

## Random coin tossing

Matthew Harris, Michael Keane

Department of Mathematics and Informatics, Department of Statistics, Probability and Operations Research, Delft University of Technology, Mekelweg 4, 2628 CD Delft, The Netherlands  
e-mail: m.d.harris@twi.tudelft.nl; keane@cw.nl

Received: 20 November 1996 / In revised form: 20 February 1997

**Summary.** A sequence of heads and tails is produced by repeatedly selecting a coin from two possible coins, and tossing it. The second coin is tossed at renewal times in a renewal process, and the first coin is tossed at all other times. The first coin is *fair* ( $\text{Prob}(\text{heads}) = 1/2$ ), and the second coin is known either to be fair, or to have known *bias*  $\theta \in (0, 1]$  ( $\text{Prob}(\text{heads}) = \frac{1}{2}(1 + \theta)$ ). Letting  $u_k := \text{Prob}(\text{There is a renewal at time } k)$ , we show that if  $\sum_{k=0}^{\infty} u_k^2 = \infty$ , we can determine, using only the sequence of heads and tails produced, if the second coin had bias  $\theta$  or 0. If  $\sum_{k=0}^{\infty} u_k^2 < 1 + \frac{1}{\theta^2}$ , we show that this is not possible.

*Mathematics Subject Classification (1991):* 60K35, 60G30, 60G42

### 1 Introduction

The research in this paper developed from the study of problems concerning random walks on scenery. We begin with a description of this to describe the context for this paper.

Fix  $d \geq 1$ , and define a *colouring* of  $\mathbb{Z}^d$  to be a map  $\xi: \mathbb{Z}^d \rightarrow \{0, 1\}$  that assigns a colour (either pink or purple, say) to each point in  $\mathbb{Z}^d$ . Let  $S_n$  be the position at time  $n$  of a simple random walk on  $\mathbb{Z}^d$  starting from the origin at time 0. We call  $\xi(S_n)$  the colour seen by the random walker at time  $n$ , and we call the sequence  $\{\xi(S_n)\}_{n=0}^{\infty}$  the *colour record* of the walk.

Consider now the situation where  $\mathbb{Z}^d$  is coloured with one of two *known* colourings  $\xi$  or  $\eta$ , and the colour record obtained is either  $\{\xi(S_n)\}_{n=0}^{\infty}$  or  $\{\eta(S_n)\}_{n=0}^{\infty}$ . One can ask when one can determine, using only the colour record, which of the colourings  $\xi$  or  $\eta$  was used, with zero probability of error.

If  $\xi(0) \neq \eta(0)$ , then the colouring at hand can be determined by using only the first colour in the colour record. Similarly, if all the neighbours of

the origin are pink in  $\xi$ , and purple in  $\eta$ , then the colouring at hand can be determined using only the second colour in the colour record. To discount such trivial cases, we say  $\xi$  and  $\eta$  are *distinguishable* if one can determine the colouring at hand ( $\xi$  or  $\eta$ ) using only the arbitrary future of the colour record.

It is not hard to show that colourings  $\eta$  obtained from  $\xi$  by translation and/or reflection in a coordinate axis are never distinguishable. Benjamini, and independently, Keane and den Hollander, conjecture that all pairs of colourings  $(\xi, \eta)$  not related as above are distinguishable.

Benjamini and Kesten (1996) study this problem when  $\xi$  and  $\eta$  are chosen randomly from all the possible colourings of  $\mathbb{Z}^d$ , and give results almost sure in the choice of  $\xi$  and  $\eta$ .

Howard (1995, 1996a, 1996b) studies the case of colourings on  $\mathbb{Z}$ . He shows that periodic colourings are distinguishable. He also shows that colourings that are obtained from periodic colourings by altering the colour at finitely many locations are also distinguishable. The random walker returns to the altered regions often enough for the “deformities” to be detected. One can ask if the detection of such deformities is possible in higher dimensions. This brings us to the related model studied in this paper.

