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On functions transforming a Wiener process into a semimartingale

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Summary. The description of the class of all functions $f = (f(t,x), t \geq 0, x \in R)$ is given, for which the transformed process $(f(t, W_t), t \geq 0)$ (where W is a standard Wiener process) is a semimartingale.

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1. Introduction and the main results

1.1. Introduction

It is known (see [1] Cinlar, Jacod, Protter, Sharp; and the previous works [2] of Brosamler and [3] of Wang) that the necessary and sufficient condition for the substitution $(f(W_t), t \geq 0)$ (of a Wiener process W) being a semimartingale is the representability of the function $f = (f(x), x \in R)$ as a difference of two convex functions. Below is given a generalization of this characterization for functions depending on the time parameter.

To formulate the main statements let us introduce the following classes of functions differentiable in generalized sense.

Introduce the measure μ on the space $(R_+ \times R, B(R_+ \times R))$ (we write R_+ and R for $[0,\infty)$ and $(-\infty,\infty)$ respectively)

$$\mu(ds, dx) = \rho(s,x)\, ds\, dx \ ,$$

where $\rho(s,x) = (2\pi s)^{-1/2} \exp(-x^2/2s)$.

Denote CB – the class of bounded continuous functions, $C^{1,2}$ – the class of functions continuously differentiable in t and twice continuously differen-

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tiable in x, C^{∞} – the class of infinitely differentiable functions and let $C_0{}^{\infty}$ be the class of infinitely differentiable functions with compact support. For functions $f \in C^{1,2}$ the L operator is defined by

$$(Lf)(s,x) = f_t(s,x) + 1/2 f_{xx}(s,x) \ ,$$

where f_t and f_{xx} are partial derivatives of the function f.

Definition 1. *We shall say that a function $f = (f(t,x), t \geq 0, x \in R)$ admits a generalized L-derivative (or belongs to the domain of definition of a generalized L-operator) w.r.t. the measure μ, if there exists a sequence of functions $(f^n, n \geq 1)$ from $C^{1,2}$ and a measurable locally μ-integrable function (Lf) such that*

$$\sup_{(s,x)\in D} |f^n(s,x) - f(s,x)| \to 0, \quad \text{as } n \to \infty \tag{1}$$

for every compact set $D \in R_+ \times R$ and for some sequence of functions $(h_k(s,x), k \geq 1)$ with properties

1) $h_k(0,x) = 1$ for each $x \in R$, $h_k(s,x) \leq h_{k+1}(s,x), h_k(s,x) \uparrow 1 - \mu$-a.e.
2) $\sigma_k = \inf\{t \colon h_k(t, W_t) \leq \lambda\}, k \geq 1$, are stopping times with $\sigma_k \uparrow \infty$ P-a.s., for some $0 < \lambda < 1$,

we have

$$\iint |(Lf^n)(s,x) - (Lf)(s,x)| h_k(s,x) \mu(ds,dx) \to 0, \quad \text{as } n \to \infty \tag{2}$$

for each $k \geq 1$.

The generalized derivatives f_t, f_x and a second order derivative f_{xx} (w.r.t. the measure μ) are defined similarly, but in the case of a first generalized derivative in x we demand

$$\iint (f_x^n(s,x) - f_x(s,x))^2 h_k(s,x) \mu(ds,dx) \to 0, \quad n \to \infty \tag{3}$$

instead of (2).

Definition 2. *We shall say that a function $f = (f(t,x), t \geq 0, x \in R)$ admits a generalized weak L-derivative (or belongs to the domain of definition of a weak generalized L-operator) w.r.t. the measure μ, if there exists a sequence of functions $(f^n, n \geq 1)$ from $C^{1,2}$ and a σ-finite signed measure ν^L on $(R_+ \times R, B(R_+ * R))$ (which is not necessarily σ-additive), such that relation* (1) *is satisfied and for some sequence of functions $(h_k(s,x), k \geq 1)$ with the properties* 1), 2) *of Definition* 1, *we have for each $k \geq 1$*

$$\iint \psi(s,x) Lf^n(s,x) h_k(s,x) \mu(ds,dx) \to \iint \psi(s,x) h_k(s,x) \nu_L(ds,dx) \text{ as } n \to \infty \tag{4}$$

for every bounded continuous function ψ.

Weak Generalized derivatives ν_t, ν_x and a second order weak derivative ν_{xx} w.r.t. the measure μ are defined similarly.

Let

$\hat{V}^L_\mu$(loc) be a class of functions having a generalized weak L-derivative in the sense of Definition 2,

V^L_μ(loc) be a class of functions having a generalized L-derivative in the sense of Definition 1, i.e. this is a class of functions from $\hat{V}^L_\mu$(loc) for which the measure ν^f_L is absolutely continuous w.r.t. the measure μ.

Denote further

$\hat{V}^{1,2}_\mu$(loc) a class of functions having a generalized weak derivative in t (ν^f_t) and a second order weak derivative in x (ν^f_{xx}) w.r.t. the measure μ,

$V^{1,2}_\mu$(loc) a class of functions having the generalized derivatives f_t, f_x, f_{xx} w.r.t. the measure μ.

The following relations are obvious

$$V^{1,2}_\mu(\text{loc}) \subset V^L_\mu(\text{loc}) \subset \hat{V}^L_\mu(\text{loc}), V^{1,2}_\mu(\text{loc}) \subset \hat{V}^{1,2}_\mu(\text{loc}) \subset \hat{V}^L_\mu(\text{loc}) \ .$$

The L operator is extended on $V^{1,2}_\mu$(loc) naturally, i.e. if $f \in V^{1,2}_\mu$(loc) then

$$(Lf)(s,x) = f_t(s,x) + (1/2)f_{xx}(s,x) \ , \tag{5}$$

with the generalized derivatives f_t, f_{xx}.

Finally, let introduce the class AD_{loc} which we define as a subclass of the class A_{loc} of processes with locally integrable variations for which increments $A_t - A_s$ are measurable w.r.t. the σ-algebra $\sigma(W_u, s \leq u \leq t)$ for each pair $s \leq t$. (In other words the elements of AD_{loc} can be considered as additive (nonhomogeneous) functionals of a Wiener process.)

1.2. Formulation of the main statements

Theorem 1. *Let $f = (f(t,x), t \geq 0, x \in R)$ be a continuous function of two variables. Then the process $(f(t, W_t), t \geq 0)$ is a semimartingale if and only if $f \in \hat{V}^L_\mu(loc)$, and it admits the decomposition*

$$f(t, W_t) = f(0, W_0) + \int_0^t f_x(s, W_s)\, dW_s + A^f_t \ , \tag{6}$$

where, $A^f \in AD_{\text{loc}}$ is uniquely determined by the relation

$$E\int_0^\infty \psi(s, W_s)\, dA^f_s = \iint \psi(s,x)\nu^f_L(ds, dx) \tag{7}$$

valid for each bounded continuous function ψ (for which the integrals in (7) *are defined*).

