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Fluctuation–dissipation equation of asymmetric simple exclusion processes

C. Landim[1,2], **H. T. Yau**[3]

[1] IMPA, Estrada Dona Castorina 110, CEP 22460 Rio de Janeiro, Brasil
[2] CNRS URA 1378, Université de Rouen, F-76128 Mont Saint Aignan, France
[3] Courant Institute, New York University, 251 Mercer Street, New York, NY 10012

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Summary. We consider asymmetric simple exclusion processes on the lattice $\mathbb{Z}^d$ in dimension $d \geq 3$. We denote by L the generator of the process, ∇ the lattice gradient, η the configuration, and w the current of the dynamics associated to the conserved quantity. We prove that the fluctuation–dissipation equation $w = Lu + D\nabla\eta$ has a solution for some function u and some constant D identified to be the diffusion coefficient. Intuitively, Lu represents rapid fluctuation and this equation describes a decomposition of the current into fluctuation and gradient of the density field, representing the dissipation. Using this result, we proved rigorously that the Green-Kubo formula converges and it can be identified as the diffusion coefficient.

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Introduction

The simple exclusion process is a system of random walks with hard core exclusion in which no two particles are allowed to be at the same site simultaneously. If the drift of the random walk is zero, this process can be considered as a model for thermal particles with hard core exclusion rule; if the drift is nonzero, it describes particles with a velocity in addition to the thermal noise and is referred to as the asymmetric simple exclusion process. The model is exactly solvable if the hard core exclusion is removed. The hard core condition persists even in the hydrodynamic limit. In the hyperbolic

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scaling the hydrodynamic equation for the asymmetric simple exclusion process is given by a viscousless Burgers equation with entropy condition ([R], [AV], [Re], [L]):

$$\partial_t u + \gamma \cdot \nabla u(1-u) = 0 \tag{0.0}$$

for some constant γ depending on the jump rate of the asymmetric simple exclusion process; here ∇ is the usual gradient. One can ask additional questions regarding the long time behavior or corrections to the hydrodynamic equation. The point of view of corrections to the hydrodynamic equation is emphasized by R. Dobrushin [D]; the long time behavior is formulated as incompressible limit in [EM] for general Hamiltonian systems. On a rigorous level, the incompressible limit is proved in [EMY1] to be

$$\partial_t u + \gamma \cdot \nabla u^2 = \nabla \cdot D \cdot \nabla u \tag{0.1}$$

for some diffusion coefficient D. Partly based on this analysis, the correction to (0.0) in the hyperbolic scaling is proved in [LOY1], [LOY2] to be

$$\partial_t u + \gamma \cdot \nabla u(1-u) = \varepsilon \nabla \cdot D \cdot \nabla u \tag{0.2}$$

with the same diffusion coefficient; here ε is the scaling parameter. One can also construct lattice gas models based on asymmetric simple exclusion process with collisions and derives the incompressible Navier-Stokes equation in the scaling limit [EMY2].

The approach of these papers involves several ingredients: a multiscale analysis, the nongradient system method originated from [V] and the relative entropy method [Y]. Denote by L the generator of the dynamics, η the configuration, w the current associated to the density field and ∇ the lattice gradient. One key step is to prove that there exists a function u (actually only approximate solutions are needed) such that the following decomposition

$$w = Lu + D\nabla\eta \tag{0.3}$$

holds for some constant D, identified to be the diffusion coefficient. Intuitively, Lu represents rapid fluctuation and the equation (0.3) describes a decomposition of the current into fluctuation and gradient of the density field, representing the dissipation. For this reason we call this equation the fluctuation–dissipation equation.

We now remark briefly on the history of recent mathematical work on the fluctuation–dissipation equation (0.3). We call a process a gradient model if the current is already a gradient; in this case the solution of the fluctuation–dissipation equation can be achieved, in a sense, with $u = 0$ and the diffusion coefficient D can be identified as a thermodynamical quantity. The first nontrivial solution of (0.3) was considered in [V] for Ginzburg-Landau models and the basic nongradient system method was introduced. Later on particle models are considered in [Q, KLO]. These models [V, Q, KLO] are reversible, i.e., the generator L is symmetric. A nonreversible model is considered in [Xu, V2] where the asymmetric part of the generator, $L - L^*$, turns out to be a bounded perturbation of the symmetric part; the resulting hydrodynamical equation is a nonlinear diffusion equation with no drift. In all these cases, the solution of (0.3) depends on the ability to perform integration by parts which in some sense converts (0.3) to what amounts to a (overdetermined) system of first order equations. This method is very general and does not need estimation of the Green's function and it requires only a bound on the spectral gap and a natural integration by parts property satisfied by all these models. For asymmetric simple exclusion processes, the asymmetric part is *not* a bounded perturbation of the symmetric part and there is *no* natural integration by parts property. Partly due to the absence of natural integration by parts property, the connection between the nongradient system method and the asymmetric simple exclusion has not been realized until [EMY1], where integration by parts is replaced by a multiscale analysis procedure which we call "multiscale integration by parts lemma".

With this lemma, it is proposed in [EMY1] to convert (0.3) to a problem of proving surjectiveness of some projection in a Hilbert space. We have found this formulation and the multiscale integration by parts lemma very useful, but its proof of the surjectiveness contains an error, to be explained in section 1 after introducing the necessary notations. In order to solve (0.3) one thus has to estimate the Green function $(\lambda - L)^{-1}$ in a precise sense. Because the full generator L is an infinite dimensional nonsymmetric operator with interactions among particles, we are not aware of any instance where such a question is addressed in an infinite dimensional setting.

This paper is organized as follows. We shall first recall the rigorous definitions of the asymmetric simple exclusion process. We then review the concept of degree which is closely related to a duality. It is well-known that a dual process exists for the *symmetric* simple exclusion process. The dual process for the asymmetric one, however, does not exist in the conventional sense. But the consideration of degree still provides very detailed information on the generator which can not be obtained otherwise. We decompose the generator according to degree. Our basic idea is to consider the part preserving the degree as the main part (or the diagonal part) and the rest (off diagonal part) as a perturbation. For this purpose, we need an estimate of the off diagonal part in terms of the diagonal part. This will be done in section 4. It should be emphasized that the off diagonal part is *not uniformly* bounded with respect to the diagonal part and one can not do perturbation theory in a naive way. The perturbative method requires cutoff estimates to be explained in section 4.

A more indirect approach will be presented in section 5 and it is based on an estimate of the resolvent equation. This proof is somewhat shorter but the first proof gives more information which, though irrelevant to the present problem, might be useful elsewhere.

Using this result, we prove rigorously that the Green-Kubo formula converges and it can be identified as the diffusion coefficient. We remark that, though Green-Kubo has been well-known for many decades, it is difficult to establish its mathematical meaning for nonreversible systems because it involves time integral from zero to infinity and space summation of current-

current correlation functions. In order to show that the Green-Kubo formula is valid, one needs at least to prove that the current-current correlation functions decay fast enough so that it is summable in space and integrable in time. We shall prove that our estimate of the fluctuation–dissipation equation carrys enough information to prove this rigorously.

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1. Notation and results

The totally asymmetric simple exclusion process is a Markov process on $\{0,1\}^{\mathbb{Z}^d}$ whose generator L acts on cylinder functions as

$$(Lf)(\eta) = \sum_{j=1}^{d} \sum_{x\in\mathbb{Z}^d} \eta_x[1-\eta_{x+e_j}][f(\eta^{x,x+e_j}) - f(\eta)] \,. \tag{1.1}$$

Here $\{e_k,\ 1\le k\le d\}$ stands for the canonical basis of $\mathbb{R}^d$, η denotes a configuration of $\{0,1\}^{\mathbb{Z}^d}$ so that η_x is equal to 1 if site x is occupied and is equal to 0 otherwise and $\eta^{x,y}$ stands for the configuration obtained from η by exchanging the occupation variables at x and y :

$$(\eta^{x,y})_z = \begin{cases} \eta_z & \text{if } z\neq x,y, \\ \eta_x & \text{if } z = y \text{ and} \\ \eta_y & \text{if } z = x \ . \end{cases}$$

For each ρ in [0,1], denote by ν_ρ the Bernoulli product measure on $\{0,1\}^{\mathbb{Z}^d}$ with density ρ and by $\langle\cdot,\cdot\rangle_\rho$ the inner product in $L^2(\nu_\rho)$. The probability measures ν_ρ are invariant for the totally asymmetric simple exclusion process. Denote by L^* (resp. S and A) the adjoint (resp. symmetric and asymmetric) part of L in $L^2(\nu_\rho)$. A simple computation shows that L^*, S and A act on cylinder functions as

$$(L^*f)(\eta) = \sum_{j=1}^{d}\sum_{x\in\mathbb{Z}^d} \eta_x[1-\eta_{x-e_j}](T_{x-e_j,x}f)(\eta) \ ,$$

$$(Sf)(\eta) = (1/2)\sum_{j=1}^{d}\sum_{x\in\mathbb{Z}^d}(T_{x,x+e_j}f)(\eta)$$

and
$$(Af)(\eta) = (1/2)\sum_{j=1}^{d}\sum_{x\in\mathbb{Z}^d}[\eta_x - \eta_{x+e_j}](T_{x,x+e_j}f)(\eta) \ ,$$

where for any bond $\{x,y\}$, $T_{x,y}$ is the operator defined by $(T_{x,y}f)(\eta) = f(\eta^{x,y}) - f(\eta)$.