A simple random walker on  $\mathbb{Z}^d$  ( $d \geq 1$ ) has two coins, and generates a sequence of heads and tails by tossing a coin before taking each step. If at the origin, the second coin is tossed, and if away from the origin, the first coin is tossed. The steps of the random walker are taken to be independent of the results of the coin tosses, and are just used to determine which coin is to be tossed. It is known that the first coin is fair ( $\text{Prob}(\text{heads}) = 1/2$ ), and the second coin is either fair, or has a *particular* bias  $\theta \in (0, 1]$ , ( $\text{Prob}(\text{heads}) = \frac{1}{2}(1 + \theta)$ ). We ask whether one can determine, just using the sequence of heads and tails obtained, whether or not the second coin had bias  $\theta$ .

The second coin is only tossed when the random walker is at the origin. In the colouring problem above, the finite regions where colourings  $\xi$  and  $\eta$  differ are only observed when the walker returns to these regions.

The simple random walk in three dimensions returns to the origin only finitely often almost surely, and thus the second coin is tossed only finitely often. We could thus not hope to determine if the second coin had bias  $\theta$ . The simple random walk in one or two dimensions does return to the origin infinitely often almost surely, and the second coin is tossed infinitely often almost surely. We prove that if  $d = 1$ , one can determine whether or not the second coin has bias  $\theta$ , but if  $d = 2$ , one cannot.

The rôle of the random walk above was just to determine the times that the second coin is tossed. We generalise this to the case when the second coin is tossed at renewal times in a renewal process. If  $u_k := \text{Prob}$  (there is a renewal at time  $k$ ), we show that if  $\sum_{k=0}^{\infty} u_k^2 = \infty$ , then we *can* determine if the second coin had bias  $\theta$ , and if  $\sum_{k=0}^{\infty} u_k^2 < 1 + \frac{1}{\theta^2}$ , then we *cannot*.

We see that this is almost a dichotomy. We believe that this should actually be a dichotomy, and conjecture that if  $\sum_{k=0}^{\infty} u_k^2 < \infty$ , then it is not possible to determine if the second coin has bias  $\theta$  for all  $\theta \in (0, 1]$ . Our

results only give this for small  $\theta$  ( $\theta \in (0, (\sum u_k^2 - 1)^{-1/2})$ ). The extension of this to  $\theta \in (0, 1]$  remains an interesting open problem.

In all of the above, we have assumed that the second coin is either fair, or has a *particular* bias  $\theta$ . One may ask, given the sequence of heads and tails produced, if one can determine the value of  $\theta \in [0, 1]$ , given no prior knowledge of its value. The methods of Howard (1995, 1996a, 1996b) give that this *is* possible if the second coin is tossed at return times to the origin of a simple random walk on  $\mathbb{Z}$ .

In the next section (Section 2) we introduce the required notation, making rigorous the above discussion, and we state the two main theorems of the paper. In Section 3 we prove Theorem 1, and in Section 4 we give an outline of Howard's methods that can be used to prove Theorem 2. In Section 5 we conclude the paper by stating some open problems.

## 2 Notation and definitions

We represent the space of sequences of coin tosses by  $X := \{-1, +1\}^{\mathbb{N}}$ . We let  $X_n : X \rightarrow \{-1, +1\}$  be the random variable defined by  $X_n(x) := x_n$  for  $x = (x_0, x_1, \dots) \in X$ , and let  $\mathcal{F} := \sigma(X_0, X_1, \dots)$ . The random variable  $X_n$  gives the  $(n+1)$ st coordinate of  $x \in X$  – the result of the  $(n+1)$ st coin toss.

We will define measures on  $(X, \mathcal{F})$ . It will be sufficient to define the measures on cylinder sets

$$[x_0, x_1, \dots, x_n] := \bigcap_{i=0}^n [X_i = x_i] \quad (1)$$

for  $n = 0, 1, \dots$ . This is because, for fixed  $n$ , the cylinder sets above generate the  $\sigma$ -algebra  $\mathcal{F}_n := \sigma(X_0, \dots, X_n)$ , and the sequence  $\mathcal{F}_{-1} := \{\emptyset, X\}$ ,  $\mathcal{F}_0, \mathcal{F}_1, \dots$  is a sequence of  $\sigma$ -algebras increasing to  $\mathcal{F}$ .