Remark. *Note, that in* (*Föllmer, Protter, and Shiryaev* [5]) *an alternative Itô formula* (6) *is given and in* (*Chitashvili and Mania* [6]) *the survey of different generalizations of Itô's formula and, in particular, the different meanings of the term A^f are presented, for the general case of a random function f and a semimartingale instead of the Wiener process.*

Theorem 2. *Let $f = (f(t,x), t \geq 0, x \in R)$ be a continuous function of two variables. Then the process $(f(t, W_t), t \geq 0)$ is an Itô process if and only if $f \in V^L_\mu(loc)$. Under this condition the decomposition*

$$f(t, W_t) = f(0, W_0) + \int_0^t f_x(s, W_s)\, dW_s + \int_0^t (Lf)(s, W_s)\, ds \ . \tag{8}$$

takes place.

Corollary 1. a) *The process* $(f(t, W_t), t \geq 0)$ *is a semimartingale of the form*

$$f(t, W_t) = M_t + A_t, \quad M \in M_{\text{loc}}, \quad A \in A_{\text{loc}} \ ,$$

with

$$\sup_{s \leq t \wedge \tau_a} EM_s^2 < \infty, \quad E(VarA)_{t \wedge \tau_a} < \infty, \qquad \text{for any } a \in R, t \geq 0 \ , \tag{9}$$

where $\tau_a = \inf\{s : |W_s| \geq a\}$, if and only if $f \in \hat{V}_\mu^L([0, T] \times [-a, a])$ for each $a \in R, T \geq 0$, (i.e. one can take $h_k(s, x) = I_{[0,k] \times [-k,k]}(s, x)$ in Definition 2).

b) The process $(f(t, W_t), t \geq 0)$ is a semimartingale with the decomposition

$$f(t, W_t) = M_t + A_t, M \in M_{\text{loc}}, A \in A_{\text{loc}} \ ,$$

such that for each $t \geq 0$

$$\sup_{s \leq t} EM_s^2 < \infty, \ E(VarA)_t < \infty \ ,$$

if and only if $f \in \hat{V}_\mu^L([0, T] \times R)$ for each $T \geq 0$, (i.e. one can take $h_k(s, x) = I_{[0,k] \times R}(s, x)$ in Definition 2).

Corollary 2. ([1], [3]) *If* $f(t, x) = f(x)$ *for all* $t \geq 0, x \in R$, *then the process* $(f(W_t), t \geq 0)$ *is a semimartingale if and only if there exists a second order generalized weak derivative* (*w.r.t. the Lebesgue measure* dx) v_{xx} *uniquely defined by the equality*

$$\int \psi_{xx} f(x)\, dx = \int \psi(x) v_{xx}^f(dx) \tag{10}$$

valid for each $\psi \in C^\infty$ with compact support, or equivalently iff the function f is representable as a difference of two convex functions on every compact interval. Besides (see [4])

$$A_t^f = \int_R L^W(t, x) v_{xx}^f(dx) \ ,$$

where $L^W(t, x)$ is a local time of a Wiener process spent at the point x.

For a domain $D \in \mathscr{B}(R_+ \times R)$ the space $W_p^{1,2}(D), p > 1$, is defined (see [7], [8]) as a completion of $C_0^\infty([0, T] * D)$ in the norm

$$||u||_{W_p^{1,2}} = \sup_{(t,x) \in \bar{D}} |u(t, x)| + ||u_t||_{L_p} + ||u_x||_{L_p} + ||u_{xx}||_{L_p} \ .$$

Denote $W_p^{1,2}(\text{loc})$ the class of functions defined on $R_+ \times R$ which belongs to the class $W_p^{1,2}(D)$ for every bounded open domain $D \in R_+ \times R$.

It follows from Hölder's inequality and from the inequality $\iint_D \rho^2(s, x)\, ds\, dx < \infty$ (which is valid for each bounded measurable domain D) that for each $p \geq 2$

$$W_p^{1,2}(\mathrm{loc}) \subset V_\mu^{1,2}(\mathrm{loc}) \ , \tag{11}$$

Corollary 3. ([7]). If $f \in W_{\mathrm{loc}}^{1,2}(p)$ for some $p \geq 2$, then $(f(t, W_t), t > 0)$ will be an Itô process and the Itô formula

$$f(t, W_t) = f(0, W_0) + \int_0^t f_x(s, W_s)\, dW_s + \int_0^t f_t(s, W_s)\, ds + \frac{1}{2}\int_0^t f_{xx}(s, W_s)\, ds$$

is valid, with the generalized derivatives f_x, f_t, f_{xx}.

The proof follows from Theorem 2 and relations (5), (11).

2. Some properties of regularizations by Gaussian kernel

Consider the sequence of functions

$$f^n(s,x) = n\int_s^{s+1/n}\int_R f(u,y)\rho(u-s, y-x)\, dy\, du \ , \tag{12}$$

where $\rho(s,x) = (2\pi s)^{-1/2}\exp\{-(x^2/2s)\}$ is a Gaussian kernel.

Here we give some properties of the function f^n.

Proposition 1. a) If $\iint_{[0,t]\times R]} f^2(s,x)\mu(ds,dx) < \infty$ for each $t > 0$, then

$$\iint_{[0,t]\times R} (f^n(s,x) - f(s,x))^2 \mu(ds,dx) \to 0, n \to \infty, t \geq 0 \ .$$

If $f = (f(t,x), t \geq 0, x \in R)$ is a continuous function of two variables, then b) for every compact D from $R_+ \times R$

$$\sup_{(s,x)\in D} |f^n(s,x) - f(s,x)| \to 0, \text{ as } n \to \infty \tag{13}$$

c) for every $t \geq 0$

$$\sup_{s\leq t} |f^n(s, W_s) - f(s, W_s)| \to 0, n \to \infty \ , \tag{14}$$

in probability.

Proof. The statements a) and b) can be proved in the same way as the similar propositions in [8]). Assertion c) is an easy consequence of b).

Proposition 2. For each $\epsilon \geq 0$

a) the function

$$f^\epsilon(s,x) = \frac{1}{\epsilon}\int_s^{s+\epsilon}\int_R f(u,y)\rho(u-s, y-x)\, dy\, du$$

belongs to the class $V_\mu^L(\mathrm{loc})$ for every bounded continuous function f, and

$$(Lf^\epsilon)(s,x) = \frac{1}{\epsilon}\int_R (f(s+\epsilon, y) - f(s,x))\rho(\epsilon, y-x)\, dy \ . \tag{15}$$

b) the process $(f^\epsilon(t, W_t), t \geq 0)$ is an Itô process and its part of bounded variation has the following form

$$A_t^\epsilon = \frac{1}{\epsilon}\int_0^t E(f(s+\epsilon, W_{s+\epsilon}) - f(s, W_s)/\mathscr{F}_s^W)\, ds \tag{16}$$

Proof. a) Let

$$f^{\epsilon,n}(s,x) = \frac{1}{\epsilon}\int_{s+1/n}^{s+\epsilon}\int_R f(u,y)\rho(u-s, y-x)\, dy\, du \ .$$

Evidently $f^{\epsilon,n} \in C^{1,2}$ and $f^{\epsilon,n} \to f^\epsilon$ uniformly on every compact (since f is bounded). Using the equality $\rho_s(u-s, y-x) + (1/2)\rho_{xx}(u-s, y-x) = 0$ we obtain that

$$(Lf^{\epsilon,n})(s,x) = \frac{1}{\epsilon}\left(\int_R f(s+\epsilon, y)\rho(\epsilon, y-x)\, dy - \int_R f(s+1/n, y)\rho(1/n, y-x)\, dy\right)$$

and it is easy to see that by the continuity of the function f

$$\int_R f(s+1/n, y)\rho(1/n, y-x)\, dy \to f(s,x), \quad n \to \infty$$

and, consequently, $Lf^{\epsilon,n} \to \frac{1}{\epsilon}\int_R (f(s+\epsilon, y) - f(s,x))\rho(\epsilon, y-x)\, dy$ in the sense of Definition 1.