For each positive integer n, denote by $\mathscr{P}_n = \mathscr{P}_n(\mathbb{Z}^d)$ the space of all finite subsets $\Lambda\subset\mathbb{Z}^d$ of cardinality n and by $|\Lambda|$ the cardinality of a finite subset Λ

of $\mathbb{Z}^d$. For two sets $\Lambda = \{a_1, \ldots, a_n\}$ and $\Omega = \{b_1, \ldots, b_n\}$ in $\mathscr{P}_n$ define the distance $d(\Lambda, \Omega)$ by

$$d(\Lambda, \Omega) = \min_\sigma \sum_{i=1}^n |a_i - b_{\sigma(i)}| \ ,$$

where the minimum is carried over all permutations σ of $\{1, \ldots, n\}$.

Fix once for all some density $0 < \rho < 1$. Denote by $\mathscr{C} = \mathscr{C}(\rho)$ the space of v_ρ-mean zero cylinder functions. For a finite subset Λ of $\mathbb{Z}^d$, denote by η_Λ the mean zero cylinder function defined by

$$\eta_\Lambda = \prod_{x \in \Lambda} (\eta_x - \rho) \ .$$

and, for $n \geq 1$, denote by $\mathscr{M}_n$ the space of cylinder functions of degree n, i.e., the space generated by all monomial s of degree n:

$$\mathscr{M}_n = \left\{ h \in \mathscr{C}; h = \sum_{\Lambda \in \mathscr{P}_n} h_\Lambda \eta_\Lambda, h_\Lambda \in \mathbb{R} \right\} \ .$$

Notice that in this definition all but a finite number of coefficients h_Λ vanish because h is assumed to be a cylinder function. Denote by $\mathscr{C}_n = \cup_{1 \leq j \leq n} \mathscr{M}_j$ the space of cylinder functions of degree less than or equal to n. All mean zero cylinder functions h can be decomposed as a finite linear combination of cylinder functions of finite degree: $\mathscr{C} = \cup_{n \geq 1} \mathscr{M}_n$.

On $\mathscr{C}$, define the semi-inner product

$$\langle\!\langle g, h \rangle\!\rangle = \langle g, h \rangle_0 = \sum_{x \in \mathbb{Z}^d} \langle \tau_x g, h \rangle_\rho = \sum_{x \in \mathbb{Z}^d} \langle \tau_x h, g \rangle_\rho \ . \tag{1.2}$$

All but a finite number of terms in this sum vanish because v_ρ is a product measure and g, h are mean zero. Denote by $\| \cdot \|_0$ the semi-norm generated by this semi-inner product and notice that for any local function g in $\mathscr{C}_n$, $\|g - \tau_x g\|_0 = 0$. Let g and h be cylinder functions of degree less than or equal to n: $g = \sum_{\Lambda, |\Lambda| \leq n} g_\Lambda \eta_\Lambda$, $h = \sum_{\Lambda, |\Lambda| \leq n} h_\Lambda \eta_\Lambda$. An elementary computation shows that

$$\langle\!\langle g, h \rangle\!\rangle_0 = \sum_{m \leq n} K_m(\rho) m^{-1} \sum_\Lambda \left(\sum_{x \in \mathbb{Z}^d} g_{x+\Lambda} \right) \left(\sum_{x \in \mathbb{Z}^d} h_{x+\Lambda} \right) \ , \tag{1.3}$$

where the second summation is carried over all subsets Λ in $\mathscr{P}_m$ that contains the origin, $K_m(\rho) = [\rho(1-\rho)]^m$ and $x + \Lambda$ is the set defined by $x + \Lambda = \{z, z - x \in \Lambda\}$. This identity leads us to introduce the following equivalence relation in $\mathscr{P}_n$.

Two sets Λ, Ω in $\mathscr{P}_n$ are said to be equivalent if there exists x in $\mathbb{Z}^d$ such that $\Omega = \Lambda + x$. In this case we write $\Lambda \sim \Omega$. Of course $\sim$ is an equivalence relation and we denote by $\tilde{\mathscr{P}}_n$ the set of equivalence classes of $\mathscr{P}_n$. With this terminology, the inner product $\langle g, f \rangle_0$ of two cylinder functions in $\mathscr{M}_n$ writes

$$\langle\!\langle g,f\rangle\!\rangle_0 = K_n(\rho)\sum_{\Lambda\in\tilde{\mathscr{P}}_n}\left(\sum_{\Omega\sim\Lambda} g_\Omega\right)\left(\sum_{\Omega\sim\Lambda} f_\Omega\right) \ . \tag{1.4}$$

It follows from the explicit formula (1.3) that the spaces $\mathscr{M}_n$ are orthogonal and a cylinder function $g = \sum_{\Lambda,|\Lambda|\le n} g_\Lambda \eta_\Lambda$ belongs to the kernel of the inner product $\langle\cdot,\rangle_0$ if and only if for all $1\le m\le n$ and Λ in $\mathscr{P}_m$, $\sum_{x\in\mathbb{Z}^d} g_{x+\Lambda} = 0$.

Denote by $\mathscr{N}_n$ (resp. $\mathscr{N}$) the kernel of the inner product in $\mathscr{C}_n$ (resp. $\mathscr{C}$), by $\mathscr{C}_n^0$ (resp. $\mathscr{C}^0$) the quotient of $\mathscr{C}_n$ (resp. $\mathscr{C}$) with respect to $\mathscr{N}_n$ (resp. $\mathscr{N}$) and by $\mathscr{R}_n$ (resp. $\mathscr{H}$) the completion of $\mathscr{C}_n^0$ (resp. $\mathscr{C}^0$). $\mathscr{H}$ and $\mathscr{R}_n$ are Hilbert spaces with the inner product (1.2). Notice that $\mathscr{R}_1$ is the one dimensional space generated by $\eta_{\{0\}} = (\eta_0 - \rho)$.

We now investigate the action of the operators S and A on the space $\mathscr{C}$. Fix a function g in $\mathscr{M}_n$: $g = \sum_{\Lambda,|\Lambda|=n} g_\Lambda\eta_\Lambda$. A simple computation shows that

$$(Sg)(\eta) = -(1/2)\sum_{j=1}^{d}\sum_{x\in\mathbb{Z}^d}\sum_{\substack{\Omega,\,|\Omega|=n-1\\ \Omega\cap\{x,x+e_j\}=\phi}}\{g_{\Omega\cup\{x+e_j\}} - g_{\Omega\cup\{x\}}\}[\eta_{\Omega\cup\{x+e_j\}} - \eta_{\Omega\cup\{x\}}] \ .$$

In particular, $S\eta_{\{0\}} = 0$ so that $S\mathscr{R}_1 = \{0\}$. In contrast, an elementary computation gives that $(\eta_x - \eta_y)^2 = (1-2\rho)\{(\eta_x-\rho)+(\eta_y-\rho)\} -2(\eta_x-\rho)(\eta_y-\rho) - 2\rho(1-\rho)$. This identity permits to decompose the asymmetric part A of the generator in two pieces M and J so that M maps $\mathscr{M}_n$ into itself and $J = J_+ + J_-$ maps $\mathscr{M}_n$ into $\mathscr{M}_{n-1}\cup\mathscr{M}_{n+1}$:

$$(Mg)(\eta) = (1/2)(1-2\rho)\sum_{j=1}^{d}\sum_{x\in\mathbb{Z}^d}\sum_{\substack{\Omega,\,|\Omega|=n-1\\ \Omega\cap\{x,x+e_j\}=\phi}}\{g_{\Omega\cup\{x+e_j\}} - g_{\Omega\cup\{x\}}\}$$
$$\times\left[\eta_{\Omega\cup\{x+e_j\}} + \eta_{\Omega\cup\{x\}}\right] \ ,$$

$$(J_+g)(\eta) = -\sum_{j=1}^{d}\sum_{x\in\mathbb{Z}^d}\sum_{\substack{\Omega,\,|\Omega|=n-1\\ \Omega\cap\{x,x+e_j\}=\phi}}\{g_{\Omega\cup\{x+e_j\}} - g_{\Omega\cup\{x\}}\}\eta_{\Omega\cup\{x,x+e_j\}} \ ,$$

$$(J_-g)(\eta) = -\rho(1-\rho)\sum_{j=1}^{d}\sum_{x\in\mathbb{Z}^d}\sum_{\substack{\Omega,\,|\Omega|=n-1\\ \Omega\cap\{x,x+e_j\}=\phi}}\{g_{\Omega\cup\{x+e_j\}} - g_{\Omega\cup\{x\}}\}\eta_\Omega \ .$$

Thus, while M maps $\mathscr{M}_n$ into $\mathscr{M}_n$, J_+ (resp. J_-) maps $\mathscr{M}_n$ into $\mathscr{M}_{n+1}$ (resp. $\mathscr{M}_{n-1}$). Let $B = S + M$ so that $L = B + J$ and B does not modify the degree of the monomials, while J changes it by 1.

