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Probability Theory and Related Fields

Nonlinear filtering and measure-valued processes

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Summary. We construct a sequence of branching particle systems with time and space dependent branching mechanisms whose expectation converges to the solution of the Zakai equation. This gives an alternative numerical method to solve the Filtering Problem.

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1 Introduction

1.1 A brief of the basic framework

Let $(\Omega, \mathscr{F}, \mathscr{F}_{t\geq 0}, P)$ be a filtered probability space where we have a $(d+m)$-dimensional standard Brownian motion $\{(W_t, V_t), \mathscr{F}_t; t \geq 0\}$ and ξ a d-dimensional, $\mathscr{F}_0$-measurable, square integrable random vector, independent of (W, V). Let also $f\colon [0,\infty) \times \mathbb{R}^d \to \mathbb{R}^d$, $\sigma\colon [0,\infty) \times \mathbb{R}^d \to \mathscr{L}(\mathbb{R}^d, \mathbb{R}^d) \cong \mathbb{R}^{d^2}, h\colon [0,\infty) \times \mathbb{R}^m \to \mathbb{R}^m$ be continuous functions which satisfy sufficient conditions to have existence and uniqueness of the solution for the following system of stochastic differential equations (cf [3] or [16]):

$$dX_t = f(t, X_t)\, dt + \sigma(t, X_t)\, dW_t \tag{1}$$

$$dY_t = h(t, X_t)\, dt + dV_t \tag{2}$$

with the initial conditions $X_0 = \xi$ and $Y_0 = 0$. The process X is usually called the signal process and Y the observation process. We denote $\mathscr{Y}_t \triangleq \sigma(Y_s, 0 \leq s \leq t)$ and $\mathscr{Y} \triangleq \sigma(Y_s, s \geq 0)$, the observation σ-fields. The fil-

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tering problem consists in determining the conditional law of the signal given the observation process, i.e., in computing

$$\pi_t(\varphi) \stackrel{\text{def}}{=} E[\varphi(X_t)|\mathscr{Y}_t], \qquad \forall t \ ,$$

where φ is a Borel bounded function on $\mathbb{R}^d$. To do this, one changes the underlying measure so that Y_t becomes a Brownian motion under the new probability measure $\tilde{P}$, independent of X and

$$\pi_t(\varphi) = \frac{\rho_t(\varphi)}{\rho_t(1)}, \qquad P - a.s. \ ,$$

where $\rho_t(\varphi) \stackrel{\text{def}}{=} \tilde{E}[\varphi(X_t)\exp(\int_0^t h^*(s,X_s)\ dY_s - \frac{1}{2}\int_0^t |h(s,X_s)|^2 ds)|\mathscr{Y}_t]$ and $\tilde{E}$ is the expectation with respect to $\tilde{P}$. By imposing stronger conditions on the coefficients and the initial conditions of (1) and (2) one proves that ρ_t uniquely satisfies the following evolution equation, called the *Zakai* equation

$$\rho_t(\varphi) = \pi_0(\varphi) + \int_0^t \rho_s(A(s)\varphi))\ ds + \int_0^t \rho_s(h^*(s)\varphi)\ dY_s, \quad a.s. \quad \forall t \ , \tag{3}$$

where $A(t) : \mathscr{D}(A) \in C_b(\mathbb{R}^d) \to C_b(\mathbb{R}^d)$ is the infinitesimal generator associated with the signal X and $\varphi \in \mathscr{D}(A)$.

For a detailed account of the filtering problem, see for instance, [3] or [16].

1.2 A brief outline of the paper

In the next section we construct a sequence of branching particle systems with wildly varying space and time dependent distributional branching generating function. This sequence is proven to be tight in the third section and it is used to prove the existence of a measure-valued branching process denoted by $\mathscr{X}$ (calligraphic X), defined on $(\Omega, \mathscr{F}, \tilde{P})$, with the property that, for every $\varphi \in \mathscr{D}(A)$, the process

$$M_\varphi(t) \stackrel{\text{def}}{=} (\mathscr{X}(t),\varphi) - (\mathscr{X}(0),\varphi) - \int_0^t (\mathscr{X}(s), A(s)\varphi)\ ds - \int_0^t (\mathscr{X}(s), h^*(s)\varphi)\ dY_s \tag{4}$$

is a square integrable martingale with respect to the filtration $\mathscr{F}_t \vee \mathscr{Y}$. From the particular construction we use, the quadratic variation of the martingale $M_\varphi(t)$ will have the form

$$\langle M_\varphi(t)\rangle = \int_0^t \left(\mathscr{X}_s, v_s\varphi^2\right)\ ds, \quad P - a.s. \tag{5}$$

where v_s is a bounded, positive function, continuous in time and $v_t \geq \frac{1}{4}$, $\forall t \in [0,1]$. The last condition insures the existence of the branching mechanism presented in the next section. In section 4 we present the connection between this process and the filtering problem. We prove that the conditional expectation of $\mathscr{X}$ given $\mathscr{Y}$ satisfies (3) and that the particle systems approximation can be used to solve numerically the filtering problem.

We feel this result is of interest for a number of reasons, some are technical and relate to the extension of the Dawson Watanabe construction ([5], [20]) of a measure valued process to a case where the expected number of offspring varies so rapidly with time that it is not a function at all but in reality is only a distribution. However, we are more excited by the potential this construction has for the numerical solution of stochastic pde's over high dimensional state spaces. We illustrate our idea in this paper by concentrating on the important example of the Zakai equation of non-linear filtering.

1.3 Problems with high-dimensional filtering

As we have set out above, the essential problem of non-linear stochastic filtering is to find the conditional distribution of X_t given the information obtained by measuring Y_s for s in some time window $[t-R, t]$. The problem has a considerable importance, but its usefulness is limited to those cases *where numerical solution is feasible*.

In the special case where the evolution of X_t is given by a linear equation and h is also linear, one has the very nice property that if one assumes a Gaussian distribution for X_0 then the conditional distribution of X_t is always Gaussian, and in consequence can be described by a finite number of parameters (its mean and covariance). This remark has enormous computational significance: the conditional distribution can be obtained by solving an ordinary differential equation for the covariance and a stochastic differential equation for the mean. This approach is the well known Kalman filter ([11], [12]).

However, it took a considerable time for the Kalman filter to be used in a routine way. The major reason for its acceptance has to be that with modern computing power it is almost a trivial exercise to solve an ODE and only slightly more difficult to solve an SDE numerically. On the negative side, there are many situations where the linear/Gaussian assumptions of this model are inappropriate and in this case it would seem attractive to apply the Zakai equation which gives a stochastic PDE for the measure (or its density) describing the conditional distribution of X_t.

This might seem like a wonderful panacea; unfortunately, in real applications X_t is often a multidimensional variable, even in four dimensions it can be a serious problem to solve a PDE and more difficult to accurately solve an SPDE, in fifty dimensions it is utterly hopeless. This has lead to attempts to find wider classes of models where the posterior distribution lies in a finite dimensional manifold (the so-called Beneš and Ocone filters, [2], [15]) but these represent a very small class.

More practical have been the approaches where linearisation can be applied recursively using the extended Kalman filter ([16]). But clearly approaches via linearisation have strong limitations if there is significant uncertainty in the observations. It has remained a serious problem to find good ways to approximate the posterior measure in the general case. It is this problem we try to address in this paper. We start with a few general remarks.

In high dimensions one of the most convenient ways to describe a measure is to generate a sample of it, in other words a sequence of points randomly chosen according to its distribution. This fact has been realised by statisticians for many years and explains the popularity of Gibbs Sampling (cf [8], [9], [10]). The reason is that often one is interested in some low dimensional marginal distribution and not the measure itself. Obtaining this directly from a density function in high dimensions is not computationally feasible, as it involves a numerical integration over the whole space. On the other hand the projection of a sample can quickly be computed, and non-parametric approaches can be used effectively to construct approximate marginal distributions.

Our idea is that it might be possible to approach the Zakai equation by creating a *sample* from the posterior measure. We do not quite succeed, but we are able to produce arbitrarily good approximations.

1.4 Constructing particle approximations

Recall ([18]) that the Dawson Watanabe measure valued process is easily constructed as a limit of branching particle systems, each particle of which moves according to the same law and branches independently of the others. Such processes are easy to simulate (particularly on parallel machines) and so the Dawson Watanabe process can be approximated numerically.

In our case, we construct a measure valued process whose expectation at any time is the conditional distribution of X_t . This also has a branching particle system approximation; moreover the particles evolve independently moving with the same law as X and branch according to a mechanism that depends on the trajectory of the particle and Y, but is independent of the events elsewhere in the system. It is also easy to simulate. It follows that one may approximate the measure valued process, and by taking independent copies of this approximation, estimate its expectation. The result is a cloud of paths, with those surviving to the current time providing an estimate for the conditional distribution of X_t.

Because we can look back along the paths that have survived and observe the historical process, we see that we are also able to update our estimate of the past behaviour of our process without serious computational difficulty.

Our approach is feasible in the sense that one can carry it out and get a return directly related to the amount of computational effort invested. However, it has to be said that the convergence could still be quite slow. We are currently investigating rates of convergence and hope to report on this at a later date. However, if we contrast this approach with the one where particles are weighted with exponentials (the classical Monte Carlo method, see for instance [6], [17], [19]), we would point out two apparent advantages over this (largely disastrous) method. Firstly, all computations done are associated with particles that carry the same weighting – one never finds oneself computing a trajectory that will obviously have a smaller weight than another. A second related point is that paths exploring unfruitful directions of exploration are rapidly killed suggesting a model akin to lemmings flowing along and reproducing heavily, but being killed if they drift away from the plausible values of the variables. This again suggests a sifting out of potentially unhelpful computation.

2 Assumptions and notations

Let $C_b(\mathbb{R}^d)$ be the space of continuous bounded functions on $\mathbb{R}^d$, $C_0(\mathbb{R}^d)$ be the space of continuous functions which vanish at infinity, $C_K(\mathbb{R}^d)$ be the space of continuous functions with compact support, $C_K^2(\mathbb{R}^d)$ be the space on continuous functions with compact support with continuous first and second partial derivatives and $C_b^2(\mathbb{R}^d)$ be the space on continuous bounded functions with continuous first and second partial derivatives.

Let $M_F(\mathbb{R}^d)$ be the space of finite measures over $\mathbb{R}^d$ endowed with the topology of weak convergence, i.e., the topology in which, $\mu_n \to \mu$ iff $(\mu_n, f) \to (\mu, f)$ for all $f \in C_b(\mathbb{R}^d)$ and $M_F'(\mathbb{R}^d)$ be the space of finite measures over $\mathbb{R}^d$ endowed with the topology of vague convergence, i.e., the topology in which, $\mu_n \to \mu$ iff $(\mu_n, f) \to (\mu, f)$ for all $f \in C_0(\mathbb{R}^d)$.

We assume that the coefficients of the system (1) + (2) satisfy the necessary Lipschitz and linear growth conditions for the solution of the Zakai equation (see [16] or [3]) to exist and be unique and that h is a continuous bounded function. We also assume that the domain $C_K^2(\mathbb{R}^d) \cup \{1\} \subseteq \mathscr{D}(A) \subset C_b(\mathbb{R}^d)$ of the infinitesimal generator $A(s)$ has the following properties:
$(*)$ For every $f \in \mathscr{D}(A)$, there exists a sequence $f_n \in \mathscr{D}(A)$ such that $f_n^2 \in \mathscr{D}(A)$ and f_n converges boundedly and pointwise to f and, respectively, Af_n converges boundedly and pointwise to Af (a sequence x_n of bounded functions converges boundedly and pointwise to x if it converges pointwise and $\sup_n \|x_n\| < \infty$).
$(**)$ There exists a sequence $\{\varphi_k\}_{k>0}$, $\varphi_k : \mathbb{R}^d \to [0, 1]$ of continuous bounded functions such that $\varphi_k, \varphi_k^2 \in \mathscr{D}(A)$, for all $s \in [0, 1]$, $x \in \mathbb{R}^d$, $|A(s)\varphi_k(x)| \leq \frac{1}{k}$ and there exists R_k and r_k such that $0 < k < r_k < R_k$, and $\varphi_k\big|_{\overline{CB(0,R_k)}} = 1$ and $\varphi_k\big|_{B(0,r_k)} = 0$

Remark 2.1 If the coefficients of the stochastic differential equation (1) are continuous, then for any $f \in C_K^2(\mathbb{R}^d) \subset \mathscr{D}(A)$, we have $f^2 \in C_K^2(\mathbb{R}^d) \subset \mathscr{D}(A)$. Moreover, for any $f \in C_b^2(\mathbb{R}^d) \cap \mathscr{D}(A)$ one can choose $f_n \in C_K^2(\mathbb{R}^d) \subset \mathscr{D}(A)$ such that f_n converges boundedly and pointwise to f and, respectively, Af_n converges boundedly and pointwise to Af. Under reasonable extra conditions (e.g. that the space-time process is Feller), if $f \in \mathscr{D}(A) \subset C_b(\mathbb{R}^d)$, then $P_t f \in C_b^2(\mathbb{R}^d) \cap \mathscr{D}(A)$ and $\lim_{t \to 0} P_t f = f$, and $\lim_{t \to 0} AP_t f = Af$ boundedly and pointwise, so condition $(*)$ is satisfied (P_t is the semigroup associated to the process X).

