Probab. Theory Relat. Fields 108, 559–571 (1997)

**Probability Theory** and Related Fields


# Regenerative embedding of Markov sets

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Received: 12 April 1996 / In revised form: 12 March 1997

**Summary.** Given a closed Markov (i.e. regenerative) set in $[0,\infty)$, we characterize the laws of the Markov sets which are regeneratively embedded into the latter. Typically, let $\Phi^{(1)}$ and $\Phi^{(2)}$ be two Laplace exponents corresponding to two regenerative laws, and $M^{(2)}$ a Markov set with exponent $\Phi^{(2)}$. There exists a Markov set $M^{(1)}$ with exponent $\Phi^{(1)}$ which is regeneratively embedded into $M^{(2)}$ if and only if $\Phi^{(1)}/\Phi^{(2)}$ is a completely monotone function. Several examples and applications are discussed.

*Mathematics Subject Classification (1991):* 60D05, 60J30

## 1 Introduction

Loosely speaking, a Markov (i.e. regenerative) set $M$ is a random closed subset of $[0,\infty)$ such that the right-hand portion of $M$ as viewed from a stopping time $T$ in this set, is independent of the left-hand portion and has the same distribution as $M$. Alternatively, a Markov set can be thought of as the closure of the set of times when some strong Markov process visits a fixed point in the state space. The precise definition that we use here is that of Maisonneuve [21], which extends the previous ones by Krylov and Yushkevich [18] and Hoffmann-Jørgensen [14]. See also Kingman [16] and Maisonneuve [19].

The motivation for the present work stems from the following tantalizing problem: Given two independent Markov sets, $M$ and $M'$, it is easily seen that their intersection $M \cap M'$ is again a Markov set. Can one describe its distribution explicitly in terms of that of $M$ and $M'$? We refer to Hawkes [12], Fitzsimmons et al. [6] and Molchanov [23] for advances on this problem.

*Key words:* Markov set, Regenerative property, Skorohod embedding, Subordinator.

A natural approach is to observe first that, roughly speaking, $M^{(1)} = M \cap M'$ is 'regeneratively embedded' into $M^{(2)} = M$ (respectively, $M^{(2)} = M'$), in the sense that the right-hand portions of the pair $(M^{(1)}, M^{(2)})$ as viewed from a stopping time $T$ in $M^{(1)}$, are independent of the left-hand portions and have the same distribution as $(M^{(1)}, M^{(2)})$. (This notion can be viewed as a special case of the general concept of regenerative systems developed by Maisonneuve [20].) This suggests the question of characterizing the class of Markov sets that can be regeneratively embedded into a given Markov set, which is the main purpose of this paper.

This question also naturally arises in the following more general setting: Suppose $Y^{(1)}$ is a strong Markov process, so the closure $M^{(1)}$ of the set of times when $Y^{(1)}$ visits a fixed point, say $y$, is a Markov set. Then let $f$ be a Markov function for $Y^{(1)}$, in the sense that $Y^{(2)} = f(Y^{(1)})$ is again a strong Markov process. The closure $M^{(2)}$ of the set of times when $Y^{(2)} = f(y)$ is also a Markov set, and $M^{(1)}$ is regeneratively embedded into $M^{(2)}$. For instance, $Y$ may be a linear Brownian motion, $y = 0$ and $f(x) = \exp\{2i\pi x\}$, so that $M^{(2)}$ is the set of times when $Y$ takes integer values. To make the connection with intersection of independent Markov sets, suppose $Y$ and $Y'$ are independent Markov processes, put $Y^{(1)} = (Y, Y')$ and let $f$ be the first projection map.

We now describe our main result. It is well-known that the distribution of a Markov set $M$ is characterized by a function $\Phi\colon [0,\infty) \to [0,\infty)$ which is the Laplace exponent of a subordinator $X$. (The relation between $M$ and $X$ is that the closed range of $X$ has the same distribution as $M$.) Given two Laplace exponents, $\Phi^{(1)}$ and $\Phi^{(2)}$, there exists a pair of Markov sets $M^{(1)}$ and $M^{(2)}$ with Laplace exponents $\Phi^{(1)}$ and $\Phi^{(2)}$, respectively, such that $M^{(1)}$ is regeneratively embedded into $M^{(2)}$ if and only if the ratio $\Phi^{(1)}/\Phi^{(2)}$ is a completely monotone function. An alternative equivalent condition is that any excessive measure for the subordinator $X^{(2)}$ is also excessive for $X^{(1)}$.

The Laplace exponent of a Markov set can be expressed via the celebrated Lévy-Khintchine formula, in terms of a drift coefficient $\mathrm{d} \geq 0$ and a measure $\Pi$ on $(0,\infty]$, called the Lévy measure. Loosely speaking, $\Pi$ describes the distribution of the gaps of $M$. The drift coefficient is zero or positive according as the Lebesgue measure of $M$ is zero or positive a.s.; one says that $M$ is light or heavy accordingly. It then seems natural to address the following question: Given a light Markov set $M$, is there a heavy Markov set $N$ with the same Lévy measure as $M$, such that $M$ is regeneratively embedded into $N$? We give a necessary and sufficient condition in terms of the Laplace exponent $\Phi$ for the positive answer. Somewhat surprisingly, many natural examples of Markov sets have this rather unexpected property.