We will define measures  $\mu_\theta$  indexed by a parameter  $\theta \in [0, 1]$ , called the *bias* of the second coin. As a first step, we introduce measures  $\mu_{\theta, \delta}$  on  $(X, \mathcal{F})$ , for  $\delta = (\delta_0, \delta_1, \dots) \in \Delta := \{0, 1\}^{\mathbb{N}}$ . Define

$$\mu_{\theta, \delta}([x_0, \dots, x_n]) := \prod_{k=0}^n \frac{1}{2} (1 + \theta x_k \delta_k) . \quad (2)$$

To understand this definition, we see that

$$\mu_{\theta, \delta}([x_0, \dots, x_n]) = \mu_{\theta, \delta}([X_0 = x_0]) \dots \mu_{\theta, \delta}([X_n = x_n]) \quad (3)$$

is a product measure. If  $\delta_k = 0$ , then  $\mu_{\theta, \delta}([X_k = +1]) = \mu_{\theta, \delta}([X_k = -1]) = \frac{1}{2}$ , and we say that the fair coin was used for the  $(k+1)$ st toss. If  $\delta_k = 1$ , then  $\mu_{\theta, \delta}([X_k = +1]) = \frac{1}{2}(1 + \theta) = 1 - \mu_{\theta, \delta}([X_k = -1])$ , and we say that a coin with bias  $\theta$  was used for the  $(k+1)$ st toss. The sequence  $\delta \in \Delta$  determines the times when the second coin with bias  $\theta$  was tossed.

To define  $\mu_\theta$  we wish to randomise  $\delta \in \Delta$ . We let  $\mathcal{G}$  be the  $\sigma$ -algebra on  $\Delta$  generated by cylinder sets on  $\Delta$  (defined as for  $X$ ), and we define a measure  $P$  on  $(\Delta, \mathcal{G})$  by defining it on cylinder sets in  $\mathcal{G}$ .

Let  $f = (f_1, f_2, \dots)$  be a probability vector, and define the sequence  $u = (u_0, u_1, \dots)$  as follows:

$$\begin{aligned} u_0 &:= 1 \\ u_n &:= \sum_{k=1}^n f_k u_{n-k} \quad n \geq 1 . \end{aligned} \quad (4)$$

We take  $f$  to be the probability distribution for the first renewal time in a renewal process, and then  $u_n$  is then the probability that there is a renewal at time  $n$ . If, for example, the renewal times are return times to the origin of a simple random walk on  $\mathbb{Z}$ , then

$$\begin{aligned} u_{2n} &= \binom{2n}{n} \left(\frac{1}{2}\right)^{2n} \\ f_{2n} &= u_{2n} - u_{2n-2} \\ u_{2n-1} &= f_{2n-1} = 0 \end{aligned} \quad (5)$$

for  $n \geq 1$ .

We now define the measure  $P$  on  $(\Delta, \mathcal{G})$  as follows: given a set  $n_1 < n_2 < \dots$  of positive integers,

$$P([\Delta_0 = \Delta_{n_1} = \dots \Delta_{n_k} = 1]) := u_{n_1} u_{n_2 - n_1} \dots u_{n_k - n_{k-1}} . \quad (6)$$

The cylinder set in (6) is the event that there are renewals at times  $0, n_1, \dots, n_k$  (and possibly at other times too).

We now let  $\mu_\theta$  be the measure  $\mu_{\theta, \delta}$  where  $\delta$  is chosen from  $\Delta$  randomly according to  $P$ . More precisely, we define

$$\mu_\theta([x_0, \dots, x_n]) := \int_{\Delta} \mu_{\theta, \delta}([x_0, \dots, x_n]) dP(\delta) \quad (7)$$

$$= \int_{\Delta} \prod_{k=0}^n \frac{1}{2} (1 + \theta x_k \Delta_k) dP . \quad (8)$$

A sequence of heads and tails generated as described in the introduction using a second coin with bias  $\theta > 0$  corresponds to drawing an  $x \in X$  according to measure  $\mu_\theta$ . Using a *fair* ( $\theta = 0$ ) second coin corresponds to drawing an  $x \in X$  according to the measure  $\mu_0$ . For  $\theta > 0$  fixed, we say that the second coin can be *determined* if the measures  $\mu_\theta$  and  $\mu_0$  are singular with respect to each other, written  $\mu_\theta \perp \mu_0$ . There then exists an  $A \in \mathcal{F}$  such that  $\mu_\theta(A) = 0$  and  $\mu_0(A^c) = 0$ , ( $A^c$  is the complement of  $A$ ) and the generating measure of  $x \in X$  can be determined with zero probability of error by determining whether or not  $x$  is in  $A$ .