Remarks 2.2 If the coefficients of the stochastic differential equation (1) satisfy the condition

$$\lim_{x\to\infty}\frac{\|f(x)\|}{\|x\|}=0, \quad \lim_{x\to\infty}\frac{\|\sigma(x)\|}{\|x\|}=0 \ ,$$

then one can prove that if

$$\varphi_k(x)=\begin{cases}0 & \text{for } \|x\|\le r_k\\ \exp\left(\frac{\|x\|^2-R_k^2}{\|x\|^2-r_k^2}\right) & \text{for } r_k<\|x\|<R_k\\ 1 & \text{for } \|x\|\ge R_k\end{cases}$$

then φ_k satisfy $(**)$ for large enough r_k and R_k.

From now on, we work under the new probability measure $\tilde{P}$ and all the expectations and conditional expectations will be considered with respect to $\tilde{P}$.

3 The construction of the particle systems

Let $\{(\mathscr{X}_n(t),\mathscr{F}_t),\ 0\le t\le 1\}$ be a sequence of branching particle systems on $(\Omega,\mathscr{F},\tilde{P})$ with values in $M_F(\mathbb{R}^d)$ defined as follows:

(*a*) *Initial condition*

1. $\mathscr{X}_n(0)$ is the occupation measure of n particles (we will denote the number of particles alive at time t by $N_n(t)$) of mass $\frac{1}{n}$, i.e.

$$\mathscr{X}_n(0)=\frac{1}{n}\sum_{i=1}^{n}\delta_{x_i^n}\ ,$$

   where $x_i^n\in\mathbb{R}^d$, for every $i,n\in\mathbb{N}$.
2. The occupation measure of the particles tends weakly to the initial distribution of the signal, i.e.

$$\lim_{n\to\infty}(\mathscr{X}_n(0),\varphi)=\tilde{E}[\varphi(\xi)]=\pi_0(\varphi) \quad \forall f\in C_b(\mathbb{R}^d)\ .$$

(*b*) *Evolution in time*

We describe the evolution of the processes in the interval $\left[\frac{i}{n},\frac{i+1}{n}\right]$, $i=0,1,\dots,n-1$.

1. At the time $\frac{i}{n}$, the process consists of the occupation measure of $N_n(\frac{i}{n})$ particles of mass $\frac{1}{n}$.
2. During the interval the particles move independently with the same law as the signal (1). Let $V(s)$, $s\in[\frac{i}{n},\frac{i+1}{n})$ be the trajectory of a generic particle in this interval.
3. At the end of the interval, each particle branches into a random number of particles with a mechanism depending on its trajectory in the interval. The mechanism is chosen so that it has finite second moment and the mean

number of offsprings for a particle given the σ-field $\mathscr{F}_{\frac{i+1}{n}-} = \sigma(\mathscr{F}_s, s < \frac{i+1}{n})$ of events up to time $\frac{i+1}{n}$ is

$$\exp\left(\int_{\frac{i}{n}}^{\frac{i+1}{n}} h^*(t, V(t))\, dY_t - \frac{1}{2}\int_{\frac{i}{n}}^{\frac{i+1}{n}} h^*h(t, V(t))\, dt\right) \tag{6}$$

and the variance is equal to $v_{\frac{i+1}{n}}$. The particles branch independently of each other.

In the description above v_s is an arbitrary bounded, positive function, continuous in time and $v_t \geq \frac{1}{4}$, $\forall t \in [0,1]$. The last condition insures the existence of the required branching mechanism. We denote by $\|v\|$ the supremum of v over the interval $[0,1]$, i.e.,

$$\|v\| = \sup_{t\in[0,1]} v_t \ .$$

Just before the $(i+1)$-th branching, we will have $N_n(\frac{i}{n})$ particles. Let us denote by $\mathscr{X}_n(\frac{i+1}{n}-)$ the state of the process just before the $(i+1)$-th branching and by $V_n^j(s)$, $s \in [\frac{i}{n}, \frac{i+1}{n})$ the trajectory of the j-th particle alive during the interval $(1 \leq j \leq N_n(\frac{i}{n}))$. Let also $q_n^j(\frac{i+1}{n})$ be the number of offsprings of the j-th particle at time $\frac{i+1}{n}$ and, since we assumed that h is a continuous bounded functions, let $\|h\|$ be the quantity

$$\|h\| = \sup_{(t,x)\in[0,\infty)\times\mathbb{R}^d} \|h(t,x)\| < \infty \ . \tag{7}$$

Remarks 3.1 We have the following relations:

i. $\tilde{E}[N_n(t)] = N_n(0) = n$, $\forall n \geq 0$, $t \in [0,1]$.
ii. $\tilde{E}[N_n^2(t)] \leq e^{\|h\|^2\frac{[nt]}{n}}n^2 + \sum_{k\leq[nt]} v_{\frac{k}{n}} e^{\|h\|^2\frac{[nt]-k}{n}}$, $\forall n \geq 0$, $t \in [0,1]$ ($[x]$ is the largest integer smaller that x).

Proof. i. N_n does not change during the intervals $(\frac{k}{n}, \frac{k+1}{n})$, $k = 0,\ldots,n-1$ so $N_n(t) = N_n(\frac{[tn]}{n})$. Therefore it suffices to prove that $\tilde{E}[N_n(\frac{i}{n})] = \tilde{E}[N_n(\frac{(i+1)}{n})]$ for $0 \leq i < n-1$. Using (6), we have

$$\begin{aligned}
\tilde{E}\left[N_n\left(\frac{i+1}{n}\right)\right] &= \tilde{E}\left[\sum_{j=1}^{N_n(\frac{i}{n})} \exp\left(\int_{\frac{i}{n}}^{\frac{i+1}{n}} h^*\big(t, V_n^j(t)\big)\, dY_t - \frac{1}{2}\int_{\frac{i}{n}}^{\frac{i+1}{n}} h^*h\big(t, V_n^j(t)\big)\, dt\right)\right] \\
&= \tilde{E}\left[\tilde{E}\left[\sum_{j=1}^{N_n(\frac{i}{n})} \exp\left(\int_{\frac{i}{n}}^{\frac{i+1}{n}} h^*\big(t, V_n^j(t)\big)\, dY_t \right.\right.\right. \\
&\qquad \left.\left.\left. - \frac{1}{2}\int_{\frac{i}{n}}^{\frac{i+1}{n}} h^*h\big(t, V_n^j(t)\big)\, dt\right)\middle|\mathscr{F}_{\frac{i}{n}}\right]\right] \\
&= \tilde{E}\left[\sum_{j=1}^{N_n(\frac{i}{n})} \tilde{E}\left[\exp\left(\int_{\frac{i}{n}}^{\frac{i+1}{n}} h^*\big(t, V_n^j(t)\big)\, dY_t\right.\right.\right.
\end{aligned}$$

$$-\frac{1}{2}\int_{\frac{i}{n}}^{\frac{i+1}{n}} h^*h\big(t, V_n^j(t)\big)dt\Bigg)\Bigg|\mathscr{F}_{\frac{i}{n}}\Bigg]\Bigg]$$

$$= E\left[N_n\left(\frac{i}{n}\right)\right]$$

since $s \to \exp\Big(\int_{\frac{i}{n}}^{s} h^*(t, V_n^j(t))\ dY_t - \frac{1}{2}\int_{\frac{i}{n}}^{s} h^*h(t, V_n^j(t))\ dt\Big)$ is an $\mathscr{F}_s$-adapted martingale.

ii. From the construction of the branching mechanism of the particles we have that

$$\tilde{E}\left[\left(q_n^j\left(\frac{(i+1)}{n}\right)\right)^2 |\mathscr{F}_{\frac{i+1}{n}-}\right]$$

$$= v_{\frac{i+1}{n}} + \left(\exp\left(\int_{\frac{i}{n}}^{\frac{i+1}{n}} h^*\big(t, V_n^j(t)\big)\ dY_t - \frac{1}{2}\int_{\frac{i}{n}}^{\frac{i+1}{n}} h^*h\big(t, V_n^j(t)\big)\ dt\right)\right)^2$$

$$\leq v_{\frac{i+1}{n}} + e^{\frac{\|h\|^2}{n}} \exp\left(\int_{\frac{i}{n}}^{\frac{i+1}{n}} 2h^*\big(t, V_n^j(t)\big)\ dY_t - \frac{1}{2}\int_{\frac{i}{n}}^{\frac{i+1}{n}} (2h)^*2h\big(t, V_n^j(t)\big)\ dt\right)$$

This inequality and the independence of the particles implies (as in i.)

$$\tilde{E}\left[\left(N_n\left(\frac{i+1}{n}\right)\right)^2\right] = \tilde{E}\left[\sum_{j=1}^{N_n\left(\frac{i}{n}\right)} \tilde{E}\left[\tilde{E}\left[\left(q_n^j\left(\frac{i+1}{n}\right)\right)^2\middle|\mathscr{F}_{\frac{i+1}{n}-}\right]\mathscr{F}_{\frac{i}{n}}\right]\right]$$

$$+ \tilde{E}\left[2\sum_{1\leq j_1<j_2\leq l}^{N_n\left(\frac{i}{n}\right)} \tilde{E}\left[\tilde{E}\left[q_n^{j_1}\left(\frac{i+1}{n}\right)\middle|\mathscr{F}_{\frac{i+1}{n}-}\right]\right.\right.$$

$$\left.\left.\times\ \tilde{E}\left[q_n^{j_2}\left(\frac{i+1}{n}\right)\middle|\mathscr{F}_{\frac{i+1}{n}-}\right]\middle|\mathscr{F}_{\frac{i}{n}}\right]\right]$$

$$\leq v_{\frac{i+1}{n}} + e^{\frac{\|h\|^2}{n}}\tilde{E}\left[N_n\left(\frac{i+1}{n}\right)\right]$$

$$+ e^{\frac{\|h\|^2}{n}}\tilde{E}\left[N_n\left(\frac{i+1}{n}\right)\left(N_n\left(\frac{i+1}{n}\right) - 1\right)\right].$$

It follows that

$$\tilde{E}\left[\left(N_n\left(\frac{i+1}{n}\right)\right)^2\right] \leq e^{\frac{\|h\|^2}{n}}\tilde{E}\left[\left(N_n\left(\frac{i}{n}\right)\right)^2\right] + v_{\frac{i+1}{n}}\ , \tag{8}$$

hence

$$\tilde{E}\left[\left(N_n\left(\frac{t}{n}\right)\right)^2\right] = \tilde{E}\left[\left(N_n\left(\frac{[tn]}{n}\right)\right)^2\right]$$

$$\leq e^{\|h\|^2\frac{[nt]}{n}}n^2 + \sum_{k\leq[nt]} v_{\frac{k}{n}} e^{\|h\|^2\frac{[nt]-k}{n}}$$

where the second inequality was obtained from (8). This completes the proof of the Remark. □