This paper is organized as follows. Preliminaries on subordinators, regenerative laws and regenerative embedding are developed in Section 2. The main results together with several examples are presented in Section 3. Proofs are given in Section 4.

## 2 Preliminaries

### 2.1 Subordinators

Let $X = (X_t : t \geq 0)$ be a -generic- subordinator. That is $X$ is a right-continuous process with values in the extended half-line $[0, \infty]$, started at $X_0 = 0$ and with independent and stationary increments. More precisely, if $\zeta = \inf\{t \geq 0 : X_t = \infty\}$ stands for the lifetime, then for every $s \geq 0$, given $\zeta > s$, the shifted process $(X_{t+s} - X_s : t \geq 0)$ is independent of $(X_t : 0 \leq t \leq s)$ and has the same law as $X$. The degenerate case when $X \equiv 0$ will be implicitly excluded in the sequel.

The law of a subordinator is specified by its Laplace exponent $\Phi$, via the identity

$$\mathbb{E}(\exp -\lambda X_t) = \exp -t\Phi(\lambda), \qquad t \geq 0, \quad \lambda > 0 \ ,$$

with the convention $\mathrm{e}^{-\infty} = 0$. The Lévy-Khintchine formula states that

$$\Phi(\lambda)/\lambda = \mathrm{d} + \int_0^\infty \mathrm{e}^{-\lambda x}\overline{\Pi}(x)\, dx \ ,$$

where $\mathrm{d} \geq 0$ is the drift coefficient and $\overline{\Pi}(x) = \Pi((x, \infty])$ the tail of the Lévy measure.

The *renewal measure* $U$ is the occupation measure of $X$, viz.

$$U(A) = \mathbb{E}\left(\int_0^\infty \mathbf{1}_{\{X_t \in A\}}\, dt\right) \ , \qquad A \in \mathscr{B}([0, \infty))$$

its Laplace transform is given by

$$\int_0^\infty \mathrm{e}^{-\lambda x} U(dx) = 1/\Phi(\lambda), \qquad \lambda > 0 \ .$$

A Radon measure $v$ on $[0, \infty)$ is called a *potential* if it can be expressed in the form $v = \mu * U$ for some Radon measure $\mu$ on $[0, \infty)$. In that case, we write $v \in Pot$. One says that $v$ is *excessive* and write $v \in Exc$ if for every real number $t \geq 0$ and every measurable function $f \geq 0$

$$\mathbb{E}^v(f(X_t), t < \zeta) \leq v(f) \ ,$$

where $\mathbb{P}^v$ refers to the law of the subordinator started with initial distribution $v$. Because $\lim_{t\to\infty} \mathbb{E}^v(f(X_t)) = 0$ for every bounded measurable function $f$ with compact support, it follows from the Riesz decomposition (cf. Theorem 16.7 in Berg and Forst [1], and also Getoor [8], chapter 2) that the notions of excessive measure and potential measure coincide for subordinators, i.e.

$$Pot = Exc = \{v = \mu * U, v \text{ and } \mu \text{ Radon measures on } [0, \infty)\} \ . \qquad (1)$$

### 2.2 Regenerative laws

This subsection is essentially an excerpt from Maisonneuve [21] and Fitzsimmons et al. [6], to whom we refer for a complete account. Denote by $\Omega^0$

the class of the closed subsets $\omega^0$ of $[0,\infty)$, endowed with the topology of Matheron [22]. In particular, $\Omega^0$ is a compact metrisable space. Recall that given a metric $\rho$ on $[0,\infty)$

$$\lim \omega_n^0 = \omega^0 \text{ in } \Omega^0 \Longleftrightarrow \lim \rho(t,\omega_n^0) = \rho(t,\omega^0) \quad \text{for every} \quad t \geq 0 \ , \tag{2}$$

see (1.2.5) in [22].

We write $M^0$ for the identity mapping on $\Omega^0$, i.e. $M^0(\omega^0) = \omega^0$. For each $t \in [0,\infty)$, define

$$d_t(\omega^0) = \inf\{\omega^0 \cap (t,\infty)\}, \quad \tau_t(\omega^0) = \{s \geq 0 : s + t \in \omega^0\} \ .$$

It is convenient to omit $\omega^0$ from the notation in the sequel. We denote by $\mathscr{G}_t^0$ the sigma-field generated by $d_t$ and $M^0 \cap [0,d_t]$. (Notice that $d_t < \infty$ if and only if $d_t$ is the maximum of $[0,d_t] \cap M^0$.) It is easily seen that $\mathscr{G}_t^0$ coincides with the sigma-field generated by $(d_s : 0 \leq s \leq t)$.

One says that a probability measure $\mathscr{Q}$ on $\Omega^0$ is a regenerative law if for each $t \in [0,\infty)$ and each bounded measurable function $f$: $\Omega^0 \to \mathbb{R}$:

$$\mathscr{Q}(f(M^0 \circ \tau_{d_t}) \mid \mathscr{G}_t^0) = \mathscr{Q}(f(M^0)) \qquad \text{on} \quad \{d_t < \infty\} \ .$$

First, we lift from [21] the following strong regenerative property.