If, for all  $B \in \mathcal{F}$ ,  $\mu_0(B) = 0 \Rightarrow \mu_\theta(B) = 0$ , we say that  $\mu_\theta$  is absolutely continuous with respect to  $\mu_0$ , and we write  $\mu_\theta \ll \mu_0$ . If  $\mu_\theta \ll \mu_0$ , then a set  $A$  as above cannot exist, and the second coin can not be determined with zero probability of error.

We can now state the main theorem of the paper.

**Theorem 1** For  $\theta \in (0, 1]$ ,

1. If  $\sum_{k=0}^{\infty} u_k^2 = \infty$ , then  $\mu_\theta \perp \mu_0$ .
2. If  $\sum_{k=0}^{\infty} u_k^2 < 1 + \frac{1}{\theta^2}$ , then  $\mu_\theta \ll \mu_0$ .

Note that if  $\sum_{k=0}^{\infty} u_k^2 < \infty$ , then  $\mu_\theta \ll \mu_0$  for all  $\theta \in (0, (\sum_{k=0}^{\infty} u_k^2 - 1)^{-1/2})$ , and the second coin can not be determined with zero probability of error if its bias is too small. Also, note that if  $\sum_{k=0}^{\infty} u_k^2 < 2$ , then  $\mu_\theta \ll \mu_0$  for all  $\theta \in (0, 1]$ .

Theorem 1 can be compared to the Kakutani (1948) dichotomy concerning two measures  $\alpha$  and  $\beta$  on  $X$  that make the coordinates  $X_n$  independent. In this case either  $\alpha \perp \beta$  or  $\alpha \ll \beta$ . Under  $\mu_\theta$ , the coordinates  $X_n$  are dependent, and Theorem 1 can be thought of as being a relative of Kakutani's result.

If the renewal times are taken to be return times to the origin of a simple random walk on  $\mathbb{Z}^2$ , then we will see that  $\sum_{k=0}^{\infty} u_k^2 < 2$ , and Theorem 1 gives us

**Corollary 1** If the renewals are returns to the origin of a simple random walk on  $\mathbb{Z}^d$ , then, for all  $\theta \in (0, 1]$ :

1. If  $d = 1$ , then  $\mu_\theta \perp \mu_0$ ,
2. If  $d \geq 2$  then  $\mu_\theta \ll \mu_0$ .

Given  $x \in X$  drawn according to either  $\mu_\theta$  ( $\theta$  known), or  $\mu_0$ , Theorem 1 tells us when the underlying measure can be determined from  $x$  with zero probability of error. One may ask for a sequence  $x \in X$  drawn according to the measure  $\mu_\theta$  for  $\theta \in [0, 1]$  unknown, can the value of  $\theta$  be determined from  $x$  with zero probability of error? We answer this for the case when the renewal times are return times to the origin of a simple random walk on  $\mathbb{Z}^d$ . From Corollary 1, we see that the answer to the above question is clearly 'no' for the case when  $d \geq 2$ . Theorem 2 tells us that the answer is 'yes' for  $d = 1$ .

**Theorem 2** If the renewals are returns to the origin of a simple random walk on  $\mathbb{Z}$ , then there exists an  $f : X \rightarrow [0, 1]$ , such that for all  $\theta \in [0, 1]$ ,  $f$  is  $\mu_\theta$  measurable, and  $\mu_\theta([f = \theta]) = 1$ .

Theorem 2 can be proved using methods in Howard (1995, 1996a, 1996b), and we only outline the proof in this paper.

