag{9.41}$$

Proof. Let

$$f_\Lambda(a) := \frac{1}{|\Lambda|}\ln\left\langle \exp\left[a\sum_{t\in\Lambda}\sigma(t)\right]\right\rangle_\Lambda^+ \ . \tag{9.42}$$

We have (see e.g. Lemma 5.1 in [Pf2])

$$P_\Lambda^+\left[\left\{\sum_{t\in\Lambda}\left(\sigma(t)-\langle\sigma(t)\rangle_\Lambda^+\right)\geq x|\Lambda|\right\}\right]\leq \exp\left(-|\Lambda|\frac{x^2}{2\sup_{a\geq 0}\frac{d^2}{da^2}f_\Lambda(a)}\right) \ . \tag{9.43}$$

GHS-inequalities give

$$\sup_{a\geq 0}\frac{d^2}{da^2}f_\Lambda(a)\leq \mathrm{var}_\Lambda^+ \ . \tag{9.44}$$

We prove 2. The proof is similar to the proof of Lemma 9.1. We define for each box $\mathscr{B}_i$ a random variable Y_i. Each box $\mathscr{B}_i$ is a translate of Λ_2; denote by $\mathscr{B}_i'$ the translate of Λ_1 by the same translation and set

$$Y_i := \frac{1}{|\Lambda_1|}\sum_{t\in\mathscr{B}_i'}\sigma(t) \ . \tag{9.45}$$

Then

$$\begin{aligned}P_\Lambda^s&\left[\left\{\left|\sum_{t\in\Lambda}\left(\sigma(t)-\langle\sigma(t)\rangle_\Lambda^s\right)\right|\geq x|\Lambda|\right\}\right]\\ &\leq P_\Lambda^s\left[\left\{\left|\sum_{j=1}^N Y_j-\frac{1}{|\Lambda_2|}\sum_{t\in\Lambda}\langle\sigma(t)\rangle_\Lambda^s\right|\geq N\left(x-2\frac{4(s+1)(2l+s+1)}{|\Lambda_2|}\right)\right\}\right] \ .\end{aligned} \tag{9.46}$$

We define a random set $\Lambda(\omega)$. Let $\gamma_1(\omega),\ldots,\gamma_n(\omega)$ be all external contours of ω such that $\mathrm{int}\,\gamma_k$ has a non-empty intersection with at least two different boxes $\mathscr{B}_k$; we set

$$\Lambda(\omega) := \bigcup_{i=1,\ldots,n}\overline{\mathrm{int}}\,\gamma_i \ . \tag{9.47}$$

By construction,

$$\Lambda(\omega)\cap\mathscr{B}_k' = \emptyset \ , \tag{9.48}$$

for all ω such that all contours are s-small. If $\hat{x} = x-8(s+1)(2l+s+1)/|\Lambda_2|$, then

$$\begin{aligned}P_\Lambda^s&\left[\left\{\left|\sum_{j=1}^N Y_j-\frac{1}{|\Lambda_2|}\sum_{t\in\Lambda}\langle\sigma(t)\rangle_\Lambda^s\right)\right|\geq N\hat{x}\right\}\right]\\ &=\sum_{\Lambda'\subset\Lambda}P_\Lambda^s\left[\left\{\left|\sum_{j=1}^N Y_j-\frac{1}{|\Lambda_2|}\sum_{t\in\Lambda}\langle\sigma(t)\rangle_\Lambda^s\right)\right|\geq N\hat{x}\right\}\mid\{\Lambda(\cdot)=\Lambda'\}\right]\\ &\quad\times P_\Lambda^s[\{\Lambda(\cdot)=\Lambda'\}] \ .\end{aligned} \tag{9.49}$$

Let Λ' be such that $P^s_\Lambda[\{\Lambda(\cdot) = \Lambda'\}] \neq 0$, so that the variables Y_i, $i = 1, \ldots, N$, are independent with respect to the probability measure $P^s_\Lambda[\cdot \mid \{\Lambda(\cdot) = \Lambda'\}]$. Using (9.36) we get

$$P^s_\Lambda\left[\left\{\left|\sum_{j=1}^N Y_j - \frac{1}{|\Lambda_2|}\sum_{t\in\Lambda}\langle\sigma(t)\rangle^s_\Lambda\right| \geq N\hat{x}\right\} \mid \{\Lambda(\cdot) = \Lambda'\}\right] \tag{9.50}$$
$$\leq P^s_\Lambda\left[\left\{\left|\sum_{j=1}^N \left(Y_j - \langle Y_j \mid \{\Lambda(\cdot) = \Lambda'\}\rangle^s_\Lambda\right)\right| \geq Ny\right\} \mid \{\Lambda(\cdot) = \Lambda'\}\right] .$$

Since the random variables Y_j are bounded by one, their variances are also bounded by one, so that we conclude by using the elementary inequality: for any $x > 0$,

$$\text{Prob}\left[\left|\sum_{i=1}^N (Y_i - \mathbb{E}[Y_i])\right| \geq nx\right] \leq \exp\left(-N\frac{x^2}{2}\right) . \tag{9.51}$$

□

Comments: 1. In the proof of (9.37) we used a trivial bound on the variances of the variables Y_j. Since there is a constraint on the size of the contours we expect that the variance goes to zero when $|\Lambda_L| \to \infty$; in such a case (9.37) can be improved.

2. To apply this proposition we need to control the quantities $\langle\sigma(t)\rangle^+_\Lambda$, var^+_Λ, $\langle\sigma(t)\rangle^{+,s}_\Lambda$ and (9.34), (9.35). We make some remarks concerning that point.

2a. Using Lemma 9.3, we obtain the following bounds,

$$\Delta^{+,s}_{\Lambda'} \leq O\left(s\frac{|\partial\Lambda'|}{|\Lambda'|}\right) + O\left(s\frac{|\partial\Lambda_1|}{|\Lambda_1|}\right) , \tag{9.52}$$

$$\Delta^{-,s}_{\Lambda''} \leq O\left(s\frac{|\partial\Lambda''|}{|\Lambda''|}\right) + O\left(s\frac{|\partial\Lambda_1|}{|\Lambda_1|}\right) . \tag{9.53}$$

2b. If $J(e) = \beta$ for all edges e and $\beta > \beta_c$ then FKG-inequalities give

$$\langle\sigma(t)\rangle^+_\Lambda \geq m^* \quad \text{and} \quad \langle\sigma(t)\rangle^-_\Lambda \leq -m^* , \tag{9.54}$$

and we can use Lemma 9.2 to estimate

$$\left|\langle\sigma(t)\rangle^+_\Lambda - m^*\right| \quad \text{or} \quad \left|\langle\sigma(t)\rangle^-_\Lambda + m^*\right| . \tag{9.55}$$

Moreover, GHS-inequalities give

$$\text{var}^-_\Lambda = \text{var}^+_\Lambda = \frac{1}{|\Lambda|}\sum_{t,t'\in\Lambda}\left\{\langle\sigma(t')\sigma(t)\rangle^+_\Lambda - \langle\sigma(t')\rangle^+_\Lambda\langle\sigma(t)\rangle^+_\Lambda\right\}$$
$$\leq \frac{1}{|\Lambda|}\sum_{t,t'\in\Lambda}\left\{\langle\sigma(t')\sigma(t)\rangle^+ - \langle\sigma(t')\rangle^+\langle\sigma(t)\rangle^+\right\}$$
$$\leq \sum_{t\in\mathbb{Z}^2}\left\{\langle\sigma(0)\sigma(t)\rangle^{+,\beta} - m^*(\beta)^2\right\} . \tag{9.56}$$

The quantity

$$\chi := \sum_{t\in\mathbb{Z}^2}\Big\{ \langle\sigma(0)\sigma(t)\rangle^{+,\beta} - m^*(\beta)^2 \Big\} \tag{9.57}$$

is called susceptibility. It coincides with the one defined by (6.2) when $\beta \leq \beta_c$. Indeed, in that latter case $m^* = 0$ and $\langle\sigma(t)\sigma(0)\rangle^{+,\beta} = \langle\sigma(t)\sigma(0)\rangle^{\beta}$. It is finite for $\beta > \beta_c$ in the 2D Ising model.

2c. Let the coupling constants $J(e)$ be given by (9.17) with $\beta > \beta_c$ and $h \geq 1$. Let $\Lambda \subset \mathbb{L}$; GHS-inequalities imply

$$\mathrm{var}_\Lambda^{-,\beta,h} = \mathrm{var}_\Lambda^{+,\beta,h} \leq \mathrm{var}_\Lambda^{+,\beta,1} \leq \chi \ . \tag{9.58}$$

Moreover, for all $t \in \mathbb{L}$, $t(2) \geq 1$ we have by FKG-inequalities (see proof of (5.20))

$$\langle\sigma(t)\rangle_{\Lambda\setminus\Sigma_0}^{-,\beta,1} \leq \langle\sigma(t)\rangle_\Lambda^{-,\beta,h} \leq \langle\sigma(t)\rangle_\Lambda^{-,\beta,1} \ , \tag{9.59}$$

so that we can use Lemma 9.2 to compare $\langle\sigma(t)\rangle_\Lambda^{-,\beta,h}$ with $-m^*$.

2d. Let the coupling constants $J(e)$ be given by (9.17) with $\beta > \beta_c$ and $0 < h \leq 1$. In that case we use

Lemma 9.4 *Let the coupling constants* $J(e)$ *be given by* (9.17) *with* $\beta > \beta_c$. *Let* $t \in \Lambda$ *such that* Λ *contains the square box*

$$\{u : |t(i) - u(i)| \leq 2s,\ i = 1,2\}, \quad s > 0 \ . \tag{9.60}$$

There exists a positive constant $\overline{a}(\beta)$ (*see Lemma* 9.2) *such that*

$$\begin{aligned}\langle\sigma(t)\rangle^{+,\beta} - 2P_\Lambda^+[\{\exists\,\gamma \text{ not } s\text{-}small\}] &\leq \langle\sigma(t)\rangle_\Lambda^{+,\beta,h} \\ &\leq \langle\sigma(t)\rangle^{+,\beta} + 2P_\Lambda^+[\{\exists\,\gamma \text{ not } s\text{-}small\}] + O(s)\exp\{-s\overline{a}(\beta)\} \ .\end{aligned} \tag{9.61}$$

Proof. Let $\mathscr{E}$ be the event: all external contours γ in ω, which have at least one edge on the boundary of Λ^*, are s-small. We have

$$\Big|\langle\sigma(t)\rangle_\Lambda^{+,\beta,h} - \langle\sigma(t)|\mathscr{E}\rangle_\Lambda^{+,\beta,h}\cdot P_\Lambda^{+,\beta,h}[\mathscr{E}]\Big| \leq P_\Lambda^{+,\beta,h}[\{\exists\,\gamma \text{ not } s\text{-}small\}] \ . \tag{9.62}$$

Since t is at a distance at least $2s$ from the boundary, then FKG-inequalities imply

$$\langle\sigma(t)|\mathscr{E}\rangle_\Lambda^{+,\beta,h} \geq \langle\sigma(t)\rangle_\Lambda^{+,\beta} \geq \langle\sigma(t)\rangle^{+,\beta} \ . \tag{9.63}$$

The lower bound follows from (9.62) and (9.63). The upper bound follows by using Lemma 9.2 to show that $\langle\sigma(t)|\mathscr{E}\rangle_\Lambda^{+,\beta} \leq \langle\sigma(t)\rangle^{+,\beta} + O(s)\exp(-s\overline{a}(\beta))$. □

There is of course a similar result with −boundary condition instead of + boundary condition. In case Λ is simply connected we can use Lemmas 8.1 and 5.6 to estimate

$$P_\Lambda^{+,\beta,h}[\{\exists\,\gamma \text{ not } s\text{-small}\}] \ . \tag{9.64}$$

To get an upper bound on $\mathrm{var}_{\Lambda}^{+,\beta,h}$ we use GKS inequalities,

$$\langle\sigma(t)\sigma(t')\rangle_{\Lambda}^{+,\beta,h} \le \langle\sigma(t)\sigma(t')\rangle_{\Lambda}^{+,\beta,1} \ , \tag{9.65}$$

and estimate by Lemmas 9.4 and 9.2 the quantity

$$\langle\sigma(t)\rangle_{\Lambda}^{+,\beta,h}\langle\sigma(t')\rangle_{\Lambda}^{+,\beta,h} - \langle\sigma(t)\rangle_{\Lambda}^{+,\beta,1}\langle\sigma(t')\rangle_{\Lambda}^{+,\beta,1} \ . \tag{9.66}$$

3. Using the above method, it is possible to improve the results on the phase of small contours given in [I2] and [SS1]. In these papers, the probability which was considered was $P_{\Lambda}^{+,s}[\frac{1}{|\Lambda|}\sum_{t\in\Lambda}\sigma(t) - m^* < -\epsilon]$. In such a case, we can apply the preceding method with $l = C(\epsilon)s$, with C sufficiently large. We then use the fact that there exist $\mu(\epsilon) > 0$ and $\epsilon'(\epsilon) > 0$ such that at least μN boxes have a magnetization at most $m^* - \epsilon'$. Using the fact that $\left\{\frac{1}{|\Lambda|}\sum_{t\in\Lambda}\sigma(t) - m^* < -\epsilon\right\}$ is decreasing we can first remove the constraint on the size of contours and then use monotonicity in the size of the box in order to reduce the discussion to the case Λ_2. The event so obtained can be estimated using the results of [S]. Choosing $s = L^b$, we get

$$P_{\Lambda}^{+,s}\left[\frac{1}{|\Lambda|}\sum_{t\in\Lambda}\sigma(t) - m^* < -\epsilon\right] \le (\exp(-O(L^b))^{O(L^{2-2b})} = \exp(-O(L^{2-b})) \tag{9.67}$$

This exponent can be shown to be optimal using the method of proof of Lemma 12.3.