b) Evidently A^ϵ is of bounded variation for every $t \geq 0$ and $(VarA^\epsilon)_t \leq \frac{2Ct}{\epsilon}$. Since

$$f^n(s,x) = n\int_s^{s+1/n} E(f(u, W_u)/W_s = x)\, du \ ,$$

by Markov property of W

$$M_t^\epsilon = f^\epsilon(t, W_t) - A_t^\epsilon = \frac{1}{\epsilon}\int_t^{t+\epsilon} E(f(u, W_u)/\mathscr{F}_t^W)\, du - \frac{1}{\epsilon}\int_0^t E(f(s+\epsilon, W_{s+\epsilon}) - f(s, W_s)/\mathscr{F}_s^W)\, ds \ ,$$

and it is easy to show that the martingale equality $E(M_t^\epsilon - M_s^\epsilon/\mathscr{F}_s^W) = 0$ holds. □

3. Condition U.T and the convergence of semimartingales

Let $(Z^n, n \geq 1)$ be a sequence of semimartingales given on some filtered probability space $(\Omega, \mathscr{F}, (\mathscr{F}_t, t \geq 0), P)$ with the canonical decomposition

$$Z_t^n = M_t^n + A_t^n, M^n \in \mathscr{M}_{\text{loc}}, A^n \in \mathscr{A}_{\text{loc}} \ ,$$

We are interested when the convergence in probability (uniformly on every compact) of the sequence $(Z^n, n \geq 1)$ implies the semimartingality of a limiting process, and, furthermore, when the martingale parts $(M^n, n \geq 1)$ converge to the martingale part of the limiting process.

Denote $\mathscr{H}$ the set of all elementary predictable processes bounded by unity.

Definition 3. *(Stricker [9]). The sequence of processes* $((Z_t^n, t \geq 0), n \geq 1)$ *satisfies the condition U.T (uniform tightness) if for each* $t > 0$ *the set*

$$\left(\int_0^t H_s\, dZ_s^n,\; H \in \mathscr{H},\; n \geq 1\right)$$

is stochastically bounded.

It was shown in (Jacubovski et al. [10]) that if $((Z_t^n, t \geq 0), n \geq 1)$ is a sequence of martingales converging weakly to some process Z and the sequence $(Z^n, n \geq 1)$ satisfies the condition U.T then the limiting process Z is also a semimartingale. Further (in Memin and Slominski [11]) it was proved that if the sequence of semimartingales $(Z^n, \geq 1)$ converges to the process Z in probability (uniformly on every compact $[0, t]$) then the U.T condition guarantees the convergence of martingale terms (and hence of terms with bounded variation) of Z^n to the martingale part of Z. For continuous semimartingales (which we consider) the U.T property is equivalent to the stochastic boundedness of the sequence of random variables $Var(A^n)_t, \langle M^n \rangle_t$ for each $t > 0$; and if $\sup_{s \leq t} |Z_s^n - Z_s| \to 0, n \to \infty$, for each $t > 0$ the U.T property is satisfied if and only if the sequence $Var(A^n)_t$ is stochastically bounded for each $t > 0$. Thus the following assertion takes place

Proposition 3. If $(Z^n, n \geq 1)$ is a sequence of continuous semimartingales and Z is some process for which

$$\sup_{s \leq t} |Z_s^n - Z_s| \to 0, \quad n \to \infty \;, \tag{17}$$

in probability for every $t \geq 0$, and

$$\lim_N \overline{\lim}_n P(Var(A^n)_t > N) = 0 \tag{18}$$

for each $t > 0$, then the process Z is also a semimartingale and

a) for each $t > 0$

$$\langle M^n - M \rangle_t \to 0, \sup_{s \leq t} |A_s^n - A_s| \to 0, \quad n \to \infty \;. \tag{19}$$

in probability,

b) there exists a subsequence of the sequence $(Z^n, n \geq 1)$ and a sequence $(\tau_k, k \geq 1)$ of stopping times with $\tau_k \to \infty$ such that

$$E\langle M^n - M \rangle_{\tau_k} \to 0, \quad E \sup_{s \leq \tau_k} |A_s^n - A_s| \to 0, \quad n \to \infty \tag{20}$$

for every $k \geq 1$.

The proof of the assertion a) one can see in Mémin and Slominski [11] (see also Lemma 2 in Chitashvili Mania [6]). The validity of b) follows from a) and from the Proposition 1 of Emery [12].

Thus, to prove that the process $(f(t, W_t), t \geq 0)$ is a semimartingale it is sufficient to show that there exists a sequence of functions $((f^n(s,x), s \geq 0, x \in R), n > 1)$ from $C^{1,2}$ for which

$$\sup_{s\le t}|f^n(s,W_s)-f(s,W_s)|\to 0,\ n\to\infty$$

for each $t\ge 0$ and, besides, the sequence of semimartingales $((f^n(s,W_s), s\ge 0), n\ge 1)$ satisfies the U.T condition. It would be noted here that the approximations (12) have the property to define semimartingale (submartingale, martingale) $(f^n(t,W_t), t\ge 0, n\ge 1)$, for the semimartingale (submartingale, martingale) $(f(t,W_t), t\ge 0)$; and, as it will be shown below, the processes $(f^n(t,W_t), t\ge 0)$ satisfy the condition (18) of Proposition 3.

4. Some properties of the class $\mathscr{AD}_{\text{loc}}$ of additive (nonhomogeneous) functionals of finite variation

Below we shall use the following property of processes from the class AD_{loc}

Lemma 1. *If $A\in AD_{\text{loc}}$ then for every $\mathscr{F}^W$-adapted bounded cadlag process Z such that $E\int_0^\infty |Z_s||dA_s|<\infty$ we have*

$$E\int_0^\infty Z_s\,dA_s = E\int_0^\infty E(Z_s/W_s)\,dA_s\ , \tag{21}$$

where $E(Z_s/W_s), s\ge 0$, is a regular modification of the process $E(Z_s/W_s)=E(Z_s/\mathscr{F}^W_{[s,\infty)})$, with $\mathscr{F}^W_{[s,\infty)}=\sigma(W_u : u\ge s)$

Proof. Let

$$A^n_t = n\int_0^t E(A_{s+1/n}-A_s/\mathscr{F}^W_s)\,ds \tag{22}$$

Applying the method used in Theorem 54 from Dellacherie [13] it can be proved that for every bounded cadlag process Z

$$E\int_0^\infty Z_s\,dA^n_s \to E\int_0^\infty Z_{s-}\,dA_s = E\int_0^\infty Z_s\,dA_s,\quad n\to\infty \tag{23}$$

(the last equality is true, because A is continuous).