Consider two cylinder functions f_i in $\mathscr{M}_i$, $i = 1, 2$. The explicit form of J_-, J_+ and M shows that J_-f_1, J_+f_1, Mf_1 vanish. Furthermore J_-f_2 is a gradient and hence it also vanishes as an element of $\mathscr{H}$. Therefore,

$$J_-\mathscr{R}_1 = J_+\mathscr{R}_1 = M\mathscr{R}_1 = J_-\mathscr{R}_2 = \{0\} \text{ in } \mathscr{H} \ . \tag{1.5}$$

We now introduce two new inner products in $\mathscr{C}$. The explicit formula for Sf permits to compute the inner product $\langle\!\langle Sf,g\rangle\!\rangle_0$ for two cylinder functions in

$\mathscr{C}_n$. Since $S\mathscr{R}_1 = \{0\}$, fix $n \ge 2$ and two cylinder functions f, g in $\mathscr{M}_n$: $g = \sum_{\Lambda,|\Lambda|=n} g_\Lambda \eta_\Lambda$, $f = \sum_{\Lambda,|\Lambda|=n} f_\Lambda \eta_\Lambda$. We have that $\langle\langle f, (-S)g \rangle\rangle_0$ is equal to

$$(1/4)K_n(\rho) \sum_{\Omega_0 \in \tilde{\mathscr{P}}_n} \sum_{\Omega_1, d(\Omega_1,\Omega_0)=1} \left(\sum_{\Lambda \sim \Omega_1} f_\Lambda - \sum_{\Lambda \sim \Omega_0} f_\Lambda \right) \left(\sum_{\Lambda \sim \Omega_1} g_\Lambda - \sum_{\Lambda \sim \Omega_0} g_\Lambda \right)$$
$$= (1/2)K_n(\rho) \sum_{j=1}^{d} \sum_{\substack{\Lambda,|\Lambda|=n-1 \\ \Lambda \cap \{0,e_j\} = \phi}} \{\tilde{f}_{\Lambda \cup \{e_j\}} - \tilde{f}_{\Lambda \cup \{0\}}\} \{\tilde{g}_{\Lambda \cup \{e_j\}} - \tilde{g}_{\Lambda \cup \{0\}}\} \ . \tag{1.6}$$

In this formula, for Λ in $\mathscr{P}_n$, $\tilde{f}_\Lambda$ stands for

$$\tilde{f}_\Lambda = \sum_{x \in \mathbb{Z}^d} f_{x+\Lambda} \ .$$

The operator S is thus symmetric and negative in $\mathscr{H}$ (in fact, it is proved in [S] that it is self adjoint). Define the inner product $\langle\langle \cdot, \cdot \rangle\rangle_1$ in $\cup_{n \ge 1} \mathscr{M}_n$ by

$$\langle\langle f, g \rangle\rangle_1 = \langle\langle f, (-S)g \rangle\rangle_0 \ .$$

The explicit formula for the quadratic form $\langle\langle f, f \rangle\rangle_1$ shows that its kernel is equal to $\mathscr{R}_1$. Denote by $\mathscr{H}_1$ the completion of $\cup_{n \ge 2} \mathscr{M}_n$ with respect to the inner product $\langle\langle \cdot, \cdot \rangle\rangle_1$ and by $\mathscr{H}_{-1}$ the dual of $\mathscr{H}_1$ with respect to $\langle\langle \cdot, \cdot \rangle\rangle_0$: $\mathscr{H}_{-1}$ is the Hilbert space generated by $\cup_{n \ge 2} \mathscr{M}_n$ and the inner product obtained by polarization of the quadratic form $\langle\langle f, f \rangle\rangle_{-1}$ defined by

$$\langle\langle f, f \rangle\rangle_{-1} = \sup_h \{ 2\langle\langle h, f \rangle\rangle_0 - \langle\langle h, h \rangle\rangle_1 \}$$
$$= \sup_h \left\{ 2 \left\langle \sum_x \tau_x f \ , \ h \right\rangle_\rho - \sum_{i=1}^{d} \left\langle \left(\nabla_{0,e_i} \sum_x \tau_x h \right)^2 \right\rangle_\rho \right\} \ , \tag{1.7}$$

where the supremum is carried over all cylinder functions h in $\cup_{n \ge 2} \mathscr{M}_n$. In this formula, for a cylinder function h, $\nabla_{0,e_i} h$ stands for $h(\eta^{0,e_i}) - h(\eta)$. Notice that (1.7) defines in principle only a semi-norm that may be infinite. It is proved in [EMY1] that (1.7) is finite for cylinder function in $\cup_{n \ge 2} \mathscr{M}_n$ and that it defines an inner product. It is not hard to prove that it defines an inner product by considering the limit

$$\lim_{\lambda \to 0} \langle\langle f, (\lambda - S)^{-1} f \rangle\rangle_0 = \langle\langle f, (-S)^{-1} f \rangle\rangle_0 \ .$$

Since the symmetric operator S preserves the degree of the monomials and the linear spaces $\{\mathscr{M}_n, n \ge 2\}$ are mutually orthogonal in $\mathscr{H}$, they are also orthogonal in $\mathscr{H}_1$ and in $\mathscr{H}_{-1}$.

Our main result is the following theorem.

Theorem 1.1. *Fix $k \ge 2$ and h in $\mathscr{C}_k$. Then,*

$$\inf_{u \in \mathscr{C}} \|h - Lu\|_{-1} = 0 \ . \tag{1.8}$$

Recall the following notations and results from [EMY]: Denote by $D(f)$ the Dirichlet form associated to the symmetric part of the generator:

$$D(f) = \langle(-S)f, f\rangle_\rho = \sum_{|x-y|=1} \langle(-S_{x,y})f, f\rangle_\rho \ , \tag{1.9}$$

where $S_{x,y}$ is the piece of the generator S restricted to jumps over the bond $\{x, y\}$: $S_{x,y}f(\eta) = f(\eta^{x,y}) - f(\eta)$. For a positive integer ℓ and for m in $\{0, 1/(2\ell+1)^d, \dots, 1\}$, denote by Λ_ℓ the cube of linear size $2\ell + 1$ centered at the origin: $\Lambda_\ell = \{-\ell, \dots, \ell\}^d$ and by $\nu_{\ell,m}$ the canonical measure on $\{0, 1\}^{\Lambda_\ell}$ with density m:

$$\nu_{\ell,m}(\xi) = \nu_\rho\left(\xi \,\middle|\, \sum_{x\in\Lambda_\ell} \eta(x) = (2\ell+1)^d m\right)$$

for each configuration ξ of $\{0, 1\}^{\Lambda_\ell}$.

Definition 1.2. *Let g be a cylinder function and denote by $s(g)$ its support*:

$$s(g) = \min\left\{\ell \in \mathbb{N};\ supp\ g \subset \{-\ell, \dots, \ell\}^d\right\} \ .$$

For each and $\ell \ge s(g)$ and m in $\{0, 1/(2\ell+1)^d, \dots, 1\}$, define the "variance" $V_\ell(g, m)$ of g with respect to $\nu_{\ell,m}$ by

$$V_\ell(g, m) = (2\ell+1)^{-d}\left\langle \sum_{|x|\le \ell - s(g)} (\tau_x g - \tilde{g}_\ell(m)), \right.$$
$$\left. (-S_\ell)^{-1} \sum_{|x|\le \ell - s(g)} (\tau_x g - \tilde{g}_\ell(m)) \right\rangle_{\nu_{\ell,m}} \ . \tag{1.10}$$

In this formula S_ℓ is the restriction to Λ_ℓ of the symmetric part of the generator L: $S_\ell = \sum_{x,y\in\Lambda_\ell} S_{x,y}$ and $\tilde{g}_\ell(m)$ is the expected value of g with respect to the canonical measure $\nu_{\ell,m}$.

If g belongs to $\cup_{n\ge 2}\mathscr{M}_n$ we define also the "variance" of g by

$$\mathbf{V}_m(g) = \limsup_{\ell\to\infty} E_{\nu_m}\left[V_\ell(g, \eta^\ell(0))\right] \ .$$

Theorem 1.3. ([*EMY1*]) *For every g in $\cup_{n\ge 2}\mathscr{M}_n$,*

$$\mathbf{V}_m(g) = \langle\!\langle g, g\rangle\!\rangle_{-1} \ .$$

Furthermore, for any $\varepsilon > 0$ there is a local function h such that

$$\mathbf{V}_m(g - Sh) \le \varepsilon \ .$$

This theorem allows us to solve the fluctuation–dissipation equation for the symmetric simple exclusion process. It is proved in [Q] when the function h is a current of a process. The general case is formulated and proved in [EMY1] using a multiscale integration by parts lemma.

We can now prove the following structure theorem :

Theorem 1.4. *Let* $\mathscr{H}^{(0)} = \{\sum_{1\le j\le d} \alpha_j[\eta(j) - \eta(0)], \alpha_j \in \mathbb{R}\}$. *Then*

$$\overline{L\mathscr{H} + \mathscr{H}^{(0)}} = \overline{\mathscr{H}} = \overline{L^*\mathscr{H} + \mathscr{H}^{(0)}} \ .$$

Proof. Fix g in $\mathscr{H}$. By Theorem 1.1, for every $\varepsilon > 0$, there exists a cylinder function h such that

$$\|g - Lh\|_{-1} \le \varepsilon \ .$$

From remark just before (1.5), the part of Lh with degree one is a gradient of the form $\sum_{1\le j\le d} \alpha_j[\eta(e_j) - \eta(0)]$. Hence,

$$\mathbf{V}\left(g - Lu - \sum_{1\le j\le d} \alpha_j[\eta(e_j) - \eta(0)]\right) = \| g - Lh \|_{-1}^2 \le \varepsilon^2 \ ,$$

what proves the theorem. □

The last result is formulated in [EMY1] and a proof was given. It is proposed in [EMY1] that the nongradient system method of [V] can be understood in Hilbert space formulation once a multiscale integration by parts lemma is used. This formulation is instructive and the multiscale integration by parts lemma is indeed useful, stated here in the form of Theorem 1.3, but the subsequent proof given there, essentially using a projection argument in a Hilbert space (proof of part (i) of Theorem 5.9 in [EMY1]), holds only for finite dimensional space.

We shall prove Theorem 1.1 in two ways. The first proof is divided in three parts. We first prove a truncated version of this result in Theorem 2.4 below. We then obtain uniform bounds of the $\mathscr{H}_1$ norm of the solution of the truncated equation. These estimates permit us to remove the cutoff in the third step.

The second proof is shorter. We first prove an a a-priori uniform bound of the $\mathscr{H}_{-1}$ norm of the solution of the equation. This is the same as the step 2 in the previous approach. Using this bound, we then prove an a priori estimate on the $\mathscr{H}$ norm of the solution. These two a priori estimates permit us to construct a solution.