10 Large deviations: lower bound

We derive a lower bound for large deviations of the magnetization of an Ising model in a finite box. It is given by the infimum of the isoperimetric problem discussed in the introduction, hence it depends on the choice of the boundary condition and the shape of the box. In the next section we show that the lower bound is optimal. We do not assume here that this isoperimetric problem has a solution. We derive a lower bound for all curves $\mathscr{C}$ which are boundaries of convex bodies with given volume. Inside $\mathscr{C}$ one has the −phase and outside the +phase. We have a uniform control of the remainder terms.

Let $r_1, r_2 \in \mathbb{N}$; we define the box $\Lambda_L = \Lambda_L(r_1, r_2)$

$$\Lambda_L := \left\{t \in \mathbb{Z}^2 : -r_1 L \le t(1) < r_1 L\,;0 \le t(2) < 2r_2 L\right\} \ . \tag{10.1}$$

We choose the coupling constants as in (9.17),

$$J(e) := \begin{cases} \beta \ge 0 & \text{if } e = \langle t,t'\rangle, t(2) \ge 0 \text{ and } t'(2) \ge 0, \\ h\beta \ge 0 & \text{if } e = \langle t,t'\rangle, t(2) \le -1 \text{ or } t'(2) \le -1 \ . \end{cases} \tag{10.2}$$

With these coupling constants the Gibbs measure μ_L^h, $h \ge 0$, is equal to the Gibbs measure in Λ_L with Λ_L^+-boundary condition, $\mu_L^h = \mu_L^+ \equiv \mu_{\Lambda_L}^+$. The case

$h < 0$ is equivalent to a $\Lambda_L^{\pm}$-boundary condition, with the same nonnegative coupling constants. By definition a configuration ω satisfies the $\Lambda_L^{\pm}$-boundary condition if

$$\omega(t) := \begin{cases} 1 & \text{if } t \notin \Lambda_L, t(2) \geq 0, \\ -1 & \text{if } t \notin \Lambda_L, t(2) < 0 \;. \end{cases} \tag{10.3}$$

The Gibbs measure $\mu_L^{\pm}$ is defined by

$$\mu_L^{\pm}(\omega) := \begin{cases} \Xi^{\pm}(\Lambda_L)^{-1} \exp(-H_{\Lambda_L}(\omega)) & \text{if } \omega(t) \text{ satisfies the } \Lambda_L^{\pm}\text{-bd. cond.,} \\ 0 & \text{otherwise.} \end{cases} \tag{10.4}$$

It is technically convenient to consider separately the cases $h \geq 0$ and $h < 0$. Below, when $h \geq 0$, we write probabilities with respect to μ_L^h by $P_L^+[\cdot]$ and when $h < 0$ by $P_L^{\pm}[\cdot]$. The functional W is denoted by W_+, resp. W_-, when $h \geq 0$, resp. $h < 0$.

Let $\beta > \beta_c$ and $m^* = m^*(\beta) > 0$ be the spontaneous magnetization. We choose m and c such that $-m^* < m < m^*$ and $0 < c < 1/2$. We define the event

$$A(m;c) := \left\{ \omega : \left| \sum_{t \in \Lambda_L} \omega(t) - m|\Lambda_L| \right| \leq |\Lambda_L| \cdot L^{-c} \right\} \;. \tag{10.5}$$

The main results of this section, Theorems 10.1 and 10.2, are lower bounds on

$$P_L^+[A(m;c)] \quad \text{and} \quad P_L^{\pm}[A(m;c)] \;, \tag{10.6}$$

valid for L large enough.

10.1 Positive boundary magnetic field

Theorem 10.1 *Assume that*

1. *The coupling constants are defined by (10.2) with* $\beta > \beta_c$ and $h > 0$. $-m^* < m < m^*$ and $c := 1/2 - \delta$, $\delta > 0$.
2. W_+ *is defined by (1.9) with*

$$\hat{\tau}(x) := \tau(x; \beta^*) \;, \tag{10.7}$$

the decay-rate of the two-point function (see Proposition 4.1), and

$$\hat{\tau}_{\text{bd}} := \tau_{\text{bd}}((1,0); \beta^*, h^*) \;, \tag{10.8}$$

the decay-rate of the boundary two-point function (see Definition 4.2). The parameter h^* *is defined by the relation*

$$\exp\{-2\beta h\} = \tanh \beta^* h^* \;. \tag{10.9}$$

Then there exists $L_0(\beta, h, m, c, Q)$ *such that, for any simple closed rectifiable curve* $\mathscr{C}$*, which is the boundary of a convex body of volume* $4r_1r_2\frac{m^*-m}{2m^*}$ in the rectangle Q, and for all $L \geq L_0$,[5]

$$P_L^+[A(m;c)] \geq \exp\left\{-L \cdot \mathrm{W}_+(\mathscr{C}) - \beta O(L^{1/2}\ln L)\right\} \ . \tag{10.10}$$

Proof. The basic strategy of the proof is taken from Section 7 in [Pf2]. Given the boundary $\mathscr{C}$ of a convex body V, we define a polygonal approximation of it. Then, by summing over all large contours passing through the vertices of the polygonal approximation we can estimate the probability of the event $A(m;c)$ in terms of the functional W_+ using Propositions 6.1 and 6.2. We divide the proof into five steps.

Step 1. Definition of a polygonal approximation of $\mathscr{C}$.

Consider a convex body V, whose boundary $\partial V = \mathscr{C}$, with given fixed volume. Let $L \in \mathbb{N}$ and set

$$\delta_L := L^{-1/2}\ln L \ . \tag{10.11}$$

Let

$$Q_L := \left\{x \in Q : \min_{y \notin Q} |y - x| \geq \delta_L\right\} \ , \tag{10.12}$$

and set $V_L := V \cap Q_L$.

We define a polygonal approximation $\mathscr{P}_L$ of ∂V_L. We first define a polygonal approximation $\mathscr{P}_L^0$. Let Δ_L be the square

$$\Delta_L := \left\{x \in \mathbb{R}^2 : |x(1)| + |x(2)| = \frac{\delta_L}{\sqrt{2}}\right\} \ , \tag{10.13}$$

and denote its four sides of length δ_L by J_1, J_2, J_3 and J_4 (counterclockwise). Since $V_L \subset Q_L$ is convex and vol $V_L \geq$ vol $V - O(\delta_L)$, there exists L_0, independent of V, such that int V_L contains a translate of Δ_L.

1. We choose four disjoint segments isometric to J_k, $k = 1, \ldots, 4$, with extremities on ∂V_L. If this is not possible, then we choose one corner isometric to $J_k \cup J_{k+1}$ with extremities on ∂V_L, but not necessarily its apex, and two disjoint segments isometric to J_m, J_n, $m, n \neq k, k+1$, as above. If this is not possible, then we choose two corners isometric to $J_k \cup J_{k+1}$ and $J_n \cup J_{n+1}$ with extremities on ∂V_L, but not necessarily their apexes. After this choice is made we construct a polygonal approximation of $\partial V_L \backslash \partial Q_L$ with a maximal number of segments of length δ_L (there are at most 8 segments of length smaller than δ_L). The resulting polygonal curve is $\mathscr{P}_L^0$.

Since $\tau(\cdot)$ is convex, Jensen's inequality implies

$$\mathrm{W}_+(\partial V_L) \geq \mathrm{W}_+(\mathscr{P}_L^0) \ . \tag{10.14}$$

[5] In (10.10) we can choose $O(L^{1/2}\ln L) \leq 75L^{1/2}\ln L$.

For each side of $\mathscr{P}_L^0 \backslash \partial Q_L$ of length δ_L we construct a box (6.5) or (6.6). Because we started our construction by fixing four segments isometric to J_k, $k = 1, \dots, 4$, all these boxes are pairwise disjoint

2. Let $[t, s] := \{x \in \mathscr{P}_L^0 : x(2) = \delta_L\}$. If $|t - s| > 0$, then we replace $[t, s]$ by the broken line from $t = (t(1), \delta_L)$ to $(t(1), 0)$, then $(t(1), 0)$ to $(s(1), 0)$ and finally from $(s(1), 0)$ to $s = (s(1), \delta_L)$. Then we subdivide the segment $(t(1), 0)$ to $(s(1), 0)$ into segments of length $\delta_L/2$ (except possibly the last one). We do a similar construction with the three other parts of $\mathscr{P}_L^0 \cap \partial Q_L$.

The polygonal approximation $\mathscr{P}_L$ of ∂V_L is given by the modification of $\mathscr{P}_L^0$ by 2.; the vertices of $\mathscr{P}_L$ are denoted by t_k. For each segment of length δ_L of $\mathscr{P}_L \cap \partial Q$, we construct a box like the box (6.25) of Proposition 6.2. We have $(\tau(x; \beta) \leq 2\beta)$

$$\mathrm{W}_+(\mathscr{C}) \geq \mathrm{W}_+(\mathscr{P}_L) - 16\beta\delta_L \ . \tag{10.15}$$

Step 2. Scaling and definition of a set of closed contours $\mathscr{G}_L$.

Let $L\mathscr{P}_L$ be the polygon obtained by scaling $\mathscr{P}_L$ by a factor L and shifting it by $(0, -1/2)$.[6]

We define a set of closed contours $\mathscr{G}_L = \{\Gamma\}$.

1. Each $\Gamma \in \mathscr{G}_L$ is closed and passes through all vertices of $L\mathscr{P}_L$ (counterclockwise). We denote by $[Lt_k, Lt_{k+1}]$ the side of $L\mathscr{P}_L$ between two consecutive vertices, Lt_k and Lt_{k+1}.
2. If there is a box B_k associated with $[Lt_k, Lt_{k+1}]$, then γ_k, the part of Γ between Lt_k and Lt_{k+1}, is contained in B_k. Otherwise $\gamma_k = \eta_k$, a fixed contour of minimal length from Lt_k to Lt_{k+1}.

The total length of the fixed part of Γ is smaller than $28L\delta_L$.

After that construction all necessary estimates have been already exposed in Sections 5, 6, 7 and 9.

Step 3. Estimation of $P_L^+[A(m; c)|\{\Gamma; \gamma \neq \Gamma s - \text{small}\}]$.

Let $\Gamma \in \mathscr{G}_L$. We estimate

$$\begin{aligned} &P_L^+[A(m; c) \,|\, \{\Gamma; \gamma \neq \Gamma\, s\text{-small}\}] \\ &\quad = 1 - P_L^+\left[\left\{\left|\sum_{t \in \Lambda_L} \omega(t) - m|\Lambda_L|\right| > |\Lambda_L| \cdot L^{-c}\right\} \middle| \left\{\Gamma; \gamma \neq \Gamma\, s\text{-small}\right\}\right] \end{aligned} \tag{10.16}$$

We use Proposition 9.1. We must estimate

$$\left\langle \sum_{t \in \Lambda_L} \sigma(t) \,|\, \{\Gamma; \gamma \neq \Gamma\, s\text{-small}\} \right\rangle \ . \tag{10.17}$$

This estimate is not difficult using Lemma 9.3. The main point is to notice that the total volume of the boxes B_k is smaller than $O(L^{3/2} \ln L)$, uniformly in V (the length of $\mathscr{C} = \partial V$ is bounded by the length of ∂Q, and thus the

[6] We suppose that we have possibly slightly modified $L\mathscr{P}_L$ so that all its vertices are in Λ_L^*.

number of sides of $\mathscr{P}_L$ is uniformly bounded by $O(L^{1/2}/\ln L)$). The difference of the volumes of LV and $L\mathscr{P}_L$ is also bounded by $O(L^{3/2}\ln L)$, uniformly in V. Therefore, we get, uniformly in V,

$$\left|\left\langle \sum_{t\in\Lambda_L}\sigma(t)\,|\,\{\Gamma;\gamma\neq\Gamma\, s\text{-small}\}\right\rangle^{+}_{\Lambda_L} - m|\Lambda_L|\right| \leq O(L^{3/2}\ln L)\ . \tag{10.18}$$

Since $0<c<1/2$, $O(L^{3/2}\ln L)$ is small compared to $|\Lambda_L|\cdot L^{-c}=O(L^{3/2+\delta})$. [7] We apply the second part of Proposition 9.1 with $s:=[L^{\delta/2}]$, $l:=[L^{1/2}]$ and we introduce a grid in Λ_L with an elementary cell congruent to Λ_2 of volume $O(L)$. We verify the hypothesis of Proposition 9.1 using Lemma 9.3. The three terms (9.34), (9.35) and $4(s+1)(2l+s+1)/|\Lambda_2|$ are of the same order $O(L^{-1/2+\delta/2})\ll O(L^{-c})$, and N is $O(L)$. We get

$$P_L^{+}[A(m;c)\,|\,\{\Gamma;\gamma\neq\Gamma\, s\text{-small}\}] \geq 1-O(\exp\{-O(L^{2\delta})\})\ . \tag{10.19}$$

Step 4. Estimation of $P_L^{+}[\{\Gamma;\gamma\neq\Gamma\, s\text{-small}\}]$.