Let φ be a continuous bounded function. Using the Remark 3.1 we get that $(\mathscr{X}_n(t), \varphi)$ is square integrable and

$$\tilde{E}\left[\left(\mathscr{X}_n\left(\frac{i+1}{n}\right), \varphi\right)\Big|\mathscr{F}_{\frac{i+1}{n}-}\right]$$
$$= \frac{1}{n}\sum_{i=1}^{N_n(\frac{i}{n})} \varphi\left(V_n^j\left(\frac{i+1}{n}\right)\right) \exp\left(\int_{\frac{i}{n}}^{\frac{i+1}{n}} h^*\big(t, V_n^j(t)\big)\, dY_t - \frac{1}{2}\int_{\frac{i}{n}}^{\frac{i+1}{n}} h^*h\big(t, V_n^j(t)\big)\, dt\right) \tag{9}$$

and also

$$\tilde{E}\left[\left(\mathscr{X}_n\left(\frac{i+1}{n}\right), \varphi\right)^2\Big|\mathscr{F}_{\frac{i+1}{n}-}\right] - \left(\tilde{E}\left[\left(\mathscr{X}_n\left(\frac{i+1}{n}\right), \varphi\right)\Big|\mathscr{F}_{\frac{i+1}{n}-}\right]\right)^2$$
$$= \frac{1}{n}\left(\mathscr{X}_n\left(\frac{i+1}{n}-\right), v_{\frac{i+1}{n}}\varphi^2\right) \ . \tag{10}$$

In between two branches the particles move according to the prescribed SDE (1), hence for t in the interval $[\frac{i}{n}, \frac{i+1}{n})$ and $\varphi \in \mathscr{D}(A)$

$$(\mathscr{X}_n(t), \varphi) = \left(\mathscr{X}_n\left(\frac{i}{n}\right), \varphi\right) + \int_{\frac{i}{n}}^{t} (\mathscr{X}_n(s), A(s)\varphi)\, ds + S_n^{\varphi,i}(t) \ , \tag{11}$$

where $\{(S_n^{\varphi,i}(t), \mathscr{F}_t), t \in [\frac{i}{n}, \frac{i+1}{n}]\}$ is a square integrable local martingale (we use again the Remark 3.1) with the quadratic variation

$$\langle S_n^{\varphi,i}\rangle(t) = \frac{1}{n}\int_{\frac{i}{n}}^{t} \left(\mathscr{X}_n(s), \sum_{j_1,j_2,k} \sigma_{j_1,k}\sigma_{j_2,k}\frac{\partial\varphi}{\partial x_{j_1}}\frac{\partial\varphi}{\partial x_{j_2}}\right) ds$$
$$= \frac{1}{n}\int_{\frac{i}{n}}^{t} (\mathscr{X}_n(s), Tr(D\varphi^*\sigma\sigma^*D\varphi))\, ds \ . \tag{12}$$

It follows that

$$(\mathscr{X}_n(t), \varphi) = (\mathscr{X}_n(0), \varphi) + \int_0^t (\mathscr{X}_n(s), A(s)\varphi)\, ds + S_n^{\varphi}(t) + M_n^{\varphi}([nt])$$
$$+ \sum_{i=1}^{[nt]}\left(\bar{E}\left[\left(\mathscr{X}_n\left(\frac{i}{n}\right), \varphi\right)\Big|\mathscr{F}_{\frac{i}{n}-}\right] - \left(\mathscr{X}_n\left(\frac{i}{n}-\right), \varphi\right)\right) \ , \tag{13}$$

where $\{(S_n^{\varphi}(t), \mathscr{F}_t), t \in [0,1]\}$ is a square integrable local martingale

$$S_n^{\varphi}(t) \triangleq S_n^{\varphi,[nt]}(t) + \sum_{i=0}^{[nt]-1} S_n^{\varphi,i}\left(\frac{i+1}{n}\right)$$

which has the quadratic variation

$$\langle S_n^{\varphi}\rangle(t) = \frac{1}{n}\int_0^t (\mathscr{X}_n(s), Tr(D\varphi^*\sigma\sigma^*D\varphi))\ ds \tag{14}$$

and $\{(M_n^{\varphi}(l), \mathscr{F}_{\frac{l+1}{n}-}),\ l = 0,1,\ldots,n\}$ is a discrete martingale

$$M_n^{\varphi}(0) \triangleq 0,$$
$$M_n^{\varphi}(l) \triangleq \sum_{i=1}^{l}\left(\left(\mathscr{X}_n\left(\frac{i}{n}\right),\varphi\right) - \bar{E}\left[\left(\mathscr{X}_n\left(\frac{i}{n}\right),\varphi\right)\Big|\mathscr{F}_{\frac{i}{n}-}\right]\right)\ .$$

and has conditional quadratic variation

$$\begin{aligned}\langle M_n^{\varphi}\rangle(l) &= \sum_{i=1}^{l}\bar{E}\left[\left(\left(\mathscr{X}_n\left(\frac{i}{n}\right),\varphi\right) - \tilde{E}\left[\left(\mathscr{X}_n\left(\frac{i}{n}\right),\varphi\right)\Big|\mathscr{F}_{\frac{i}{n}-}\right]\right)^2\Big|\mathscr{F}_{\frac{i}{n}-}\right]\\ &= \frac{1}{n}\sum_{i=1}^{l}\left(\mathscr{X}_n\left(\frac{i}{n}-\right), v_{\frac{i}{n}}\varphi^2\right)\end{aligned} \tag{15}$$

Remarks 3.2 The process $M_n^{\varphi}(l)$ is a martingale also with respect to the larger filtration $\mathscr{F}_{\frac{l+1}{n}-}\vee\mathscr{Y}$.

Using (9) and (13), we can express the process $(\mathscr{X}_n(t),\varphi)$ as

$$\begin{aligned}(\mathscr{X}_n(t),\varphi) &= (\mathscr{X}_n(0),\varphi) + \int_0^t (\mathscr{X}_n(s),A(s)\varphi)\ ds + S_n^{\varphi}(t) + M_n^{\varphi}([nt])\\ &\quad + \sum_{i=1}^{[nt]}\frac{1}{n}\sum_{j=1}^{N_n\left(\frac{i-1}{n}\right)}\varphi\left(V_n^j\left(\frac{i}{n}\right)\right)\times\left(\exp\left(\int_{\frac{i-1}{n}}^{\frac{i}{n}} h^*\big(s,V_n^j(s)\big)\ dY_s\right.\right.\\ &\quad \left.\left. -\frac{1}{2}\int_{\frac{i-1}{n}}^{\frac{i}{n}} h^*h\big(s,V_n^j(s)\big)\ ds\right) - 1\right)\ .\end{aligned} \tag{16}$$

Then applying Ito's rule to the exponential in the last term of (16) and exploiting the fact that Y is a Brownian motion, we get

$$\begin{aligned}(\mathscr{X}_n(t),\varphi) =&(\mathscr{X}_n(0),\varphi) + \int_0^t (\mathscr{X}_n(s),A(s)\varphi)\ ds + S_n^{\varphi}(t) + M_n^{\varphi}([nt])\\ &+\frac{1}{n}\int_0^{\frac{[nt]}{n}}\sum_{j=1}^{N_n\left(\frac{[sn]}{n}\right)}\varphi\left(V_n^j\left(\frac{[sn]+1}{n}\right)\right)B_n^s\big(V_n^j,s\big)h^*\big(s,V_n^j(s)\big)\ dY_s\end{aligned} \tag{17}$$

where

$$B_n^s\big(V_n^j,p\big) = \exp\left(\int_{\frac{[sn]}{n}}^{p} h^*\big(r,V_n^j(r)\big)\ dY_r - \frac{1}{2}\int_{\frac{[sn]}{n}}^{p} h^*h\big(r,V_n^j(r)\big)\ dr\right) \tag{18}$$

4 The existence of the process $\mathcal{X}$

We show first that the sequence $\{\mathcal{X}_n\}_{n>0}$ is tight in $D_{M'_F(\mathbb{R}^d)}[0,1]$ endowed with the Skorohod topology and then that it 'stays mostly' within a compact set. More precisely, we will prove that there exists a sequence of compact sets $K_k \in \mathbb{R}^d$ such that, for every $\varepsilon > 0$

$$\lim_{k\to\infty} \limsup_{n\to\infty} \tilde{P}(\exists t \in [0,1] \quad \text{s.t.} \quad (\mathcal{X}_n(t), I_{CK_k}) > \varepsilon) = 0 \tag{19}$$

The two properties will ensure that the sequence $\{\mathcal{X}_n\}_{n>0}$ is tight in the space $D_{M_F(\mathbb{R}^d)}[0,1]$ endowed with the Skorohod topology. So first we prove the tightness over the space $D_{M'_F(\mathbb{R}^d)}[0,1]$. For this, it is sufficient to prove (cf. [18]) that the processes $\{(\mathcal{X}_n(s), \varphi_i),\ s \in [0,1]\}$ form a tight sequence, where $\{\varphi_i\}_{i\geq 0}$ is defined as follows: φ_0 is the constant function 1 and $\{\varphi_i; i > 0\}$ is a dense set in $C_0(\mathbb{R}^d)$ (we will take them to be in $\mathscr{D}(A)$ with compact support). In order to prove that $\{(\mathcal{X}_n(s), \varphi_i), s \in [0,1]\}$ is a tight sequence for every $i \geq 0$, we use the following theorem (cf. [1])

Theorem 4.1 [Aldous] *Let $\{a_n\}$ be a sequence of real valued processes with cádlág paths such that*

(i) $\{a_n(t)\}$ is tight on the line for each $t \in [0,1]$.
(ii) For any arbitrary sequence of stopping times $\{\tau_n\}_{n\geq 0}$ (with respect to the natural filtration of $\{a_n\}$) and any sequence $\{\delta_n\}_{n\geq 0}$ of positive real numbers with $\lim_{n\to\infty} \delta_n = 0$, we have

$$\lim_{n\to\infty} a_n(\tau_n + \delta_n) - a_n(\tau_n) = 0 \textit{ in probability} .$$

Then $\{a_n\}$ is tight.

Condition (i) follows from the Proposition 4.2.

Proposition 4.2 *For every $t \in [0,1]$ we have*

$$\lim_{k\to\infty} \sup_{n\geq 0} \tilde{P}\left(\sup_{0\leq s\leq t} (\mathcal{X}_n(s), 1) > k \right) = 0 . \tag{20}$$

Proof. Since

$$\tilde{P}\left(\sup_{0\leq s\leq t} (\mathcal{X}_n(s), 1) > k \right) \leq \frac{\tilde{E}\left[\left(\sup_{0\leq s\leq t} (\mathcal{X}_n(s), 1) \right)^2 \right]}{k^2} , \tag{21}$$

it is enough to prove that $\sup_{n\geq 0} \tilde{E}[(\sup_{0\leq s\leq t}(\mathcal{X}_n(s), 1))^2]$ is finite. Let us denote by

$$\psi_n(t) \triangleq \tilde{E}\left[\left(\sup_{0\leq s\leq t} (\mathcal{X}_n(s), 1) \right)^2 \right] .$$

From (17) we obtain

$$\psi_n(t) \le 3(\mathscr{X}_n(0),1)^2 + 3\tilde{E}\left[\left(\sup_{0\le i\le [nt]}\left|M_n^1(i)\right|\right)^2\right]$$
$$+\frac{3}{n^2}\tilde{E}\left[\left(\sup_{0\le p\le \frac{[tn]}{n}}\left|\int_0^p \sum_{j=1}^{N_n\left(\frac{[sn]}{n}\right)} B_n^s\left(V_n^j,s\right)h^*\left(s,V_n^j(s)\right)\,dY_s\right|\right)^2\right] \quad (22)$$

We prove that ψ_n is bounded from above uniformly in n, by exploiting (22) and using the Gronwall inequality. For this we give an upper bound for each of the three terms of the right hand side of the inequality (22) of the form $\alpha + \beta\int_0^t \psi_n(s)\,ds$.