### 3 Proof of Theorem 1

We prove Theorem 1 using a martingale argument. Define

$$\rho_n(x) := \frac{\mu_\theta([x_0, \dots, x_n])}{\mu_0([x_0, \dots, x_n])} \quad (9)$$

for  $X \ni x = (x_0, x_1, \dots)$ , so that

$$\mu_\theta(A) = \int_A \rho_n(x) d\mu_0 \quad (10)$$

for  $A \in \mathcal{F}_n$ . We see that  $\rho_n$  is just the Radon-Nikodým derivative of  $\mu_\theta$  with respect to  $\mu_0$  restricted to  $\mathcal{F}_n$ . Now  $\rho_n$  is a positive  $\mu_0$  martingale with filtration  $\mathcal{F}_n$  and has a finite limit  $\rho$   $\mu_0$  almost surely. To define  $\rho$  on the whole of  $X$ , define

$$\rho(x) := \limsup_{n \rightarrow \infty} \rho_n(x) . \quad (11)$$

The Lebesgue decomposition of  $\mu_\theta$  with respect to  $\mu_0$  can be written

$$\mu_\theta(A) = \mu_\theta^c(A) + \mu_\theta^\perp(A) \quad (12)$$

for  $A \in \mathcal{F}$ , with

$$\mu_\theta^c(A) := \int_A \rho d\mu_0 \quad (13)$$

$$\mu_\theta^\perp(A) := \mu_\theta(A \cap [\rho = \infty]) . \quad (14)$$

It is clear that  $\mu_\theta^c \ll \mu_0$  and, as  $\mu_0([\rho = \infty]) = 0$ , we have  $\mu_\theta^\perp \perp \mu_0$ .

We prove the first part of Theorem 1 by showing that  $\mu_0([\rho = 0]) = 1$ . If this is the case,  $\mu_\theta = \mu_\theta^c$  and  $\mu_\theta \perp \mu_0$ .

To prove the second part of Theorem 1, we show that  $\rho_n$  is uniformly integrable. If this is the case, then

$$\mu_\theta(A) = \lim_{n \rightarrow \infty} \int_A \rho_n d\mu_0 = \int_A \rho d\mu_0 \quad (15)$$

for  $A$  in the  $\pi$ -system  $\bigcup_n \mathcal{F}_n$ , and then for all  $A \in \mathcal{F}$ . This means that  $\mu_\theta = \mu_\theta^c$  and  $\mu_\theta \ll \mu_0$ .

### 3.1 Frequent renewals

In this section we prove the first part of Theorem 1. We begin by proving the following preliminary lemma.

**Lemma 1**  $\mu_0([\rho = 0]) = 0$  or 1.

*Proof* To prove Lemma 1 we show that the event  $[\rho = 0]$  is a tail event, and use the Kolmogorov 0-1 Law. Define

$$\mathcal{F}'_m := \sigma(X_m, X_{m+1}, \dots) \quad (16)$$

and let

$$\mathcal{T} := \bigcap_{m=0}^{\infty} \mathcal{F}'_m \quad (17)$$

be the tail  $\sigma$ -algebra. We show that the event  $[\rho = 0] \in \mathcal{T}$ . Consider

$$x = (x_0, x_1, \dots, x_{n-1}, x_n, x_{n+1}, \dots) \in [\rho = 0]$$

and define

$$y := (x_0, x_1, \dots, x_{n-1}, -x_n, x_{n+1}, \dots) .$$

Then

$$\begin{aligned} \frac{1-\theta}{1+\theta} \limsup_{m \rightarrow \infty} \int_{\Delta} \prod_{k=0}^m (1 + \theta x_k \Delta_k) dP &\leq \limsup_{m \rightarrow \infty} \int_{\Delta} \prod_{k=0}^m (1 + \theta y_k \Delta_k) dP \\ &\leq \frac{1+\theta}{1-\theta} \limsup_{m \rightarrow \infty} \int_{\Delta} \prod_{k=0}^m (1 + \theta x_k \Delta_k) dP \end{aligned} \quad (18)$$

Thus,  $\rho(y) = 0$  if and only if  $\rho(x) = 0$ , and the event  $[\rho = 0]$  is independent of the  $n$ th coordinate. This means that  $[\rho = 0] \in \bigcap_{m=0}^{\infty} \mathcal{F}'_m = \mathcal{F}$ . As  $X_0, X_1, \dots$  are independent under  $\mu_0$ , the Kolmogorov 0-1 Law proves the lemma.  $\square$