Define

$$\Lambda_L(\operatorname{ext}\Gamma) := \Lambda_L\backslash\overline{\operatorname{int}}\,\Gamma,\quad \Lambda_L(\operatorname{int}\Gamma) := \Lambda_L\backslash\overline{\operatorname{ext}}\,\Gamma\ . \tag{10.20}$$

We have

$$P_L^{+}[\{\Gamma;\gamma\neq\Gamma\, s\text{-small}\}] = w^*(\Gamma)\frac{Z^{+,s}(\Lambda_L(\operatorname{ext}\Gamma))Z^{+,s}(\Lambda_L(\operatorname{int}\Gamma))}{Z^{+}(\Lambda)}\ , \tag{10.21}$$

where $Z^{+,s}(\Lambda')$ is defined as $Z^{+}(\Lambda')$ in (8.7), but by summing only over s-small contours. $Z^{+}(\Lambda)=Z(\Lambda_L^*)$ by Lemma 8.1; although $\Lambda_L(\operatorname{ext}\Gamma)$ is not simply connected, any $\Lambda_L(\operatorname{ext}\Gamma)^*$-compatible family of s-small closed contours is $\Lambda_L(\operatorname{ext}\Gamma)^+$-compatible, and consequently we also have $Z^{+,s}(\Lambda_L(\operatorname{ext}\Gamma))=Z^s(\Lambda_L(\operatorname{ext}\Gamma)^*)$. Dividing and multiplying by $Z(\Lambda_L^*|\Gamma)$, we can express $P_L^{+}[\{\Gamma;\gamma\neq\Gamma\, s\text{-small}\}]$ as

$$q_{\Lambda_L^*}(\Gamma;\beta^*,h^*)\cdot\langle\{\gamma\, s\text{-small}\}\rangle_{\Lambda_L(\operatorname{ext}\Gamma)^*}\cdot\langle\{\gamma\ s\text{-small}\}\rangle_{\Lambda_L(\operatorname{int}\Gamma)^*}\ . \tag{10.22}$$

Lemma 5.6 implies (if diameter $d(\gamma)\leq s$, then γ is s-small)

$$\langle\{\gamma\, s\text{-small}\}\rangle_{\Lambda_L(\operatorname{ext}\Gamma)^*}\geq 1-O\Big(L^{2+\delta}\exp\Big\{-\alpha L^{\delta/2}\Big\}\Big)\ . \tag{10.23}$$

A similar estimate holds for $\langle\{\gamma\, s\text{-small}\}_{\Lambda_L(\operatorname{int}\Gamma)^*}$. Summarizing these estimates, we get

$$P_L^{+}[A(m;c)] \geq \Big(1-O\Big(L^{2+\delta}\exp\{-\alpha L^{\delta/2}\}\Big)\Big)\sum_{\Gamma\in\mathscr{G}_L} q_{\Lambda_L^*}(\Gamma;\beta^*,h^*)\ . \tag{10.24}$$

Step 5. Estimation of $P_L^{+}[A(m;c)]$ in terms of the functional W_+.

It remains to control the sum over $\Gamma\in\mathscr{G}_L$. Lemmas 5.4 and 5.3 give

[7] This is the reason for allowing fluctuations of the magnetization of order $|\Lambda_L|\cdot L^{-c}$

$$\begin{aligned}\sum_{\Gamma\in\mathscr{G}_L} q_{\Lambda_L^*}(\Gamma) &\geq \sum_{\Gamma=\{\gamma_i\}\in\mathscr{G}_L} \prod_i q_{\Lambda_L^*}(\gamma_i) \\ &\geq \sum_{\Gamma=\{\gamma_i\}\in\mathscr{G}_L} \prod_i q_{\mathbb{L}}(\gamma_i) \ . \end{aligned} \tag{10.25}$$

We use the last part of Lemma 5.3 to replace $q_{\mathbb{L}}(\gamma_k)$ by $q(\gamma_k)$ whenever $\delta\gamma_k = \{Lt_k, Lt_{k+1}\}$, with $t_k, t_{k+1} \notin \Sigma_0$. By definition of $\mathscr{G}_L$, the sums over all γ_i, which are not fixed, are independent, so that we can estimate them using Propositions 6.1 and 6.2 with $a := c_1 \ln L$, c_1 large enough. Using (7.2), (7.3) and Proposition 7.1 we can find constants c_2 and c_3 such that

$$\sum_{\substack{\gamma_k:\delta\gamma_k=\{Lt_k,Lt_{k+1}\}\\ \gamma_k \text{ inside } B_k}} q(\gamma_\mathrm{k}) \geq (1-O(L^{-c_2}))\frac{\exp\{-\mathrm{W}_+([Lt_k, Lt_{k+1}])\}}{|Lt_{k+1}-Lt_k|^{c_3}} \ . \tag{10.26}$$

We have $O(L^{1/2}/\ln L)$ boxes B_k, the total length of the fixed part of Γ is smaller than $28L^{1/2}\ln L$; if we replace $q(\gamma_k)$, by $\exp\{-\mathrm{W}_+([Lt_k, Lt_{k+1}])\}$, then we make an error at most $\exp\{2\beta|\gamma_k|\}$. Taking into account (10.15), this proves the theorem. □

10.2 Nonpositive boundary magnetic field

Theorem 10.2 *Assume that*

1. *The coupling constants are defined by* (10.2) *with* $\beta > \beta_c$ and $h \leq 0$. $-m^* < m < m^*$ *and* $c := 1/2 - \delta$, $\delta > 0$.
2. W_- *is defined by* (1.9) *with*

$$\hat{\tau}(x) := \tau(x;\beta^*) \ , \tag{10.27}$$

the decay-rate of the two-point function (*see Proposition* 4.1.5.), *and*

$$\hat{\tau}_{\mathrm{bd}} := -\tau_{\mathrm{bd}}((1,0);\beta^*,h^*) \ , \tag{10.28}$$

τ_{bd} *being the decay-rate of the boundary two-point function.*
The parameter h^* *is defined by the relation*

$$\exp\{-2\beta|h|\} = \tanh\beta^* h^* \ . \tag{10.29}$$

Then there exists $L_0(\beta,h,m,c,Q)$ *such that, for any simple closed rectifiable curve* $\mathscr{C}$, *which is the boundary of a convex body of volume* $4r_1r_2\frac{m^*-m}{2m^*}$ *in the rectangle* Q, *and for all* $L \geq L_0$,[8]

$$P_L^+[A(m;c)] \geq \exp\Big\{-L\cdot \mathrm{W}_-(\mathscr{C}) - \beta O(L^{1/2}\ln L)\Big\} \ . \tag{10.30}$$

[8] In (10.30) we can choose $O(L^{1/2}\ln L) \leq 75L^{1/2}\ln L$.

The proof of Theorem 10.2 is similar to that of Theorem 10.1. In the case $h = 0$, there are two simple modifications to make. First, we use the remark following the proof of Lemma 5.6. Second, we do not introduce boxes for the sides of $\mathscr{P}_L$, which are along the lower horizontal boundary of Λ_L^*.

Let $h < 0$. There is one important difference, which we discuss now. If ω satisfies a $\Lambda_L^{\pm}$-boundary condition then there is *always* an open contour with a fixed left-hand end-point t_1^* and a fixed right-hand end-point t_2^*. We denote this particular contour by Γ^*.

Definition 10.1 *A family of compatible contours* $\underline{\gamma}$ *in* Λ_L^* *is* $\Lambda_L^{\pm}$-**compatible** *if and only if there exists a configuration* ω *satisfying the* $\Lambda_L^{\pm}$*-boundary condition, such that* $\underline{\gamma}$ *is the family of contours of* ω.

The normalization constant $\Xi(\Lambda_L)^{\pm}$ appearing in the definition of $\mu_L^{\pm}$ can be written as

$$\begin{aligned}\Xi(\Lambda_L)^{\pm} &= \sum_{\omega} \prod_{\substack{e=\langle t,t'\rangle:\\ e\cap\Lambda_L\neq\emptyset}} \exp\{J(e)[\sigma(t)(\omega)\sigma(t')(\omega) - 1]\} \\ &= \sum_{\substack{\underline{\gamma}:\\ \Lambda_L^{\pm}\text{-comp.}}} \prod_{\gamma\in\underline{\gamma}} \prod_{e^*\in\gamma} \exp\{-2J(e)\} \\ &= \sum_{\substack{\underline{\gamma}:\\ \Lambda_L^{\pm}\text{-comp.}}} \prod_{\gamma\in\underline{\gamma}} w^*(\gamma) \ . \end{aligned} \tag{10.31}$$

We set

$$Z^{\pm}(\Lambda_L|\underline{\gamma}';J) := \sum_{\substack{\underline{\gamma}:\ \underline{\gamma}\cup\underline{\gamma}'\\ \Lambda_L^{\pm}\text{-comp}}} w^*(\underline{\gamma}) \ . \tag{10.32}$$

Since Λ_L is simply connected we have the important identity

$$\frac{Z^{\pm}(\Lambda_L;J)}{Z^{+}(\Lambda_L;J)} = \sum_{\Gamma^*} w^*(\Gamma^*)\frac{Z^{\pm}(\Lambda_L|\Gamma^*;J)}{Z^{+}(\Lambda_L;J)} = \langle\sigma(t_1^*)\sigma(t_2^*)\rangle_{\Lambda_L^*}^{J^*} \ . \tag{10.33}$$

This quantity can be controlled by Propositions 4.2 and 7.1.

Proof of Theorem 10.2. We first construct the polygonal approximation $\mathscr{P}_L$ as in the proof of Theorem 10.1. Let $I := \mathscr{P}_L \cap \{x \in Q : x(2) = 0\}$. If $I = \emptyset$, then we subdivide the $\{x \in Q : x(2) = 0\}$ into segments of length $\delta_L/2$ and introduce boxes like in Proposition 6.2. The open contour Γ^* is constrained to pass though the extremities of these segments and to stay inside these boxes. We can repeat the proof of Theorem 10.1 since the construction of Theorem 10.1 does not interfere with the open contour in that case.

Suppose now that $I = [a, b]$. We define a new polygonal line $\mathscr{P}'_L$. $\mathscr{P}'_L$ goes from the bottom left corner of Q up to a along $\{x \in Q : x(2) = 0\}$, then it follows $\mathscr{P}_L \setminus I$ up to b, and finally goes along $\{x \in Q : x(2) = 0\}$ up to the bottom right corner of Q. The proof is essentially the same as that of Theorem 10.1, $\mathscr{P}'_L$ replacing the polygonal line $\mathscr{P}_L$.

Dividing and multiplying by $Z^+(\Lambda_L; J)$ and using identity (10.33) we can conclude. Since $\langle\sigma(t_1^*)\sigma(t_2^*)\rangle^{J^*}_{\Lambda_L^*}$ of (10.33) appears in the denominator the relevant functional is now W_-. □

11 Large deviations: upper bound

By Theorems 10.1 and 10.2, for L large enough,

$$P_L^+[A(m;c)] \geq \exp\Big\{-L\cdot \mathrm{W}^*(m) - \beta O(L^{1/2}\ln L)\Big\} \ , \tag{11.1}$$

where

$$\mathrm{W}^*(m) := \inf\Big\{\mathrm{W}(\mathscr{C}) : \mathscr{C}\subset Q,\ \mathrm{vol}\,\mathscr{C} = 4r_1r_2\frac{m^*-m}{2m^*}\Big\} \tag{11.2}$$

and

$$\mathrm{W} = \begin{cases} \mathrm{W}_+ & + \text{ boundary condition,} \\ \mathrm{W}_- & \pm \text{ boundary condition.}\end{cases} \tag{11.3}$$

We show that the leading term of the lower bound is optimal. To do this we analyze the measures in terms of large contours. The basic idea is to make a coarse-grained description of the large contours. We consider separately the cases of positive and negative boundary fields. The basic estimates come from Lemmas 5.4, 5.5 and Proposition 9.1. As pointed out in the first comment following the proof of that proposition, (9.37) is not a sharp bound. For that reason we prove optimality only for

$$A(m;c) = \left\{\omega : \left|\sum_{t\in\Lambda_L}\omega(t) - m|\Lambda_L|\right| \leq |\Lambda_L|\cdot L^{-c}\right\} \ , \tag{11.4}$$

with $c = 1/4-\delta$, $\delta>0$ instead of $c = 1/2-\delta$, $\delta>0$.

11.1. Positive boundary magnetic field

For $h\geq 0$ and $\mathscr{C}\subset Q$ we have $\hat{\tau}_{\mathrm{bd}}\geq 0$.[9] Hence

$$\mathrm{W}_+(\mathscr{C}) \geq \int_0^r \hat{\tau}(\dot{u}^+(t),\dot{v}^+(t))\,dt \ , \tag{11.5}$$

where $(u^+(t), v^+(t))$ is a parametrization of the curve $\mathscr{C}_+ := \mathscr{C}\backslash w_Q$.

Let $r_1, r_2\in\mathbb{N}$ and $\Lambda_L = \Lambda_L(r_1,r_2)$ be the box

$$\Lambda_L = \{t\in\mathbb{Z}^2 : -r_1L\leq t(1) < r_1L;\ 0\leq t(2) < 2r_2L\} \ . \tag{11.6}$$

[9] See preamble of part II.

The constant c is fixed, $c = 1/4 - \delta$, $\delta > 0$. The cut-off for small contours is $(\delta > \delta' > 0)$

$$s := [L^{\delta'}] \ . \tag{11.7}$$

In each configuration ω with Λ_L^+-boundary condition we denote the large contours by $\Gamma_1, \Gamma_2, \ldots$. They are all closed. We choose a total order on $(\mathbb{Z}^2)^*$:

$$t < t' \Longleftrightarrow t(2) < t'(2) \quad \text{or} \quad t(2) = t'(2) \quad \text{and} \quad t(1) < t'(1). \tag{11.8}$$

The unit-speed parametrization of Γ_i, $s \mapsto \Gamma_i(s)$, is chosen so that it is counterclockwise and $\Gamma_i(0)$ is the first point of Γ_i. The coarse-grained description of Γ_i consists of defining a sequence of points of $(\mathbb{Z}^2)^*$, $S_i = (t_{i0}, t_{i1}, \ldots, t_{in_i})$. The procedure is similar to the one used in the proof of Lemma 5.6, but here we must treat the points of Γ_i on the line $\{t \in (\mathbb{Z}^2)^* : \ t(2) = -1/2\}$ with special care. If Γ_i does not touch the line $\{t \in (\mathbb{Z}^2)^* : \ t(2) = -1/2\}$, then we do a coarse-graining like in [Pf2], points 1. to 5. below. Otherwise we mark the last points of $\Gamma_i \cap \{t(2) = -1/2\}$ before Γ_i leaves the tube

$$\left\{t \in (\mathbb{Z}^2)^* : \ -1/2 \leq t(2) \leq [L^{\delta'}]\right\} \ , \tag{11.9}$$

and we mark the first points of $\Gamma_i \cap \{t(2) = -1/2\}$, after Γ_i enters the tube (11.9).