The first term
We have

$$(\mathscr{X}_n(0),1)^2 = \frac{N_n(0)^2}{n^2} = 1 \ . \quad (23)$$

The second term
Doob's maximal inequality (cf [13], pp. 14) gives us the following upper bound:

$$\tilde{E}\left[\left(\sup_{0\le i\le [nt]}\left|M_n^1(i)\right|\right)^2\right] \le 4\tilde{E}\left[\left(M_n^1([nt])\right)^2\right] = 4\tilde{E}\left[\left(\langle M_n^1\rangle([nt])\right)^2\right]$$
$$\le \frac{4\|v\|}{n}\sum_{i=1}^{[nt]}\tilde{E}\left[\left(\mathscr{X}\left(\frac{i}{n}-\right),1\right)\right] \le 4\|v\|\frac{[nt]}{n} \le 4\|v\| \ . \quad (24)$$

The third term
We find first an upper bound for $\tilde{E}[(\sum_{j=1}^{N_n(\frac{[sn]}{n})} B_n^s(V_n^j,s))^2]$. We have that

$$\tilde{E}\left[\left(\sum_{j=1}^{N_n\left(\frac{[sn]}{n}\right)} B_n^s\left(V_n^j,s\right)\right)^2\right]$$
$$= \tilde{E}\left[\sum_{j_1,j_2=1}^{N_n\left(\frac{[sn]}{n}\right)} \tilde{E}\left[B_n^s\left(V_n^{j_1},s\right)B_n^s\left(V_n^{j_2},s\right)\Big|\mathscr{F}_{\frac{[sn]}{n}}\right]\right]$$
$$\le \tilde{E}\left[\sum_{j_1,j_2=1}^{N_n\left(\frac{[sn]}{n}\right)} e^{\frac{\|h\|^2}{n}}\tilde{E}\left[\exp\left(\int_{\frac{[sn]}{n}}^s \left(h^*\left(p,V_n^{j_1}(p)\right)+h^*\left(p,V_n^{j_2}(p)\right)\right)\,dY_p\right.\right.\right.$$
$$\left.\left.\left.-\frac{1}{2}\int_{\frac{[sn]}{n}}^s \left|h\left(p,V_n^{j_1}(p)\right)+h\left(p,V_n^{j_2}(p)\right)\right|^2 dp\right)\Big|\mathscr{F}_{\frac{[sn]}{n}}\right]\right]$$

which gives us, as in Remark 3.1

$$\tilde{E}\left[\left(\sum_{j=1}^{N_n\left(\frac{[sn]}{n}\right)} B_n^s\left(V_n^j, s\right)\right)^2\right] \le e^{\|h\|^2}\tilde{E}\left[\left(N_n\left(\frac{[sn]}{n}\right)\right)^2\right] \ . \tag{25}$$

Now using Burkholder-Davis-Gundy inequality and (25), we find

$$\tilde{E}\left[\left(\sup_{0\le p\le\frac{[tn]}{n}}\left|\int_0^p \sum_{j=1}^{N_n\left(\frac{[sn]}{n}\right)} B_n^s\left(V_n^j, s\right)h^*\left(s, V_n^j(s)\right)\,dY_s\right|\right)^2\right]$$

$$\le \int_0^{\frac{[tn]}{n}} \tilde{E}\left[\left|\sum_{j=1}^{N_n\left(\frac{[sn]}{n}\right)} B_n^s\left(V_n^j, s\right)h\left(s, V_n^j(s)\right)\right|^2\right] ds$$

$$\le 4e^{\|h\|^2}\|h\|^2\int_0^t \tilde{E}\left[\left(N_n\left(\frac{[sn]}{n}\right)\right)^2\right] ds \ .$$

The last inequality gives the following upper bound on the third term of (22)

$$4e^{\|h\|^2}\|h\|^2\int_0^t \psi_n(s)\,ds \ , \tag{26}$$

where K_2 is a constant independent of n.

From (22), (23), (24) and (26) we obtain

$$\psi_n(t) \le (3 + 12\|v\|) + 12e^{\|h\|^2}\|h\|^2\int_0^t \psi_n(s)\,ds \ .$$

Finally, using the *Gronwall inequality* (see, for instance [13] pp. 287) we find that $\psi_n(t) \le c(t)$, where

$$c(t) \stackrel{\text{def}}{=} (3 + 12\|v\|)e^{\frac{4\|h\|^2 e^{\|h\|^2}}{1+4\|v\|}t}, \quad t \in [0,1] \ .$$

So also $\sup_{n\ge1}\tilde{E}[(\sup_{0\le s\le t}(\mathscr{X}_n(s), 1))^2] \le c(t)$ which finishes the proof of the proposition. □

Remarks 4.3 Using a similar argument one can prove that, $\forall p \ge 1$, there exists a function $c_p\colon [0,1] \to \mathbb{R}_+$, such that

$$\sup_{n\ge1}\tilde{E}\left[\left(\sup_{0\le s\le t}(\mathscr{X}_n(s), 1)\right)^p\right] \le c_p(t), \quad t \in [0,1] \ . \tag{27}$$

We prove now that the processes $(\mathscr{X}_n(t), \varphi_i)$ satisfy condition (ii) of Theorem (4.1). Since $C_k^2(\mathbb{R}^d) \subset \mathscr{D}(A)$ is dense in $C_0(\mathbb{R}^d)$ (under the uniform norm) we can take the functions φ_i, $i \ge 1$ from this set.

Proposition 4.4 *For any arbitrary sequence of stopping times* $\{\tau_n\}_{n\ge0}$ *any positive real sequence* $\{\delta_n\}_{n\ge0}$ *with* $\lim_{n\to\infty}\delta_n = 0$ *and* $\varphi \in C_K^2(R^d) \cup \{1\}$, *we have*

$$\lim_{n\to\infty} \tilde{E}\left[|(\mathscr{X}_n(\tau_n+\delta_n),\varphi)-(\mathscr{X}_n(\tau_n),\varphi)|^2\right]=0 \tag{28}$$

and hence

$$\lim_{n\to\infty} \tilde{P}(|(\mathscr{X}_n(\tau_n+\delta_n),\varphi)-(\mathscr{X}_n(\tau_n),\varphi)|\geq \varepsilon)=0\ . \tag{29}$$

for all $\varepsilon > 0$.

Proof. Let a and b be the following quantities

$$a \stackrel{\text{def}}{=} \sup_{\{(t,x)\in[0,1]\times\mathbb{R}^d\}} \|A(t)\varphi(t,x)\| < \infty$$

$$b \stackrel{\text{def}}{=} \sup_{\{(t,x)\in[0,1]\times\mathbb{R}^d\}} |Tr(D\varphi^*\sigma\sigma^* D\varphi)| < \infty.$$

Obviously, if φ is the constant function 1, then $a=b=0$. Using (17) we get

$$\begin{aligned}
&\tilde{E}\left[|(\mathscr{X}_n(\tau_n+\delta_n),\varphi)-(\mathscr{X}_n(\tau_n),\varphi)|^2\right]\\
&\quad\leq 4\tilde{E}\left[\left(\int_{\tau_n}^{\tau_n+\delta_n}(\mathscr{X}_n(s),A(s)\varphi)\ ds\right)^2\right]\\
&\qquad+4\tilde{E}\left[\left(S_n^{\varphi}(\tau_n+\delta_n)-S_n^{\varphi}(\tau_n)\right)^2\right]+4\tilde{E}\left[\left(M_n^{\varphi}([n(\tau_n+\delta_n)])-M_n^{\varphi}([n\tau_n])\right)^2\right]\\
&\qquad+4\tilde{E}\left[\left(\int_{\frac{[n\tau_n]}{n}}^{\frac{[n(\tau_n+\delta_n)]}{n}}\sum_{j=1}^{N_n\left(\frac{[sn]}{n}\right)}\varphi\left(V_n^j\left(\frac{[sn]+1}{n}\right)\right)B_n^s\left(V_n^j,s\right)h^*\left(V_n^j(s)\right)\ dY_s\right)^2\right]
\end{aligned}\tag{30}$$

We have, consecutively,

$$\begin{aligned}
\tilde{E}\left[\left(\int_{\tau_n}^{\tau_n+\delta_n}(\mathscr{X}_n(s),A(s)\varphi)\ ds\right)^2\right] &\leq \delta_n\tilde{E}\left[\int_{\tau_n}^{\tau_n+\delta_n}(\mathscr{X}_n(s),A(s)\varphi)^2\ ds\right]\\
&\leq \delta_n a^2\tilde{E}\left[\left(\sup_{0\leq s\leq 1}(\mathscr{X}_n(s),1)\right)^2((\tau_n+\delta_n)-\tau_n)\right]\\
&\leq \delta_n^2 a^2 c(1)
\end{aligned}\tag{31}$$

$$\begin{aligned}
\tilde{E}\left[\left(S_n^{\varphi}(\tau_n+\delta_n)-S_n^{\varphi}(\tau_n)\right)^2\right] &\leq K\tilde{E}[\langle S_n^{\varphi}\rangle(\tau_n+\delta_n)-\langle S_n^{\varphi}\rangle(\tau_n)]\\
&\leq \frac{K}{n}\tilde{E}\left[\int_{\tau_n}^{\tau_n+\delta_n}(\mathscr{X}_n(s),Tr(D\varphi^*\sigma\sigma^*D\varphi))\right]ds\\
&\leq \frac{Kb}{n}\tilde{E}\left[\sup_{0\leq s\leq 1}(\mathscr{X}_n(s),1)((\tau_n+\delta_n)-\tau_n)\right]\\
&\leq \frac{Kb}{2n}(1+c(1))\delta_n
\end{aligned}\tag{32}$$

$$\tilde{E}\Big[\big(M_n^{\varphi}([n(\tau_n+\delta_n)]) - M_n^{\varphi}([n\tau_n])\big)^2\Big]$$

$$= \tilde{E}\Big[\big(M_n^{\varphi}([n(\tau_n+\delta_n)]) - M_n^{\varphi}([n\tau_n])\big)^2\Big] = \frac{1}{n}\tilde{E}\left[\sum_{[n\tau_n]+1}^{[n(\tau_n+\delta_n)]}\left(\mathscr{X}_n\left(\frac{i}{n}-\right), v_{\frac{i}{n}}\varphi^2\right)\right]$$

$$\leq \frac{\|v\|\|\varphi\|^2}{n}\tilde{E}\Big[\sup_{0\leq s\leq 1}(\mathscr{X}_n(s),1)([n(\tau_n+\delta_n)]-[n\tau_n])\Big]$$

$$\leq \|v\|\|\varphi\|^2\frac{1+c(1)}{2}\left(\delta_n+\frac{1}{n}\right) \tag{33}$$

$$\tilde{E}\left[\left(\frac{1}{n^2}\int_{\frac{[n\tau_n]}{n}}^{\frac{[n(\tau_n+\delta_n)]}{n}}\sum_{j=1}^{N_n\left(\frac{[sn]}{n}\right)}\varphi\left(V_n^j\left(\frac{[sn]+1}{n}\right)\right)B_n^s(V_n^j,s)h^*\big(s,V_n^j(s)\big)\,dY_s\right)^2\right]$$

$$= \tilde{E}\left[\int_{\frac{[n\tau_n]}{n}}^{\frac{[n(\tau_n+\delta_n)]}{n}}\frac{1}{n^2}\left(\sum_{j=1}^{N_n\left(\frac{[sn]}{n}\right)}\varphi\left(V_n^j\left(\frac{[sn]+1}{n}\right)\right)B_n^s(V_n^j,s)h^*(s,V_n^j(s))\right)^2 ds\right]$$

$$\leq \|h\|^2\|\varphi\|^2\tilde{E}\left[\frac{\left(\sup_{0\leq s\leq 1}\sum_{j=1}^{N_n\left(\frac{[sn]}{n}\right)}B_n^s(V_n^j,s)\right)^2}{n^2}\left(\frac{[n(\tau_n+\delta_n)]}{n}-\frac{[n\tau_n]}{n}\right)\right]$$

$$\leq \|\varphi\|^2\|h\|^2c'(1)\left(\delta_n+\frac{1}{n}\right) \tag{34}$$

where $c'(1)$ is obtained similarly to $c(1)$ as an uniform upper bound for

$$\tilde{E}\left[\frac{\left(\sup_{0\leq s\leq 1}\sum_{j=1}^{N_n\left(\frac{[sn]}{n}\right)}B_n^s(V_n^j,s)\right)^2}{n^2}\right]$$

The inequalities (31), (32), (33), (34) imply that all the terms from the right hand side of (30) tend to 0 when n goes to ∞, hence $\tilde{E}[|(\mathscr{X}_n(\tau_n+\delta_n),\varphi) - (\mathscr{X}_n(\tau_n),\varphi)|^2]$ tends to 0 as well. □

We prove now that the sequence satisfies (19). For this, we need the following two results.