2. First proof of Theorem 1.1

We begin this section with some comments on the degree preserving operators S and M. Recall that for each $n \ge 1$, we denote by $\mathscr{R}_n$ the Hilbert space generated by $\mathscr{M}_n$ and the inner product $\langle\!\langle \cdot, \cdot \rangle\!\rangle_0$ introduced in (1.4). Denote by $\mathscr{R}_{n,1}$ (resp. $\mathscr{R}_{n,-1}$) the Hilbert space generated by $\mathscr{M}_n$ and the inner product $\langle\!\langle \cdot, \cdot \rangle\!\rangle_1$ defined in (1.6) (resp. the quadratic form $\langle\!\langle \cdot, \cdot \rangle\!\rangle_{-1}$ defined in (1.7)). A simple computation shows that for f in $\mathscr{M}_n$, $\langle\!\langle f, f \rangle\!\rangle_1 \le 2dn \ \langle\!\langle f, f \rangle\!\rangle_0$.

In particular, $\mathscr{R}_{n,-1} \subset \mathscr{R}_n \subset \mathscr{R}_{n,1}$ and $\langle\langle f, f \rangle\rangle_0 \le 2dn \ \langle\langle f, f \rangle\rangle_{-1}$ for all f in $\mathscr{R}_{n,-1}$.

The explicit formula for M permits to compute the inner product $\langle\langle Mf, g \rangle\rangle_0$ for every f, g in $\mathscr{M}_n$. It is given by

$$\begin{aligned} &\frac{1-2\rho}{2} K_n(\rho) \sum_{\Omega_0 \in \tilde{\mathscr{P}}_n} \sum_{\Omega_1} \left(\sum_{\Lambda \sim \Omega_1} f_\Lambda - \sum_{\Lambda \sim \Omega_0} f_\Lambda \right) \left(\sum_{\Lambda \sim \Omega_1} g_\Lambda + \sum_{\Lambda \sim \Omega_0} g_\Lambda \right) \\ &\quad = \frac{1-2\rho}{2} K_n(\rho) \sum_{j=1}^{d} \sum_{\substack{\Lambda, |\Lambda| = n-1 \\ \Lambda \cap \{0, e_j\} = \phi}} \{ \tilde{f}_{\Lambda \cup \{e_j\}} - \tilde{f}_{\Lambda \cup \{0\}} \} \{ \tilde{g}_{\Lambda \cup \{e_j\}} + \tilde{g}_{\Lambda \cup \{0\}} \} \ . \end{aligned} \tag{2.1}$$

In the first line, the second sum is carried over all set Ω_1 at distance 1 from Ω_0. In particular, M is an antisymmetric operator :$\langle\langle Mf, g \rangle\rangle_0 = -\langle\langle f, Mg \rangle\rangle_0$.

On $\mathscr{R}_n$ the operators S and M are bounded : there exists a constant $C(d)$ depending only on the dimension such that

$$\langle\langle Sf, Sf \rangle\rangle_0 \le C(d) n^2 \langle\langle f, f \rangle\rangle_0 \text{ and } \langle\langle Mf, Mf \rangle\rangle_0 \le C(d) n^2 \langle\langle f, f \rangle\rangle_0$$

for all f in $\mathscr{R}_n$. Moreover, S is bounded operator from $\mathscr{H}_{-1}$ to $\mathscr{H}$: $\langle\langle Sf, Sf \rangle\rangle_{-1} = \langle\langle f, f \rangle\rangle_1$ and, by Lemma (4.2), M is a bounded operator from $\mathscr{R}_n$ to $\mathscr{R}_{n,-1}$: $\langle\langle Mf, Mf \rangle\rangle_{-1} \le C(d) n \langle\langle f, f \rangle\rangle_0$ for some finite constant depending only on the dimension. Hence, Sf and Mf belong to $\mathscr{R}_{n,-1}$ for any function f in $\mathscr{R}_n$.

Theorem (1.1) would not be difficult to prove if the operator L was symmetric. This is the content of the next result where we show that the range of S is dense in $\mathscr{H}_{-1}$.

Lemma 2.1. *For each $k \ge 2$ and h in $\mathscr{M}_k$,*

$$\inf_{f \in \mathscr{M}_k} \| Sf - h \|_{-1} = 0 \ .$$

Proof. Fix $k \ge 2$ and h in $\mathscr{M}_k$. A direct computation shows that S is a bounded operator on $\mathscr{R}_k$. In particular, for any $\lambda > 0$, the operator $(\lambda - S)^{-1}$ is well defined and has norm bounded by λ^{-1} (cf. [Li]). Consider, therefore, for $\lambda > 0$, the solution f_λ of the equation

$$(\lambda - S) f_\lambda = h \ . \tag{2.2}$$

We have seen at the beginning of this section that Sf_λ belongs to $\mathscr{H}_{-1}$. Since by Lemma 4.1 h also belongs to $\mathscr{H}_{-1}$, so does f_λ. Moreover, by equation (2.2), $\|h\|_{-1}^2$ is equal to

$$\lambda^2 \langle\langle f_\lambda, f_\lambda \rangle\rangle_{-1} \ + 2\lambda \langle\langle f_\lambda, f_\lambda \rangle\rangle_0 + \langle\langle f_\lambda, f_\lambda \rangle\rangle_1$$

so that $\|\lambda f_\lambda\|_{-1}$ and $\|f_\lambda\|_1$ are uniformly bounded in λ. In particular, there exists a subsequence λ_n for which $\lambda_n f_{\lambda_n}$ (resp. f_{λ_n}) converges weakly in $\mathscr{H}_{-1}$ (resp. in $\mathscr{H}_1$) to some limit denoted by F_{-1} (resp. F_1). We claim that $F_{-1} = 0$.

Indeed, let g be a function in $\mathscr{M}_k$ that belongs therefore to $\mathscr{H}_1 \cap \mathscr{H}_{-1}$. By definition,

$$\langle\!\langle F_{-1}, g \rangle\!\rangle_0 = \lim_{n\to\infty} \lambda_n \langle\!\langle f_{\lambda_n}, g \rangle\!\rangle_0 = 0 \ ,$$

because f_{λ_n} converges weakly to F_1 in $\mathscr{H}_1$. It follows from this identity and formula (1.7) for the $\mathscr{H}_{-1}$ norm of F_1 that $F_{-1} = 0$ in $\mathscr{H}_{-1}$.

It is now easy to deduce from this convergence the existence of a sequence $\lambda_{n,j}$, $1 \le j \le n$, such that $(1/n)\sum_{1\le j\le n} \lambda_{n,j} f_{\lambda_{n,j}}$ converges strongly to 0 in $\mathscr{H}_{-1}$. Since $f_{\lambda_{n,j}}$ is the solution of $(\lambda_{n,j} - S) f_{\lambda_{n,j}} = h$, we obtain that

$$S\left(\frac{1}{n}\sum_{1\le j\le n} f_{\lambda_{n,j}}\right) + h$$

converges strongly to 0 in $\mathscr{H}_{-1}$. Since $\mathscr{M}_k$ is dense in $\mathscr{R}_{k,1}$ and S is a bounded operator from $\mathscr{H}$ to $\mathscr{H}_{-1}$, we may replace $f_{\lambda_{n,j}}$ by a cylinder function in $\mathscr{M}_k$. This concludes the proof of the lemma. □

The same argument and some estimates on the asymmetric operator M permit to show that the range of $S + M$ is dense in $\mathscr{H}_{-1}$.

Theorem 2.2. *For every $k \ge 2$ and every h in $\mathscr{M}_k$,*

$$\inf_{f\in\mathscr{M}_k} \|(S+M)f - h\|_{-1} = 0 \ .$$

The proof of this result requires some notation. Denote by $\mathscr{Q}_n$ the configurations of $\mathbb{N}^{\mathbb{Z}^d}$ with n particles. Notice that $\mathscr{P}_n$ can be considered as the subset of $\mathscr{Q}_n$ consisting of all configurations of $\mathscr{Q}_n$ with at most one particle per site. We shall now extend to $\mathscr{Q}_n$ the structure defined in $\mathscr{P}_n$.

Denote by ζ, ξ the configurations of $\mathscr{Q}_n$ and by $\sim$ the equivalence relation defined by $\zeta \sim \xi$ if there exists x in $\mathbb{Z}^d$ such that $\zeta(z) = \xi(z + x)$ for every z in $\mathbb{Z}^d$. Let $\tilde{\mathscr{Q}}_n$ the quotient of $\mathscr{Q}_n$ with respect to this equivalence relation.

On $\mathscr{Q}_n$ consider the operator $S^{(i)}$ (resp. $M^{(i)}$) that corresponds to nearest neighbor symmetric (resp. totally asymmetric) independent random walks:

$$(S^{(i)}f)(\zeta) = (1/2)\sum_{|y|=1}\sum_{x\in\mathbb{Z}^d} \zeta(x)[f(\sigma^{x,y}\zeta) - f(\zeta)] \ ,$$

$$(M^{(i)}f)(\zeta) = \sum_{j=1}^{d}\sum_{x\in\mathbb{Z}^d} \zeta(x)[f(\sigma^{x,x+e_j}\zeta) - f(\zeta)] \ ,$$

where $\sigma^{x,y}\zeta$ is the configuration obtained from ζ letting one particle jump from x to y:

$$(\sigma^{x,x+y}\zeta)(z) = \begin{cases} \zeta(x) - 1 & \text{if } z = x, \\ \zeta(y) + 1 & \text{if } z = y, \\ \zeta(z) & \text{otherwise} \ . \end{cases}$$

Notice that $(S^{(i)}$ is the discrete Laplacian on $(\mathbb{Z}^d)^n$. Hereafter up to the end of the proof of Lemma 2.2, i stands for independent.