**Lemma 1** *Denote by $\underline{M}^0$ the subset of $M^0$ which consists of points which either are isolated in $M^0$ or are right-accumulation points in $M^0$. Let $\mathscr{Q}$ be a regenerative law and $T$ an $(\mathscr{G}_{s+}^0)$-stopping time such that $T \in \underline{M}^0$ a.s. on $\{T < \infty\}$. Then*

$$\mathscr{Q}(f(M^0 \circ \tau_T) \mid \mathscr{G}_{T+}^0) = \mathscr{Q}(f(M^0)) \qquad \text{on} \quad \{T < \infty\}$$

*for every bounded measurable function $f$: $\Omega^0 \to \mathbb{R}$.*

Second, we rephrase Theorem 3.4 of [6] on limit of regenerative laws.

**Lemma 2** *Suppose that $(\mathscr{Q}_n : n \in \mathbb{N})$ is a sequence of regenerative laws on $\Omega^0$ which converges weakly towards some probability measure $\mathscr{Q}$ on $\Omega^0$. Then $\mathscr{Q}$ is a regenerative law.*

We next turn our attention to the connection between regenerative laws and subordinators. When $X$ is a subordinator, the closure of its range $\{X_t : 0 \leq t < \zeta\}$ viewed as a random closed subset of $[0,\infty)$, induces a regenerative law on $\Omega^0$. Conversely, suppose $\mathscr{Q}$ is a regenerative law on $\Omega^0$ with $d_0 = 0$ $\mathscr{Q}$-a.s. (that is to say that $M^0$ is perfect, $\mathscr{Q}$-a.s.). Then there exists an $\mathscr{Q}$-a.s. continuous increasing $(\mathscr{G}_s^0)$-adapted process $L = (L_s : s \geq 0)$, which increases exactly on $M^0$, and which has the property of additivity, namely $L_{t+s} = L_t + L_s \circ \tau_t$. We call $L$ a local time on $M^0$. Applying the strong regenerative property of Lemma 1, it is easily seen that for every $t \in [0,\infty)$, conditionally on $\{d_t < \infty\}$, the process $L \circ \tau_{d_t}$ is independent of $\mathscr{G}_{d_t+}^0$ and has the same law as $L$. The inverse $X_\cdot = \inf\{s \geq 0 : L(s) > \cdot\}$ is a subordinator, and finally $M^0$ coincides with the closed range of $X$, a.s.

If $\mathscr{Q}$ is a regenerative law with $d_0 > 0$ $\mathscr{Q}$-a.s. (that is to say that all the points in $M^0$ are isolated, $\mathscr{Q}$-a.s.), then the same feature as above holds, except that the local time is then an integer-valued process and the inverse local time an increasing random walk. Of course, one can always transform an increasing random walk into a subordinator via a time-substitution based on an independent Poisson process.

If two subordinators $X$ and $X'$ are such that their closed ranges have the same regenerative law $\mathcal{Q}$, then their Laplace exponents are proportional. Among those, there is a unique $\Phi$ which fulfills the -arbitrary- normalization condition $\Phi(1) = 1$. We will then refer to $\Phi$ as the Laplace exponent of $\mathcal{Q}$; and by extension, to $U$, d, $\Pi$..., as the renewal measure, drift coefficient, Lévy measure..., of $\mathcal{Q}$.

We finally lift from [6] (see Proposition 3.9 there) the following characterization of convergence of regenerative laws in terms of Laplace exponents.

**Lemma 3** *Let $\mathcal{Q}, \mathcal{Q}_1, \mathcal{Q}_2, \cdots$ be regenerative laws with Laplace exponents $\Phi, \Phi_1, \Phi_2 \cdots$. Then $\mathcal{Q}_n$ converges weakly towards $\mathcal{Q}$ if and only if $\Phi_n$ converges pointwise towards $\Phi$.*

### *2.3 Regenerative embedding*

Using (2), it is easy to verify that

$$\Omega = \left\{\omega = \left(\omega^{(1)}, \omega^{(2)}\right) \in \Omega^0 \times \Omega^0 : \omega^{(1)} \subseteq \omega^{(2)}\right\}$$

is a closed subset of $\Omega^0 \times \Omega^0$. As a consequence, $\Omega$ is a compact metrisable space. We write $M^{(1)}(\omega) = \omega^{(1)}$, $M^{(2)}(\omega) = \omega^{(2)}$ for the canonical projections on $\Omega$, $M = \left(M^{(1)}, M^{(2)}\right)$, and for every $t \geq 0$:

$$d_t^{(1)}(\omega) = d_t\left(\omega^{(1)}\right), \quad \tau_t(\omega) = \left(\tau_t\left(\omega^{(1)}\right), \tau_t\left(\omega^{(2)}\right)\right) .$$

We denote by $\mathcal{G}_t$ the sigma-field generated by $d_t^{(1)}$, $M^{(1)} \cap \left[0, d_t^{(1)}\right]$ and $M^{(2)} \cap \left[0, d_t^{(1)}\right]$; it is easy to check that $(\mathcal{G}_t)_{t\geq 0}$ is a filtration.

We now introduce the notion of *regenerative embedding laws* on $\Omega$ by mimicking the definition of regenerative laws on $\Omega^0$: A probability measure $\mathcal{P}$ on $\Omega$ is called a regenerative embedding law if for each $t \in [0, \infty)$ and each bounded measurable function $f$: $\Omega \to \mathbb{R}$

$$\mathcal{P}\left(f\left(M \circ \tau_{d_t^{(1)}}\right) \middle| \mathcal{G}_t\right) = \mathcal{P}(f(M)) \qquad \text{on} \quad \left\{d_t^{(1)} < \infty\right\} .$$

Just as in [21], one can establish the following variation of Lemma 1.