Proposition 4.5 *Let* $\varphi \in \mathscr{D}(A) \subset C_b(\mathbb{R}^d)$ *such that* $\varphi, \varphi^2 \in \mathscr{D}(A)$. *Then*

$$\lim_{n\to\infty}\tilde{E}\left[\sup_{s,t\in[0,1]}\left(\frac{1}{n}\int_{\frac{[ns]}{n}}^{\frac{[nt]}{n}}\sum_{j=1}^{N_n\left(\frac{[rn]}{n}\right)}\varphi\left(V_n^j\left(\frac{[rn]+1}{n}\right)\right)B_n^r(V_n^j,r)h^*(r,V_n^j(r))\,dY_r.\right.\right.$$
$$\left.\left.-\int_s^t(\mathscr{X}_n(r),h^*(r)\varphi)\,dY_r\right)^2\right] = 0 \tag{35}$$

Proof. It is enough to prove that

$$\lim_{n\to\infty} \tilde{E}\left[\sup_{t\in[0,1]}\left(\frac{1}{n}\int_0^{\frac{[nt]}{n}}\sum_{j=1}^{N_n\left(\frac{[rn]}{n}\right)} \varphi\left(V_n^j\left(\frac{[rn]+1}{n}\right)\right)B_n^r(V_n^j,r)h^*(r,V_n^j(r))\,dY_r\right.\right.$$
$$\left.\left. - \int_0^t (\mathscr{X}_n(r), h^*(r)\varphi)\,dY_r\right)^2\right] = 0 \qquad (36)$$

Firstly, we observe that last integral can be taken from 0 to $\frac{[nt]}{n}$ without changing the limit. Then, using (25) and Burkholder-Davis-Gundy inequality, we get

$$\tilde{E}\left[\sup_{t\in[0,1]}\left(\frac{1}{n}\int_0^{\frac{[nt]}{n}}\sum_{j=1}^{N_n\left(\frac{[rn]}{n}\right)} \varphi\left(V_n^j\left(\frac{[rn]+1}{n}\right)\right)(B_n^r(V_n^j,r)-1)h^*(r,V_n^j(r))\,dY_r\right)^2\right]$$
$$\le \frac{k}{n^2}\int_0^1 \tilde{E}\left[\left|\sum_{j=1}^{N_n\left(\frac{[rn]}{n}\right)} \varphi\left(V_n^j\left(\frac{[rn]+1}{n}\right)\right)B_n^r(V_n^j,r)-1)h^*(r,V_n^j(r))\right|^2 dr\right]$$
$$\le k\|h\|^2\|\varphi\|^2\int_0^1 \tilde{E}\left[\frac{1}{n^2}\left(\int_{\frac{[rn]}{n}}^r \sum_{j=1}^{N_n\left(\frac{[rn]}{n}\right)} B_n^r(V_n^j,p)h^*(s,V_n^j(p))\,dY_p\right)^2\right] dr$$
$$\le \frac{1}{2}k\|h\|^2\|\varphi\|^2 c(1)(e^{\frac{1}{n}}-1). \qquad (37)$$

Thus one can eliminate $B_n^r(V_n^j,r)$ from the first term of (35) without changing the limit. After these 2 transformations, (35) becomes

$$\lim_{n\to\infty} \tilde{E}\left[\sup_{t\in[0,1]}\left(\frac{1}{n}\int_0^{\frac{[nt]}{n}}\sum_{j=1}^{N_n\left(\frac{[rn]}{n}\right)}\left(\varphi\left(V_n^j\left(\frac{[rn]+1}{n}\right)\right)-\varphi(V_n^j(r))\right)h^*(r,V_n^j(r))\,dY_r\right)^2\right]$$

Using once again Burkholder-Davis-Gundy inequality, we find the following upper bound for the terms of the sequence

$$k\|h\|^2 c(1)\left(\frac{\|A\varphi\|^2}{n^2}+\frac{C\|A\varphi^2-2\varphi A\varphi\|}{n}\right)$$

which completes our proof (we used the classical identity $Tr(D\varphi^*\sigma\sigma^*D\varphi^*) = A\varphi^2 - 2\varphi A\varphi$). □

Proposition 4.6 *For φ_k defined as in the assumption $(**)$ in section 2, there exists an uniform constant M such that*

$$\limsup_{n\to\infty} \tilde{E}[(\mathscr{X}_n(t),\varphi_k)] \le (\pi_0,\varphi_k) + \frac{M}{k} \qquad (38)$$

for all $t \in [0,1]$.

Proof. Using equation (17), we have that

$$(\mathscr{X}_n(t),\varphi_k) =(\mathscr{X}_n(0),\varphi_k) + \int_0^t (\mathscr{X}_n(s),A(s)\varphi_k)\; ds + S_n^{\varphi_k}(t) + M_n^{\varphi_k}([nt]) + \frac{1}{n}\int_0^{\frac{[nt]}{n}} \sum_{j=1}^{N_n\left(\frac{[sn]}{n}\right)} \varphi_k\left(V_n^j\left(\frac{[sn]+1}{n}\right)\right) B_n^s(V_n^j,s)h^*(s,V_n^j(s))\; dY_s\,. \quad (39)$$

Since for φ_k as above $S_n^{\varphi_k}(t)$ and $M_n^{\varphi_k}([nt])$ are martingales with mean zero and also the last term has mean zero, we have that

$$\limsup_{n\to\infty}\tilde{E}[(\mathscr{X}_n(t),\varphi_k)] \le \limsup_{n\to\infty}\tilde{E}[(\mathscr{X}_n(0),\varphi_k)] + \frac{t}{k}\limsup_{n\to\infty}\tilde{E}\left[\sup_{s\in[0,1]}(\mathscr{X}_n(s),1)\right]$$

and since $\lim_{n\to\infty}\tilde{E}[(\mathscr{X}_n(0),\varphi_k)] = (\mathscr{X}_n(0),\varphi_k) = (\pi_0,\varphi_k)$ and $\tilde{E}\big[\sup_{s\in[0,1]}(\mathscr{X}_n(s),1)\big] \le \frac{1+c(1)}{2}$ we have our claim. □

We want to prove that there exists a sequence of compact sets K_k, such that, for all $\varepsilon > 0$,

$$\lim_{k\to\infty}\limsup_{n\to\infty}\tilde{P}(\exists t\in[0,1],(\mathscr{X}_n(s),I_{CK_k})\ge\varepsilon) = 0$$

which is equivalent to proving that

$$\lim_{k\to\infty}\limsup_{n\to\infty}\tilde{P}(\sup_{t\in[0,1]}(\mathscr{X}_n(s),I_{CK_k})\ge\varepsilon) = 0 \quad (40)$$

which, in turn, is implied by (using Chebychev's inequality)

$$\lim_{k\to\infty}\limsup_{n\to\infty}\tilde{E}\left[\sup_{t\in[0,1]}(\mathscr{X}_n(t),I_{CK_k})^2\right] = 0\;. \quad (41)$$

Proposition 4.7 *For* $K_k \triangleq \overline{B(0,R_k)}$, *where* R_k *was defined in assumption* $(**)$, (41) *holds.*

Proof. Since $I_{CK_k}\le\varphi_k$ it is enough to prove that

$$\lim_{k\to\infty}\limsup_{n\to\infty}\tilde{E}\left[\sup_{t\in[0,1]}(\mathscr{X}_n(t),\varphi_k)^2\right] = 0\;. \quad (42)$$

We have

$$(\mathscr{X}_n(t),\varphi_k) =(\mathscr{X}_n(0),\varphi_k) + \int_0^t (\mathscr{X}_n(s),A(s)\varphi_k)\; ds + S_n^{\varphi_k}(t) + M_n^{\varphi_k}([nt]) + \frac{1}{n}\int_0^{\frac{[nt]}{n}} \sum_{j=1}^{N_n\left(\frac{[sn]}{n}\right)} \varphi_k\left(V_n^j\left(\frac{[sn]+1}{n}\right)\right) B_n^s(V_n^j,s)h^*(s,V_n^j(s))\; dY_s\;. \quad (43)$$

Since $\mathscr{X}_n(0)$ is convergent to π_0 we have that

$$\lim_{k\to\infty}\limsup_{n\to\infty}\tilde{E}\left[(\mathscr{X}_n(0),\varphi_k)^2\right]=\lim_{k\to\infty}(\pi_0,\varphi_k)^2=0\ . \tag{44}$$

From the definition of the functions φ_k we have (as in the proof of the previous proposition) that

$$\lim_{k\to\infty}\limsup_{n\to\infty}\tilde{E}\left[\sup_{t\in[0,T]}\left(\int_0^t(\mathscr{X}_n(s),A(s)\varphi_k)\ ds\right)^2\right]=0\ . \tag{45}$$

Using Burkholder-Davis-Gundy inequality

$$\begin{aligned}\lim_{k\to\infty}\limsup_{n\to\infty}\tilde{E}&\left[\sup_{t\in[0,T]}\left(S_n^{\varphi_k}(t)\right)^2\right]\\&\le\lim_{k\to\infty}\limsup_{n\to\infty}\frac{K_2}{n}\tilde{E}\left[\int_0^T\left(\mathscr{X}(r),A\varphi_n^2-2\varphi_nA\varphi_n\right)\ dr\right]=0\end{aligned} \tag{46}$$

and also

$$\lim_{k\to\infty}\limsup_{n\to\infty}\tilde{E}\left[\sup_{t\in[0,T]}\left(M_n^{\varphi_k}(t)\right)^2\right]\le\lim_{k\to\infty}\limsup_{n\to\infty}K_2\|v\|\tilde{E}\left[\int_0^T(\mathscr{X}_n(r),\varphi_k^2)\ dr\right]\ .$$

Since $\varphi_k^2\le\varphi_k$, we have, using Fatou's lemma

$$\lim_{k\to\infty}\limsup_{n\to\infty}\tilde{E}\left[\sup_{t\in[0,T]}\left(M_n^{\varphi_k}(t)\right)^2\right]\le\lim_{k\to\infty}K_2\|v\|\int_0^T\limsup_{n\to\infty}\tilde{E}[(\mathscr{X}_n(r),\varphi_k)]\ dr \tag{47}$$

From (38) and (47) we obtain that

$$\lim_{k\to\infty}\limsup_{n\to\infty}\tilde{E}\left[\sup_{t\in[0,T]}\left(M_n^{\varphi_k}(t)\right)^2\right]=0\ . \tag{48}$$

Using (35) we obtain that

$$\begin{aligned}&\lim_{k\to\infty}\limsup_{n\to\infty}\tilde{E}\left[\sup_{t\in[0,T]}\left(\frac{1}{n}\int_0^{\frac{[nt]}{n}}\sum_{j=1}^{N_n(\frac{[sn]}{n})}\varphi_k\left(V_n^j\left(\frac{[sn]+1}{n}\right)\right)B_n^s(V_n^j,s)h^*(s,V_n^j(s))\ dY_s\right)^2\right]\\&=\lim_{k\to\infty}\limsup_{n\to\infty}\tilde{E}\left[\sup_{t\in[0,T]}\left(\int_0^t(\mathscr{X}_n(r),h^*(r)\varphi_k)\ dY_r\right)^2\right]\\&\le k\|h\|^2\lim_{k\to\infty}\limsup_{n\to\infty}^{2}\int_0^t\tilde{E}\left[(\mathscr{X}_n(r),\varphi_k)^2\right]\ dr\ .\end{aligned} \tag{49}$$

Let now $\Psi(T)\triangleq\lim_{k\to\infty}\limsup_{n\to\infty}\tilde{E}\left[\sup_{t\in[0,T]}(\mathscr{X}_k(t),\varphi_n)^2\right]$. From (43), (44), (45), (46), (48), (49) and Fatou's lemma we obtain that there exists a constant K such that

$$\Psi(T) \le K \int_0^T \Psi(s)\, ds$$

for all $T \in [0,1]$ which implies our claim, using, once again, Gronwall's inequality. □

We know now that the sequence $\mathscr{X}_n$ is tight in $D_{M_F(\mathbb{R}^d)}[0,1]$, hence relatively compact. Then $(\mathscr{X}_n, Y)$ is relatively compact in the space $D_{M_F(\mathbb{R}^d)\times\mathbb{R}^m}[0,1]$. Let $(\mathscr{X}, Y)$ be the limit process of one of its convergent subsequences (to avoid even more cumbersome notation we re-index this sequence as $\{(\mathscr{X}_n, Y)\}_{n\ge 0}$). We will show that $\mathscr{X}$ is a solution of the 'martingale problem' (4) + (5). We need first several useful results.