Since $A\in AD_{\text{loc}}$ and W is a Markov process we have

$$E(A_{s+1/n}-A_s/\mathscr{F}^W_s) = E(A_{s+1/n}-A_s/W_s) \tag{24}$$

and therefore (taking into account (22) and Lemma 8.3 of [20])

$$E\int_0^\infty Z_s\,dA^n_s = E\int_0^\infty E(Z_s/W_s)\,dA^n_s \tag{25}$$

Evidently the relation (21) follows from (23) by passage to the limit in (25). □

Corollary 5. ([1] Cinlar et al.). Let $A,B\in AD_{\text{loc}}$. If for every bounded measurable function $\psi(s,x)$

$$\psi(s,W_s)\,dA_s = 0 (a.s) \text{ implies } \psi(s,W_s)\,dB_s = 0\ (a.s)\ , \tag{26}$$

then $dB_s \ll dA_s$ and there exists a measurable function $g(s,x)$ such that

$$B_t = \int_0^t g(s, W_s)\, dA_s \ (a.s.)$$

in particular, if $A \in AD_{\text{loc}}$ and $dA_s \ll ds$ then there exists a measurable function $a(s,x)$ for which

$$A_t = \int_0^t a(s, W_s)\, ds \ (a.s.)$$

Proof. Without loss of generality we can assume that $A, B \in AD$. If $E \int_0^\infty Z_s\, dA_s = 0$ for some bounded cadlag adapted process Z, then Lemma 1 implies that $E \int_0^\infty E(Z_s/W_s)\, dA_s = 0$ and since there exists a measurable function $\psi(s,x)$ for which $\psi(s, W_s) = E(Z_s/W_s)$ $(a.s.)$ from (26) we have that $E \int_0^\infty E(Z_s/W_s)\, dB_s = 0$. Therefore using again Lemma 1 we get that $E \int_0^\infty Z_s\, dB_s = 0$ and hence $dB_t \ll dA_t$. Thus there exists a predictable process y such that $B_t = \int_0^t y_s\, dA_s$ and from Lemma 1 we obtain the equality $B_t = \int_0^t E(y_s/W_s)\, dA_s = \int_0^t \psi(s, W_s)\, dA_s$ for some measurable function ψ. □

The following statement for the construction of localizing moments for additive functionals is similar in character to those used in [14] (Revuz), [1] (Cinlar et al.) (Lemma 4.7) and [15] (Höhnle and Sturm).

Lemma 2. *(localization lemma) Let* $A = (A_t, t \geq 0) \in AD_{\text{loc}}$ *and* $(\tau_k, k \geq 1)$ *be the localizing sequence of stopping times of the process A (i.e.* $\tau_k \uparrow \infty$ *and* $E(varA)_{\tau_k} < \infty$ *for every* $k \geq 1$*).*

Let, for some $0 < \lambda < 1$,

$$h_k(s,x) = E(I_{[s<\tau_k]}/W_s = x) \ .$$

Then the sequence of functions $h_k(s,x), k \geq 1)$ *satisfies the conditions 1), 2) of Definition 1 and for each* $k \geq 1$

$$E \int_0^{\sigma_k} |dA_s| \leq (1/\lambda) E \int_0^\infty h_k(s, W_s) |dA_s| = (1/\lambda) E \int_0^{\tau_k} |dA_s| \ , \qquad (27)$$

where

$$\sigma_k = \inf\{t \colon h_k(t, W_t) \leq \lambda\} \ .$$

Proof. Evidently, $h_k \leq h_{k+1}$, for every $k \geq 1$ and $h_k \uparrow 1$, since $\tau_k \uparrow \infty$.

Let us show that the random variables $\sigma_k, k \geq 1$, are stopping times, for each $k \geq 1$, such that for every t

$$P(\sigma_k > t) \to 1, k \to \infty \ .$$

Consider the process $h_k(s, W_s) = E(I_{[s<\tau_k]}/W_s), s > 0$. This process is a submartingale with respect to the inverse flow of σ-algebras $(F^W_{[t,\infty)}, t \geq 0)$. In fact, by the Markovian property of the Wiener process W

$$E(I_{[s<\tau_k]}/\mathscr{F}^W_{[s,\infty)}) = E(I_{[s<\tau_k]}/W_s)$$

which gives that for $s \geq t$

$$E(h_k(t,W_t)/\mathscr{F}^W_{[s,\infty)}) = E(E(I_{[t<\tau_k]}/\mathscr{F}^W_{[t,\infty)})/\mathscr{F}^W_{[s,\infty)})$$
$$= E(I_{[t<\tau_k]}/\mathscr{F}^W_{[s,\infty)}) \geq E(I_{[s<\tau_k]}/\mathscr{F}^W_{[s,\infty)}) = h_k(s,W_s) \; .$$

The function $Eh_k(s,W_s) = P(\tau_k > s)$ is right continuous in s and, hence, there exists a right continuous modification of the submartingale $(h_k(s,W_s), s \geq 0)$. It follows from this that the variable σ_k is a stopping time (for each $k \geq 1$) and from Doob's inequality for supermartingales we have

$$P(\sigma_k > t) = P(\inf_{s\leq t} h_k(s,W_s) > \lambda)$$
$$= 1 - P\left(\sup_{s\leq t}(1-h_k(s,W_s)) \geq 1-\lambda\right) \geq 1-(1/(1-\lambda))P(\tau_k \leq t) \to 1 \; ,$$

since $\tau_k \to \infty$ as $k \to \infty$.

Let us show now the validity of (27).

Since $(s < \sigma_k) \subseteq (h_k(s,W_s) > \lambda)$, using Lemma 1 we have

$$\int_0^{\sigma_k} |dA_s| = \int_0^\infty I_{[s\leq\sigma_k)}|dA_s| \leq E\int_0^\infty I_{[h_k(s,W_s)>\lambda]}|dA_s|$$
$$\leq \frac{1}{\lambda}\int_0^\infty h_k(s,W_s)|dA_s| = \frac{1}{\lambda}\int_0^\infty E(I_{[s\leq\tau_k)}/W_s)|dA_s| = \frac{1}{\lambda}E\int_0^{\tau_k}|dA_s|$$

and relation (27) is valid. □

Sometimes the following definition, equivalent to Definition 1, is useful.

Definition 1′. *A function $f = (f(t,x), t \geq 0, x \in R)$ admits a generalized L-derivative (w.r.t. the measure μ), if there exists a sequence of functions $(f^n, n \geq 1)$ from $C^{1,2}$, satisfying the relation (1), and a measurable locally μ-integrable function (Lf) such that for some sequence of bounded measurable domains $(D_k, k \geq 1)$ with*

1′) $(0,x) \in D_1$ *for each* $x \in R$, $D_k \subseteq D_{k+1}, k \geq 1, \cup_k D_k = R_+ \times R$, *$\mu$-a.e.*
2′) $u_k = \inf\{t\colon (t,W_t) \notin D_k\}$ *are stopping times with* $u_k \uparrow \infty$ *we have for each* $k \geq 1$

$$\iint_{D_k} |(Lf^n)(s,x) - (Lf)(s,x)|\mu(ds,dx) \to 0, \quad \text{as } n \to \infty \tag{28}$$

Proposition 4. Definition 1 and Definition 1′ are equivalent.