Now consider the integral  $A(n)$ :

$$\begin{aligned} A(n) &:= \int_X \rho_n(x) \rho_n(-x) d\mu_0(x) \\ &= \int_X \int_{\Delta} \int_{\Delta'} \prod_{k=0}^n (1 + \theta x_k \Delta_k) (1 - \theta x_k \Delta'_k) dP' dP d\mu_0(x) \\ &= \int_{\Delta} \int_{\Delta'} \prod_{k=0}^n \int_X (1 + \theta x_k \Delta_k - \theta x_k \Delta'_k - \theta^2 \Delta_k \Delta'_k) d\mu_0(x) dP' dP \\ &= \int_{\Delta} \int_{\Delta'} \prod_{k=0}^n (1 - \theta^2 \Delta_k \Delta'_k) dP' dP . \end{aligned} \quad (19)$$

The product space  $(\Delta \times \Delta, \mathcal{G} \times \mathcal{G}, P \times P)$  is represented as  $((\Delta, \Delta'), (\mathcal{G}, \mathcal{G}'), (P, P'))$ , so the probability space  $(\Delta', \mathcal{G}', P')$  above is just a copy of  $(\Delta, \mathcal{G}, P)$ . The third line above follows from the independence of the  $x$  coordinates under the  $\mu_0$  measure and the fact that  $x_k^2 = 1$  for all  $k$ . The last line is obtained by observing that  $\int_X x_k d\mu_0(x) = 0$ .

We observe that the simultaneous renewal times of two independent renewal processes are renewal times of a new renewal process, and then

$$P \times P([\Delta_k \Delta'_k = 0 \forall k > 0]) = \begin{cases} (\sum_{k=0}^{\infty} u_k^2)^{-1} & \text{if } \sum u_k^2 < \infty \\ 0 & \text{if } \sum u_k^2 = \infty . \end{cases} \quad (20)$$

(See, for example, Kingman (1972), Theorem 1.5.) If  $\sum_{k=0}^{\infty} u_k^2 = \infty$ , we see  $P \times P([\Delta_k \Delta'_k = 1 \text{ for some } k > 0]) = 1$ , and as  $P \times P$  represents a renewal process,

$$P \times P([\Delta_k \Delta'_k = 1 \text{ for infinitely many } k]) = 1 . \quad (21)$$

and

$$P \times P\left(\left[\prod_{k=0}^n (1 - \theta^2 \Delta_k \Delta'_k) \rightarrow 0\right]\right) = 1 . \quad (22)$$

As  $\prod_{k=0}^n (1 - \theta^2 \Delta_k \Delta'_k)$  is bounded from above by 1, it follows that  $A(n) \rightarrow 0$ . Now, from Fatou's Lemma, we obtain

$$\begin{aligned} 0 &= \liminf_n \int_X \rho_n(x) \rho_n(-x) d\mu_0(x) \\ &\geq \int_X \liminf_n \rho_n(x) \rho_n(-x) d\mu_0(x) \\ &= \int_X \rho(x) \rho(-x) d\mu_0(x) . \end{aligned} \quad (23)$$

The last equality holds as we know that  $\rho_n \rightarrow \rho$   $\mu_0$  almost surely. As  $\rho_n$  is a positive martingale, this implies

$$\int_X \rho(x) \rho(-x) d\mu_0(x) = 0 . \quad (24)$$

It is now clearly not the case that  $\mu_0([\rho = 0]) = 0$ , so, by Lemma 1, we must have that  $\mu_0([\rho = 0]) = 1$ , and  $\mu_\theta \perp \mu_0$ .  $\square$

*Remark 1* The fact that the  $P$  originates from a renewal process has not been used. All that was used was

$$P \times P([\Delta_k \Delta'_k = 1 \text{ for infinitely many } k]) = 1 . \quad (25)$$

If  $P$  is a measure on  $(\Delta, \mathcal{G})$  that satisfies (25), then  $\mu_\theta \perp \mu_0$ .

### 3.2 Infrequent renewals

In this section we prove the second part of Theorem 1. We show that if  $\sum_{k=0}^\infty u_k^2 < 1 + \frac{1}{\theta^2}$ , then  $\rho_n$  is an  $L_2$ -bounded martingale, and therefore uniformly integrable. This, as already mentioned, will give us that  $\mu_\theta \ll \mu_0$ .