Proposition 4.8 *Let φ be a continuous bounded function. Then, for all $p \ge 1$*

$$\tilde{E}[(\sup_{t\in[0,1]} |(\mathscr{X}(t),\varphi)|)^p] < \infty \tag{50}$$

Proof. Since $f_k : D_{\mathbb{R}}[0,1] \to \mathbb{R}$, $f_k(a) \triangleq (\sup_{t\in[0,1]} |a_t|)^p \wedge k > 0$ is a bounded continuous function on $D_{\mathbb{R}}[0,1]$ and the process $t \to (\mathscr{X}_n(t),\varphi)$ converges in distribution to the process $t \to (\mathscr{X}(t),\varphi)$, we have that, for all $k > 0$

$$\begin{aligned}
\tilde{E}\left[(\sup_{t\in[0,1]} |(\mathscr{X}(t),\varphi)|)^p \wedge k\right] &= \lim_{k\to\infty} \tilde{E}\left[\left(\sup_{t\in[0,1]} |(\mathscr{X}(t),\varphi)|\right)^p \wedge k\right] \\
&= \lim_{k\to\infty} \lim_{n\to\infty} \tilde{E}\left[\left(\sup_{t\in[0,1]} |(\mathscr{X}_n(t),\varphi)|\right)^p \wedge k\right] \\
&\le \|\varphi\|^p c_p(1)
\end{aligned} \tag{51}$$

where c_p is the function defined in Remark 4.3.

Proposition 4.9 *The process $\mathscr{X}$ has continuous paths in $M_F(\mathbb{R}^d)$.*

Proof. With a similar proof to the one in Proposition 4.4 one shows that for $\varphi \in C_K^2(\mathbb{R}^d)$ and for all $\varepsilon > 0$

$$\lim_{\delta\to 0} \limsup_{n\to\infty} \tilde{E}\left[\left(\sup_{s,t\in[0,1],|s-t|\le\delta} |(\mathscr{X}_n(s),\varphi) - (\mathscr{X}_n(t),\varphi)|\right)^2\right] = 0 \ . \tag{52}$$

Using the fact that $\sup_{n\ge 1} \tilde{E}[(\sup_{0\le s\le 1}(\mathscr{X}_n(s),1))^2] < \infty$ (see proof of Proposition 4.2), one then extends (52) to all $\varphi \in C_0(\mathbb{R}^d)$ by taking a sequence of functions $\varphi_n \in C_K^2(\mathbb{R}^d)$ that converges uniformly to φ. This implies that for all $\varphi \in C_0(\mathbb{R}^d)$ and for all $\varepsilon > 0$, one has

$$\lim_{\delta\to 0} \limsup_{n\to\infty} \tilde{P}\left(\sup_{s,t\in[0,1],|s-t|\le\delta} |(\mathscr{X}_n(s),\varphi) - (\mathscr{X}_n(t),\varphi)| \ge \varepsilon\right) = 0 \ . \tag{53}$$

Based on Theorem 15.5 from [4], (53) implies that the real valued process $t \to (\mathscr{X}(t),\varphi)$ is continuous $\tilde{P}$-a.s.. for all $\varphi \in C_0(\mathbb{R}^d)$ and hence the process

$t \to \mathscr{X}(t)$ is continuous as a process with values in $M_{F'}(\mathbb{R}^d)$. Since $t \to \mathscr{X}(t)$ is a genuine $D_{M_F(\mathbb{R}^d)}([0,1])$ process, we get that it is continuous as a $M_F(\mathbb{R}^d)$-valued process. □

Since the limit process is continuous, the one dimensional projections of the sequence – $\mathscr{X}_n(t)$ – are convergent in distribution to $\mathscr{X}(t)$ and, in particular, the sequence $(\mathscr{X}_n(t), \varphi)$ is convergent in distribution to $(\mathscr{X}(t), \varphi)$ for any $\varphi \in C_b(\mathbb{R}^d)$.

Proposition 4.10 *Let φ be a continuous bounded function. Then, for all $p \geq 1$*

$$\lim_{n\to\infty} \tilde{E}[|(\mathscr{X}_n(t), \varphi)|^p] = \tilde{E}[|(\mathscr{X}(t), \varphi)|^p] \ . \tag{54}$$

Proof. The proposition follows from the fact that $(\mathscr{X}_n(t), \varphi)$ is convergent in distribution to $(\mathscr{X}(t), \varphi)$, by using the uniform integrability of the sequence and Remark 4.3. □

We are now able to prove that $\mathscr{X}$ satisfies the martingale problem (4) + (5).

Theorem 4.11 *For $\varphi \in \mathscr{D}(A)$ the process $\{(M^\varphi(t), \mathscr{F}_t \vee \mathscr{Y}), t \in [0,1]\}$ where*

$$M^\varphi(t) \triangleq (\mathscr{X}(t), \varphi) - (\mathscr{X}(0), \varphi) - \int_0^t (\mathscr{X}(s), A(s)\varphi)\ ds - \int_0^t (\mathscr{X}(s), h^*(s)\varphi)\ dY_s$$

is a square integrable martingale with the quadratic variation

$$\langle M^\varphi \rangle(t) = \int_0^t (\mathscr{X}(s), v_s\varphi^2)\ ds \ .$$

Proof. We will use the idea contained in the Theorem 8.2 from [7]. Let $\mathscr{M}$ be a separating subset of the set of continuous bounded functions on $M_F(\mathbb{R}^d)$ and $\mathscr{N}$ be a separating subset of $C_b(\mathbb{R}^m)$. We want to prove that for all $\varphi \in \mathscr{D}(A)$

$$\tilde{E}\Big[(M^\varphi(t) - M^\varphi(s))\Pi_{i=1}^m k_i(X(t_i))\Pi_{j=1}^{m'} k'_j(Y(t_j))\Big] = 0 \tag{55}$$

and

$$\tilde{E}\left[((M^\varphi(t) - M^\varphi(s))^2 - \int_s^t \left(\mathscr{X}(r), v_r\varphi^2\right) dr)\Pi_{i=1}^m k_i(X(t_i))\Pi_{j=1}^{m'} k'_j(Y(t'_j))\right] = 0 \tag{56}$$

for all $m, m' \geq 0$, $0 \leq t_1 < t_2 < \ldots < t_m \leq s \leq t$, $0 \leq t'_1 < t'_2 < \ldots < t'_m \leq 1$, $k_1, \ldots, k_m \in \mathscr{M}$ and $k'_1, \ldots, k'_m \in \mathscr{N}$. We prove only (55), since (56) can be done analogously. From the definition of M^φ, (55) is equivalent to

$$\tilde{E}[((\mathscr{X}(t), \varphi) - (\mathscr{X}(s), \varphi) - \int_s^t (\mathscr{X}(r), A(r)\varphi)\ ds - \int_s^t (\mathscr{X}(r), h^*(r)\varphi)\ dY_r)$$
$$\times \Pi_{i=1}^m k_i(X(t_i))\Pi_{j=1}^{m'} k'_j(Y(t'_j))] = 0 \tag{57}$$

We only need to show (57) for φ with the property that $\varphi^2 \in \mathscr{D}(A)$ since using the property $(*)$ of $A(s)$ and the dominated convergence theorem we can extend this to an arbitrary $\varphi \in \mathscr{D}(A)$. Using a proof analogous with the one used in Proposition 4.10 one shows, consecutively, that since $(\mathscr{X}_n, Y)$ converges in distribution to $(\mathscr{X}, Y)$

$$\lim_{n\to\infty} \tilde{E}\Big[(\mathscr{X}_n(t),\varphi)\Pi_{i=1}^m k_i(X_n(t_i))\Pi_{j=1}^{m'} k_j'(Y(t_j))\Big] \\ = \tilde{E}\Big[(\mathscr{X}(t),\varphi)\Pi_{i=1}^m k_i(X(t_i))\Pi_{j=1}^{m'} k_j'(Y(t_j))\Big] \tag{58}$$

$$\lim_{n\to\infty} \tilde{E}\Big[(\mathscr{X}_n(s),\varphi)\Pi_{i=1}^m k_i(X_n(t_i))\Pi_{j=1}^{m'} k_j'(Y(t_j))\Big] \\ = \tilde{E}\Big[(\mathscr{X}(s),\varphi)\Pi_{i=1}^m k_i(X(t_i))\Pi_{j=1}^{m'} k_j'(Y(t_j))\Big] \tag{59}$$

$$\lim_{n\to\infty} \tilde{E}\left[\int_s^t (\mathscr{X}_n(r),A(r)\varphi)\; dr\Pi_{i=1}^m k_i(X_n(t_i))\Pi_{j=1}^{m'} k_j'(Y(t_j))\right] \\ = \tilde{E}\left[\int_s^t (\mathscr{X}(r),A(r)\varphi)\; dr\Pi_{i=1}^m k_i(X(t_i))\Pi_{j=1}^{m'} k_j'(Y(t_j))\right] . \tag{60}$$

Using theorem 2.2 from [14], we have that, since $(\mathscr{X}_n, Y)$ converges in distribution to $(\mathscr{X}, Y)$ also $(\mathscr{X}_n, Y, \int_0^t (\mathscr{X}_n(s), h^*(s)\varphi)\; dY_s)$ converges in distribution to $(\mathscr{X}, Y, \int_0^t (\mathscr{X}(s), h^*(s)\varphi)\; dY_s)$ and using (35) and, once again, an argument similar to the one used in Proposition 4.10, we have that

$$\lim_{n\to\infty} \tilde{E}\left[\frac{1}{n}\int_{\frac{[ns]}{n}}^{\frac{[nt]}{n}} \sum_{j=1}^{N_n\left(\frac{[rn]}{n}\right)} \varphi\left(V_n^j\left(\frac{[rn]+1}{n}\right)\right) B_n^r(V_n^j, r) h^*(r, V_n^j(r))\; dY_r \right. \\ \left. \times\, \Pi_{i=1}^m k_i(X_n(t_i))\Pi_{j=1}^{m'} k_j'(Y(t_j))\right] \\ = \tilde{E}\left[\int_s^t (\mathscr{X}(s), h^*(s)\varphi)\; dY_s \Pi_{i=1}^m k_i(X(t_i))\Pi_{j=1}^{m'} k_j'(Y(t_j))\right] . \tag{61}$$

Since $\varphi^2 \in \mathscr{D}(A)$, we have that $Tr(D\varphi^*\sigma\sigma^* D\varphi) = A\varphi^2 - 2\varphi A\varphi \in C_b(\mathbb{R}^d)$ and hence S_n^φ is a square integrable martingale such that

$$E\Big[(S_n^\varphi)^2(p)\Big] = E\Big[\langle S_n^\varphi\rangle^2(p)\Big] \le \frac{\|A\varphi^2 - 2\varphi A\varphi\|}{n}$$

and hence

$$\lim_{n\to\infty} \tilde{E}\Big[\big(S_n^\varphi(t) - S_n^\varphi(s)\big)\Pi_{i=1}^m k_i(X_n(t_i))\Pi_{j=1}^{m'} k_j'(Y(t_j))\Big] = 0 . \tag{62}$$

From (58),(59),(60),(61) and (62) we obtain that

$$\tilde{E}\Big[(M^\varphi(t) - M^\varphi(s))\Pi_{i=1}^m k_i(X(t_i))\Pi_{j=1}^{m'} k_j'(Y(t_j))\Big] \\ = \lim_{n\to\infty} \tilde{E}\Big[(M_n^\varphi(t) - M_n^\varphi(s))\Pi_{i=1}^m k_i(X_n(t_i))\Pi_{j=1}^{m'} k_j'(Y(t_j))\Big] = 0 . \tag{63}$$

□

Remark 4.12 The martingale M^{φ} is a martingale also with respect to the initial filtration $\mathscr{F}_t$ and its conditional expectation with respect to $\mathscr{Y}$ is 0.

With this we conclude the existence of the process with the properties described in the introduction.

Remark 4.13 The normalised occupation measure μ_n of a sequence of points chosen randomly with the distribution π_0 will almost surely converge (weakly) to π_0. Therefore the entire construction is valid when $\mathscr{X}_n(0)$ is taken to be μ_n. The readers may find the arguments in this paper more intuitive if they have in mind this initial data.

At this point in time, we have not had the energy required to prove the uniqueness of the solution of the filtered martingale problem (4) + (5), although we believe this to be unique. Uniqueness is not central to our overall objective, achieved in the next section, where we show that, given Y, the (conditional) mean of $\mathscr{X}_n$ converges almost surely to the unique solution of the Zakai equation and hence to the unnormalised distribution of the signal.