Proof. If f is generalized L-differentiable in the sense of Definition 1, then, evidently, the sequence $D_k = \{(s,x)\colon h_k(s,x) > \lambda < 1\}$ possesses properties 1′), 2′) (obviously $u_k = \inf\{t\colon (t,W_t) \notin D_k\} = \inf\{t\colon h_k(t,W_t) \leq \lambda\} = \sigma_k \uparrow \infty$) and for each $k \geq 1$

$$\iint_{D_k} |Lf^n(s,x) - (Lf)(s,x)|\mu(ds,dx)$$
$$\leq (1/\lambda)\iint |(Lf^n)(s,x) - (Lf)(s,x)|h_k(s,x)\mu(ds,dx) \to \infty \; ,$$

as $n \to \infty$.

Conversely, let $(D_k, k \geq 1)$ be a sequence with properties $1'), 2')$. Then the sequence of functions $h_k(s,x) = E(I_{[u_k \geq s]}/W_s = x)$ satisfies conditions 1), 2) of Definition 1, since according to Lemma 2 $u_k \uparrow \infty$ implies that $\sigma_k = \inf\{t \colon h_k(t, W_t) \leq \lambda\} \uparrow \infty$ and using Lemma 1 and equality $I_{(s<u_k)} \leq I_{D_k}(s, W_s)$ we have

$$\iint |(Lf)^n(s,x) - (Lf)(s,x)| h_k(s,x)\mu(ds, dx)$$
$$\leq \iint_{D_k} |(Lf^n)(s,x) - (Lf)(s,x)|\mu(ds, dx) \ . \quad \square$$

5. Proof of Theorem 1 and its corollaries

Proof of Theorem 1. Sufficiency. Let us show that if $f \in \hat{V}_\mu^L(\text{loc})$ then the process $(f(t, W_t), t \geq 0)$ is a semimartingale with the decomposition (6).

Let $(h_k(s,x), k \geq 1)$ be a localizing sequence of functions from Definition 2 of the class $\hat{V}_\mu^L(\text{loc})$ and let $(\sigma_k, k \geq 1)$ be corresponding stopping times defined by

$$\sigma_k = \inf(t \leq k \colon h_k(t, W_t) \leq \lambda < 1) \ .$$

Since $\sigma_k \to \infty$, to prove that the process $(f(t, W_t), t \geq 0)$ is a semimartingale it is sufficient to show that the process $(f(t \wedge \sigma_k, W_{t \wedge \sigma_k}), t \geq 0)$ is a semimartingale for every $k \geq 1$. For convenience we shall omit the index k.

Let $(f^n, n \geq 1)$ be the sequence of smooth functions from Definition 2. Evidently, for every $n \geq 1$ the process $Z^n = (f^n(t \wedge \sigma, W_{t \wedge \sigma}), t \geq 0)$ is a semimartingale with the decomposition

$$f^n(t \wedge \sigma, W_{t \wedge \sigma}) = f^n(0, W_0) + M_t^n + A_t^n \ ,$$

where

$$M_t^n = \int_0^{t \wedge \sigma} f_x^n(s, W_s)\, dW_s, \ A_t^n = \int_0^{t \wedge \sigma} (Lf^n)(s, W_s)\, ds \ .$$

It follows from the relation (1) that for each $t \geq 0$

$$\sup_{s \leq t} |f^n(s \wedge \sigma, W_{s \wedge \sigma}) - f(s \wedge \sigma, W_{s \wedge \sigma})| \to 0, \quad n \to \infty \tag{29}$$

in probability and according to Proposition 3 it is sufficient to show that for each $t > 0$

$$\lim_N \sup_n P((VarA^n)_{t \wedge \sigma} > N) = 0 \ . \tag{30}$$

We have

$$E(VarA^n)_{t\wedge\sigma} = E\int_0^{t\wedge\sigma} |(Lf^n)(s,W_s)|ds \le \frac{1}{\lambda}E\int_0^{t\wedge\sigma} h_k(s,W_s)|(Lf^n)(s,W_s)|ds$$
$$\le \frac{1}{\lambda}\iint |(Lf^n)(s,x)|h_k(s,x)\mu(ds,dx) \ ,$$

but

$$\iint \psi(s,x)(Lf^n)(s,x)h_k(s,x)\mu(ds,dx) \to \iint \psi(s,x)h_k(s,x)\nu_L^f(ds,dx) \ , \quad (31)$$

as $n\to\infty$, for all $\psi\in CB$ and, hence, for each $k\ge 1$

$$\sup_n \iint |(Lf^n)(s,x)|h_k(s,x)\mu(ds,dx) < \infty \ . \quad (32)$$

Therefore the process $(f(t\wedge\sigma_k, W_{t\wedge\sigma_k}), t\ge 0)$ is a semimartingale for each $k\ge 1$ and, consequently, the process $(f(t,W_t), t\ge 0)$ is also a semimartingale.

Let us show now that the function f is generalized differentiable in x. Let

$$Z_t = f(t,W_t) = M_t + A_t \quad (33)$$

be the minimal decomposition of the semimartingale $Z_t = f(t,W_t)$, i.e. $M\in\mathscr{M}_{\text{loc}}(\mathscr{F}^Z)$, $A\in\mathscr{A}_{\text{loc}}(\mathscr{F}^Z)$.

Since the sequence of semimartingales $((f^n(t,W_t), t\ge 0), n\ge 1)$ satisfies the $U.T$ condition and converges to the semimartingale $(f(t,W_t), t\ge 0)$ in probability uniformly on every compact, then it follows from Proposition 3.b) that there exists a subsequence of the sequence f^n (for convenience we use the same index n for the subsequence) and a sequence $(\tau_k, k\ge 1)$, of stopping times with $\tau_k\uparrow\infty$, $\quad k\uparrow\infty$ such that, for every $k\ge 1$

$$E\langle M^n - M\rangle_{\tau_k} \to 0, \ E\sup_{s\le\tau_k}|A_s^n - A_s| \to 0, \quad n\to\infty \ . \quad (34)$$

Let us define a new sequence of localizing functions $\tilde{h}_k(s,x) = E(I_{[s<\tau_k]}/W_s = x)$ and let $C_k = ((s,x) : \tilde{h}_k(s,x) > \lambda)$ for some $0<\lambda<1$. Then according to Lemma 2

$$\tilde{\sigma}_k = \inf\{t\colon \tilde{h}_k(t,W_t)\le\lambda\} = \inf\{t\colon (t,W_t)\notin C_k\}, k\ge 1 \ ,$$

are stopping times with $\tilde{\sigma}_k\uparrow\infty$. It follows from Lemma 1 and (34) that for each $k\ge 1$

$$\iint_{C_k} (f_x^n(s,x) - f_x^m(s,x))^2\mu(ds,dx)$$
$$\le (1/\lambda)\int_0^\infty\int_R (f_x^n(s,x) - f_x^m(s,x))^2\tilde{h}_k(s,x)\mu(ds,dx)$$
$$= (1/\lambda)E\int_0^{\tau_k} (f_x^n(s,W_s) - f_x^m(s,W_s))^2\,ds \to 0, \quad n\to\infty, \quad m\to\infty \ . \quad (35)$$

Evidently, for each $k\ge 1$, there exists a function $f_x^{(k)}$ for which

$$\iint_{C_k} (f_x^n(s,x) - f_x^{(k)}(s,x))^2\mu(ds,dx) \to 0, \quad n\to\infty \ . \quad (36)$$