1. We set $t_{i0} := \Gamma_i(0)$.
2. If $t_{i0}(2) = -1/2$, then go to 6. Otherwise go to 3.
3. Let s_1 be the first integer time such that Γ_i is outside the square $B(t_{i0}; [L^{\delta'}])$. We set $t_{i1} := \Gamma_i(s_1)$.
4. Let s_2 be the first integer time greater than s_1 such that Γ_i is outside the square $B(t_{i1}; [L^{\delta'}])$. We set $t_{i2} := \Gamma_i(s_2)$.
5. The procedure is iterated until it stops.
6. If $t_{i0}(2) = -1/2$, then there exists $s \in \mathbb{N}$ such that $\Gamma_i(s)(2) = -1/2$. We set $t_{i1} := \Gamma_i(s_1)$ such that s_1 is the largest integer time with the property
$$\Gamma_i(s_1)(2) = -1/2 \quad \text{and} \quad \Gamma_i(s)(2) \leq [L^{\delta'}] \quad \forall s \in [0, s_1] \ . \tag{11.10}$$
7. If for all $s > s_1$ $\Gamma_i(s)(2) \neq -1/2$, then apply the procedure 3. to 5. to the part of Γ_i defined by $\{\Gamma_i(s) : s \geq s_1\}$. Otherwise go to 8.
8. Let s_2 be the first integer time greater than s_1, such that $\Gamma_i(s_2)(2) > [L^{\delta'}]$. We set $t_{i2} := \Gamma_i(s_2)$. Let s^* be the first integer time greater than s_2 such that $\Gamma_i(s^*)(2) = -1/2$. Apply the procedure 3. to 5. to the part of Γ_i defined by $\{\Gamma_i(s) : s_2 \leq s \leq s^*\}$. Then apply the procedure starting at 2. to the part of Γ_i defined by $\{\Gamma_i(s) : s \geq s^*\}$.

Let $S := (t_1, \ldots, t_n)$ be an ordered sequence of points and $\mathscr{P}(S)$ be the corresponding closed polygonal line with vertices $(t_1, \ldots, t_n)$. To each Γ_i we associate a closed polygonal line $\mathscr{P}(\Gamma_i)$:

$$\mathscr{P}(\Gamma_i) := \mathscr{P}(S_i) \ , \tag{11.11}$$

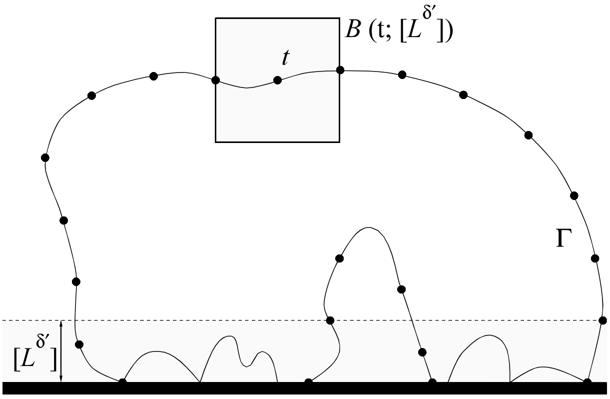


Fig. 1. Coarse-graining of a large contour Γ touching the bottom wall; the dots represent the sequence of points $S_i = \{t_{i0}, \ldots, t_{in_i}\}$

where $S_i = (t_{i0}, t_{i1}, \ldots, t_{in_i})$ is the ordered sequence of points defined by the above procedure. We set

$$B(S_i) := \left\{ t \in \Lambda_L : \ t(2) \leq [L^{\delta'}] \right\} \bigcup_{t_{ij} \in S_i} \left(B(t_{ij}; [L^{\delta'}]) \cap \Lambda_L \right) \ . \tag{11.12}$$

By definition, if Γ is a large contour with $\mathscr{P}(\Gamma) = \mathscr{P}(S_i)$, then Γ is inside $B(S_i)$. W_+ is defined as in Theorem 10.1 and we set

$$\mathrm{W}_+(S_1, \ldots, S_k) := \sum_{j=1}^{k} \mathrm{W}_+(\mathscr{P}(S_j)) \ . \tag{11.13}$$

We estimate $P_L^+[\{S_1, \ldots, S_k\}]$. We use the following remarks below. Whenever $\Gamma(s_j)(2) = t_j(2) \neq -1/2$ or $\Gamma(s_{j+1})(2) = t_{j+1}(2) \neq -1/2$,

$$\{\Gamma(s) : s_j < s < s_{j+1}\} \cap \{t \in (\mathbb{Z}^2)^* : t(2) = -1/2\} = \emptyset \ , \tag{11.14}$$

so that Lemma 5.5 applies. On the other hand, if $t_j(2) = -1/2$ and $t_{j+1}(2) = -1/2$, then the second part of Lemma 5.1 applies. Therefore (use Lemma 8.1, $Z^{+,s}(\Lambda|\underline{\Gamma}) \leq Z^+(\Lambda|\underline{\Gamma}) \leq Z(\Lambda^*|\underline{\Gamma})$ and $Z^+(\Lambda) = Z(\Lambda^*)$),

$$\begin{aligned} P_L^+[\{S_1, \ldots, S_k\}] &= \sum_{\substack{\underline{\Gamma}:\mathscr{P}(\Gamma_i)=\mathscr{P}(S_i) \\ i=1,\ldots,k}} w^*(\underline{\Gamma}) \frac{Z^{+,s}(\Lambda|\underline{\Gamma})}{Z^+(\Lambda)} \\ &\leq \sum_{\substack{\underline{\Gamma}:\mathscr{P}(\Gamma_i)=\mathscr{P}(S_i) \\ i=1,\ldots,k}} q_{\Lambda^*}(\underline{\Gamma}) \\ &\leq \exp\{-\mathrm{W}_+(S_1, \ldots, S_k)\} \ . \end{aligned} \tag{11.15}$$

Let ω_{Γ_i} be the unique configuration satisfying the Λ_L^+-boundary condition having Γ_i as single contour. The **interior** of $\mathscr{P}(S_i)$ is

$$\operatorname{Int}\mathscr{P}(S_i) := \{t \in \Lambda_L : \omega_{\Gamma_i}(t) = -1\} \backslash B(S_i) \ , \tag{11.16}$$

where Γ_i is any contour such that $\mathscr{P}(\Gamma_i) = \mathscr{P}(S_i)$. The **volume** of $\mathscr{P}(S_i)$ is

$$\operatorname{Vol}\mathscr{P}(S_i) := |\operatorname{Int}\mathscr{P}(S_i)| \ . \tag{11.17}$$

The **closure** of $\operatorname{Int}\mathscr{P}(S_i)$ is

$$\overline{\operatorname{Int}}\,\mathscr{P}(S_i) := \operatorname{Int}\mathscr{P}(S_i) \cup B(S_i) \ . \tag{11.18}$$

In a similar way let $\omega_{\underline{\Gamma}}$ be the unique configuration satisfying the Λ_L^+-boundary condition having $\underline{\Gamma} := (\Gamma_1, \Gamma_2, \ldots, \Gamma_k)$ as set of contours. The **interior** of $\underline{S} := (S_1, \ldots, S_k)$ is

$$\operatorname{Int}\underline{S} := \{t \in \Lambda_L : \omega_{\underline{\Gamma}}(t) = -1\} \backslash \bigcup_i B(S_i) \ , \tag{11.19}$$

where $\underline{\Gamma} := (\Gamma_1, \ldots, \Gamma_k)$ is any set of contours such that $\mathscr{P}(\Gamma_i) = \mathscr{P}(S_i)$, $i = 1, \ldots, k$. The **phase volume** of $\underline{S}$ is

$$\alpha(\underline{S})|\Lambda_L| := |\operatorname{Int}\underline{S}| \ . \tag{11.20}$$

Lemma 11.1 *We assume that the coupling constants are defined by* (10.2), $\beta > \beta_c$, *and that* W_+ *is defined as in Theorem 10.1. Then for any* $\eta < \delta'$ *and* $T > 0$

$$P_L^+\left[\left\{\sum_{j\geq 1} \mathrm{W}_+(\mathscr{P}(S_j)) \geq T\right\}\right] \leq \exp\left\{-T[1 - O(L^{\eta-\delta'})]\right\} \ . \tag{11.21}$$

The proof of Lemma 11.1 is a special case of that of Lemma 11.4.

Lemma 11.2 *We assume that the coupling constants are defined by* (10.2), $\beta > \beta_c$. *Let* $c = 1/4 - \delta$, $\delta > 0$ and $-m^* < m < m^*$. *For any* $\eta > 0$

$$P_L^+\left[\left\{\left|\alpha(\underline{S}) - \frac{m^* - m}{2m^*}\right| \geq \frac{1+\eta}{2m^* L^c}\right\} | A(m;c)\right] \leq \exp\{-O(L)\} \ , \tag{11.22}$$

provided L is large enough.

Proof. We set

$$E(m;c) := \left\{\left|\alpha(\underline{S}) - \frac{m^* - m}{2m^*}\right| \geq \frac{1+\eta}{2m^* L^c}\right\} \ . \tag{11.23}$$

We partition $E(m;c)$ into sets indexed by the set of their large contours. Let

$$[\underline{\Gamma}] := \{\omega : \underline{\Gamma} \text{ is the family of large contours of } \omega\} \ . \tag{11.24}$$

We write

$$P_L^+[E(m;c)|A(m;c)] = \sum_{\substack{\underline{\Gamma}: \\ [\underline{\Gamma}] \subset E(m;c)}} P_L^+[A(m;c)|[\underline{\Gamma}]] \cdot \frac{P_L^+[[\underline{\Gamma}]]}{P_L^+[A(m;c)]} \ . \tag{11.25}$$

Since (11.1) and Lemma 11.1 hold we can find a constant K such that

$$\begin{aligned} &P_L^+\left[E(m;c)\cap\left\{\sum_i \mathrm{W}_+(\mathscr{P}(S_i))\geq KL\right\}\,|\,A(m;c)\right] \\ &\quad\leq P_L^+\left[\left\{\sum_i \mathrm{W}_+(\mathscr{P}(S_i))\geq KL\right\}\,|\,A(m;c)\right]\leq \exp\{-O(L)\}\ . \end{aligned} \tag{11.26}$$

It is sufficient to control in (11.25) the terms with $\underline{\Gamma}$ such that

$$\sum_i \mathrm{W}_+(\mathscr{P}(S_i))\leq KL\ . \tag{11.27}$$

From now on we suppose that this condition is satisfied in the rest of the proof. Therefore the total length of the polygonal lines is at most $O(L)$. Suppose that $\underline{\Gamma}=\{\Gamma_1,\ldots,\Gamma_k\}$ and that $\mathscr{P}(\Gamma_j)=\mathscr{P}(S_j)$, $j=1,\ldots,k$. Each Γ_j is inside some set $B(S_j)$. Since the total length of the polygonal lines is $O(L)$,

$$\left|\bigcup_i B(S_i)\right|\leq O(L^{1+\delta'})\ . \tag{11.28}$$

We introduce $\alpha(\underline{\Gamma})$ and $\Lambda(\underline{\Gamma})$ (see Section 8):

$$\alpha(\underline{\Gamma})|\Lambda_L| := |\{t\in\Lambda_L : \omega_{\underline{\Gamma}}(t)=-1\}|\ ; \tag{11.29}$$

$$\Lambda(\underline{\Gamma}) := \Lambda_L\backslash\left(\overline{\mathrm{int}}\,\Gamma\cap\overline{\mathrm{ext}}\,\Gamma\right)\ . \tag{11.30}$$

If we compare $\alpha(\underline{\Gamma})$ of (11.20) with $\alpha(\underline{S})$, then

$$|\alpha(\underline{S})-\alpha(\underline{\Gamma})||\Lambda_L|\leq\left|\bigcup_i B(S_i)\right|\leq O(L^{1+\delta'})\ . \tag{11.31}$$

If $\star$ is the boundary condition given by any $\omega\in[\underline{\Gamma}]$, then

$$P_L^+[A(m;c)|[\underline{\Gamma}]]=P^{\star,s}_{\Lambda(\underline{\Gamma})}[A(m;c)]\ . \tag{11.32}$$

From Lemma 9.3 and (11.31), we have

$$\left\langle\sum_{t\in\Lambda_L}\sigma(t)\right\rangle^{\star,s}_{\Lambda(\underline{\Gamma})} = m^*|\Lambda_L|(1-2\alpha(\underline{\Gamma}))\pm O(L^{1+\delta'}) \tag{11.33}$$

$$= m^*|\Lambda_L|(1-2\alpha(\underline{S}))\pm O(L^{1+\delta'})\ . \tag{11.34}$$

Since

$$\begin{aligned} \sum_{t\in\Lambda_L}\sigma(t)(\omega)-m|\Lambda_L| &= \left(\sum_{t\in\Lambda_L}\sigma(t)(\omega)-\left\langle\sum_{t\in\Lambda_L}\sigma(t)\right\rangle^{\star,s}_{\Lambda(\underline{\Gamma})}\right) \\ &\quad+\left(\left\langle\sum_{t\in\Lambda_L}\sigma(t)\right\rangle^{\star,s}_{\Lambda(\underline{\Gamma})}-m|\Lambda_L|\right)\ , \end{aligned} \tag{11.35}$$

we have for every $\omega \in A(m;c)$ and L large enough,

$$\begin{aligned}\left|\sum_{t\in\Lambda_L}\sigma(t)(\omega)-\left\langle\sum_{t\in\Lambda_L}\sigma(t)\right\rangle^{\star,s}_{\Lambda(\underline{\Gamma})}\right| &\geq \left|\left\langle\sum_{t\in\Lambda_L}\sigma(t)\right\rangle^{\star,s}_{\Lambda(\underline{\Gamma})} - m|\Lambda_L|\right| \\ &\quad - \left|\sum_{t\in\Lambda_L}\sigma(t)(\omega)-m|\Lambda_L|\right| \\ &\geq \frac{1+\eta}{L^c}|\Lambda_L| - O(L^{1+\delta'}) - \frac{|\Lambda_L|}{L^c} \\ &\geq \frac{\eta}{2}\frac{|\Lambda_L|}{L^c} \quad . \end{aligned} \tag{11.36}$$