Let  $R_n$  be the  $L_2$ -norm of  $\rho_n$ . Analogous to (19), we obtain

$$\begin{aligned} R_n &:= \int_X \rho_n^2(x) d\mu_0(x) \\ &= \int_\Delta \int_{\Delta'} \prod_{k=0}^n (1 + \theta^2 \Delta_k \Delta'_k) dP' dP . \end{aligned} \quad (26)$$

Note that the terms in the above product have value either 1 or  $1 + \theta^2$ , the latter when  $\Delta_k \Delta'_k = 1$ . If  $\sum_{k=0}^\infty u_k^2 < \infty$ , then (20) gives that

$$\alpha := P \times P([\Delta_k \Delta'_k = 1 \text{ for some } k > 0]) = 1 - \left( \sum_{k=0}^\infty u_k^2 \right)^{-1} < 1 , \quad (27)$$

and, as  $P \times P$  represents a renewal process, we see that

$$P \times P([\Delta_k \Delta'_k = 1 \text{ for exactly } m \text{ values of } k > 0]) = \alpha^m (1 - \alpha) . \quad (28)$$

Clearly

$$R_\infty := \int_{\Delta} \int_{\Delta'} \prod_{k=0}^{\infty} (1 + \theta^2 \Delta_k \Delta'_k) dP' dP \quad (29)$$

is an upper bound for  $R_n$ ,  $n \geq 0$ , and

$$\begin{aligned} R_\infty &= \sum_{m=0}^{\infty} (1 + \theta^2)^{m+1} P \times P(\Delta_k \Delta'_k = 1 \text{ for exactly } m \text{ values of } k > 0) \\ &= \sum_{m=0}^{\infty} (1 + \theta^2)^m \alpha^m (1 - \alpha) (1 + \theta^2) \\ &< \infty \end{aligned} \quad (30)$$

if  $\alpha(1 + \theta^2) < 1$ . We note the extra factor  $(1 + \theta^2)$  in the above is due to the fact that  $\Delta_0 \Delta'_0 = 1$  always. From the definition of  $\alpha$  (equation (28)), we see that the condition that  $\alpha(1 + \theta^2) < 1$  is equivalent to the condition on  $\sum u_k^2$  given in the theorem. We have thus shown that in this case,  $\rho_n$  is an  $L_2$ -bounded martingale, and hence  $\mu_\theta \ll \mu_0$ .  $\square$

### 3.3 The simple random walk special case

In this section we prove Corollary 1. For  $u_n$  as defined in (5) (the renewal times are taken to be return times to the origin of a simple random walk on  $\mathbb{Z}$ ), we see that  $\sum_{k=0}^{\infty} u_k^2 = \infty$ , and Theorem 1 proves the first part of Corollary 1.

If the renewal times are taken to be return times to the origin of the simple random walk on  $\mathbb{Z}^2$ , it is standard (see Feller (1957) page 328) that

$$\begin{aligned} u_{2n} &= \left( \binom{2n}{n} \left(\frac{1}{2}\right)^{2n} \right)^2 \\ u_{2n+1} &= 0 \end{aligned} \quad (31)$$

for  $n \geq 0$ . Then

$$\begin{aligned} \sum_{k=0}^{\infty} u_k^2 &= \sum_{k=0}^{\infty} \left( \binom{2n}{n} \left(\frac{1}{2}\right)^{2n} \right)^4 \\ &\leq 1 + \sum_{k=1}^{\infty} \frac{1}{\pi^2 n^2} \\ &= \frac{7}{6} . \end{aligned} \quad (32)$$

The inequality is obtained from bounds on  $n!$  obtained by Robbins (1955) (see also Feller (1957)). Thus  $\sum_{k=0}^{\infty} u_k^2 < 2$ , and as noted after the statement of Theorem 0, this implies that  $\mu_\theta \ll \mu_0$ . The value of  $\sum_{k=0}^{\infty} u_k^2$  is clearly not larger than  $7/6$  for the cases when the renewal times are taken to return times

to the origin of the simple random walk on  $\mathbb{Z}^d$  for  $d \geq 3$ , completing the proof of the corollary.  $\square$

#### 4 Calculation of $\theta$

In this section we sketch the proof of Theorem 2. The methods used are those of Howard (1995, 1996a, 1996b).