**Lemma 4** *Denote by $\underline{M}^{(1)}$ the subset of $M^{(1)}$ which consists of points which either are isolated in $M^{(1)}$ or are right-accumulation points in $M^{(1)}$. Let $\mathcal{P}$ be a regenerative embedding law and $T$ an $(\mathcal{G}_{s+})$-stopping time such that $T \in \underline{M}^{(1)}$ a.s. on $\{T < \infty\}$. Then*

$$\mathcal{P}(f(M \circ \tau_T) \mid \mathcal{G}_{T+}) = \mathcal{P}(f(M)) \qquad on \quad \{T < \infty\}$$

*for every bounded measurable function $f$: $\Omega \to \mathbb{R}$.*

Plainly, when $\mathscr{P}$ is a regenerative embedding law on $\Omega$, its first marginal on $\Omega^0$ is a regenerative law. We now arrive at the central notion of this work:

**Definition 1** (Embedding of regenerative laws) *Let $\mathscr{Q}^{(1)}$ and $\mathscr{Q}^{(2)}$ be two regenerative laws on $\Omega^0$. We say that $\mathscr{Q}^{(1)}$ is embedded into $\mathscr{Q}^{(2)}$ and write $\mathscr{Q}^{(1)} \prec \mathscr{Q}^{(2)}$ if there exists a regenerative embedding law $\mathscr{P}$ on $\Omega$ with marginal distributions $\mathscr{Q}^{(1)}$ and $\mathscr{Q}^{(2)}$.*

Repeating essentially the same arguments as given in section 3 of [6] yields the following variation of Lemma 2.

**Lemma 5** *Suppose that $(\mathscr{P}_n : n \in \mathbb{N})$ is a sequence of regenerative embedding laws on $\Omega$ which converges weakly towards some probability measure $\mathscr{P}$ on $\Omega$. Then $\mathscr{P}$ is a regenerative embedding law.*

Thanks to the normalization for the Laplace exponent of a regenerative law, the drift coefficient d of a regenerative law $\mathscr{Q}$ coincides with the expected Lebesgue measure of $M \cap [0, \tau]$, where $\tau$ is an independent exponential variable with mean 1. One says that $\mathscr{Q}$ is *light* if $\mathrm{d} = 0$, and is *heavy* otherwise. This motivates the following -dual- definitions.

**Definition 2** (Thinning) *Let $\mathscr{Q}$ be a heavy regenerative law. Denote by $\Phi$ its Laplace exponent and by $\mathrm{d} > 0$ its drift coefficient. Then $\Phi^{(-\mathrm{d})}(\lambda) = -\mathrm{d}\lambda + \Phi(\lambda)$ is the Laplace exponent of a subordinator with zero drift; denote the corresponding regenerative law by $\mathscr{Q}^{(-\mathrm{d})}$. We say that $\mathscr{Q}$ can be thinned if $\mathscr{Q}^{(-\mathrm{d})}$ is embedded in $\mathscr{Q}$.*

**Definition 3** (Thickening) *Let $\mathscr{Q}$ be a light regenerative law with Laplace exponent $\Phi$. For every $\mathrm{d} > 0$, write $\Phi^{(\mathrm{d})}(\lambda) = \mathrm{d}\lambda + \Phi(\lambda)$ and $\mathscr{Q}^{(\mathrm{d})}$ for the regenerative law associated with $\Phi^{(\mathrm{d})}$. We say that $\mathscr{Q}$ can be thickened if $\mathscr{Q}$ embedded in $\mathscr{Q}^{(\mathrm{d})}$ for every $\mathrm{d} > 0$.*

## 3 Results and examples

We first characterize the class of regenerative laws which can be embedded into a given one, in the sense of Definition 1.

**Theorem 1** *Let $\mathscr{Q}^{(1)}$ and $\mathscr{Q}^{(2)}$ be two regenerative laws, with Laplace exponents $\Phi^{(1)}$ and $\Phi^{(2)}$, respectively. Then $\mathscr{Q}^{(1)}$ is embedded into $\mathscr{Q}^{(2)}$ if and only if $\Phi^{(1)}/\Phi^{(2)}$ is a completely monotone function.*

*Examples* 1. Suppose $\mathscr{Q}^{(1)}$ is Gamma and $\mathscr{Q}^{(2)}$ is stable, i.e. $\Phi^{(1)}(\lambda) = a \log (1 + \lambda/b)$ for some parameters $a, b > 0$, and $\Phi^{(2)}(\lambda) = \lambda^{\beta}$ for some $\beta \in (0, 1)$. Then the derivative of $\Phi^{(1)}/\Phi^{(2)}$ is

$$\frac{a}{(b+\lambda)\lambda^{\beta}} - \frac{a\beta \log(1+\lambda/b)}{\lambda^{\beta+1}} \sim a\lambda^{-\beta}\left(\frac{1}{b} - \frac{\beta}{b}\right) \qquad \text{as} \quad \lambda \to 0+$$

is positive is some neighbourhood of $\lambda = 0$. Therefore, a Gamma Markov set cannot be regeneratively embedded into a stable Markov set.