Consequently

$$P_L^+[A(m;c)|[\underline{\Gamma}]] \leq P^{\star,s}_{\Lambda(\underline{\Gamma})}\left[\left|\sum_{t\in\Lambda_L}\sigma(t)(\omega)-\left\langle\sum_{t\in\Lambda_L}\sigma(t)\right\rangle^{\star,s}_{\Lambda(\underline{\Gamma})}\right| \geq \frac{\eta}{2}|\Lambda_L|L^{-c}\right] \ . \tag{11.37}$$

We estimate (11.37) by Proposition 9.1. We introduce a grid composed of squares whose sides have length $[L^{1/4}]$.[10] Notice that the cells of the grid are much larger than the boxes used for defining the coarse-grained procedure. There are $O(L^{3/2})$ squares of the grid in Λ_L. There are at most $O(L^{1-\delta'})$ squares of the grid, which have a non-empty intersection with $\cup_j B(S_j)$. The squares of the grid not intersecting $\cup_j B(S_j)$ play the role of the boxes $\mathscr{B}_i$ in Proposition 9.1. The term $4(s+1)(2l+s+1)/|\Lambda_2|$ is $O(L^{-1/4+\delta'}) \ll O(L^{-c})$ for L large enough. The same is true for (9.34) and (9.35) as a consequence of Lemma 9.3 and of the upper bound $O(L)$ on the total length of the polygonal lines. Proposition 9.1 implies that

$$P_L^+[A(m;c)|[\underline{\Gamma}]] \leq \exp\{-O(L^{1+2\delta})\} \ , \tag{11.38}$$

provided L is large enough. □

Theorem 11.1 *Assume that*

1. *The coupling constants are defined by* (10.2) *with* $\beta > \beta_c$ *and* $h \geq 0$. $-m^* < m < m^*$ *and* $c := 1/4 - \delta$, $\delta > 0$.
2. W_+ *is defined as in Theorem* 10.1.
3. $\mathrm{W}_+^*(m)$ *is defined by*

$$\mathrm{W}_+^*(m) := \inf\left\{\mathrm{W}_+(\mathscr{C}) : \mathscr{C} \subset Q,\ \mathrm{vol}\,\mathscr{C} = 4r_1r_2\frac{m^*-m}{2m^*}\right\} \ . \tag{11.39}$$

Let $0 < \delta'$, *such that* $\delta' + \delta/2 < 1/8$ *and* $0 < \eta < \delta'$. *We set*

[10] Because of comment 1. on Proposition 9.1, this choice is essentially optimal, as can be checked using remark 3 preceding the proof of Proposition 9.1, and comment 2 on Proposition 9.1. It is at that point that we need $c = 1/4 - \delta$.

$$A(m;c) := \left\{ \omega : \left| \sum_{t\in\Lambda_L} \omega(t) - m|\Lambda_L| \right| \le |\Lambda_L| \cdot L^{-c} \right\} \ ; \tag{11.40}$$

$$E_1(m;c) := \left\{ \left| \alpha(\underline{S}) - \frac{m^* - m}{2m^*} \right| < \frac{1+\eta}{2m^* L^c} \right\} \ ; \tag{11.41}$$

$$E_2(m;c) := \left\{ \sum_i \mathrm{W}_+(S_i) \le L \cdot \mathrm{W}_+^*(m) \Big[1 + O(L^{\eta-\delta'}) \Big] \right\} \ . \tag{11.42}$$

Then, for L large enough,

$$P_L^+[E_1(m;c) \cap E_2(m;c) | A(m;c)] \ge 1 - \exp\Big\{ -O(L^{1+\eta-\delta'}) \Big\} \tag{11.43}$$

and[11]

$$\left| \frac{1}{L} \ln P_L^+[A(m;c)] + \mathrm{W}_+^*(m) \right| \le O(L^{\eta-\delta'}) \ . \tag{11.44}$$

Proof. The first affirmation follows from Theorem 10.1, Lemma 11.1 and Lemma 11.2. We prove the second affirmation. For L large enough Theorem 10.1 implies that

$$-1/L \ln P_L^+[A(m;c)] \le \mathrm{W}_+^*(m) + O(L^{-1/4+\varepsilon/2}) \ , \tag{11.45}$$

with $0 < \varepsilon < \delta$. Let $\widetilde{E}_1(m;c)$ be the complementary event of $E_1(m;c)$. We have

$$\begin{aligned} P_L^+[A(m;c)] &= P_L^+[A(m;c) \cap E_1] + P_L^+[A(m;c) \cap \widetilde{E}_1] \\ &= P_L^+[A(m;c) \cap E_1] + P_L^+[\widetilde{E}_1 | A(m;c)] \cdot P_L^+[A(m;c)] \ . \end{aligned} \tag{11.46}$$

Therefore, setting $A = A(m;c)$,

$$(1 - P_L^+[\widetilde{E}_1|A]) \cdot P_L^+[A] \le P_L^+[E_1] \ . \tag{11.47}$$

The inequality

$$\sum_i \mathrm{Vol}\,\mathscr{P}(S_i) \ge \alpha(\underline{S})|\Lambda_L| \ge \Big(\frac{m^* - m}{2m^*} - \frac{1+\eta}{2m^* L^c} \Big) |\Lambda_L| \tag{11.48}$$

implies that

$$\sum_i \mathrm{W}_+(\mathscr{P}(S_i)) \ge \mathrm{W}_+^* \Big(m + \frac{1+\eta}{L^c} \Big) L \ . \tag{11.49}$$

Let $V_1 \subset Q$ be a convex body realizing the minimum $\mathrm{W}_+^*(m + (1+\eta)/L^c)$ and $V_2 \subset Q$ be a disk of volume $(1+\eta)/2m^* L^c$. We can choose these convex bodies so that their union is a set of volume $|Q|(m^* - m)/2m^*$. Thus

[11]The weaker statement $\lim_{L\to\infty} 1/L \ln P_L^+[A(m;c)] = -\mathrm{W}_+^*(m)$ can be proven without using the lower bounds on the two-point function obtained by McCoy and Wu.

$$\mathrm{W}_+^*\left(m + \frac{1+\eta}{L^c}\right) + \mathrm{W}_+(\partial V_2) \geq \mathrm{W}_+^*(m) \ . \tag{11.50}$$

Therefore

$$(1 - P_L^+[\widetilde{E}_1|A]) \cdot P_L^+[A] \leq P_L^+\left[\left\{\sum_i \mathrm{W}_+(\mathscr{P}(S_i)) \geq \mathrm{W}_+^*\left(m + \frac{1+\eta}{L^c}\right)L\right\}\right]$$
$$\leq P_L^+\left[\left\{\sum_i \mathrm{W}_+(\mathscr{P}(S_i)) \geq \mathrm{W}_+^*(m)L - \mathrm{W}_+(\partial V_2)L\right\}\right] \ . \tag{11.51}$$

Lemma 11.1 implies that for L large enough

$$-1/L \ln P_L^+[A(m;c)] \geq \mathrm{W}_+^*(m) - O(L^{\eta-\delta'}) \ . \tag{11.52}$$

□

11.2 Negative boundary magnetic field

The remarks of Subsection 10.2 apply. By definition the open contour Γ^* is a large contour. We associate to Γ^* a sequence of points $S^* := (t_{*0}, \dots, t_{*N})$ using the same procedure as for the other contours. $\mathscr{P}(S^*)$ is the open polygonal line with vertices S^*. We thus obtain a family $(\mathscr{P}(S_1), \dots, \mathscr{P}(S_q), \mathscr{P}(S^*), \mathscr{P}(S_1'), \dots, \mathscr{P}(S_p'))$ of polygonal lines. We have distinguished between the polygonal lines with no edge belonging to the line $\{t \in \mathbb{R}^2 : t(2) = -1/2\}$, which are denoted by $(\mathscr{P}(S_1), \dots, \mathscr{P}(S_q))$, and the other ones denoted by $(\mathscr{P}(S_1'), \dots, \mathscr{P}(S_p'))$. We will now associate to the set of polygonal lines $(\mathscr{P}(S^*), \mathscr{P}(S_1'), \dots, \mathscr{P}(S_p'))$ a new set of *closed* polygonal lines $(\mathscr{P}(S_{q+1}), \dots, \mathscr{P}(S_k))$. This is done in the following way:

1. Consider the family of polygonal lines $(\mathscr{P}(S^*), \mathscr{P}(S_1'), \dots, \mathscr{P}(S_p'))$; let $\mathscr{E}^*$ be the set of edges formed by all edges of $(\mathscr{P}(S^*), \mathscr{P}(S_1'), \dots, \mathscr{P}(S_p'))$, which belong to the line $\{t \in \mathbb{R}^2 : t(2) = -1/2\} \cap \Lambda^*$. Remove $\mathscr{E}^*$ from the set of all edges of $(\mathscr{P}(S^*), \mathscr{P}(S_1'), \dots, \mathscr{P}(S_p'))$.
2. Close the polygonal lines obtained in 1. by adding the set

$$(\{t \in \mathbb{R}^2 : t(2) = -1/2\} \cap \Lambda^*) \setminus \mathscr{E}^* \ . \tag{11.53}$$

This defines a set of closed polygonal lines denoted $\mathscr{P}(S_{q+1}), \dots, \mathscr{P}(S_k)$.

Remark: We do not modify the large contours. The relation between the family $(S_1, \dots, S_k)$ and the large contours of the configuration is that these contours must be compatible with the original family $(S_1, \dots, S_q, S^*, S_1', \dots, S_p')$.

Notice that the above construction is such that we have the identity

$$\mathrm{W}_-(S_1, \dots, S_k) = \mathrm{W}_+\left(S_1, \dots, S_q, S^*, S_1', \dots, S_p'\right) - \tau_{\mathrm{bd}}(2r_1L + 1) \tag{11.54}$$

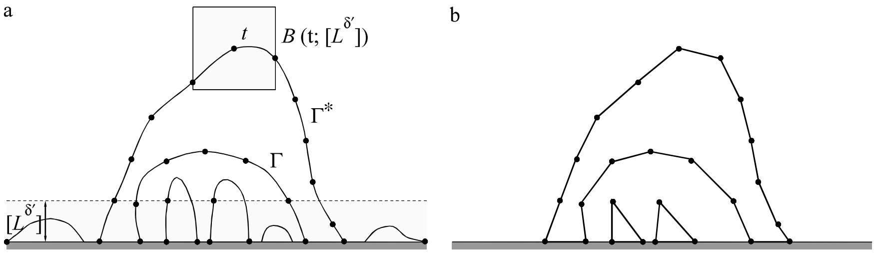


Fig. 2. a Coarse-graining of a large contour Γ touching the lower wall and of the open contour Γ^*; the dots represent the sequence of points obtained by the coarse-graining procedure described. **b** The *three* resulting closed polygonal lines

where

$$\mathrm{W}_-(S_1, \ldots, S_k) := \sum_{i=1}^{k} \mathrm{W}_-(\mathscr{P}(S_i)) \ . \tag{11.55}$$

Lemma 11.3 *In the setting described above, there exists a constant* K_2 *such that*

$$P_L^{\pm}[\{S_1, \ldots, S_k\}] \leq K_2 L^{3/2} \exp\{-\mathrm{W}_-(S_1, \ldots, S_k)\} \ . \tag{11.56}$$

Proof. We write $P_L^{\pm}[\{S_1, \ldots, S_k\}]$ as a quotient

$$P_L^{\pm}[\{S_1, \ldots, S_k\}] =: \frac{Z^{\pm}(\Lambda_L | S_1, \ldots, S_k)}{Z^{\pm}(\Lambda_L)} \ . \tag{11.57}$$

Dividing and multiplying by $Z^+(\Lambda_L)$ we must consider the quotients

$$\frac{Z^{\pm}(\Lambda_L | S_1, \ldots, S_k)}{Z^+(\Lambda_L)}, \quad \frac{Z^{\pm}(\Lambda_L)}{Z^+(\Lambda_L)} \ . \tag{11.58}$$

The first quotient is estimated using Lemmas 5.4, 5.5 and the above remark,

$$\frac{Z^{\pm}(\Lambda_L | S_1, \ldots, S_k)}{Z^+(\Lambda_L)} \leq \exp\Big\{-\mathrm{W}_+\Big(S_1, \ldots, S_q, S^*, S'_1, \ldots, S'_p\Big)\Big\} \tag{11.59}$$

The second quotient is estimated as in subsection 10.2, using Proposition 7.1,

$$\frac{Z^{\pm}(\Lambda_L)}{Z^+(\Lambda_L)} = \langle \sigma(t_1^*) \sigma(t_2^*) \rangle_{\Lambda_L^*} \geq C(2r_1 L)^{-3/2} \exp\big\{-\tau_{\mathrm{bd}}\big(t_2^* - t_1^*\big)\big\} \ . \tag{11.60}$$

These inequalities give, using (11.54),

$$P_L^{\pm}[\{S_1, \ldots, S_k\}] \leq C^{-1}(2r_1)^{3/2} L^{3/2} \exp\{-\mathrm{W}_-(S_1, \ldots, S_k)\} \ . \tag{11.61}$$

□

Lemma 11.4 *We assume that the coupling constants are defined by* (10.2), $\beta > \beta_c$, *and that* W_- *is defined as in Theorem* 10.2. *Then for any* $\eta < \delta' < \delta$ *and* $T > 0$

$$P_L^{\pm}\left[\left\{\sum_{j\geq 1} \mathrm{W}_-(\mathscr{P}(S_j)) \geq T\right\}\right] \leq \exp\left\{-T\left[1-O\left(L^{\eta-\delta'}\right)\right]+O\left(L^{1+\eta-\delta'}\right)\right\} \ . \tag{11.62}$$

Proof. We start by an entropy estimate. Let $\mathscr{N}(x,k)$ be the number of integer solutions of $1 \leq \alpha_1 \leq \ldots \leq \alpha_k \leq x$, $\sum_{i=1}^k \alpha_i = x$, k fixed, and $\mathscr{N}(x)$ the number of integer solutions of $1 \leq \alpha_1 \leq \ldots \leq \alpha_k \leq x$ and $\sum_{i=1}^k \alpha_i = x$, k arbitrary. For large x

$$\mathscr{N}(x) \sim \frac{1}{4\sqrt{3}x}\exp\left(2\pi\sqrt{x/6}\right) \ . \tag{11.63}$$