Denote by $\mathscr{C}_i$ the collection of functions $f : \mathscr{Q}_n \to \mathbb{R}$ of finite support, i.e., f belongs to $\mathscr{C}_i$ if there exists a positive integer ℓ such that $f(\zeta) = 0$ if $\zeta(x) > 0$ for some x not in Λ_ℓ. On $\mathscr{C}_i$ consider the inner product $\langle\langle \cdot, \cdot \rangle\rangle_{i,0}$ defined by

$$\langle\langle g, h \rangle\rangle_{i,0} = \sum_{\zeta \in \tilde{\mathscr{Q}}_n} \left(\sum_{\xi \sim \zeta} f(\zeta) \right) \left(\sum_{\xi \sim \zeta} g(\zeta) \right) .$$

We may now define the Hilbert space $\mathscr{H}_{i,0}$ generated by $\mathscr{C}_i$ and this inner product. An elementary computations shows that $S^{(i)}$ (resp. $M^{(i)}$) is a symmetric (resp. antisymmetric) operator with respect to this inner product. We may thus define the Hilbert spaces $\mathscr{H}_{i,1}$ and $\mathscr{H}_{i,-1}$ obtained from the operator $S^{(i)}$. The inner product are respectively denoted by $\langle\langle \cdot, \cdot \rangle\rangle_{i,1}$ and $\langle\langle \cdot, \cdot \rangle\rangle_{i,-1}$.

Every function f in $L^2(\mathscr{P}_n)$ can be extended to $\mathscr{Q}_n$ by setting $f(\zeta) = 0$ for every configuration ζ not in $\mathscr{P}_n$. Clearly, with this convention, $\langle\langle f, f \rangle\rangle_0 = \langle\langle f, f \rangle\rangle_{i,0}$ for every f in $L^2(\mathscr{P}_n)$. Moreover,

Lemma 2.3. *There exists an universal constant C such that*

$$\langle\langle g, g \rangle\rangle_1 \le \langle\langle g, g \rangle\rangle_{i,1} \le Cn \langle\langle g, g \rangle\rangle_1 ,$$
$$(Cn)^{-1} \langle\langle g, g \rangle\rangle_{-1} \le \langle\langle g, g \rangle\rangle_{i,-1} \le \langle\langle g, g \rangle\rangle_{-1}$$

for any cylinder function g in $\mathscr{M}_n$.

Proof. We only have to prove the first set of inequalities since the second is deduced from the first by duality. The first inequality is trivial from the definition of $S^{(i)}$. To prove the second inequality, recall from section 4 the definition of the set $R_{n,1}$ and notice that

$$\langle\langle g, g \rangle\rangle_{1,i} \le \langle\langle g, g \rangle\rangle_1 + \langle\langle (1 - w_1) g, (1 - w_1) g \rangle\rangle_0 .$$

The second inequality is now easily obtained from Corollary 4.11. □

We now return to Theorem 2.2.

Proof of Theorem 2.2. For each positive λ, consider the solution f_λ of

$$(\lambda - S - M) f_\lambda = h . \tag{2.3}$$

Taking the inner product with respect to f_λ on both sides of the equation, by Schwarz inequality we get that

$$\lambda \langle\langle f_\lambda, f_\lambda \rangle\rangle_0 + \langle\langle f_\lambda, f_\lambda \rangle\rangle_1 = \langle\langle f_\lambda, h \rangle\rangle_0 \le \|f_\lambda\|_1 \|h\|_{-1}$$

because M is asymmetric and h belongs to $\mathscr{H}_{-1}$ by Lemma 4.1. In particular, $\|f_\lambda\|_1$ is bounded above by $\|h\|_{-1}$.

With the notation introduced above we may rewrite equation (2.3) as

$$(\lambda - S^{(i)} - M^{(i)}) f_\lambda = h + E_i f_\lambda . \tag{2.4}$$

where $E_i = (S - S^{(i)}) + (M - M^{(i)})$. Since Mf_λ and Sf_λ belong to $\mathscr{H}_{-1}$, by Lemma 2.3, they also belong to $\mathscr{H}_{i,-1}$. On the other hand, by the same reasons presented just after (2.1), $M^{(i)}f_\lambda$ and $S^{(i)}f_\lambda$ belong to $\mathscr{H}_{i,-1}$. Therefore, all terms in (2.4) belong to $\mathscr{H}_{i,-1}$. The $\mathscr{H}_{i,-1}$ norm of $h + E_i f_\lambda$ is equal to

$$\begin{aligned}\lambda^2 \langle\!\langle f_\lambda, (-S^{(i)})^{-1} f_\lambda \rangle\!\rangle_{i,0} &+ 2\lambda \langle\!\langle f_\lambda, f_\lambda \rangle\!\rangle_{i,0} - 2\lambda \langle\!\langle f_\lambda, (-S^{(i)})^{-1} M^{(i)} f_\lambda \rangle\!\rangle_{i,0} \\ &- 2\langle\!\langle f_\lambda, M^{(i)} f_\lambda \rangle\!\rangle_{i,0} + \langle\!\langle f_\lambda, (-S^{(i)}) f_\lambda \rangle\!\rangle_{i,0} \\ &+ \langle\!\langle M^{(i)} f_\lambda, (-S^{(i)})^{-1} M^{(i)} f_\lambda \rangle\!\rangle_{i,0} \ .\end{aligned}$$

It is easy to check that $M^{(i)}$ and $S^{(i)}$ commute and thus $(S^{(i)})^{-1}$ and $M^{(i)}$ also commute. In particular, the third term on the first line vanishes because $M^{(i)}$ is antisymmetric, for the same reasons, the first term on the second line vanishes. Since all other terms are positive,

$$\lambda^2 \langle\!\langle f_\lambda, (-S^{(i)})^{-1} f_\lambda \rangle\!\rangle_{i,0} \ \le \ \langle\!\langle h + E_i f_\lambda, (-S^{(i)})^{-1} h + E_i f_\lambda \rangle\!\rangle_{i,0} \ . \qquad (2.5)$$

The right hand side is bounded above by $2\|h\|_{i,-1}^2 + 2\|E_i f_\lambda\|_{i,-1}^2$. By Lemma 2.3, $\|h\|_{i,-1}^2$ is less than or equal to $\|h\|_{-1}^2$.

For a positive integer ℓ, denote by $W_{n,\ell}$ the subsets of $\mathbb{Z}^d$ in $\mathscr{P}_n$ whose elements are at least at distance ℓ:

$$W_{n,\ell} = \{A = \{a_1, \dots, a_n\} \in \mathscr{P}_n; |a_i - a_j| \ge \ell \text{ for } i \ne j\} \qquad (2.6)$$

and by $w_\ell = w_{n,\ell}$ the indicator function of this set. Since S, $S^{(i)}$ and M, $M^{(i)}$ coincide on the sets Λ belonging to $W_{n,2}$, $E_i f_\lambda = (1 - w_3) E_i f_\lambda$. Therefore, by the variational formula for the $\mathscr{H}_{i,-1}$ norm, Schwarz inequality and a version of Corollary 4.11 for independent random walks, $\|E_i f_\lambda\|_{i,-1}^2$ is bounded above by $Cn\|E_i f_\lambda\|_{i,1}^2$. A direct computation similar to the one performed to prove Lemma 4.3 shows that this expression is less than or equal to $Cn^3\|f_\lambda\|_{i,1}^2$, which is bounded by $Cn^4\|f_\lambda\|_1^2$, in virtue of Lemma 2.3. In conclusion, it follows from the estimate deduced in the beginning of this proof, Lemma 2.3 and (2.5) that

$$\lambda^2 \langle\!\langle f_\lambda, f_\lambda \rangle\!\rangle_{-1} \ \le \ Cn\lambda^2 \langle\!\langle f_\lambda, (-S^{(i)})^{-1} f_\lambda \rangle\!\rangle_{i,0} \le Cn^5 \langle\!\langle h, h \rangle\!\rangle_{-1}$$

for some constant C depending on the dimension only. We may now repeat the arguments presented at the end of the proof of Lemma 2.1 to conclude. □

For $n \ge 1$, let $\pi_n \colon \mathscr{H} \to \mathscr{R}_n$ denote the projection onto the space generated by the monomials of degree n and let $P_n = \sum_1^n \pi_n$. To restrict the operator J to $\mathscr{R}_n$, define J_n as $P_n J P_n$. Notice that

$$\langle\!\langle g, Jg \rangle\!\rangle_0 \ = \ \langle\!\langle g, J_n g \rangle\!\rangle_0 = 0$$

The leftmost expression vanishes because $J = A - M$ and both A and M are asymmetric operators. On the other hand, by definition of J_n, $\langle\!\langle g, J_n g \rangle\!\rangle_0 = \langle\!\langle P_n g, JP_n g \rangle\!\rangle_0$ that vanishes because we just showed that J is an asymmetric operator.

In section 5, using perturbative arguments, we shall extend the previous result to the operator L_n defined by

$$L_n = S + M + P_n J P_n \ .$$

Theorem 2.4. *Fix $k \geq 2$ and h in $\mathscr{C}_k$. For any $n \geq k$,*

$$\inf_{u \in \mathscr{C}_n} \|h - L_n u\|_{-1} = 0 \ . \tag{2.7}$$

We need the following estimates from [SVY] on the $\mathscr{H}_1$ norm of the solution to the equation (2.7) :

Lemma 2.5 (A priori estimate). *Fix $k \geq 2$, h in $\mathscr{C}_k$ and $n \geq k$. Consider u a solution of equation (2.7) in the sense that u belongs to $\mathscr{C}_n$ and $\|h - L_n u\|_{-1} \leq e^{-n}$. Then, for every $m \geq 1$*

$$\sum_{j=2}^{n} j^m \|\pi_j u\|_1^2 \leq C(k,m)\{1 + \|h\|_{-1}\}$$

for some finite constant $C(k,m)$ depending only on k and m.