2. Suppose $\mathscr{Q}^{(1)}$ is Stable($\alpha$) for some $\alpha \in (0,1)$. Then $\Phi^{(1)}/\Phi^{(2)}$ is the Laplace transform of the fractional derivative of order $\alpha$ of $U^{(2)}$. In other words, a Stable($\alpha$) Markov set can be regeneratively embedded into a given Markov set if and only if the $\alpha$-fractional derivative of the renewal measure of the latter is a Radon measure on $[0,\infty)$.

3. Suppose $\mathscr{Q}^{(2)}$ is Stable($\beta$) for some $\beta \in (0,1)$. Then $\Phi^{(1)}/\Phi^{(2)}$ is the Laplace transform of the fractional derivative of order $1-\beta$ of $\mathrm{d}^{(1)}\delta_0 + \overline{\Pi}^{(1)}$. In other words, a Markov set can be regeneratively embedded into a Stable($\beta$) Markov set if and only if its drift coefficient is zero and the $(1-\beta)$-fractional derivative of the tail of its Lévy measure is a Radon measure on $[0,\infty)$.

We stress that, thanks to (1) and Bernstein's theorem, the complete monotonicity condition in Theorem 1 for the ratio of two Laplace exponents can be expressed as follows in terms of excessive measures and potentials (in the obvious notation):

$$\Phi^{(1)}/\Phi^{(2)} \text{ is completely monotone} \iff U^{(2)} \in Pot^{(1)} \iff Exc^{(2)} \subseteq Exc^{(1)} \ .$$

To this end, we recall that Markov processes having the same excessive measures have been characterized by Getoor and Glover [9]. In a different direction, it is also interesting to mention that the complete monotonicity condition also appears in connection with the maximum principle for convolution kernels; see Itô [15], Hirsch [13] and the references therein.

Then, we present a property of subordinators which provides a further insight on the structure induced by regenerative embedding for Markov sets.

**Proposition 1** *Suppose that* $\Phi^{(1)}/\Phi^{(2)}$ *is completely monotone. Then, on some probability space endowed with a filtration* $(\mathscr{F}_t)_{t\geq 0}$, *there are an* $(\mathscr{F}_t)$*-adapted subordinator* $X^{(2)}$ *with Laplace exponent* $\Phi^{(2)}$, *and two subordinators* $\sigma$ *and* $X^{(1)}$, *such that for each* $s \geq 0$, $\sigma_s$ *is an* $(\mathscr{F}_t)$*-stopping time, and*

$$X^{(1)} = X^{(2)} \circ \sigma \ .$$

It is interesting to relate Proposition 1 with the concept of subordination in the sense of Bochner [3]. When $\sigma$ is independent of $X^{(2)}$, Proposition 1 means that $X^{(1)}$ is subordinated to $X^{(2)}$, and then $\Phi^{(1)} = \kappa \circ \Phi^{(2)}$ where $\kappa$ denotes the Laplace exponent of $\sigma$. To check that, in this case, $\Phi^{(1)}/\Phi^{(2)}$ is completely monotone, we recall that $\kappa(\lambda)/\lambda$ is completely monotone and that $\Phi^{(2)}$ has a completely monotone derivative. According to Criterion 2 on page 441 in Feller [4], $(\kappa \circ \Phi^{(2)})/\Phi^{(2)} = \Phi^{(1)}/\Phi^{(2)}$ is completely monotone. In general, $\sigma$ is not independent of $X^{(2)}$, and Proposition 1 can be viewed as a solution to a continuous version of the Skorohod embedding problem, which consists of exhibiting two subordinators with Laplace exponents $\Phi^{(1)}$ and $\Phi^{(2)}$, respectively, such that the first is obtained from the second by some time-substitution. We refer to Fitzsimmons [5] and Shih [25] for related works.

We finally turn our attention to thickening and thinning of regenerative laws.

**Corollary 1** *The following assertions are equivalent*:

(*a*) $\mathcal{Q}$ *is a heavy regenerative law that can be thinned.*
(*b*) *The renewal measure U is absolutely continuous on* $[0, \infty)$ *and possesses a continuous monotone decreasing density u*: $[0, \infty) \to (0, \infty)$.
(*c*) $\kappa(\lambda) = \lambda/\Phi(\lambda)$ *is the Laplace exponent of a subordinator with zero drift and finite Lévy measure* (*i.e. of a compound Poisson process*).

**Corollary 2** *The following conditions are equivalent*:

(*a*) $\mathcal{Q}$ *is a light regenerative law that can be thickened.*
(*b*) *The renewal measure U is absolutely continuous on* $(0, \infty)$, *possesses a monotone decreasing density u*: $(0, \infty) \to [0, \infty)$, *and either* $u(0+) = \infty$ *or U has an atom at* 0.
(*c*) $\kappa(\lambda) = \lambda/\Phi(\lambda)$ *is the Laplace exponent of a subordinator with either positive drift or infinite Lévy measure* (*i.e. not of a compound Poisson process*).

The equivalences (b) $\Leftrightarrow$ (c) in Corollaries 1 and 2 are doubtless well-known (see e.g. [10]); however we shall present the simple proof for the sake of completeness.

We now conclude this section presenting a number of natural examples in which the conditions of Corollary 2 are fulfilled.

*Examples.* 1. Suppose that the drift is zero and the tail of the Lévy measure $\overline{\Pi}$ is log convex. Then the condition (b) of Corollary 2 holds. Indeed, this is Theorem 2.1 of Hawkes [11].