Let us consider k polygonal lines $\mathscr{P}(S_1),\ldots,\mathscr{P}(S_k)$, where $S_i = (t_{i0}, t_{i1}, \ldots, t_{in_i})$. $L^{O(N)}$ is a rough estimate of the number of families of k polygonal lines with $n_1 + \cdots + n_k = N$. Therefore the number of families of polygonal lines with $n_1 + \cdots + n_k = N$, k arbitrary, is bounded by

$$\sum_k \mathscr{N}(N,k)L^{O(N)} = \exp\{NO(\ln L)\} \ . \tag{11.64}$$

Suppose that

$$\mathrm{W}_-(S_1,\ldots,S_k) = T' \equiv T'_+ - T'_- \ , \tag{11.65}$$

where T'_+, resp. T'_-, is the positive, resp. negative, part of the functional W_-. The total number N of vertices of the polygonal lines $\mathscr{P}(S_i)$, $i = 1,\ldots,k$, can be bounded by T'_+,

$$N \leq T'_+ K L^{-\delta'} \ , \tag{11.66}$$

for K large enough. Since $|T'_-|$ is at most $O(L)$, taking into account (11.66),

$$\begin{aligned}
P_L^{\pm}[\{S_1,\ldots,S_k\}] &\leq \exp\{-\mathrm{W}_-(S_1,\ldots,S_k)\}K_2L^{3/2} \\
&= \exp\left\{-T'_+ + T'_- + NL^{\eta}\right\}K_2L^{3/2}\exp\{-NL^{\eta}\} \\
&\leq \exp\left\{-T'_+\left(1-O\left(L^{-\delta'+\eta}\right)\right)+T'_-\right\}K_2L^{3/2}\exp\{-NL^{\eta}\} \\
&\leq \exp\left\{-\mathrm{W}_-(S_1,\ldots,S_k)\left(1-O\left(L^{-\delta'+\eta}\right)\right)+O\left(L^{1-\delta'+\eta}\right)\right\} \\
&\quad \cdot K_2L^{3/2}\exp\{-NL^{\eta}\} \ .
\end{aligned} \tag{11.67}$$

Therefore,

$$\begin{aligned}
&P_L^{\pm}\left[\left\{\sum_{j\geq 1}\mathrm{W}_-(\mathscr{P}(S_j)) \geq T\right\}\right] = \sum_{k\geq 1}\sum_{S_1,\ldots,S_k} P_L^{\pm}[\{\mathrm{W}_-(S_1,\ldots,S_k) \geq T\}] \\
&\quad \leq \exp\left\{-T\left[1-O\left(L^{\eta-\delta'}\right)\right]+O\left(L^{1-\delta'+\eta}\right)\right\} \\
&\qquad \times \sum_{N\geq 1} K_2L^{3/2}\exp\{NO(\ln L) - NL^{\eta}\} \\
&\quad \leq \exp\left\{-T\left[1-O\left(L^{\eta-\delta'}\right)\right]+O\left(L^{1-\delta'+\eta}\right)\right\} \ .
\end{aligned} \tag{11.68}$$

□

Defining $\alpha(S_1,\dots S_k) := \alpha(S_1,\dots,S_q,S^*,S'_1,\dots,S'_p)$, the next lemma is proven in the same way as Lemma 11.2.

Lemma 11.5 *We assume that the coupling constants are defined by* (10.2), *$\beta > \beta_c$. Let $c = 1/4 - \delta$, $\delta > 0$ and $-m^* < m < m^*$. For any $\eta > 0$*

$$P_L^{\pm}\left[\left\{\left|\alpha(\underline{S}) - \frac{m^* - m}{2m^*}\right| \geq \frac{1+\eta}{2m^* L^c}\right\} \,|\, A(m;c)\right] \leq \exp\{-O(L)\} \ , \tag{11.69}$$

provided L is large enough.

Theorem 11.2 *Assume that*

1. *The coupling constants are defined by* (10.2) *with $\beta > \beta_c$ and $h < 0$. -$m^* < m < m^*$ and $c := 1/4 - \delta$, $\delta > 0$.*
2. W_- *is as in Theorem* 10.2.
3. $W_-^*(m)$ *is defined by*

$$W_-^*(m) := \inf\left\{W_-(\mathscr{C}) : \mathscr{C} \subset Q,\ \mathrm{vol}\,\mathscr{C} = 4r_1r_2\frac{m^* - m}{2m^*}\right\} \ . \tag{11.70}$$

Let $0 < \delta'$, such that $\delta' + \delta/2 < 1/8$ and $0 < \eta < \delta'$. We set

$$A(m;c) := \left\{\omega : \left|\sum_{t\in\Lambda_L} \omega(t) - m|\Lambda_L|\right| \leq |\Lambda_L| \cdot L^{-c}\right\} \ ; \tag{11.71}$$

$$E_1(m;c) := \left\{\left|\alpha(\underline{S}) - \frac{m^* - m}{2m^*}\right| < \frac{1+\eta}{2m^* L^c}\right\} \ ; \tag{11.72}$$

$$E_2(m;c) := \left\{\sum_i W_-(S_i) \leq L \cdot W_-^*(m)\left[1 + O(L^{\eta-\delta'})\right]\right\} \ . \tag{11.73}$$

Then, for L large enough,

$$P_L^{\pm}[E_1(m;c) \cap E_2(m;c) \,|\, A(m;c)] \geq 1 - \exp\left\{-O(L^{1+\eta-\delta'})\right\} \tag{11.74}$$

and[12]

$$\left|\frac{1}{L}\ln P_L^{\pm}[A(m;c)] + W_-^*(m)\right| \leq O\left(L^{\eta-\delta'}\right) \ . \tag{11.75}$$

12 Macroscopic droplet

In this last section we consider the limit of the lattice spacing going to zero. We suppose that $\beta > \beta_c$, $h \in \mathbb{R}$, $-m^* < m < m^*$ and $c = 1/4 - \delta > 0$ are fixed. We define the canonical Gibbs measure $\langle\cdot|m\rangle_L(\beta,h)$ by

[12]The weaker statement $\lim_{L\to\infty} 1/L \ln P_L^{\pm}[A(m;c)] = -W_-^*(m)$ can be proven without using the lower bounds on the two-point function obtained by McCoy and Wu.

$$\langle\cdot|m\rangle_L(\beta,h) := \begin{cases} \langle\cdot|A(m;c)^+_L(\beta,h) & \text{if } h \geq 0, \\ \langle\cdot|A(m;c)\rangle^\pm_L(\beta,|h|) & \text{if } h < 0 \ . \end{cases} \tag{12.1}$$

Probability with respect to that measure is denoted by Prob[$\cdot$]. In this section we treat both cases $h \geq 0$ and $h < 0$ simultaneously. We set

$$\langle\cdot\rangle_L(\beta,h) := \begin{cases} \langle\cdot\rangle^+_L(\beta,h) & \text{if } h \geq 0, \\ \langle\cdot\rangle^\pm_L(\beta,|h|) & \text{if } h < 0 \ . \end{cases} \tag{12.2}$$

As in the preceding section a contour is small if and only if it can be put inside a translate of the box $B(0;[L^{\delta'}])$, $0 < \delta' < \delta$. The specific choice of δ' is made later on; δ' is small. We do the analysis in the box $\Lambda_L(r_1,r_2)$ and at the end we scale everything by $1/L$ and take the limit of the lattice spacing going to zero.

Let $C \subset \mathbb{Z}^2$; **the empirical magnetization in** C is

$$m_C(\omega) := \frac{1}{|C|}\sum_{t\in C}\sigma(t)(\omega) \ . \tag{12.3}$$

Let $0 < a < 1$; we introduce a grid $\mathscr{L}(a)$ in Λ_L made of cells which are translates of the square box $B(0;[L^a])$. The specific choice of a is made later on; a is close to 1. In most of the cells the empirical magnetization is close to m^* or $-m^*$ with high probability (see Theorem 12.1).

The polygonal lines which we will consider in this section are constructed as in Section 11; in particular they are defined using the same intermediate scale $L^{\delta'}$.

Let $\mu > 0$ so that $a + \mu < 1$; we say that a **polygonal line is small** if $\overline{\text{Int}}\,\mathscr{P}(S_i)$ can be put inside a translate of the box $B(0;[L^{a+\mu}])$; otherwise the polygonal line is **large**. We partition the cells of $\mathscr{L}(a)$ into four sets. A cell C is **polluted** if

$$\left| C \cap \left(\bigcup_{\mathscr{P}(S)\text{small}} \overline{\text{Int}}\,\mathscr{P}(S) \right) \right| \geq L^{2a-\eta''} \ , \tag{12.4}$$

with η'' a small positive number to be chosen later on. A cell of $\mathscr{L}(a)$ is an **interface-cell** if it is not polluted and it has a non-empty intersection with $B(S_i)$ for some large polygonal line $\mathscr{P}(S_i)$, where in this section

$$B(S_i) := \bigcup_{t_{ij}\in S_i}\left(B(t_{ij};[L^{\delta'}]) \cap \Lambda_L\right) \ . \tag{12.5}$$

A cell of $\mathscr{L}(a)$ is called a **phase-cell** if it is neither polluted nor an interface-cell and it is entirely contained inside Λ_L. The remaining cells are called **boundary cells**.

Lemma 12.1 *Let $\omega \in E_1(m;c) \cap E_2(m;c)$ and suppose $\delta' < a$, $a + \mu < 1 - \eta''$. Then, uniformly in ω,*

$$\#\{C\,cell\,of\,\mathscr{L}(a) : C\,is\,polluted\} \leq O\Big(L^{1-a+\mu+\eta''}\Big)$$
$$\#\{C\,cell\,of\,\mathscr{L}(a) : C\,is\,an\,interface\text{-}cell\} \leq O\big(L^{1-a}\big)$$
$$\#\{C\,cell\,of\,\mathscr{L}(a) : C\,is\,a\,boundary\text{-}cell\} \leq O\big(L^{1-a}\big)$$

Proof. We estimate the total volume of the region containing small polygonal lines. We partition the small polygonal lines into families. The first family contains all small polygonal lines $\mathscr{P}(S)$ with $\mathrm{Int}\,\mathscr{P}(S) = \emptyset$. We then partition the remaining polygonal lines into families so that for each family

$$[L^{a+\mu}]^2 \leq \left| \bigcup_{\mathscr{P}(S)} \mathrm{Int}\mathscr{P}(S) \right| \leq 10[L^{a+\mu}]^2 \tag{12.6}$$

(except possibly for the last family which may not satisfy the lower bound). The total length of the members of a family satisfying the latter inequalities is at least $K_3 L^{a+\mu}$ (isoperimetric inequality). Since the total length of the polygonal lines is at most $K'L$, we have at most $O(L^{1-a-\mu})$ families. Consequently, the total volume of these small polygonal lines is bounded by $O(L^{1+a+\mu})$. The volume of $B(\underline{S})$ is bounded by $O(L^{1+\delta'})$. Hence the total volume of the closure of the interior of these small polygonal lines is at most $O(L^{1+a+\mu})$.

The number of polluted cells is therefore at most $O(L^{1+a+\mu})/L^{2a-\eta''} = O(L^{1-a+\mu+\eta''})$. To count the number of interface-cells we estimate the number of points we need in order to make a coarse-grained description of large polygonal lines using a reference box $B(0;[L^a])$ according to the method of the previous sections. Since the total length of the polygonal lines is at most $K'L$ and $\delta' < a$, the total number of interface-cells is at most $4K'L^{1-a}$.

The number of boundary cells is bounded by $O(L^{1-a})$. □

Let $\varepsilon(L)$ be a positive decreasing function such that $\lim_{L\to\infty} \varepsilon(L) = 0$ (see Lemma 12.2). Notice that a phase-cell cannot be surrounded by a small polygonal line; otherwise it would be polluted. We define the event E_3: in any phase-cell C the empirical magnetization satisfies

$$|m_C(\omega) - m^*| \leq \varepsilon(L) \ , \tag{12.7}$$

if the phase-cell is outside all external large contours or inside an even number of large contours, otherwise

$$|m_C(\omega) + m^*| \leq \varepsilon(L) \ . \tag{12.8}$$

Theorem 12.1 *Let $\beta > \beta_c$, $h \in \mathbb{R}$, $-m^* < m < m^*$ and $c = 1/4 - \delta > 0$. Let $\langle \cdot | m \rangle_L(\beta, h)$ be the canonical Gibbs state. Let E_1 and E_2 be the events defined in Theorems* 11.1 *or* 11.2. *Let $\eta' > 0$ be such that $2a - \delta' - 3\eta' > 1$. Then there exists a positive constant κ (see* (12.42)) *such that for L large enough*

$$\mathrm{Prob}[E_3 \,|\, E_1 \cap E_2] \geq 1 - \exp\{-O(L^\kappa)\} \tag{12.9}$$

and

$$\mathrm{Prob}[E_3 \cap E_1 \cap E_2] \geq 1 - \exp\{-O(L^{\kappa})\} \ . \tag{12.10}$$

Proof. Let $A \equiv A(m;c)$, E_3^c complementary event to E_3, and $E_{1,2} := E_2 \cap E_1$. By definition

$$\begin{aligned}\mathrm{Prob}\big[E_3^c | E_{1,2}\big] &= \frac{\langle E_3^c \cap E_{1,2} \,|\, m\rangle_L}{\langle E_{1,2} \,|\, m\rangle_L} \\ &= \frac{\langle E_3^c \cap E_{1,2} \cap A\rangle_L}{\langle E_{1,2} \cap A\rangle_L} \\ &= \langle A \,|\, E_3^c \cap E_{1,2}\rangle_L \frac{\langle E_3^c \,|\, E_{1,2}\rangle_L}{\langle A \,|\, E_{1,2}\rangle_L} \\ &\leq \frac{\langle E_3^c \,|\, E_{1,2}\rangle_L}{\langle A \,|\, E_{1,2}\rangle_L} \ .\end{aligned} \tag{12.11}$$

The numerator and denominator are estimated in the following lemmas.