5 Application to the nonlinear filtering problem

The process $\mathscr{X}$ is the solution of the 'filtered' martingale problem (4) + (5). It follows that for $\varphi \in \mathscr{D}(A)$ (and the uniform square integrability of $(\mathscr{X}(t), \varphi)$, $(\mathscr{X}(t), A\varphi)$ and $(\mathscr{X}(t), h^*\varphi)$)

$$\tilde{E}[(\mathscr{X}(t),\varphi)|\mathscr{Y}_t] = \tilde{E}[(\mathscr{X}(0),\varphi)|\mathscr{Y}_0] + \int_0^t \tilde{E}[(\mathscr{X}(s),A\varphi)|\mathscr{Y}_s]\, ds + \int_0^t \tilde{E}[(\mathscr{X}(t),h^*\varphi)|\mathscr{Y}_s]\, dY_s \tag{64}$$

In establishing (64), we used the fact that for every integrable $\mathscr{F}_t$-measurable random variable $\mathscr{A}$ we have $\tilde{E}[\mathscr{A}|\mathscr{Y}] = \tilde{E}[\mathscr{A}|\mathscr{Y}_t]$ (since Y is a Brownian motion) and if $\{U_t; t \geq 0\}$ is an $\mathscr{F}_t$-progressively measurable process such that $\tilde{E}\int_0^t U_s^2 dt < \infty, \forall s \geq 0$ then

$$\tilde{E}\left[\int_0^t U_s dY_s|\mathscr{Y}_t\right] = \int_0^t \tilde{E}[U_s|\mathscr{Y}_s]\, dY_s$$
$$\tilde{E}\left[\int_0^t U_s ds|\mathscr{Y}_t\right] = \int_0^t \tilde{E}[U_s|\mathscr{Y}_s]\, ds$$

A proof of these observation can be found in [16]. One can also obtain the corresponding evolution equation for time dependent φ.

Let $\mathscr{X}_n^{Y(\omega)}$ and $\mathscr{X}^{Y(\omega)}$ be the processes $\mathscr{X}_n$ and, respectively, $\mathscr{X}$ *given* the observation path $Y_{\cdot}(\omega)$. Let also $\tilde{E}_\omega$ be the corresponding expectations given $Y_{\cdot}(\omega)$, $\mathscr{X}_n^{\omega}(t) = \tilde{E}_\omega[\mathscr{X}_n^{Y(\omega)}(t)]$, i.e., the measure obtained by integrating the measure valued random variable $\mathscr{X}_n^{Y(\omega)}(t)$ (this is, actually, what we are computing in numerical applications) and $Z^{\omega}(t) = \tilde{E}_{\omega}[\mathscr{X}^{Y(\omega)}(t)]$. Using Fubini's theorem, we have

$$(\mathscr{Z}^{\omega}(t), \varphi) = \tilde{E}_{\omega}\left[\left(\mathscr{X}^{Y(\omega)}(t), \varphi\right)\right] = \tilde{E}[(\mathscr{X}(t), \varphi)|\mathscr{Y}](\omega) \tag{65}$$

$$\left(\mathscr{Z}_n^{\omega}(t), \varphi\right) = \tilde{E}_{\omega}\left[\left(\mathscr{X}_n^{Y(\omega)}(t), \varphi\right)\right] = \tilde{E}[(\mathscr{X}_n(t), \varphi)|\mathscr{Y}](\omega) \tag{66}$$

Using (65), the evolution equation (64) becomes

$$(\mathscr{Z}(t), \varphi) = (\mathscr{Z}(0), \varphi) + \int_0^t (\mathscr{Z}(s), A\varphi)\, ds + \int_0^t (\mathscr{Z}(t), h^*\varphi)\, dY_s \tag{67}$$

From (67) and the fact that we assumed from the beginning that the solution of the Zakai equation is unique, we deduce the following

Theorem 5.1 *The unnormalised conditional distribution of the signal X given the observation coincide with the conditional expectation of $\mathscr{X}$ given the observation.*

The next theorem is the cornerstone of the numerical algorithm. It shows that, in order to approximate the unnormalised conditional distribution ρ_t, we construct the process $\mathscr{X}_n$ up to time t (where n is taken so that the error is as small as we want), keeping the observation path fixed, and then compute its (conditional) expectation.

Theorem 5.2 *There exists $\bar{\Omega} \in \Omega$ with $\tilde{P}(\bar{\Omega}) = 1$ such that for every $\omega \in \bar{\Omega}$ we have $\lim_{n\to\infty} \mathscr{Z}_n^{\omega}(t) = \rho_t^{Y_\cdot(\omega)}$, i.e.,*

$$\lim_{n\to\infty} \left(\mathscr{Z}_n^t(\omega), \varphi\right) = \rho_t^{Y_\cdot(\omega)}(\varphi) \tag{68}$$

for every φ continuous bounded function ($\rho_t^{Y_\cdot(\omega)}$ is the unnormalised distribution of the signal given the observation path $Y_\cdot(\omega)$).

Proof. Let M be a set containing a countable collection of $C_0^{\infty}(\mathbb{R}^d)$ functions, uniformly dense in $C_0(\mathbb{R}^d)$ and the constant function 1. To prove the theorem, we only need to show that, for every function in M,

$$\lim_{n\to\infty} \left(\mathscr{Z}_n^{\omega}(t), \varphi\right) = \rho_t^{Y_\cdot(\omega)}(\varphi), \qquad \tilde{P} - a.s. \; . \tag{69}$$

(to simplify the notation we will omit the ω variable from now on). For this we use the solution of the following backward Itô equation

$$\begin{aligned} d\psi_s(x) &= -A(s)\psi_s(x) - h^*(s,x)\psi_s(x)\, dY_s \\ \psi_t(x) &= \varphi(x) \end{aligned} \tag{70}$$

where $\varphi \in M$. From [3], pp. 126–134 or [16], we obtain that equation (70) has a unique solution in appropriate spaces of solutions and $\rho_t(\psi_t) = \rho_0(\psi_0)$. Since ψ_0 is continuous and bounded $\tilde{P} - a.s.$, it follows that $\lim_{n\to\infty}(\mathscr{Z}_n(0), \psi_0) = (\pi_0, \psi_0) = \rho_0(\psi_0)$, $\tilde{P}$-a.s.. Hence, in order to show (69), we need to prove that

$$\lim_{n\to\infty}(\mathscr{Z}_n(t),\psi_t)-(\mathscr{Z}_n(0),\psi_0)=0,\qquad \tilde{P}-a.s.\ . \tag{71}$$

The first step is to prove that

$$\left(\mathscr{Z}_n\left(\left[\frac{nt}{n}\right]\right),\psi_{\left[\frac{nt}{n}\right]}\right)=(\mathscr{Z}_n(0),\psi_0),\qquad \tilde{P}-a.s. \tag{72}$$

and then that

$$\lim_{n\to\infty}(\mathscr{Z}_n(t),\psi_t)-\left(\mathscr{Z}_n\left(\left[\frac{nt}{n}\right]\right),\psi_{\left[\frac{nt}{n}\right]}\right)=0,\qquad \tilde{P}-a.s.\ . \tag{73}$$

We have that

$$\left(\mathscr{Z}_n\left(\left[\frac{nt}{n}\right]\right),\psi_{\left[\frac{nt}{n}\right]}\right)-(\mathscr{Z}_n(0),\psi_0)$$
$$=\sum_{i=1}^{\left[\frac{nt}{n}\right]}\tilde{E}\left[\left(\mathscr{X}_n\left(\frac{i}{n}\right),\psi_{\frac{i}{n}}\right)\middle|\mathscr{Y}\right]-\tilde{E}\left[\left(\mathscr{X}_n\left(\frac{i-1}{n}\right),\psi_{\frac{i-1}{n}}\right)\middle|\mathscr{Y}\right]$$

and

$$\tilde{E}\left[\left(\mathscr{X}_n\left(\frac{i}{n}\right),\psi_{\frac{i}{n}}\right)\middle|\mathscr{Y}\right]-\tilde{E}\left[\left(\mathscr{X}_n\left(\frac{i-1}{n}\right),\psi_{\frac{i-1}{n}}\right)\middle|\mathscr{Y}\right]$$
$$=\tilde{E}\left[\sum_{j=1}^{N_n\left(\frac{i-1}{n}\right)}\psi_{\frac{i}{n}}\left(V_n^j\left(\frac{i}{n}\right)\right)q_n^j\left(\frac{i}{n}\right)-\psi_{\frac{i-1}{n}}\left(V_n^j\left(\frac{i-1}{n}\right)\right)\middle|\mathscr{Y}\right]\ .$$

Since the number of offsprings $q_n^j(\frac{i}{n})$ of the particle V_n^j is independent of the 'future' of $Y_{\frac{i}{n}}$, we have that

$$\begin{aligned}\tilde{E}\left[q_n^j\left(\frac{i}{n}\right)|\mathscr{F}_{\frac{i}{n}}\vee\mathscr{Y}\right]&=\tilde{E}\left[q_n^j\left(\frac{i}{n}\right)\middle|\mathscr{F}_{\frac{i}{n}}\vee\mathscr{Y}_{\frac{i}{n}}\right]\\&=\tilde{E}\left[q_n^j\left(\frac{i}{n}\right)\middle|\mathscr{F}_{\frac{i}{n}}\right]\\&=e^{\int_{\frac{i-1}{n}}^{\frac{i}{n}}h^*\left(V_n^j(s)\right)\,dY_s-\frac{1}{2}\int_{\frac{i-1}{n}}^{\frac{i}{n}}h^*h\left(V_n^j(s)\right)\,ds}\ .\end{aligned}$$

Hence

$$\tilde{E}\left[\sum_{j=1}^{N_n\left(\frac{i-1}{n}\right)}\psi_{\frac{i}{n}}\left(V_n^j\left(\frac{i}{n}\right)\right)q_n^j\left(\frac{i}{n}\right)-\psi_{\frac{i-1}{n}}\left(V_n^j\left(\frac{i-1}{n}\right)\right)\middle|\mathscr{Y}\right]$$
$$=\tilde{E}\left[\sum_{j=1}^{N_n\left(\frac{i-1}{n}\right)}\psi_{\frac{i}{n}}\left(V_n^j\left(\frac{i}{n}\right)\right)\tilde{E}\left[q_n^j\left(\frac{i}{n}\right)\middle|\mathscr{F}_{\frac{i}{n}}\vee\mathscr{Y}\right]-\psi_{\frac{i-1}{n}}\left(V_n^j\left(\frac{i-1}{n}\right)\right)\middle|\mathscr{Y}\right]$$
$$=\tilde{E}\left[\sum_{j=1}^{N_n\left(\frac{i-1}{n}\right)}\psi_{\frac{i}{n}}\left(V_n^j\left(\frac{i}{n}\right)\right)e^{\int_{\frac{i-1}{n}}^{\frac{i}{n}}h^*\left(V_n^j(s)\right)\,dY_s-\frac{1}{2}\int_{\frac{i-1}{n}}^{\frac{i}{n}}h^*h\left(V_n^j(s)\right)\,ds}-\psi_{\frac{i-1}{n}}\left(V_n^j\left(\frac{i-1}{n}\right)\right)\middle|\mathscr{Y}\right]$$

$$= \tilde{E}\left[\sum_{j=1}^{N_n\left(\frac{i-1}{n}\right)} \tilde{E}\left[\psi_{\frac{i}{n}}\left(V_n^j\left(\frac{i}{n}\right)\right)e^{\int_{\frac{i-1}{n}}^{\frac{i}{n}} h^*\left(V_n^j(s)\right)\,dY_s - \frac{1}{2}\int_{\frac{i-1}{n}}^{\frac{i}{n}} h^* h\left(V_n^j(s)\right)\,ds}\right.\right.$$
$$\left.\left. - \psi_{\frac{i-1}{n}}\left(V_n^j\left(\frac{i-1}{n}\right)\right)\Big|\mathscr{Y}\vee\mathscr{F}_{\frac{i}{n}}\right]\Big|\mathscr{Y}\right] \ . \tag{74}$$

We prove that

$$\tilde{E}\left[\psi_{\frac{i}{n}}\left(V_n^j\left(\frac{i}{n}\right)\right)e^{\int_{\frac{i-1}{n}}^{\frac{i}{n}} h^*\left(V_n^j(s)\right)\,dY_s - \frac{1}{2}\int_{\frac{i-1}{n}}^{\frac{i}{n}} h^* h\left(V_n^j(s)\right)\,ds}\right.$$
$$\left. -\psi_{\frac{i-1}{n}}\left(V_n^j\left(\frac{i-1}{n}\right)\right)\Big|\mathscr{Y}\vee\mathscr{F}_{\frac{i-1}{n}}\right] = 0 \ . \tag{75}$$