Since $f_x^{(r)} = f_x^{(k)}$ on the set C_k for all $r \geq k$, one can define the function f_x coinciding with $f^{(k)}$ on the set C_k such that

$$\iint_{C_k} (f_x^n(s,x) - f_x(s,x))^2 \mu(ds,dx) \to 0, \quad n \to \infty \ . \tag{37}$$

Thus, taking into account Proposition 4, we can conclude that the function f is locally generalized differentiable in x w.r.t. the measure μ. Besides the martingales $(M_t, t \geq 0)$ and $(\int_0^t f_x(s,W_s), dW_s, t \geq 0)$ are indistinguishable. In fact, since

$$\begin{aligned} &E\left\langle \int_0^\cdot f_x^n(s,W_s)\,dW_s - \int_0^\cdot f_x(s,W_s)\,dW_s)\right\rangle_{\tilde\sigma_k} \\ &\leq \iint_{C_k} (f_x^n(s,x) - f_x(s,x))^2 \mu(ds,dx) \to 0, \quad n \to \infty \ , \end{aligned} \tag{38}$$

from (34) we have that

$$E\left\langle \int_0^\cdot f_x^n(s,W_s)\,dW_s - M\right\rangle_{\tau_k} \to 0, \quad n \to \infty \ , \tag{39}$$

and, consequently,

$$E\left\langle \int_0^\cdot f_x(s,W_s)\,dW_s - M\right\rangle_{\tilde\sigma_k \wedge \tau_k} = 0 \tag{40}$$

for each $k \geq 1$. Now taking into account that $\tilde\sigma_k \uparrow \infty, \tau_k \uparrow \infty$ we obtain the indistinguishability of the processes M and $\int f_x\,dW$.

Thus the semimartingale $(f(t,W_t), t \geq 0)$ can be represented in the form

$$f(t,W_t) = f(0,W_0) + \int_0^t f_x(s,W_s)\,dW_s + A_t^f \ , \tag{41}$$

where f_x is a generalized derivative in x w.r.t. the measure μ and, evidently, $A^f \in AD_{\text{loc}}$.

The proof of necessity. Let the process $(f(t,W_t),\ t \geq 0))$ be a continuous semimartingale and let

$$Z_t = f(t,W_t) = M_t + A_t \tag{42}$$

be its minimal decomposition.

We first assume that the semimartingale $(f(t,W_t),\ t \geq 0)$ is bounded, i.e.

$$\sup_t |f(t,W_t)| < C \tag{43}$$

Let $(\tau_k, k \geq 1)$ be a sequence of stopping times (with $\tau_k \uparrow \infty,\ k \to \infty$) for which

$$E(VarA)_{\tau_k} < \infty, \quad E\langle M\rangle_{\tau_k} < \infty \tag{44}$$

for every $k \geq 1$.

Consider the sequence of functions $(f^n, n \geq 1)$

$$f^n(s,x) = n\int_s^{s+1/n}\int_R f(u,y)\rho(u-s,y-x)\,dy\,du \tag{45}$$

It follows from Proposition 2 that the process $(f^n(t,W_t), t \geq 0)$ is a semimartingale (for every $n \geq 1$) with the decomposition

$$f^n(t,W_t) = M_t^n + A_t^n \ , \tag{46}$$

where M^n is a martingale and the part of bounded variation has the form

$$A_t^n = n\int_0^t E(f(s+1/n, W_{s+1/n}) - f(s,W_s)/W_s)\,ds \ . \tag{47}$$

According to Proposition 1c)

$$\sup_{s\leq t}|f^n(s,W_s) - f(s,W_s)| \to 0, \quad n\to\infty \tag{48}$$

in probability for every $t \geq 0$.

Let us show that the family of semimartingales $(f^n(t,W_t), t \geq 0, n \geq 1)$ satisfies the $U.T$ condition. In fact, it is sufficient to prove that for every $k \geq 1$

$$\sup_n E(VarA^n)_{\tau_k} < \infty \tag{49}$$

From (47) it follows that

$$\begin{aligned}(VarA^n)_{\tau_k} &= n\int_0^{\tau_k}|E(f(s+1/n, W_{s+1/n}) - f(s,W_s)/\mathscr{F}_s^W)|\,ds \\ &= n\int_{\tau_k}^{\tau_k+1/n}|E(f(u,W_u) - f(u-1/n, W_{u-1/n})/\mathscr{F}_{u-1/n}^W)|\,du \\ &\quad + n\int_{1/n}^{\tau_k}|E(f(u,W_u) - f(u-1/n, W_{u-1/n})/\mathscr{F}_{u-1/n}^W)|\,du \\ &\leq 2C + n\int_{1/n}^{\tau_k}|E(A_u - A_{u-1/n}/\mathscr{F}_{u-1/n}^W)|\,du \ ,\end{aligned}$$

since $\sup_t|f(t,W_t)| < C$ and

$$E(f(t,W_t) - f(s,W_s)/\mathscr{F}_s^W) = E(A_t - A_s/\mathscr{F}_s^W)$$

on the set $s < t \leq \tau_k$.

Therefore

$$E(VarA^n)_{\tau_k} \leq 2C + nE\int_{1/n}^{\tau_k}|A_u - A_{u-1/n}|\,du \leq 2C + nE\int_{1/n}^{\tau_k}\int_{u-1/n}^{u}|\,dA_s|\,du$$

Now using Fubini's theorem

$$E\int_{1/n}^{\tau_k}\int_{u-1/n}^{u}|dA_s|\,du = E\int_0^{\tau_k}\int_{s\vee 1/n}^{(s+1/n)\wedge\tau_k}du|\,dA_s|$$

and taking into account the inequality $(s+1/n)\wedge\tau_k - s\vee 1/n \leq 1/n$ we obtain that

$$E(VarA^n)_{\tau_k} \leq 2C + E(VarA)_{\tau_k} \tag{50}$$

for every $n \geq 1$.

Thus the conditions of Proposition 3 are fulfilled and therefore there exists a subsequence of the sequence $(f^n, n \geq 1)$ and a sequence $(\tau_k, k \geq 1)$ of stopping times with $\tau_k \to \infty$ (for convenience we use the same indexeis for this subsequence) such that

$$E\langle M^n - M\rangle_{\tau_k} \to 0, \quad n \to \infty \tag{51}$$

$$E \sup_{s \leq \tau_k} |A_s^n - A_s| \to 0, \quad n \to \infty \tag{52}$$

for every $k \geq 1$.

Evidently the process $(A_t^n, t \geq 0)$ belongs to the class AD_{loc} for each n and it follows from the relation (52) that the process $(A_t, t \geq 0)$ also belongs to the same class.

Define a measure $v_A(ds, dx)$ on the Borel sets from $R_+ \times R$. For measurable positive function $\psi = (\psi(s,x), s \geq 0, x \in R)$ with $E\int_0^\infty \psi(s, W_s)|dA_s| < \infty$ let

$$v_A(\psi) = E\int_0^\infty \psi(s, W_s)\, dA_s \tag{53}$$

The measure v_A is σ-finite on $\mathscr{B}(R_+ * R)$, since for the sequence of domains $(D_k, k \geq 1)$

$$D_k = ((s,x) : E(I_{[s<\tau_k]}/W_s = x) > \lambda),\ 0 < \lambda < 1 \ ,$$

we have that $\cup_k D_k = R_+ \times R$ and

$$|v_A(D_k)| = E\int_0^\infty I_{D_k}(s, W_s)|dA_s| \leq (1/\lambda)E\int_0^\infty E(I_{[s<\tau_k]}/W_s)|dA_s|$$
$$= (1/\lambda)E\int_0^{\tau_k} |dA_s| < \infty \ ,$$

which follows from Lemma 1, since $A \in AD_{\text{loc}}$.