Lemma 12.2 *Let*

$$\lim_{L\to\infty} \frac{\max(L^{-\eta'}, L^{-\eta''})}{\varepsilon(L)} = 0 \ . \tag{12.12}$$

1. *If the phase-cell C is outside all external large contours or inside an even number of large contours, then for L large enough*

$$\langle \{|m_C(\omega) - m^*| \geq \varepsilon(L)\} \,|\, E_{1,2})\rangle_L \leq \exp\Big\{-O(L^{2a-2\delta'-2\eta'})\varepsilon(L)^2\Big\} \ . \tag{12.13}$$

2. *If the phase-cell C is inside an odd number of large contours, then for L large enough*

$$\Big\langle \{|m_C(\omega) + m^*| \geq \varepsilon(L)\} \,|\, E_{1,2})\Big\rangle_L \leq \exp\Big\{-O(L^{2a-2\delta'-2\eta'})\varepsilon(L)^2\Big\} \ . \tag{12.14}$$

Proof. We prove 1. Let $\underline{\Gamma}$ be a family of large contours; $E(\underline{\Gamma})$ is the set of configurations with $\underline{\Gamma}$ as family of large contours. $\underline{\Gamma}$ has a coarse-grained description $\underline{S}$. $E(\underline{S})$ is the set of configurations such that the large contours have the coarse-grained description $\underline{S}$.

Let $\underline{S}$ such that $E(\underline{S}) \subset E_{1,2}$; It is sufficient to prove that

$$\langle \{|m_C(\omega) - m^*| \geq \varepsilon(L)\} \,|\, E(\underline{\Gamma})\rangle_L \leq \exp\Big\{-O(L^{2a-2\delta'-2\eta'})\varepsilon(L)^2\Big\} \ , \tag{12.15}$$

with $O(L^{2a-2\delta'-2\eta'})$ uniform in $\underline{\Gamma}$ such that $E(\underline{\Gamma}) \subset E(\underline{S}) \subset E_{1,2}$. Let

$$C^* := C \cap \left(\bigcup_{\mathscr{P}(S)\ \mathrm{small}} \overline{\mathrm{Int}}\, \mathscr{P}(S) \right) \ . \tag{12.16}$$

For L large enough (use $\varepsilon(L) \gg L^{-\eta''}$)

$$\langle\{|m_C(\omega) - m^*| \geq \varepsilon(L)\} \,|\, E(\underline{\Gamma})\rangle_L \\ \leq \langle\{|m_{C\backslash C^*}(\omega) - m^*| \geq 2\varepsilon(L)/3\}|E(\underline{\Gamma})\rangle_L \ . \tag{12.17}$$

We have $C \backslash C^* \subset \Lambda(\underline{\Gamma})$ (see (11.30)) and consequently

$$\left\langle\left\{\left|m_{C\backslash C^*}(\omega) - m^*\right| \geq 2\varepsilon(L)/3\right\} \,|\, E(\underline{\Gamma})\right\rangle_L \\ = \left\langle\left\{\left|m_{C\backslash C^*}(\omega) - m^*\right| \geq 2\varepsilon(L)/3\right\}\right\rangle^{\star,s}_{\Lambda(\underline{\Gamma})} \ , \tag{12.18}$$

$\langle\cdot\rangle^{\star,s}_{\Lambda(\underline{\Gamma})}$ being the Gibbs measure in $\Lambda(\underline{\Gamma})$ with $\star$ boundary condition (see Section 11.1), conditioned on the fact that there are only small contours. Using Lemmas 9.2, 9.3 or 9.4 we get

$$\left|\left\langle m_{C\backslash C^*}(\omega)\right\rangle^{\star,s}_{\Lambda(\underline{\Gamma})} - m^*\right| \leq \exp\left\{-O\left(L^{\delta'}\right)\right\} \ . \tag{12.19}$$

We apply Proposition 9.1 with $l = L^{\delta'+\eta'}$ and use $\varepsilon(L) \gg L^{-\eta'}$. The number of cells of $\mathscr{L}(\delta' + \eta')$, which have a non-empty intersection with $B(S_i)$, $S_i \in \underline{S}$, is bounded by $O(L^{1-\delta'})$; indeed, there are at most $K_1 L^{1-\delta'}$ vertices for the polygonal lines $\mathscr{P}(\underline{S})$; around each such vertex t the box $B(t; [L^{\delta'+\eta'}])$ contains one box of $B(S_i)$, isometric to the box $B(t; [L^{\delta'}])$, which is used in the coarse-grained procedure; each box $B(t; [L^{\delta'+\eta'}])$ intersects at most four cells of the grid $\mathscr{L}(\delta' + \eta')$. The total volume of these boxes is at most $O(L^{1+\delta'+2\eta'})$, which is small compared to $L^{2a}\varepsilon(L)$. The same is true for the boxes of the grid $\mathscr{L}(\delta' + \eta')$ intersecting the boundary of the cell C. Since $2a - 2\delta' - 2\eta' > 1 - \delta'$, the number of cells of $\mathscr{L}(\delta' + \eta')$, which are inside the cell C and do not intersect any $B(S_i)$, is $O(L^{2a-2\delta'-2\eta'})$; we have for L large enough (Proposition 9.1)

$$\left\langle\left\{\left|m_{C\backslash C^*}(\omega) - m^*\right| \geq 2\varepsilon(L)/3\right\}\right\rangle^{\star,s}_{\Lambda(\underline{\Gamma})} \\ \leq \left\langle\left\{\left|m_{C\backslash C^*}(\omega) - \left\langle m_{C\backslash C^*}(\omega)\right\rangle^{\star,s}_{\Lambda(\underline{\Gamma})}\right| \geq \varepsilon(L)/2\right\}\right\rangle^{\star,s}_{\Lambda(\underline{\Gamma})} \\ \leq \exp\left\{-O\left(L^{2a-2\delta'-2\eta'}\right)\varepsilon(L)^2\right\} \ . \tag{12.20}$$

□

Lemma 12.3 *For L large enough*

$$\left\langle A(m;c) \,|\, E_{1,2}\right\rangle_L \geq \exp\left\{-O\left(L^{2-c-\delta'}\right)\right\} \ . \tag{12.21}$$

Proof. Let $\underline{\Gamma}$ be given, $E(\underline{\Gamma}) \subset E_{1,2}$. It is sufficient to prove that

$$\langle A(m;c) \,|\, E(\underline{\Gamma})\rangle_L \geq \exp\left\{-O\left(L^{2-c-\delta'}\right)\right\} \ , \tag{12.22}$$

uniformly in $\underline{\Gamma} \subset E(\underline{S}) \subset E_{1,2}$. All contours $\gamma \notin \underline{\Gamma}$ in $\omega \in E(\underline{\Gamma})$ are s-small, $s = [L^{\delta'}]$. Since $E(\underline{\Gamma}) \subset E_{1,2}$ the phase volume $\alpha(\underline{S})$ satisfies

$$\left|\alpha(\underline{S}) - \frac{m^* - m}{2m^*}\right| < \frac{1+\eta}{2m^* L^c} \ , \tag{12.23}$$

with η some fixed positive number smaller than δ'. We have $|\Lambda_L \backslash \Lambda(\underline{\Gamma})| \leq 2K'L^{1+\delta'}$; hence

$$\left| \left\langle \sum_{t \in \Lambda_L} \sigma(t) \,|\, E(\underline{\Gamma}) \right\rangle_L - \left\langle \sum_{t \in \Lambda(\underline{\Gamma})} \sigma(t) \right\rangle^{\star,s}_{\Lambda(\underline{\Gamma})} \right| \leq O\left(L^{1+\delta'}\right) . \tag{12.24}$$

We have

$$\left\langle \sum_{t \in \Lambda_L} \sigma(t) \,|\, E(\underline{\Gamma}) \right\rangle_L = m^\star |\Lambda_L| (1 - 2\alpha(\underline{S})) \pm O\left(L^{1+\delta'}\right) . \tag{12.25}$$

Therefore

$$\left| \left\langle \sum_{t \in \Lambda(\underline{\Gamma})} \sigma(t) \right\rangle^{\star,s}_{\Lambda(\underline{\Gamma})} - m|\Lambda_L| \right| \leq \frac{1 + 2\eta}{L^c} |\Lambda_L| , \tag{12.26}$$

for L large enough. If

$$\left| \left\langle \sum_{t \in \Lambda(\underline{\Gamma})} \sigma(t) \right\rangle^{\star,s}_{\Lambda(\underline{\Gamma})} - m|\Lambda_L| \right| \leq \frac{1 - \eta}{L^c} |\Lambda_L| , \tag{12.27}$$

then, using Proposition 9.1,

$$\begin{aligned} \langle A(m;c) \,|\, E(\underline{\Gamma}) \rangle_L &\geq 1 - P^{\star,s}_{\Lambda(\underline{\Gamma})} \left[\left\{ \left| \sum_{t \in \Lambda(\underline{\Gamma})} \sigma(t)(\omega) - \left\langle \sum_{t \in \Lambda(\underline{\Gamma})} \sigma(t) \right\rangle^{\star,s}_{\Lambda(\underline{\Gamma})} \right| > \frac{\eta}{2L^c} |\Lambda_L| \right\} \right] \\ &> \frac{1}{2} , \end{aligned} \tag{12.28}$$

if L is large enough. We can therefore suppose that

$$\left| \left\langle \sum_{t \in \Lambda(\underline{\Gamma})} \sigma(t) \right\rangle^{\star,s}_{\Lambda(\underline{\Gamma})} - m|\Lambda_L| \right| > \frac{1 - \eta}{L^c} |\Lambda_L| . \tag{12.29}$$

To be specific we consider the case $(0 < \epsilon \leq 3\eta)$

$$\left\langle \sum_{t \in \Lambda(\underline{\Gamma})} \sigma(t) \right\rangle^{\star,s}_{\Lambda(\underline{\Gamma})} = m|\Lambda_L| + \frac{1 - \eta + \varepsilon}{L^c} |\Lambda_L| . \tag{12.30}$$

In this case, the mean magnetization is too large in $\Lambda(\underline{\Gamma})$. Let Λ^+ be the component of $\Lambda(\underline{\Gamma})$ where the $\star$ boundary condition corresponds to $+$ boundary condition. We construct a region $\Delta \subset \Lambda^+$ of suitable volume and we impose zero magnetization inside Δ in order to reduce the total magnetization. First let us compute the volume of Δ. It is specified by the condition

$$\left\langle \sum_{t \in \Lambda(\underline{\Gamma}) \backslash \Delta} \sigma(t) \right\rangle^{\star,s}_{\Lambda(\underline{\Gamma})} = m|\Lambda_L| , \tag{12.31}$$

that is,

$$\left\langle \sum_{t\in\Lambda(\underline{\Gamma})} \sigma(t) \right\rangle^{\star,s}_{\Lambda(\underline{\Gamma})} - \left\langle \sum_{t\in\Delta} \sigma(t) \right\rangle^{\star,s}_{\Lambda(\underline{\Gamma})} = m|\Lambda_L| + \frac{1-\eta+\varepsilon}{L^c}|\Lambda| - |\Delta| m^* = m|\Lambda_L| \ , \tag{12.32}$$

which implies that

$$|\Delta| = \frac{1-\eta+\varepsilon}{m^* L^c}|\Lambda| \ . \tag{12.33}$$

We now show that we can construct Δ as a union of cubes which are translate of $B(0,[L^{\delta'}])$ so that all contours inside these boxes are small. We introduce the grid $\mathscr{L}(\delta')$. The number of cells of $\mathscr{L}(\delta')$ which intersect some $B(S_i)$ is bounded by $O(L^{1-\delta'})$. The total number of cells of $\mathscr{L}(\delta')$ is $O(L^{2-2\delta'})$ so that it is always possible to find $O(L^{2-c-2\delta'})$ cells not intersecting any $B(S_i)$, provided L is large enough. Let $0 < \delta'' < \delta'$. Inside each selected cells $\mathscr{B}_j$ there is in the center a translate $\mathscr{B}'_j$ of the box $B(0,[L^{\delta'} - L^{\delta''}])$. We define the event $\tilde{A}$:

1. all contours which have a non-empty intersection with $\Lambda(\underline{\Gamma})\backslash\Delta$ or with at least two $\mathscr{B}_j$ are $L^{\delta''}$-small;
2.
$$\left| \sum_{t\in\Lambda(\underline{\Gamma})\backslash\Delta} \sigma(t) - m|\Lambda_L| \right| \leq |\Lambda_L|/2L^c \ ; \tag{12.34}$$
3. for each box $\mathscr{B}'_j$ we have
$$\left| \sum_{t\in\mathscr{B}'_j} \sigma(t) \right| \leq \left|\mathscr{B}'_j\right|/L^{c\delta'} \ . \tag{12.35}$$

By definition $\tilde{A} \subset A(m;c)$. Therefore

$$\langle A(m;c)\,|\,E(\underline{\Gamma})\rangle_L \geq \langle \tilde{A}\rangle^{\star,s}_{\Lambda(\underline{\Gamma})} \ . \tag{12.36}$$

Let $\tilde{A}_{1,2}$ be the event defined by conditions 1. and 2. only. Then

$$\langle \tilde{A}\rangle^{\star,s}_{\Lambda(\underline{\Gamma})} = \langle \tilde{A} \mid \tilde{A}_{1,2}\rangle^{\star,s}_{\Lambda(\underline{\Gamma})} \langle \tilde{A}_{1,2}\rangle^{\star,s}_{\Lambda(\underline{\Gamma})} \ . \tag{12.37}$$