Proof. Fix $m \geq 1$, a cylinder function u in $\mathscr{C}_n$ such that $\|h - L_n u\|_{-1} \leq e^{-n}$ and denote $\pi_j u$ by u_j. To prove the lemma we shall estimate $\sum_{1 \leq j \leq n} (a + j^m) \langle\!\langle u_j, Lu \rangle\!\rangle_0$ for all $a > 0$. Fix $1 \leq j \leq n$ and adopt the convention that $u_0 = u_{n+1} = 0$. Since $L = S + A$, A is asymmetric and S preserves the degree of a monomial while A changes it by at most 1,

$$\begin{aligned} \langle\!\langle u_j, Lu \rangle\!\rangle_0 &= \langle\!\langle u_j, Su_j \rangle\!\rangle_0 + \langle\!\langle u_j, Au_{j+1} \rangle\!\rangle_0 + \langle\!\langle u_j, Au_{j-1} \rangle\!\rangle_0 \\ &= \langle\!\langle u_j, Su_j \rangle\!\rangle_0 + \langle\!\langle u_j, Au_{j+1} \rangle\!\rangle_0 - \langle\!\langle u_{j-1}, Au_j \rangle\!\rangle_0 \ . \end{aligned}$$

Therefore, by summation by parts and the convention that $u_0 = u_{n+1} = 0$, $\sum_{1 \leq j \leq n} (a + j^m) \ \langle\!\langle u_j, Lu \rangle\!\rangle_0$ is equal to

$$\begin{aligned} &\sum_{j=1}^{n} (a + j^m) \langle\!\langle u_j, Su_j \rangle\!\rangle_0 + \sum_{j=1}^{n} (a + j^m) \big\{ \langle\!\langle u_j, Au_{j+1} \rangle\!\rangle 0 - \langle\!\langle u_{j-1}, Au_j \rangle\!\rangle_0 \big\} \\ &\quad = \sum_{j=1}^{n} (a + j^m) \langle\!\langle u_j, Su_j \rangle\!\rangle_0 + \sum_{j=1}^{n} [j^m - (j+1)^m] \langle\!\langle u_j, Au_{j+1} \rangle\!\rangle_0 \ . \end{aligned}$$

Since $A = M + J$ and M preserves the degree, $\langle\!\langle u_j, Au_{j+1} \rangle\!\rangle_0 = \langle\!\langle u_j, Ju_{j+1} \rangle\!\rangle_0$. By Schwarz inequality, the absolute value of this expression is bounded above by $(1/2\gamma) \langle\!\langle u_j, u_j \rangle\!\rangle_1 + (\gamma/2) \langle\!\langle Ju_{j+1}, Ju_{j+1} \rangle\!\rangle_{-1}$ for every $\gamma > 0$. By Corollary 4.5 below, $\langle\!\langle Ju_{j+1}, Ju_{j+1} \rangle\!\rangle_{-1}$ is bounded above by $Cj \langle\!\langle u_{j+1}, u_{j+1} \rangle\!\rangle_1$ for some universal constant C. Therefore, choosing $\gamma = j^{-1/2}$,

$$|\langle\!\langle u_j, Au_{j+1} \rangle\!\rangle_0| \leq C\sqrt{j} \big\{ \langle\!\langle u_j, u_j \rangle\!\rangle_1 + \langle\!\langle u_{j+1}, u_{j+1} \rangle\!\rangle_1 \big\}$$

so that

$$\left|\sum_{j=1}^{n}[j^m-(j+1)^m]\langle\!\langle u_j, Au_{j+1}\rangle\!\rangle_0\right| \le C(m)\sum_{j=1}^{n} j^{m-(1/2)}\langle\!\langle u_j,u_j\rangle\!\rangle_1$$

for some constant C depending only on m.

On the other hand, $\sum_{1\le j\le n}(a+j^m)\langle\!\langle u_j, Lu\rangle\!\rangle_0$ is equal to

$$\sum_{j=1}^{n}(a+j^m)\big\{\langle\!\langle u_j, L_nu-h\rangle\!\rangle_0+\langle\!\langle u_j,h\rangle\!\rangle_0\big\}\ . \tag{2.8}$$

By Schwarz inequality and the assumption on u, $\langle\!\langle u_j, L_nu-h\rangle\!\rangle_0$ is bounded above by $(1/4)\langle\!\langle u_j,u_j\rangle\!\rangle_1+e^{-n}$. In contrast, since the spaces $\mathscr{M}_j$ are orthogonal, by Schwarz inequality, $\langle\!\langle u_j,h\rangle\!\rangle_0=\langle\!\langle u_j,h_j\rangle\!\rangle_0$ is bounded above by $(1/4)\langle\!\langle u_j,u_j\rangle\!\rangle_1+\langle\!\langle h_j,h_j\rangle\!\rangle_{-1}$. Since by assumption h belongs to $\mathscr{C}_k$, the absolute value of (2.8) is bounded above by

$$(1/2)\sum_{j=1}^{n}(a+j^m)\langle\!\langle u_j,u_j\rangle\!\rangle_1+\sum_{j=1}^{n}(a+j^m)e^{-n}+\sum_{j=1}^{k}(a+j^m)\langle\!\langle h_j,h_j\rangle\!\rangle_{-1}$$
$$=(1/2)\sum_{j=1}^{n}(a+j^m)\langle\!\langle u_j,u_j\rangle\!\rangle_1\ +\ C(a,m,k)\{1+\langle\!\langle h,h\rangle\!\rangle_{-1}\}$$

for some constant $C(a,m,k)$ depending only on a, k and m.

Recollecting the two previous estimates and since by definition $\langle\!\langle u,u\rangle\!\rangle_1=\langle\!\langle u,(-S)u\rangle\!\rangle_0$, we have that

$$\sum_{j=1}^{n} j^m\left\{\frac{1}{2}+\frac{a}{2j^m}-\frac{C(m)}{\sqrt{j}}\right\}\langle\!\langle u_j,u_j\rangle\!\rangle_1\le C(a,m,k)\{1+\langle\!\langle h,h\rangle\!\rangle_{-1}\}\ .$$

To conclude the proof of the lemma it remains to choose $a=a(m)$ large enough so that $(1/2)+(a/2j^m)-\{C(m)/\sqrt{j}\}\ge 1/4$ for every $j\ge 1$. □

Proof of Theorem 1.1. Fix $n\ge 2$. By Theorem 2.4 and Lemma 2.5, there exists u in $\mathscr{C}_n$ such that $\|h-L_nu\|_{-1}\le e^{-n}$ and $\sum_{1\le j\le n} j^2\|\pi_j u\|_1^2\le C(h)$. Since u belongs to $\mathscr{C}_n$ and the generator modifies the degree of a monomial by at most 1, $Lu=P_{n+1}Lu$. Therefore, $Lu=L_nu+\pi_{n+1}Lu=L_nu+J_+u_n$ because S and M preserve the degree of monomials and J_- reduces it by 1. In particular, by definition of u and Schwarz inequality, $\|Lu-h\|_{-1}^2$ is bounded above by $2\|J_+u_n\|_{-1}^2+2e^{-n}$. By Corollary 4.5 and the estimate of the $\mathscr{H}_1$ norm of u_n, $\|J_+u_n\|_{-1}^2\le n\|u_n\|_1^2\le C(h)n^{-1}$. In conclusion, we showed that $\|Lu-h\|_{-1}^2$ is bounded above by $2C(h)n^{-1}+2e^{-n}$, what concludes the proof of the theorem. □

3. Second proof of Theorem 1.1

Recall that we denote by L_n the operator L restricted to the space of functions of degree n: $L_n=S+M+P_nJP_n$. We have already seen in the previous section that the operator $J_n=P_nJP_n$ is antisymmetric. A simple computation shows that it is bounded in $\mathscr{R}_n$ for every $n \geq 1$. In particular, for any $\lambda > 0$, the operator $(\lambda - L_n)^{-1}$ is well defined and has norm bounded by λ^{-1}. Fix $k \geq 2$, h in $\mathscr{C}_k$, $n \geq k$ and consider the solution $f_{\lambda,n}$ of the equation $(\lambda - L_n) f_{\lambda,n} = h$. Our first task is to obtain estimates, uniform in n, of the various norms of $f_{\lambda,n}$.

With the same proof of Lemma 2.5, we deduce

Lemma 3.1. *For every $\lambda > 0$, $n \geq k$ and $m \geq 1$,*

$$\lambda \sum_{j=1}^{n} j^m \|\pi_j f_{\lambda,n}\|_0^2 + \sum_{j=1}^{n} j^m \|\pi_j f_{\lambda,n}\|_1^2 \leq C(m,k) \|h\|_{-1}^2$$

for some finite constant $C(m,k)$ depending only on m and k.

Lemma 3.2. *For each fixed n and λ,*

$$\lambda \|f_{\lambda,n}\|_{-1} \leq C(k) \|h\|_{-1}$$

for some finite constant C depending only on k.