Since the function $f^n(s,x)$ belongs to the domain of definition of L operator for each $n \geq 1$ (Proposition 2a), the assertion of the sufficiency part of this theorem implies that the semimartingale $(f^n(t, W_t), t \geq 0)$ admits the decomposition

$$f^n(t, W_t) = f^n(0, W_0) + \int_0^t f_x^n(s, W_s), dW_s + \int_0^t (Lf^n)(s, W_s)\, ds \ . \tag{54}$$

Therefore, it follows from (46), (54) and from the uniqueness of the canonical decomposition of semimartingales that

$$A_t^n = \int_0^t (Lf^n)(s, W_s)\, ds \ . \tag{55}$$

Since from (50), (52) we have that, for every $k \geq 1$,

$$E\int_0^{\tau_k} \psi(s, W_s)\, dA_s^n \to E\int_0^{\tau_k} \psi(s, W_s)\, dA_s \tag{56}$$

for every bounded continuous function ψ and according to equality (55) and Lemma 1

$$E\int_0^{\tau_k}\psi(s,W_s)\,dA_s^n=\iint\psi(s,x)(Lf^n)(s,x)h_k(s,x)\mu(ds,dx)\ , \tag{57}$$

we obtain that, for each $k\geq 1$,

$$J_k^n(\psi)=\iint\psi(s,x)(Lf^n)(s,x)h_k(s,x)\mu(ds,dx)\to E\int_0^{\tau_k}\psi(s,W_s)\,dA_s \tag{58}$$

for every bounded continuous function ψ. Thus, for every $k\geq 1$ $J_k^n(\psi)$ is a sequence of linear bounded functionals on CB which converges for each $\psi\in CB$. Therefore (see e.g. Th. 1. II.1 of [16]) the functional J_k defined by $J_k(\psi)=\lim_{n\to\infty}J_k^n(\psi)$ is also linear bounded functional on CB and, hence, it is representable in the form (see e.g. Th. 2. Ch. IV.6 of [17])

$$J_k(\psi)=\iint\psi(s,x)v^k(ds,dx) \tag{59}$$

where v^k is a finite measure on $\mathscr{B}(R_+\times R)$.

Evidently the relations (56)–(59) imply the equality

$$E\int_0^{\tau_k}\psi(s,W_s)\,dA_s=\iint\psi(s,x)v^k(ds,dx) \tag{60}$$

and, hence measure v^k is absolutely continuous with respect to the measure $v_{|A|}$ which is defined by $v_{|A|}(\psi)=E\int_0^\infty\psi(s,W_s)\,dVar(A)_s$. Therefore, it follows from the definition of the measure v_A that $v^k(ds,dx)=h_k(s,x)v_A(ds,dx)$ and from relations (58), (60) we obtain that, for each $k\geq 1$,

$$\iint\psi(s,x)(Lf^n)(s,x)h_k(s,x)\mu(ds,dx)\to\iint\psi(s,x)h_k(s,x)v_A(ds,dx) \tag{61}$$

for every bounded continuous function ψ.

Since

$$\sup_{(s,x)\in D}|f^n(s,x)-f(s,x)|\to 0,\quad n\to\infty \tag{62}$$

on every compact $D\in R_+\times R$ (Proposition 1) and f^n itself belongs to the class $V_\mu^L(\mathrm{loc})$ for every $n\geq 1$, then it is easy to prove (using the diagonal sequence) the existence of a sequence of functions from $C^{1,2}$ with the same properties ((61), (62)), hence, $f\in\hat{V}_\mu^L(\mathrm{loc})$ and the measure $v_A^f(ds,dx)$ is a generalized weak L-derivative of the function f, i.e. $v_L^f=v_A^f$.

Evidently the relations (56), (61) imply the equality

$$E\int_0^\infty\psi(s,W_s)\,dA_s=\iint\psi(s,x)v_L(ds,dx) \tag{63}$$

valid for every bounded continuous function ψ.

Let us show that this equality uniquely determines the process $A\in\mathscr{A}\mathscr{D}_{\mathrm{loc}}$. Let some $B\in\mathscr{A}\mathscr{D}_{\mathrm{loc}}$ also satisfies the relation (63). Then we have that

$$E\int_0^{\tau_k}\psi(s,W_s)\,dA_s = E\int_0^{\tau_k}\psi(s,W_s)\,dB_s \tag{64}$$

for every bounded continuous function ψ.

If $\psi(t,x)$ is a bounded function such that the composite process $(\psi(t,W_t), t\geq 0)$ is right-continuous, then we can approximate this function by bounded continuous functions in the sense of Proposition 1c) and, therefore, one can conclude that the equality (64) is valid for every bounded function ψ for which the process $\psi(t,W_t)$ is right-continuous.

Since for each cadlag process Y there exists a cadlag modification of the process $\psi(t,W_t) = E(Y_t/W_t) = E(Y_t/\mathscr{F}^W_{[t,\infty[})$ (M. Rao [18]), (i.e. for such a function $\psi(s,x) = E(Y_s/W_s = x)$ the equality (64) is true) and since $A, B \in \mathscr{A}\mathscr{D}_{\text{loc}}$ it follows from Lemma 1 and equality (64) that for every adapted bounded cadlag process X

$$\begin{aligned} E\int_0^{\tau_k} X_s\,dA_s &= E\int_0^{\infty} E(I_{(s\leq\tau_k)}X_s/W_s)\,dA_s \\ &= E\int_0^{\infty} E(I_{(s\leq\tau_k)}X_s/W_s)\,dB_s = E\int_0^{\tau_k} X_s\,dB_s \end{aligned}$$

which implies that the processes A and B are indistinguishable.

Thus, it was proved that if the process $(f(t,W_t), t\geq 0)$ is a bounded continuous semimartingale, then the function f belongs to the class $\hat{V}^L_\mu(\text{loc})$. To get rid of the assumption of boundedness of $(f(t,W_t), t > 0)$, consider the family of functions $f_C = \min(\max((-C), f(t,x)), C)$. Obviously, since $(f(t,W_t), t\geq 0)$ is a continuous semimartingale, the process $(f_C(t,W_t), t\geq 0)$ is a continuous bounded semimartingale for every C and the function

$$f^n_C(s,x) = n\int_s^{s+1/n}\int_R f_C(u,y)P(s,x,u,dy)\,du$$

belongs to the class $V^L_\mu(\text{loc})$ for each $n\geq 1, C > 0$. Hence, the function f_C belongs to the class $\hat{V}^L_\mu(\text{loc})$ for every $C > 0$ and, consequently (since f_C converges to f uniformly on every compact), the function f belongs to the same class. □

Remark. *It follows from the proof of this theorem that if f admits a generalized weak L-derivative then there exists a generalized first derivative in x of the function f.*

Proof of Theorem 2. Sufficiency. Let $f \in V^L_\mu(\text{loc})$. Then $f \in \hat{V}^L_\mu(\text{loc}), v^f_L(ds,dx) \ll \mu(ds,dx)$ and from Theorem 1 we have that the process $(f(t,W_t), t\geq 0)$ is a semimartingale with the decomposition

$$f(t,W_t) = f(0,W_0) + \int_0^t f_x(s,W_s)\,dW_s + A^f_t\ , \tag{65}$$

and the process $A^f \in AD_{\text{loc}}$ is uniquely determined by the relation

$$\mathsf{E}\int_0^\infty \psi(s, W_s)\, dA_s^f = \iint \psi(s,x)\nu_L^f(ds, dx) \ , \tag{66}$$

valid for every bounded continuous ψ.