2. Suppose $M$ is the zero set of some regular diffusion process on $\mathbb{R}$ started at the origin. Then $M$ is a Markov set, and is light whenever 0 is not a sticky point for the diffusion. According to the spectral theory of M.G. Krein, the renewal measure of $M$ is absolutely continuous and possesses a completely monotone density; see Kotani and Watanabe [17]. In particular, condition (b) of Corollary 2 holds.

3. Let $\mu$ be some sigma-finite Borel measure on $(0, \infty)$, and consider a family of random open intervals $\{(x_i, x_i + \ell_i) : i \in I\}$, where the point $(x_i, \ell_i) \in \mathbb{R}_+^2$ is issued from a Poisson measure with intensity $dx \otimes \mu(d\ell)$. Let $M$ be the subset of $[0, \infty)$ left uncovered by these random intervals, viz.

$$M = [0, \infty) - \bigcup_{i \in I} (x_i, x_i + \ell_i) \ .$$

See Fitzsimmons et al. [7] for details. Then $M$ is a light Markov set whenever $\int (1 \wedge \ell)\mu(d\ell) = \infty$ (Proposition 1 in [7]) and its renewal measure is absolutely continuous on $(0, \infty)$ with a convex decreasing density (Theorem 1 in [7]). Thus condition (b) of Corollary 2 holds.

4. Suppose that $M$ is the ladder time set of some real Lévy process, i.e. the set of times when the Lévy process reaches a new supremum. Then $M$ is a Markov set, and is light whenever 0 is regular for $(-\infty, 0)$. Moreover, the Laplace exponent of $M$ has $\Phi(\lambda) = \lambda/\widehat{\Phi}(\lambda)$, where $\widehat{\Phi}$ stands for the Laplace exponent of the ladder times of the dual Lévy process. See Equation (VI.3) in [2]. Thus condition (c) of Corollary 2 is fulfilled.

## 4 Proofs

*Proof of Theorem 1:* For simplicity, we will focus on the case when both $M^{(1)}$ and $M^{(2)}$ are perfect (i.e. have no isolated points) a.s., or equivalently when both $X^{(1)}$ and $X^{(2)}$ are not compound Poisson processes. The arguments for the case when at least one of the subordinators is a compound Poisson process are very similar, and sometimes even easier.

Suppose first that there is a regenerative embedding law $\mathscr{P}$ with marginals $\mathscr{Q}^{(1)}$ and $\mathscr{Q}^{(2)}$. Let $M^{(1)}$, $M^{(2)}$ and $(\mathscr{G}_s)_{s\geq 0}$ be as in sub-section 2.3. Denote by $L^{(i)}$ the local time on $M^{(i)}$ (which is a continuous $(\mathscr{G}_s)$-adapted process), and by $X^{(i)}$ the inverse local time, so that $X^{(i)}$ is a subordinator with Laplace exponent $\Phi^{(i)}$ $(i = 1, 2)$. For every $\lambda > 0$, we have

$$\frac{1}{\Phi^{(i)}(\lambda)} = \mathscr{P}\left(\int_0^\infty \exp\left\{-\lambda X_t^{(i)}\right\} dt\right) = \mathscr{P}\left(\int_0^\infty \mathrm{e}^{-\lambda t}\, dL_t^{(i)}\right) .$$

On the one hand, the uniqueness – up to some constant factor – of the local time on $M^{(1)}$ ensures the existence of a real number $k \geq 0$ such that the following identity between Stieltjes measures on $[0, \infty)$ holds a.s.

$$\mathbf{1}_{\{t\in M^{(1)}\}}\, dL_t^{(2)} = k\, dL_t^{(1)} .$$

On the other hand, the strong regenerative property of Lemma 4 enables us to apply Maisonneuve's exit-system formula (cf. [20]) as follows:

$$\begin{aligned}
&\mathscr{P}\left(\int_0^\infty \mathrm{e}^{-\lambda t} dL_t^{(2)}\right)\\
&= \mathscr{P}\left(\int_{M^{(1)}} \mathrm{e}^{-\lambda t} dL_t^{(2)}\right) + \mathscr{P}\left(\sum_{s\geq 0} \int_{X_{s-}^{(1)}}^{X_s^{(1)}} \mathrm{e}^{-\lambda t} dL_t^{(2)}\right)\\
&= k\mathscr{P}\left(\int_0^\infty \mathrm{e}^{-\lambda t} dL_t^{(1)}\right) + \mathscr{P}\left(\sum_{s\geq 0} \mathrm{e}^{-\lambda X_{s-}^{(1)}} \left(\int_0^{d_0^{(1)}} \mathrm{e}^{-\lambda t} dL_t^{(2)}\right) \circ \tau_{X_{s-}^{(1)}}\right)\\
&= \left(\int_0^\infty \mathrm{e}^{-\lambda t} dL_t^{(1)}\right)\left(k + \mathscr{P}^\star\left(\int_0^{d_0^{(1)}} \mathrm{e}^{-\lambda t} dL_t^{(2)}\right)\right) ,
\end{aligned}$$

where $\mathscr{P}^\star$ denotes the excursion measure away from the homogeneous set $M^{(1)}$.