Proof. It follows from estimate (4.3), Corollary 4.5 and Lemma 3.1 that $L_n f_{\lambda,n}$ belongs to $\mathscr{H}_{-1}$. In particular, $f_{\lambda,n}$ belongs to $\mathscr{H}_{-1}$ because $h \in \mathscr{H}_{-1}$ and $f_{\lambda,n} = \lambda^{-1}(h + L_n f_{\lambda,n})$. Moreover, from the equation $(\lambda - L_n) f_{\lambda,n} = h$, we obtain that $\|h\|_{-1}^2$ is equal to

$$\lambda^2 \langle\!\langle f_{\lambda,n}, f_{\lambda,n} \rangle\!\rangle_{-1} + 2\lambda \langle\!\langle f_{\lambda,n}, f_{\lambda,n} \rangle\!\rangle_0 - 2\lambda \langle\!\langle f_{\lambda,n}, A_n f_{\lambda,n} \rangle\!\rangle_{-1}$$
$$- 2 \langle\!\langle f_{\lambda,n}, A_n f_{\lambda,n} \rangle\!\rangle_0 + \langle\!\langle f_{\lambda,n}, f_{\lambda,n} \rangle\!\rangle_1 + \langle\!\langle A_n f_{\lambda,n}, A_n f_{\lambda,n} \rangle\!\rangle_{-1} \ ,$$

where A_n stands for $M + J_n$. The first term on the second line vanishes because A_n is antisymmetric. Since all but the third term on the first line are positives,

$$\lambda^2 \langle\!\langle f_{\lambda,n}, f_{\lambda,n} \rangle\!\rangle_{-1} \leq \|h\|_{-1}^2 + 2\lambda \langle\!\langle f_{\lambda,n}, A_n f_{\lambda,n} \rangle\!\rangle_{-1} \ . \tag{3.1}$$

To conclude the proof of the lemma it remains to estimate the second term on the right hand side of this inequality.

On the one hand, we claim that for all $n \geq k$,

$$2\lambda \left| \langle\!\langle f_{\lambda,n}, J_n f_{\lambda,n} \rangle\!\rangle_{-1} \right| \leq \frac{\lambda^2}{4} \|f_{\lambda,n}\|_{-1}^2 + C(k) \|h\|_{-1}^2 \tag{3.2}$$

for some constant $C(k)$ depending only on k. Indeed, $2\lambda \langle\!\langle f_{\lambda,n}, J_n f_{\lambda,n} \rangle\!\rangle_{-1}$ is equal to

$$2\lambda \sum_{\substack{1 \leq j,k \leq n \\ |j-k|=1}} \langle\!\langle \pi_j f_{\lambda,n}, J \pi_k f_{\lambda,n} \rangle\!\rangle_{-1} \leq (\lambda^2/4) \sum_{j=1}^{n} \|\pi_j f_{\lambda,n}\|_{-1}^2 + 16 \sum_{j=1}^{n} \|J \pi_j f_{\lambda,n}\|_{-1}^2 \ .$$

Since the spaces $\{\mathscr{M}_j, j \geq 2\}$ are orthogonal in $\mathscr{H}_{-1}$, the first term on the right hand side is just $(\lambda^2/4)\|f_{\lambda,n}\|_{-1}^2$. The second, by Corollary 4.5, is bounded above by $C_0 \sum_{j=1}^{n} j \|\pi_j f_{\lambda,n}\|_1^2$ for some universal constant C_0. In virtue of Lemma 3.1, this expression is less than or equal to $C_1(k)\|h\|_{-1}^2$, what proves (3.2).

On the other hand, we claim that

$$2\lambda\big|\langle\!\langle f_{\lambda,n}, Mf_{\lambda,n}\rangle\!\rangle_{-1}\big| \le \frac{\lambda^2}{4}\|f_{\lambda,n}\|_{-1}^2 + C(d,k)\|h\|_{-1}^2 \tag{3.3}$$

for some constant depending only on the degree k and the dimension d. Indeed, since M preserves the degree, $2\lambda\langle\!\langle f_{\lambda,n}, Mf_{\lambda,n}\rangle\!\rangle_{-1}$ is equal to

$$2\lambda\sum_{j=1}^{n}\langle\!\langle \pi_j f_{\lambda,n}, M\pi_j f_{\lambda,n}\rangle\!\rangle_{-1} \ .$$

By Lemma 4.12, with $\gamma = \lambda/8$, this expression is bounded above by

$$(\lambda^2/4)\sum_{j=1}^{n}\pi_j f_{\lambda,n}, \langle\!\langle \pi_j f_{\lambda,n}\rangle\!\rangle_{-1} + C\sum_{j=1}^{n} j^3\langle\!\langle \pi_j f_{\lambda,n}, \pi_j f_{\lambda,n}\rangle\!\rangle_1$$

for some constant $C(d)$ depending exclusively on the dimension d. The first term is just $(\lambda^2/4)\|f_{\lambda,n}\|_{-1}^2$ because the spaces $\mathscr{M}_j$ are orthogonal in $\mathscr{H}_{-1}$. By Lemma 3.1, the second term is bounded by $C(k)\|h\|_{-1}^2$, what proves claim (3.3).

Estimates (3.2) and (3.3) together with inequality (3.1) conclude the proof of the lemma. □

We are now ready to prove Theorem 1.1

Proof of Theorem 1.1. In the previous two lemmas, we showed that the solution $f_{\lambda,n}$ of the equation $(\lambda - L_n)f_{\lambda,n} = h$ is such that

$$\lambda\|f_{\lambda,n}\|_0^2 + \lambda^2\|f_{\lambda,n}\|_{-1}^2 + \|f_{\lambda,n}\|_1^2 \le C(k)\|h\|_{-1}^2$$

for some finite constant $C(k)$ depending only on k. Since these estimates are uniform over n, we may consider a subsequence, still denoted by n for convenience, so that $f_{\lambda,n}$ converges weakly in $\mathscr{H}_{-1}$, $\mathscr{H}$ and $\mathscr{H}_1$ to a function f_λ. It is straightforward to check that f_λ is a solution of $(\lambda - L)f_\lambda = h$. Moreover, from the previous estimates we obtain that

$$\lambda^2\|f_\lambda\|_{-1}^2 + \|f_\lambda\|_1^2 \le C(k)\|h\|_{-1}^2 \ .$$

We may now consider a subsequence λ_k for which f_{λ_k} converges weakly in $\mathscr{H}_1$ to some F_1 and $\lambda_k f_{\lambda_k}$ converges weakly in $\mathscr{H}_{-1}$ to some F_{-1}. For g in $\mathscr{H}_1 \cap \mathscr{H}_{-1}$, we have that

$$\langle\!\langle g, F_{-1}\rangle\!\rangle_0 = \lim_{k\to\infty}\lambda_k\langle\!\langle g, f_{\lambda_k}\rangle\!\rangle_0 \ = \ 0$$

because f_{λ_k} converges weakly in $\mathscr{H}_1$ to F_1. This shows that $\lambda_k f_{\lambda_k}$ converges weakly to 0 in $\mathscr{H}_{-1}$. From this convergence, it follows that there exists a sequence $\lambda_{n,j}$, $1 \le j \le n$, such that $(1/n)\sum_{1\le j\le n}\lambda_{n,j} f_{\lambda_{n,j}}$ converges strongly to 0 in $\mathscr{H}_{-1}$. Since $f_{\lambda_{n,j}}$ is the solution of $(\lambda_{n,j} - L)f_{\lambda_{n,j}} = h$, we obtain that

$$L\left(\frac{1}{n}\sum_{1\le j\le n} f_{\lambda_{n,j}}\right) + h$$

converges strongly to 0 in $\mathscr{H}_{-1}$, what concludes the proof of the theorem because $f_{\lambda_{n,j}}$ belongs to $\mathscr{H}$ and may thus be approximated in $\mathscr{H}_{-1}$ by linear combinations of cylinder functions. □

4. Estimates on the asymmetric part of the generator

We start this section with some remarks concerning the operators M, S, J and the Hilbert spaces $\mathscr{H}$, $\mathscr{H}_1$ and $\mathscr{H}_{-1}$. Recall from (1.9) the definition of the Dirichlet form D. For a fixed cylinder function g in $\mathscr{M}_n$: $g = \sum_{\Lambda;|\Lambda|=n} g_\Lambda \eta_\Lambda$, the Dirichlet form of g writes

$$D(g) = (1/4)K_n(\rho)\sum_{j=1}^{d}\sum_{x}\sum_{\substack{\Lambda,|\Lambda|=n-1\\ \Lambda\cap\{x,x+e_j\}=\phi}} \left\{g_{\Lambda\cup\{x+e_j\}} - g_{\Lambda\cup\{x\}}\right\}^2 . \tag{4.1}$$

Moreover, every cylinder function h in $\mathscr{M}_k$ has a finite $\mathscr{H}_1$ norm equal to

$$\langle\!\langle h, h\rangle\!\rangle_1 = \lim_{\ell\to\infty}(2\ell)^{-d} D\left(\sum_{|x|\le \ell - s_h}\tau_x h\right) , \tag{4.2}$$

where s_h is the linear size of the support of h : $s_h = \min\{k,\ h$ is measurable with respect to $\mathscr{F}_k = \sigma(\eta(x), x\in\Lambda_k)\}$ so that $\tau_x h$ is $\mathscr{F}_\ell$ measurable for all x in $\Lambda_{\ell-s_h}$.

Next result is a restatement of Lemma 5.1 in [EMY1].

Lemma 4.1. *In dimension 3 or higher every cylinder function h in $\mathscr{M}_k$, $k\ge 2$, has finite $\mathscr{H}_{-1}$ norm and*

$$\langle\!\langle h, h\rangle\!\rangle_{-1} = \lim_{\ell\to\infty}(2\ell)^{-d}\sup_f\left\{2\left\langle\sum_{|x|\le\ell-s_h}\tau_x h, f\right\rangle_\rho - D_\ell(f)\right\} ,$$

where the supremum is carried over all cylinder functions f measurable with respect to $\mathscr{F}_\ell$ and $D_\ell(f)$ is the Dirichlet form restricted to Λ_ℓ:

$$D_\ell(f) = \sum_{\substack{x,y\in\Lambda_\ell\\ |x-y|=1}}\langle(-S_{x,y})f, f\rangle_\rho .$$

We turn now to the proof of some estimates on the asymmetric operators M and J used throughout the article.