The term $\langle \tilde{A} \mid \tilde{A}_{1,2}\rangle^{\star,s}_{\Lambda(\underline{\Gamma})}$ is estimated using Theorems 10.1 and 10.2. Denote by $\underline{\gamma}(\omega)$ all external contours in ω which have a non-empty intersection with $\Lambda(\underline{\Gamma})\backslash\Delta$ or with at least two $\mathscr{B}_j$, and by $\tilde{A}_{1,2}(\underline{\gamma}')$ the set of $\omega \in \tilde{A}_{1,2}$ such that $\underline{\gamma}(\omega) = \underline{\gamma}'$. Then

$$\langle \tilde{A} \mid \tilde{A}_{1,2}\rangle^{\star,s}_{\Lambda(\underline{\Gamma})} = \sum_{\underline{\gamma}'} \left\langle \tilde{A} \mid \tilde{A}_{1,2}(\underline{\gamma}') \right\rangle^{\star,s}_{\Lambda(\underline{\Gamma})} \frac{\langle \tilde{A}_{1,2}(\underline{\gamma}')\rangle^{\star,s}_{\Lambda(\underline{\Gamma})}}{\langle \tilde{A}_{1,2}\rangle^{\star,s}_{\Lambda(\underline{\Gamma})}} \ . \tag{12.38}$$

Under the condition $\widetilde{A}_{1,2}(\underline{\gamma}')$ local events, which are $\mathscr{F}_{\mathscr{B}'_j}$-measurable for different j, become independent. Since the boxes $\mathscr{B}'_j$ are isometric to $B(0,[L^{\delta'}-L^{\delta''}])$ there is no condition on the contours inside these boxes. In each box we have a large deviation as in Theorems 10.1 and 10.2 with $m=0$ and $\tilde{L}=[L^{\delta'}-L^{\delta''}]$ instead of L. Therefore, applying these theorems with $\mathscr{C}$ a Wulff shape in the center of each $\mathscr{B}'_j$,

$$\langle \tilde{A} \,|\, \tilde{A}_{1,2}\rangle^{\star,s}_{\Lambda(\underline{\Gamma})} \geq \exp\Big\{-O\Big(L^{\delta'}\Big)\cdot O\Big(L^{2-c-2\delta'}\Big)\Big\} \\ \geq \exp\Big\{-O\Big(L^{2-c-\delta'}\Big)\Big\} \ . \tag{12.39}$$

Proposition 9.1 and Lemma 5.6 imply that $\lim_{L\to\infty}\langle\tilde{A}_{1,2}\rangle^{*,s}_{\Lambda(\underline{\Gamma})}=1$. Indeed, let $\chi(\delta')$ be the event that all contours are $L^{\delta'}$-small and $\chi(\delta'')$ the event that all contours are $L^{\delta''}$-small. Then

$$\langle\tilde{A}_{1,2}\rangle^{\star,s}_{\Lambda(\underline{\Gamma})} \geq \langle\tilde{A}_{1,2}\,\chi(\delta'')\rangle^{\star,s}_{\Lambda(\underline{\Gamma})} \\ = \Big\langle\widetilde{A}_{1,2}\,|\,\chi(\delta'')\Big\rangle^{\star}_{\Lambda(\underline{\Gamma})}\frac{\langle\chi(\delta'')\rangle^{\star}_{\Lambda(\underline{\Gamma})}}{\langle\chi(\delta')\rangle^{\star}_{\Lambda(\underline{\Gamma})}} \ . \tag{12.40}$$

Lemma 5.6 implies that the numerator and denominator of the quotient tend to 1 as $L\to\infty$; Proposition 9.1 implies that $\langle\widetilde{A}_{1,2}\,|\,\chi(\delta'')\rangle^{*}_{\Lambda(\underline{\Gamma})}$ tends to 1 as $L\to\infty$. □

We now conclude the proof of Theorem 12.1.

Recall that $\varepsilon(L)\gg L^{-\eta'}$ and $c=1/4-\delta>0$; from Lemmas 12.2 and 12.3

$$\mathrm{Prob}[E_3|E_1\cap E_2]\geq 1-\exp\{-O(L^{\kappa})\} \tag{12.41}$$

follows, if we can find a such that $1>a>0$, δ' such that $0<\delta'<\delta$ and $0<\eta'$ so that the hypothesis of Theorem 12.1 is satisfied and

$$\kappa := 2a-\delta'-4\eta'-2+c>0 \ . \tag{12.42}$$

(12.42) is equivalent to

$$a>1-\frac{c}{2}+\frac{\delta'}{2}+2\eta' \ , \tag{12.43}$$

which is true for suitable a, δ' and η'. The last affirmation

$$\mathrm{Prob}[E_3\cap E_1\cap E_2]\geq 1-\exp\{-O(L^{\kappa})\} \tag{12.44}$$

is a consequence of (12.41) and Theorems 11.1 and 11.2. □

12.1 Continuum limit

We consider the model in the box Λ_L and scale everything by $1/L$, so that after scaling the box is the rectangle Q. We define the set of macroscopic droplets at equilibrium as

$$\mathscr{D}(m) := \left\{ \mathscr{V} \subset Q : |\mathscr{V}| = \frac{m^* - m}{2m^*}|Q|, \quad \mathrm{W}(\partial\mathscr{V}) = \mathrm{W}^*(m) \right\} . \tag{12.45}$$

For each $\mathscr{V} \in \mathscr{D}(m)$ we have a magnetization profile,

$$\rho_{\mathscr{V}}(x) := \begin{cases} m^* & \text{if } x \in Q \backslash \mathscr{V}, \\ -m^* & \text{if } x \in \mathscr{V} . \end{cases} \tag{12.46}$$

Let f be a real-valued function on Q; we set

$$d_1(f, \mathscr{D}(m)) := \inf_{\mathscr{V} \in \mathscr{D}(m)} \int_Q dx |f(x) - \rho_{\mathscr{V}}(x)| . \tag{12.47}$$

For each ω we define a magnetization profile $\rho_L(x;\omega)$ on Q. We subdivide the box Λ_L by the cells of the grid $\mathscr{L}(a)$. In each cell C we define the empirical magnetization $m_C(\omega)$. Then

$$\rho_L(x;\omega) := m_C(\omega) \text{ if } Lx \in C \tag{12.48}$$

where Lx is the point $x \in Q$ scaled by L.

Let $\omega \in E_{1,2,3} := E_1 \cap E_2 \cap E_3$ and let $\mathscr{P}(\underline{S}) = \{\mathscr{P}(S_i)(\omega) : i = 1, \ldots, k\}$ be the polygonal lines defined by the configuration ω. Using these polygonal lines scaled by $1/L$ we define a set $V(\underline{S}) \subset Q$ with the following properties (see Theorems 11.1 and 11.2)

1. The set $V(\underline{S}) \supset \mathrm{Int}\,\underline{S}$ and its volume is such that

$$\left| |V(\underline{S})| - \frac{m^* - m}{2m^*}|Q| \right| \leq \frac{1+\eta}{2m^* L^c}|Q| ; \tag{12.49}$$

2. The boundary $\partial V(\underline{S})$ of $V(\underline{S})$ is such that $\partial V(\underline{S}) \subset \cup_i \mathscr{P}(S_i)$ and

$$\mathrm{W}(\partial V(\underline{S})) \leq \mathrm{W}^*(m) + O\big(L^{\eta-\delta}\big) . \tag{12.50}$$

In the generic case the boundary of the set $V(\underline{S})$ has several connected components. We define an auxiliary connected set $\hat{V}(\underline{S})$ by translating some of these components so that $\hat{V}(\underline{S})$ has the same volume as $V(\underline{S})$, its boundary is connected and therefore can be parametrized by a single Lipschitz map $t \mapsto (u(t), v(t))$, and $\mathrm{W}(\partial V(\underline{S})) = \mathrm{W}(\partial\hat{V}(\underline{S}))$. We compare the set $\hat{V}(\underline{S})$ with the droplets in $\mathscr{D}(m)$. Given two sets $F \subset Q$ and $G \subset Q$ their distance is

$$d(F, G) := \max\left\{ \sup_{s \in F} \inf_{t \in G} |s - t|, \ \sup_{t \in G} \inf_{s \in F} |s - t| \right\} . \tag{12.51}$$

The following lemma, inspired by Corollary 3.2 in [DP], shows that one component of $\hat{V}(\underline{S})$ is close to a droplet of $\mathscr{D}(m)$ and that the total volume of the other components is small.

Lemma 12.4 *Let* $\varepsilon > 0$. *There exists a function* $\delta(\varepsilon)$ *with* $\lim_{\varepsilon \to 0} \delta(\varepsilon) = 0$ *such that if* $V \subset Q$ *has the properties*

1. *the boundary of* V *is parametrized by a unit-speed Lipschitz parametrization* $t \mapsto (u(t), v(t))$,

2. *the volume of* V *is larger than* $|Q|(m^* - m)/2m^* - \varepsilon$,
3. $\mathrm{W}(\partial V) \le \mathrm{W}^*(m) + \varepsilon$,

then

$$\inf_{\mathscr{V} \in \mathscr{D}(m)} d(\mathscr{V}, V) \le \delta(\varepsilon) \ . \tag{12.52}$$

Proof. Suppose that there exists $\delta' > 0$, V_n, $n \in \mathbb{N}$, and $\varepsilon_n \downarrow 0$ such that

$$\inf_{\mathscr{V} \in \mathscr{D}(m)} d(\mathscr{V}, V_n) \ge \delta' \quad \forall n \ . \tag{12.53}$$

Let $t \mapsto (u_n(t), v_n(t))$ be the unit-speed Lipschitz parametrization of the boundary of V_n. We choose the parametrization in such way that

$$|V_n| = \frac{1}{2} \int_{\partial V_n} (v_n' u_n - u_n' v_n) \ . \tag{12.54}$$

By our hypothesis the length of the boundary ∂V_n is uniformly bounded, so that we can parametrize all boundaries ∂V_n by maps defined on a single interval $I \subset \mathbb{R}$ (we still denote the parametrizations by $(u_n(t), v_n(t))$). Since the parametrizations are Lipschitz with a Lipschitz constant bounded by one, the maps $t \mapsto (u_n(t), v_n(t))$ are equicontinuous. By Ascoli's Theorem we can extract a uniformly convergent subsequence so that $(u^*(t), v^*(t)) = \lim_k (u_{n_k}(t), v_{n_k}(t))$ is the boundary of a set V^* with volume

$$|V^*| = \lim_{k \to \infty} |V_{n_k}| \ge |Q|(m^* - m)/2m^* \ . \tag{12.55}$$

By the uniform convergence of the sequence we have

$$\liminf_{k \to \infty} [\hat{\tau}_{\mathrm{bd}} - \hat{\tau}((1,0))] |\partial V_{n_k} \cap w_Q| \ge [\hat{\tau}_{\mathrm{bd}} - \hat{\tau}((1,0))] \left|\partial V^* \cap w_Q\right| \ , \tag{12.56}$$

since $[\hat{\tau}_{\mathrm{bd}} - \hat{\tau}((1,0))] \le 0$. A classical theorem (see e.g. [Da] Chapter 3) gives

$$\liminf_{k \to \infty} \int_I \hat{\tau}(\dot{u}_{n_k}(t), \dot{v}_{n_k}(t))\, dt \ge \int_I \hat{\tau}(\dot{u}^*(t), \dot{v}^*(t))\, dt, \tag{12.57}$$

since $\hat{\tau}$ is convex. Therefore

$$\mathrm{W}(\partial V^*) \le \lim_{k \to \infty} \mathrm{W}(\partial V_{n_k}) \le \mathrm{W}^*(m) \ , \tag{12.58}$$

thus $V^* \in \mathscr{D}(m)$, which contradicts the existence of δ'. □

Corollary 12.1 *Under the hypothesis of Lemma* 12.4, *if* ε *is small enough, then one connected component of* V *is at distance at most* $\delta(\varepsilon)$ *from a droplet of* $\mathscr{D}(m)$ *and the total volume of the remaining components is at most* $O(\delta(\varepsilon))$.

Theorem 12.2 *Let* $\beta > \beta_c$, $h \in \mathbb{R}$, $-m^* < m < m^*$ *and* $c = 1/4 - \delta > 0$. *Let* $\langle \cdot | m \rangle_L(\beta, h)$ *be the canonical Gibbs state. Then there exists a positive function* $\overline{\varepsilon}(L)$, $\lim_{L \to \infty} \overline{\varepsilon}(L) = 0$, *and* $\kappa > 0$ (*see* (12.42)) *such that for* L *large enough*

$$\mathrm{Prob}[\{d_1(\rho_L(\cdot;\omega), \mathscr{D}(m)) \le \overline{\varepsilon}(L)\}] \ge 1 - \exp\{-O(L^\kappa)\} \ . \tag{12.59}$$

Proof. Let $\omega \in E_{1,2,3}$ and let $\mathscr{P}(\underline{S}) = \{\mathscr{P}(S_i)(\omega) : i = 1, \dots, k\}$ be the polygonal lines defined by the configuration ω. We define $V(\underline{S}) \subset Q$ with properties (12.49) and (12.50) as above and set

$$\rho_L(x;\underline{S}) := \begin{cases} m^* & \text{if } x \in Q\backslash V(\underline{S}), \\ -m^* & \text{if } x \in V(\underline{S}) \ . \end{cases} \tag{12.60}$$

There exist two positive numbers μ and η'' (see Lemma 12.1),

$$\mu + \eta'' < 1 - a \ , \tag{12.61}$$

such that, if $\omega \in E_{1,2,3}$ and $\mathscr{P}(\underline{S})(\omega) = \mathscr{P}(\underline{S})$, then uniformly in $\omega \in E_{1,2,3}$

$$\int_Q dx|\rho_L(x;\omega) - \rho_L(x;\underline{S})| \leq O(L^{a-1+\mu+\eta''}) + \varepsilon(L)|Q| + O(L^{a-1}) \ . \tag{12.62}$$

The first term on the right hand side is the contribution coming from the polluted cells, the second term from the phase-cells and the last one from the interface-cells and boundary-cells. We define

$$d_1(L) := \sup_{\omega \in E_{1,2,3}} d_1(\rho_L(\cdot;\underline{S}(\omega)), \mathscr{D}(m)) \ . \tag{12.63}$$

Then Lemma 12.4 and Corollary 12.1 imply that $\lim_{L\to\infty} d_1(L) = 0$. Theorem 12.2 follows by choosing

$$\bar{\varepsilon}(L) := O(L^{a-1+\mu+\eta''}) + \varepsilon(L)|Q| + O(L^{a-1}) + d_1(L) \ . \tag{12.64}$$

□

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