Since V_n^j is a Markov process, we have that

$$\tilde{E}\left[\psi_{\frac{i}{n}}\left(V_n^j\left(\frac{i}{n}\right)\right)e^{\int_{\frac{i-1}{n}}^{\frac{i}{n}} h^*\left(V_n^j(s)\right)\,dY_s - \frac{1}{2}\int_{\frac{i-1}{n}}^{\frac{i}{n}} h^* h\left(V_n^j(s)\right)\,ds}\Big|\mathscr{Y}\vee\mathscr{F}_{\frac{i-1}{n}}\right]$$
$$= \tilde{E}\left[\psi_{\frac{i}{n}}\left(V_n^j\left(\frac{i}{n}\right)\right)e^{\int_{\frac{i-1}{n}}^{\frac{i}{n}} h^*\left(V_n^j(s)\right)\,dY_s - \frac{1}{2}\int_{\frac{i-1}{n}}^{\frac{i}{n}} h^* h\left(V_n^j(s)\right)\,ds}\Big|\mathscr{Y}\vee\sigma\left(V_n^j\left(\frac{i-1}{n}\right)\right)\right] \ . \tag{76}$$

We compute first

$$R(x) = \tilde{E}_{\frac{i-1}{n},x}\left[\varphi\left(V_n^j(t)\right)e^{\int_{\frac{i-1}{n}}^{t} h^*\left(V_n^j(s)\right)\,dY_s - \frac{1}{2}\int_{\frac{i-1}{n}}^{t} h^* h\left(V_n^j(s)\right)\,ds}\Big|\mathscr{Y}\right]$$

where the expectation $\tilde{E}_{\frac{i-1}{n},x}$ is taken with respect to the probability $\tilde{P}_{\frac{i-1}{n},x}$ and $\tilde{P}_{\frac{i-1}{n},x}$ is taken so that V_n^j start at time $\frac{i-1}{n}$ from x. This will imply that the conditional expectation of $\varphi(V_n^j(t))e^{\int_{\frac{i-1}{n}}^{t} h^*(V_n^j(s))\,dY_s - \frac{1}{2}\int_{\frac{i-1}{n}}^{t} h^* h(V_n^j(s))\,ds}$ given $\mathscr{Y}\vee\sigma(V_n^j(\frac{i-1}{n}))$ (and, consequently, given $\mathscr{Y}\vee\mathscr{F}_{\frac{i-1}{n}}$) is $R(V_n^j(\frac{i-1}{n}))$. Using the fact that $\rho_t(\varphi) = \rho_{\frac{i-1}{n}}(\psi_{\frac{i-1}{n}})$, we find

$$\tilde{E}_{\frac{i-1}{n},x}\left[\varphi\left(V_n^j(t)\right)e^{\int_{\frac{i-1}{n}}^{t} h^*\left(V_n^j(s)\right)\,dY_s - \frac{1}{2}\int_{\frac{i-1}{n}}^{t} h^* h\left(V_n^j(s)\right)\,ds}|\mathscr{Y}\right]$$
$$= \tilde{E}_{\frac{i-1}{n},x}\left[\psi_{\frac{i-1}{n}}\left(V_n^j\left(\frac{i-1}{n}\right)\right)\Big|\mathscr{Y}\right]$$
$$= \psi_{\frac{i-1}{n}}(x)$$

Hence

$$\tilde{E}\left[\varphi\left(V_n^j(t)\right)e^{\int_{\frac{i-1}{n}}^{t} h^*\left(V_n^j(s)\right)\,dY_s - \frac{1}{2}\int_{\frac{i-1}{n}}^{t} h^* h\left(V_n^j(s)\right)\,ds}\Big|\mathscr{Y}\vee\mathscr{F}_{\frac{i-1}{n}}\right] = \psi_{\frac{i-1}{n}}\left(V_n^j\left(\frac{i-1}{n}\right)\right) \tag{77}$$

Similarly

$$\tilde{E}\left[\varphi\left(V_n^j(t)\right)e^{\int_{\frac{i}{n}}^{t} h^*\left(V_n^j(s)\right)\,dY_s - \frac{1}{2}\int_{\frac{i}{n}}^{t} h^* h\left(V_n^j(s)\right)\,ds}\Big|\mathscr{Y}\vee\mathscr{F}_{\frac{i}{n}}\right] = \psi_{\frac{i}{n}}\left(V_n^j\left(\frac{i}{n}\right)\right) \tag{78}$$

From (77) and (78) we get that

$$\tilde{E}\left[\psi_{\frac{i}{n}}\left(V_n^j\left(\frac{i}{n}\right)\right)e^{\int_{\frac{i-1}{n}}^{\frac{i}{n}} h^*\left(V_n^j(s)\right)\,dY_s-\frac{1}{2}\int_{\frac{i-1}{n}}^{\frac{i}{n}} h^*h\left(V_n^j(s)\right)\,ds}\middle|\mathscr{Y}\vee\mathscr{F}_{\frac{i-1}{n}}\right]$$

$$= \tilde{E}\left[\tilde{E}\left[\varphi\left(V_n^j(t)\right)e^{\int_{\frac{i}{n}}^{t} h^*\left(V_n^j(s)\right)\,dY_s-\frac{1}{2}\int_{\frac{i}{n}}^{t} h^*h\left(V_n^j(s)\right)\,ds}\middle|\mathscr{Y}\vee\mathscr{F}_{\frac{i}{n}}\right]\right.$$

$$\left.\times\, e^{\int_{\frac{i-1}{n}}^{\frac{i}{n}} h^*\left(V_n^j(s)\right)\,dY_s-\frac{1}{2}\int_{\frac{i-1}{n}}^{\frac{i}{n}} h^*h\left(V_n^j(s)\right)\,ds}\middle|\mathscr{Y}\vee\mathscr{F}_{\frac{i-1}{n}}\right]$$

$$= \psi_{\frac{i-1}{n}}\left(V_n^j\left(\frac{i-1}{n}\right)\right)$$

which proves (75). The identity (72) follows now from (74) and (75).

In the analysis above we considered V_n^j defined up to time t, although in the description of the branching system it is not, but obviously we can attach 'an extension' from $\frac{i}{n}$ to t, satisfying the same SDE and independent of Y.

We prove now (73). Using Ito's formula we have that

$$(\mathscr{Z}_n(t),\psi_t)-\left(\mathscr{Z}_n\left(\left[\frac{nt}{n}\right]\right),\psi_{\left[\frac{nt}{n}\right]}\right)$$

$$=\frac{1}{n}\tilde{E}\left[\sum_{j=1}^{N_n\left(\left[\frac{nt}{n}\right]\right)}\int_{\left[\frac{nt}{n}\right]}^{t} h^*\left(V_n^j(s)\right)\psi\left(V_n^j(s)\right)\,dY_s|\mathscr{Y}\right]$$

$$=\frac{1}{n}\tilde{E}\left[\sum_{j=1}^{N_n\left(\left[\frac{nt}{n}\right]\right)}\int_{\left[\frac{nt}{n}\right]}^{t}\tilde{E}\left[h^*\left(V_n^j(s)\right)\psi\left(V_n^j(s)\right)|\mathscr{Y}\vee\mathscr{F}_{\left[\frac{nt}{n}\right]}\right]\,dY_s\middle|\mathscr{Y}\right]\,. \tag{79}$$

Hence

$$\tilde{E}\left[\left((\mathscr{Z}_n(t),\psi_t)-\left(\mathscr{Z}_n\left(\left[\frac{nt}{n}\right]\right),\psi_{\left[\frac{nt}{n}\right]}\right)\right)^4\right]$$

$$\le\frac{1}{n^4}\tilde{E}\left[\left(\sum_{j=1}^{N_n\left(\left[\left(\frac{nt}{n}\right)\right]\right)}\int_{\left[\frac{nt}{n}\right]}^{t}\tilde{E}\left[h^*\left(V_n^j(s)\right)\psi\left(V_n^j(s)\right)\middle|\mathscr{Y}\vee\mathscr{F}_{\left[\frac{nt}{n}\right]}\right]\,dY_s\right)^4\right]$$

$$=\frac{1}{n^4}\tilde{E}\left[\sum_{j_1,j_2,j_3,j_4=1}^{N_n\left(\left[\left(\frac{nt}{n}\right)\right]\right)}\tilde{E}\left[\prod_{k=1}^{4}\int_{\left[\frac{nt}{n}\right]}^{t}\tilde{E}\left[h^*\left(V_n^{j_k}(s)\right)\psi\left(V_n^{j_k}(s)\right)\middle|\mathscr{Y}\vee\mathscr{F}_{\left[\frac{nt}{n}\right]}\right]\,dY_s\middle|\mathscr{F}_{\left[\frac{nt}{n}\right]}\right]\right]$$

Using once again an argument based on the Gronwall inequality we obtain that

$$\tilde{E}\left(\tilde{E}\left[\psi\left(V_n^j(s)\right)\middle|\mathscr{Y}\vee\mathscr{F}_{\left[\frac{nt}{n}\right]}\right]\right)^4\middle|\left[\mathscr{F}_{\left[\frac{nt}{n}\right]}\right]\le m\|\varphi\|^4,\quad\forall s\in\left[\left[\frac{nt}{n}\right],t\right]$$

and from this, using integration by parts, we find the following upper bound

$$\tilde{E}\left[\prod_{k=1}^{4}\int_{\left[\frac{nt}{n}\right]}^{t}\tilde{E}\left[h^*\left(V_n^{j_k}(s)\right)\psi\left(V_n^{j_k}(s)\right)|\mathscr{Y}\vee\mathscr{F}_{\left[\frac{nt}{n}\right]}\right]dY_s\middle|\mathscr{F}_{\left[\frac{nt}{n}\right]}\right]\le\frac{M\|h\|^4\|\varphi\|^4}{n^2}$$

where the constant M is independent of n. It follows that

$$\tilde{E}\left[\left((\mathscr{Z}_n(t), \psi_t) - \left(\mathscr{Z}_n\left(\left[\frac{nt}{n}\right]\right), \psi_{\left[\frac{nt}{n}\right]}\right)\right)^4\right] \leq \frac{Mc_4(t)\|h\|^4\|\varphi\|^4}{n^2} \tag{80}$$

(c_4 is the function defined in Remark 4.3). Finally from (80) we obtain (73) by a Borel-Cantelli type argument. □

Remarks 5.3 The expectation $\mathscr{Z}_n(t)$ of the process $\mathscr{X}_n^Y(t)$, i.e., the process $\mathscr{X}_n(t)$ with fixed observation path $Y_\cdot$, converges almost surely to the unnormalised conditional distribution of the signal $\rho_t^{Y_\cdot}$. One can prove the following: let $\mathscr{X}_n^{Y,1}(t), \ldots, \mathscr{X}_n^{Y,m(n)}(t)$ be $m(n)$ independent copies of $\mathscr{X}_n^Y(t)$, then, for $m(n) = O(n^\alpha)$ and $\alpha > 0$,

$$\lim_{n\to\infty}\left(\frac{1}{m(n)}\sum_{i=1}^{m(n)} \mathscr{X}_n^{Y,i}(t) - \mathscr{Z}_n(t)\right) = 0, \quad \tilde{P} - \text{a.s.}.$$

So if we take $m(n)$ independent copies of the system consisting of n initial particles of mass $\frac{1}{n}$ and let them evolve and branch at times $\frac{k}{n}$, by averaging them we obtain an approximation of the unnormalised conditional distribution of the signal.

An alternative way of looking at the above approximation procedure is to start with proportionally more particles so that, if the time step is $\frac{1}{n}$, we start with $m(n) \times n$ initial particles of mass $\frac{1}{m(n)\times n}$. In this way we see that the measure valued process we have constructed is, in some sense, extremal, and that, if we introduce slightly longer interbranching times relative to the number of particles one starts with initially, one would get convergence to the solution of the Zakai equation.

In this paper we have proved the *existence* of a solution to the filtered martingale problem (4)+ (5). This is an extension of the classical Dawson-Watanabe construction. Averaging the particle approximations over independent evolutions leads to numerical approximation of the Zakai equation.

A sequel in preparation to this paper will look at the numerical *effectiveness* of this method and closely related approaches.

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