We consider the case when the renewals are return times to the origin of a simple random walk on  $\mathbb{Z}$ . Thus  $u_n$  is as in (5). We define

$$U_n := \sum_{k=1}^n u_k \quad (33)$$

to be the expected number of returns to the origin from time 1 up to and including time  $n$ . This is also the expected number of times that the second coin is tossed between these times. Thus

$$N_n(x) := \frac{1}{U_n} \sum_{k=1}^n x_k \quad (34)$$

has expectation  $\theta$  under measure  $\mu_\theta$ . This is true for all  $\theta \in [0, 1]$ .

It can be shown that  $\liminf_{n \rightarrow \infty} \frac{U_n}{\sqrt{n}} > 0$ , a property of the  $u_n$  in (5), implies that  $N_n(x)$  is almost independent of  $N_m(x)$  for  $n$  much larger than  $m$ , and that  $N_n(x)$  has bounded variance under  $\mu_\theta$ , uniform in  $n$  and  $\theta$ . A dependent weak law of large numbers gives the existence of a sequence  $a_k$ ,  $a_1 < a_2 < \dots$ , such that

$$M_n(x) := \frac{1}{n} \sum_{i=1}^n N_{a_i}(x) \rightarrow \theta \quad (35)$$

in probability. The same sequence  $a_i$  works for all  $\theta \in [0, 1]$ . Taking a further subsequence gives almost sure convergence.  $\square$

As  $\liminf_{n \rightarrow \infty} \frac{U_n}{\sqrt{n}} > 0$  does not follow from  $\sum_{n=0}^{\infty} u_n^2 = \infty$ , we have not shown that  $\theta$  can be determined with zero probability of error more generally when  $\sum_{n=0}^{\infty} u_n^2 = \infty$ . It remains an interesting question as to whether or not this can be done.

#### 5 Questions

Theorem 1 is unsatisfactory in the sense that the situation where  $1 + \frac{1}{\theta^2} \leq \sum_{k=0}^{\infty} u_k^2 < \infty$  is not covered. We believe that Theorem 1 should actually be a dichotomy, and the second part should be replaced by

$$\text{if } \sum_{k=0}^{\infty} u_k^2 < \infty, \text{ then } \mu_\theta \ll \mu_0 . \quad (36)$$

To show that  $\mu_\theta \ll \mu_0$ , we showed that  $\rho_n$  is an  $L_2$ -bounded martingale. It is conceivable that for  $\theta \in (0, 1]$  such that  $1 + \frac{1}{\theta^2} \leq \sum_{k=0}^{\infty} u_k^2 < \infty$ ,  $\rho_n$  is  $L_{1+\epsilon}$  bounded for some  $\epsilon \in (0, 1]$ , and it would also follow that  $\mu_\theta \ll \mu_0$ . We have not been able to show this.

We could consider the situation in which we know the second coin has bias either  $\theta$  or  $\varphi \in (0, 1]$ . We may then ask when  $\mu_\theta \ll \mu_\varphi$ , and when  $\mu_\theta \perp \mu_\varphi$ . More generally, we could ask when the bias of the second coin can be determined, with zero probability of error, from the sequence of heads and tails with no prior knowledge about its bias. This would generalise Theorem 2.

## 6 References

- [1] Benjamini, I., Kesten, H.: Distinguishing sceneries by observing the scenery along a random walk path. *J. Anal. Math.*, **69**, 97–135 (1996)
- [2] Feller, W.: *An introduction to Probability: Theory and Examples*. New York: Wiley (1957)
- [3] Howard, D.: The orthogonality of measures induced by random walks with scenery. PhD thesis, Courant Inst. Math. Sciences, New York, NY. (1995)
- [4] Howard, D.: Detecting defects in periodic scenery by random walks on  $\mathbb{Z}$ , *Random Structures Algorithms*, **8** no 1, 59–74 (1996a)
- [5] Howard, D.: Orthogonality of measures induced by random walks with scenery *Combin. Probab. Comput.*, **5** no 3, 247–256 (1996b)
- [6] Kakutani, S.: On equivalence of infinite product measures. *Ann. of Math. II. Ser* **49**, 214–224 (1948)
- [7] Kingman, J.: *Regenerative Phenomena*, John Wiley and Sons (1972)
- [8] Robbins, H.: A remark on Stirling's formula, *American Mathematical Monthly*, **62**, 26–29 (1955)