Putting the pieces together, we get that

$$\frac{1}{\Phi^{(2)}(\lambda)} = \frac{1}{\Phi^{(1)}(\lambda)} \mathscr{L}\mu(\lambda) ,$$

where $\mathscr{L}\mu$ is the Laplace transform of the measure

$$\mu(dt) = k\delta_0(dt) + \mathscr{P}^\star\left(\mathbf{1}_{\{t\leq d_0^{(1)}\}}\, dL_t^{(2)}\right), \quad t \geq 0 .$$

Conversely, suppose that $\Phi^{(1)}/\Phi^{(2)} = \mathscr{L}\mu$ for some Radon measure $\mu$ on $[0, \infty)$. Because the Laplace transform of the renewal measure $U^{(i)}$ is $1/\Phi^{(i)}$, we see by Laplace inversion that $U^{(2)}$ is a potential for the subordinator $X^{(1)}$. In particular, $U^{(2)} \in Exc^{(1)}$.

First fix a step $\eta > 0$, and let $P_\eta^{(1)}$ be the law of $X_\eta^{(1)}$. The preceding observation ensures Rost's balayage inequality:

$$\delta_0 * U^{(2)} \geq P_\eta^{(1)} * U^{(2)} \ .$$

We appeal to the well-known result of Rost [24] on Skorohod's embedding problem. There exists a family $(S(t) : 0 \leq t \leq 1)$ of stopping times – in the natural filtration of $X^{(2)}$ – such that if $v$ is an independent random variable with a uniform distribution on $[0, 1]$, then $X_{S(v)}^{(2)}$ has the same law as $X_\eta^{(1)}$.

Using the independence and homogeneity of the increments of subordinators and an immediate induction, we can construct a probability space with a filtration $(\mathscr{F}_t)_{t\geq 0}$, an $(\mathscr{F}_t)$-adapted subordinator $X^{(2)}$, and an increasing sequence of stopping times $T = (T(k) : k = 0, 1, \cdots)$ such that:

- $T(0) = 0$, $T(1) = S(v)$.
- For each integer $k$, conditionally on $\{T(k) < \infty\}$, the shifted pair $\big(X^{(2)}, T\big) \circ \theta_{T(k)}$ is independent of $\mathscr{F}_{T(k)}$ and has the same law as $\big(X^{(2)}, T\big)$, where

$$X^{(2)} \circ \theta_T(k) = \left(X_{t+T(k)}^{(2)} - X_{T(k)}^{(2)} : t \geq 0\right) \ ,$$

$$T \circ \theta_{T(k)} = (T(k+j) - T(k) : j = 0, 1, \cdots) \ .$$

This entails that the discrete set $\big\{X_{T(k)}^{(2)} : k = 0, 1, \cdots\big\}$ is regeneratively embedded in the closed range of $X^{(2)}$. If we denote by $\mathscr{P}_\eta$ the corresponding law on $\Omega$, the marginals of $\mathscr{P}_\eta$ are clearly $\mathscr{Q}^{(1,\eta)}$ and $\mathscr{Q}^{(2)}$, where $\mathscr{Q}^{(1,\eta)}$ is the regenerative law associated with the increasing random walk with step distribution $P_\eta^{(1)}$. In other words, the Laplace exponent of $\mathscr{Q}^{(1,\eta)}$ is

$$\Phi^{(1,\eta)}(\lambda) = \frac{1 - \exp\big\{-\eta\Phi^{(1)}(\lambda)\big\}}{1 - \exp\big\{-\eta\Phi^{(1)}(1)\big\}}, \qquad \lambda > 0 \ .$$

(Recall that the Laplace exponent evaluated at $\lambda = 1$ equals 1.)

Next, we let $\eta$ tend to $0+$; notice that

$$\lim_{\eta\to 0+} \Phi^{(1,\eta)}(\lambda) = \Phi^{(1)}(\lambda), \qquad \lambda > 0 \ . \tag{3}$$

Because $\Omega$ is a compact metrisable space, we may extract from $\big(\mathscr{P}_\eta : \eta > 0\big)$ a sequence that converges; let $\mathscr{P}$ denote the limit. By Lemma 5, we know that $\mathscr{P}$ is a regenerative embedding law. By (3) and Lemma 2, its first marginal is $\mathscr{Q}^{(1)}$. Its second marginal is plainly $\mathscr{Q}^{(2)}$. This shows that $\mathscr{Q}^{(1)} \prec \mathscr{Q}^{(2)}$. □

*Proof of Proposition 1:* We use the same setting as in the proof of Theorem 1. Fix $t > 0$ and notice that $X_t^{(1)}$ necessarily takes values in the set $\underline{M}^{(1)}$ of points in $M^{(1)}$ which are not isolated on their right (because $X^{(1)}$ is strictly increasing and right-continuous, and $M^{(1)}$ is the closed range of $X^{(1)}$). Moreover $X_t^{(1)}$ is clearly an $(\mathscr{G}_{s+})$-stopping time.

Put $\sigma_t = L^{(2)} \circ X_t^{(1)}$; notice that $\sigma_t$ is $\left(\mathscr{G}_{X_t^{(1)}}\right)$-adapted. Applying the additivity of the local time and the strong regenerative property of Lemma 4, it is immediately seen that, conditionally on $\left\{X_t^{(1)} < \infty\right\}$, the process

$$L^{(2)} \circ \tau_{X_t^{(1)}} = \left(L^{(2)}_{X_t^{(1)}+s} - \sigma_t : s \geq 0\right)$$

is independent of $\mathscr{G}_{X_t^{(1)}}$ and has the same law as $L^{(2)}$. This entails that $\sigma$ is a subordinator. Finally, we have

$$X^{(1)} = X^{(2)} \circ L^{(2)} \circ X^{(1)} = X^{(2)} \circ \sigma \ ,$$

where the first identity stems from the fact that $X_t^{(1)} \in \underline{M}^{(2)}$ a.s. □

*Proof of Corollary 1:* It is well-known that the drift d is positive if and only if the renewal measure $U$ is absolutely continuous and has a continuous density $u\colon [0,\infty) \to (0,\infty)$ with $u(0) = 1/\mathrm{d}$; see e.g. Theorem III.5 in [2]. Recall that the Laplace transform of the renewal measure $U$ is $1/\Phi$.