Evidently the process $(\int_0^t (Lf)(s, W_s)\, ds, t \geq 0)$ also satisfies (66) and, hence, the processes $(A_t^f, t \geq 0)$ and $(\int_0^t (Lf)(s, W_s)\, ds, t \geq 0)$ are indistinguishable.

Necessity. Let $f(t, W_t)$ be an Itô process. Then from Theorem 1 $f \in \hat{V}_\mu^L(\mathrm{loc})$, the process $f(t, W_t)$ admits the decomposition (65) and the process $A^f \in AD_{\mathrm{loc}}$ is uniquely determined by the relation (66). Since $dA_s^f \ll ds$, by the corollary of Lemma 1, a measurable function $g(s,x)$ exists such that $A_t^f = \int_0^t g(s, W_s)\, ds$.

Therefore from the relation (66) we have that

$$\iint \psi(s,x)\nu_L^f(ds, dx) = E\int_0^\infty \psi(s, W_s) g(s, W_s)\, ds = \iint \psi(s,x) g(s,x)\rho(s,x)\, ds\, dx$$

for every $\psi \in CB$ which implies that $\nu_L^f(ds, dx) \ll \mu(ds, dx)$ and, hence, $f \in V_\mu^L(\mathrm{loc})$. $\square$

Proof of Corollary 1. Since in this case the localizing sequence of stopping times $(\tau_k, k \geq 1)$ has the form

$$\tau_k = \inf\{t : |W_t| \geq k\} \wedge k \ , \tag{67}$$

using the formula (see e.g. [19]) for the common distribution function of $\sup_{0 \leq s \leq t} |W_s|$ and W_t

$$P\left\{\sup_{0 \leq s \leq t} |W_s| < a, W_t \in [c, d]\right\} = \int_c^d \sum_{i=-\infty}^{\infty} (-1)^i \rho(t, 2ia - x)\, dx$$

for $a \geq 0$, $[c, d] \in [-a, a]$, it is easy to see that

$$\begin{aligned} h_k(s,x) &= E(I_{(\tau_k \geq s)}/W_s = x) \\ &= I_{[0,k]\times[-k,k]}(s,x) P\left(\sup_{0 \leq u \leq s} |W_u| < k/W_s = x\right) \\ &= I_{[0,k]\times[-k,k]}(s,x) \sum_{i=-\infty}^{\infty} \exp\{-4ik(x - ik)\} \end{aligned} \tag{68}$$

and, hence, $h_k(s,x)$ is a bounded continuous function, which is zero if $(s,x) \notin (0,k) \times (-k,k)$ and is strictly positive if $(s,x) \in (0,k) \times (-k,k)$. Therefore it follows from Theorem 1 that if the process $f(t, W_t)$ is a semimartingale of the form (9) then there exists a sequence of functions $(f^n, n \geq 1)$ from $C^{1,2}$ converging (uniformly on every compact) to f and a signed measure ν_L, finite on every compact (hence $\nu_L(ds, dx)$ will be a σ-additive measure according to Theorem 3 (Ch. IV.6) of [17]), such that for every $k \geq 1$

$$\iint \psi(s,x)(Lf^n)(s,x)\mu(ds, dx) \to \iint \psi(s,x)\nu_L(ds, dx)$$

for every bounded continuous function ψ finite on $[0,k] \times [-k,k]$, which is equivalent to $f \in V_\mu^L([0,T] \times [-a,a])$ for any $T > 0$ and $a > 0$. It follows from the proof of the sufficiency part of Theorem 1 that if $f \in V_\mu^L([0,T] \times [-a,a])$ for any $T > 0$ and $a > 0$ then $f(t, W_t)$ will be a semimartingale of the form (9). Proof of the assertion b), evidently, follows from a). □

Proof of Corollary 2. If $f(t,x) = f(x)$ for every $t \geq 0$ and if the process $(f(W_t), t \geq 0)$ is a continuous semimartingale, then, evidently $(f = f(x), x \in R)$ will be a continuous function (really, let $(a_n, n \geq 1)$ be a sequence of real numbers converging to some $a_0 \in R$ and let $\tau_n = \inf\{t : W_t = a_n\}, n \geq 0$. Obviously, $\tau_n \to \tau_0$, $W_{\tau_n} = a_n$ for every $n \geq 0$ and the continuity of the process $(f(W_t), t \geq 0)$ implies that $f(a_n) = f(W_{\tau_n}) \to f(W_{\tau_0}) = f(a_0), n \to \infty$).

Therefore for the regularizations

$$f^n(s,x) = n \int_s^{s+1/n} \int_R f(y) \rho(u - s, y - x)\, dy\, du$$
$$= \int_R f(y) \left(n \int_0^{1/n} \rho(u, y - x) du \right) dy = f^n(x)$$

we have that

$$\sup_{x \in D} |f^n(x) - f(x)| \to 0, \quad n \to \infty$$

on every compact $D \in R$ and besides

$$(Lf^n)(s,x) = f_{xx}^n(x) \ .$$

Since in this case, according to [1], the localizing sequence of stopping times $(\tau_k, k \geq 1)$ of the semimartingale $f(W_t)$ has the form (67), it follows from (68) that

$$h_k(s,x)\rho(s,x) = I_{[0,k]\times[-k,k]}(s,x)\rho_k(s,x) \ ,$$

where $\rho_k(s,x) = \sum_{i=-\infty}^{\infty} \rho(s, x - 2ik)$.

$$\iint \psi(s,x)(Lf^n)(s,x)h_k(s,x)\mu(ds,dx) = \int_{-k}^{k} f_{xx}^n(x) \left(\psi(x) \int_0^k \rho_k(s,x)\, ds \right) dx$$

and $(\int_0^k \rho_k(s,x)\, ds, x \in R)$ is continuous function finite on $[-k,k]$, which is strictly positive on $(-k,k)$. Therefore, it follows from Theorem 1 that for every continuous function φ which is finite on $[-k,k]$ there exists a limit

$$\lim_{n \to \infty} \int f_{xx}^n(x) \varphi(x)\, dx$$

and, hence, there exists a signed measure v_{xx}, bounded on every compact (therefore the restriction of $v_{xx}(dx)$ on every compact is a σ-additive measure according to Theorem 3 (Ch. IV.6) of [17]), such that

$$\int f^n_{xx}(x)\varphi(x)\,dx \to \int \varphi(x)\nu_{xx}(dx)$$

for every continuous function φ with compact support, which is equivalent to (15) and to the representability of the function f as a difference of two convex functions on every compact interval $[-k,k]$. Finally note that the process

$$A^f_t = \int_R L^W(t,x)\nu_{xx}(dx)\ ,$$

evidently, satisfies the relation (7). □

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