(a) $\Leftrightarrow$ (b) Suppose first that $\mathscr{Q}$ is heavy and can be thinned. By Theorem 1 and Bernstein's theorem, there is a measure $\mu$ on $[0,\infty)$ with Laplace transform

$$\mathscr{L}\mu(\lambda) = \frac{\Phi^{(-\mathrm{d})}(\lambda)}{\Phi(\lambda)} = 1 - \frac{\mathrm{d}\lambda}{\Phi(\lambda)} \ . \tag{4}$$

Since 1 is the Laplace transform of the Dirac point mass at 0, (4) forces the renewal density $u$ to be monotone decreasing.

Conversely, suppose that the renewal measure has monotone decreasing and bounded density on $[0,\infty)$. Then we know that $\mathscr{Q}$ is heavy. Because $u(0) = 1/\mathrm{d}$, we see that the right-hand side of (4) is the Laplace transform of a Radon measure on $(0,\infty)$. According to Theorem 1 for $\Phi^{(1)} = \Phi^{(-\mathrm{d})}$ and $\Phi^{(2)} = \Phi$, $\mathscr{Q}$ can be thinned.

(b) $\Leftrightarrow$ (c) When (b) holds, the renewal density $u$ can be thought of as the tail of the Lévy measure of some subordinator with zero drift and finite Lévy measure (because $u(0) < \infty$). If $\kappa$ denotes its Laplace exponent, then

$$\kappa(\lambda)/\lambda = \mathscr{L}u(\lambda) = \int_0^\infty \mathrm{e}^{-\lambda x} u(x)\, dx = 1/\Phi(\lambda) \ ,$$

which establishes (c).

Conversely, if (c) holds, then

$$1/\Phi(\lambda) = \kappa(\lambda)/\lambda = \int_0^\infty \mathrm{e}^{-\lambda x} \overline{v}(x)\, dx \ ,$$

where $\overline{v}$ stands for the tail of the Lévy measure of $\kappa$. By Laplace inversion, $\overline{v}(x)\, dx = U(dx)$, and (b) holds. □

*Proof of Corollary 2:* (a) $\Rightarrow$ (b) We use obvious notation. We know by hypothesis that for every $\mathrm{d} > 0$, $\mathscr{Q}^{(\mathrm{d})}$ can be thinned. By Corollary 1, this entails that the renewal density $u^{(\mathrm{d})}$ is decreasing. Since $1/\Phi^{(\mathrm{d})}$ converges pointwise towards $1/\Phi$ as d tends to $0+$, the renewal measure $U^{(\mathrm{d})}$ converges weakly towards the renewal measure $U$, and this implies that $U$ has a monotone decreasing density on $(0,\infty)$. Since the drift d is zero, either $U$ has an atom at 0 or $u$ is unbounded.

(b) $\Rightarrow$ (c) By hypothesis, we know that there is a monotone decreasing function $u\colon (0,\infty) \to [0,\infty)$ with Laplace transform

$$\mathscr{L}u(\lambda) = \int_0^\infty \mathrm{e}^{-\lambda x} u(x)\, dx = \frac{1}{\Phi(\lambda)} - \frac{1}{\Phi(\infty)} ,$$

that is

$$\frac{1}{\Phi(\lambda)} = \frac{1}{\Phi(\infty)} + \mathscr{L}u(\lambda) .$$

Since $\int_0^1 u(x)\, dx < \infty$, $u$ can be thought of as the tail of the Lévy measure of some subordinator with drift $1/\Phi(\infty)$; denote the Laplace exponent of the latter by $\kappa$. We then have

$$\frac{\kappa(\lambda)}{\lambda} = \frac{1}{\Phi(\infty)} + \mathscr{L}u(\lambda) = \frac{1}{\Phi(\lambda)} .$$

Plainly $\kappa$ cannot be bounded, as either $U$ has an atom at 0 or $u$ is unbounded; this establishes (c).

(c) $\Rightarrow$ (a) Because $\lim_{\lambda\to\infty} \kappa(\lambda) = \infty$, $\Phi(\lambda) = o(\lambda)$ as $\lambda \to \infty$, which implies that the drift of $\Phi$ is zero. Using the identity $\Phi(\lambda)\kappa(\lambda) = \lambda$, we get for every $\mathrm{d} > 0$

$$\frac{\Phi(\lambda)}{\Phi(\lambda) + \mathrm{d}\lambda} = \frac{1}{1 + \mathrm{d}\kappa(\lambda)} .$$

the right-hand side is the Laplace transform of the 1-resolvent measure of the subordinator with Laplace exponent $\mathrm{d}\kappa$. By Theorem 1, this proves (a). $\square$

*Acknowledgement*. It is a pleasure to thank Pat Fitzsimmons for several insightful and stimulating comments on the first draft of this work.

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