Lemma 4.2. *For every $n\ge 2$ and every function f in $\mathscr{R}_n$,*

$$\langle\!\langle Mf, Mf\rangle\!\rangle_{-1} \le 2(1-2\rho)^2 nd\langle\!\langle f,f\rangle\!\rangle_0 \cdot \tag{4.3}$$

Proof. Fix a function f in $\mathscr{M}_n$. Recall from (2.1) the explicit formula for the inner product $\langle\!\langle Mf, g\rangle\!\rangle_0$ for some cylinder function g in $\mathscr{M}_n$. By Schwarz inequality,

$$\langle\!\langle Mf, g\rangle\!\rangle_0 \le (1-2\rho)\left\{\frac{\gamma}{2}\|g\|_1^2 + \frac{dn}{\gamma}\|f\|_0^2\right\}$$

for every $\gamma > 0$. The statement of the lemma for cylinder functions in $\mathscr{M}_n$ follows from this inequality and formula (1.7) for the $\mathscr{H}_{-1}$ norm. To extend it to functions in $\mathscr{R}_n$ one just need to recall that $\mathscr{M}_n$ is dense in $\mathscr{R}_n$ and that M is a bounded operator on $\mathscr{R}_n$. □

Lemma 4.3. *There exists a constant C depending only on the dimension d such that for every $n \ge 2$,*

$$\langle\!\langle SMf, SMf\rangle\!\rangle_1 \le Cn^2\langle\!\langle Sf, Sf\rangle\!\rangle_1 \text{ and } \langle\!\langle Mf, Mf\rangle\!\rangle_1 \le Cn^2\langle\!\langle f, f\rangle\!\rangle_1$$

for every f in $\mathscr{R}_n$.

Proof. This statement follows straightforwardly from the explicit form (1.6) for the inner product $\langle\!\langle\cdot,\cdot\rangle\!\rangle_1$ and Schwarz inequality. □

We investigate now the antisymmetric operator J. This result was obtained in collaboration with S. Sethuraman and S. Varadhan.

Theorem 4.4. *There exists an universal constant C such that for every $n \ge 2$ and cylinder functions h in $\mathscr{M}_{n+1}$,* g in $\mathscr{M}_n$

$$\langle h, J_+ g\rangle_\rho \le C\sqrt{n}\{\gamma D(h) + \gamma^{-1}D(g)\}$$

for every $\gamma > 0$. Moreover, if both functions h and J_+g are *$\mathscr{F}_\ell$-measurable we may replace on the right hand side $D(h)+D(g)$ by $D(h)+D_\ell(g)$.*

Before proving this theorem, we deduce an estimate repeatedly used in the previous section.

Corollary 4.5. *There exists an universal constant C such that for every $k \ge 2$ and cylinder function g in $\mathscr{C}_k$,*

$$\langle\!\langle Jg, Jg\rangle\!\rangle_{-1} \le C\rho(1-\rho)k\langle\!\langle Sg, Sg\rangle\!\rangle_{-1} = Ck\langle\!\langle g, g\rangle\!\rangle_1 \ .$$

Proof. Since the spaces $\mathscr{M}_j$, $2 \le j \le k$, are orthogonal in $\mathscr{H}_{-1}$, assume without loss of generality that g belongs to $\mathscr{M}_j$ for some $2 \le j \le k$. By Lemma 4.1,

$$\langle\!\langle lJg, Jg\rangle\!\rangle_{-1} = \lim_{\ell\to\infty}(2\ell+1)^{-d}\sup_f\left\{2\left\langle\sum_{|x|<\ell-s_g}\tau_x Jg, f\right\rangle_\rho - D_\ell(f)\right\} \ .$$

Since both $\tau_x Jg$ and f are $\mathscr{F}_\ell$-measurable, by Theorem 4.4, for each fixed ℓ,

$$2\left\langle \sum_{|x|<\ell-s_g} \tau_x Jg, f \right\rangle_\rho \le CkD\left(\sum_{|x|<\ell-s_g} \tau_x g \right) + D_\ell(f)$$

for some universal constant C. In particular, the last supremum is bounded above by

$$Ck(2\ell+1)^{-d} D\left(\sum_{|x|<\ell-s_g} \tau_x g \right) \ .$$

By (4.2), as $\ell \uparrow \infty$, this expression converges to $Ck\langle\!\langle g, g\rangle\!\rangle_1 = Ck \langle\!\langle Sg, Sg\rangle\!\rangle_{-1}$. □

Corollary 4.6. *There exists an universal constant C such that for every $k \ge 2$ and cylinder functions g, f in $\mathscr{C}_k$,*

$$\langle\!\langle Jg, f\rangle\!\rangle_{-1} \le C\sqrt{k}\|g\|_1 \|f\|_1 \ .$$

Proof. Since the spaces $\mathscr{M}_j$, $2 \le j \le k$, are orthogonal in $\mathscr{H}_{-1}$, assume without loss of generality that g belongs to $\mathscr{M}_j$ and f belongs to $\mathscr{M}_{j+1}$ for some $2 \le j \le k-1$. In this case, by definition of the inner product $\langle\!\langle \cdot, \cdot \rangle\!\rangle_0$,

$$\langle\!\langle J_+ g, f\rangle\!\rangle_0 = \lim_{\ell\to\infty} (2\ell)^{-d} \left\langle \sum_{|x|\le \ell-s_f} J\tau_x f, \sum_{|y|\le \ell - s_g} \tau_y g \right\rangle_\rho \ .$$

By Theorem 4.4 and identity (4.2), this expression is bounded above by

$$\begin{aligned} &C\sqrt{n} \lim_{\ell\to\infty} (2\ell)^{-d} \left\{ \gamma D\left(\sum_{|x|\le\ell-s_f} J\tau_x f \right) + \gamma^{-1} D\left(\sum_{|y|\le \ell-s_g} \tau_y g \right) \right\} \\ &\quad = C\sqrt{n}\left\{ \gamma \langle\!\langle f, f\rangle\!\rangle_1 + \gamma^{-1}\gamma\langle\!\langle g, g\rangle\!\rangle_1 \right\} \ . \end{aligned}$$

To conclude the proof of the corollary it remains to optimize in γ. □

Proof of Theorem 4.4. Fix $n \ge 2$ and cylinder functions h in $\mathscr{M}_{n+1}$ and g in $\mathscr{M}_n$: $h = \sum_{\Lambda, |\Lambda|=n+1} h_\Lambda \eta_\Lambda$, $g = \sum_{\Lambda, |\Lambda|=n} g_\Lambda \eta_\Lambda$. By the explicit formula for $J_+ g$ obtained in the previous section,

$$\langle h, J_+ g\rangle_\rho = K_{n+1}(\rho) \sum_{j=1}^d \sum_x \sum_{\substack{\Lambda, |\Lambda|=n-1 \\ \Lambda\cap\{x, x+e_j\}=\phi}} \left\{ g_{\Lambda\cup\{x+e_j\}} - g_{\Lambda\cup\{x\}} \right\} h_{\Lambda\cup\{x,x+e_j\}} \ . \tag{4.4}$$

By Schwarz inequality, this expression is bounded above by

$$\frac{1}{2\gamma}K_{n+1}(\rho)\sum_{j=1}^{d}\sum_{x}\sum_{\substack{\Lambda;|\Lambda|=n-1\\ \Lambda\cap\{x,x+e_j\}=\phi}}\Big\{g_{\Lambda\cup\{x+e_j\}}-g_{\Lambda\cup\{x\}}\Big\}^2$$

$$+\frac{\gamma}{2}K_{n+1}(\rho)\sum_{j=1}^{d}\sum_{x}\sum_{\substack{\Lambda;|\Lambda|=n-1\\ \Lambda\cap\{x,x+e_j\}=\phi}}h^2_{\Lambda\cup\{x,x+e_j\}}$$

for every $\gamma > 0$. By (4.1), the first line is just $\{\{\rho(1-\rho)\}/2\gamma\}D(g)$.

To conclude the proof of the theorem, it remains to show that the second line is bounded above by $C\gamma\rho(1-\rho)nD(h)$ and choose $\gamma = n^{-1/2}\gamma_0$. This estimate is the content of the next result.

Note that in the case where h and g are $\mathscr{F}_\ell$-measurable, on the right hand side of (4.4), we may restrict the summation over all sites x such that x and $x+e_j$ belong to Λ_ℓ because otherwise $h_{\Lambda\cup\{x,x+e_j\}}=0$. Thus we obtain an estimate with $D_\ell(g)$ in place of $D(g)$. □

Theorem 4.7. *There is a constant $C > 0$ independent of n such that*

$$CK_n(\rho)\sum_{j=1}^{d}\sum_{x\in\mathbb{Z}^d}\sum_{\substack{\Lambda;|\Lambda|=n-2\\ \Lambda\cap\{x,x+e_j\}=\phi}}h^2_{\Lambda\cup\{x,x+e_j\}}\le(n-1)D(h)$$

for every function h in $L^2(\mathscr{P}_n)$.

The proof of Theorem 4.7 is divided in several lemmas. We start with a Schwarz inequality. For each function $g\colon\mathscr{P}_n\to\mathbb{R}$ in $L^2(\mathscr{P}_n)$ ($\sum_{|\Lambda|=n} g_\Lambda^2<\infty$), denote by ρ_1 and ρ_2 the one and two point functions:

$$\rho_1(x)=\sum_{\substack{\Lambda;|\Lambda|=n\\ x\in\Lambda}}g_\Lambda^2=\sum_{\substack{\Lambda;|\Lambda|=n-1\\ \Lambda\cap\{x\}=\phi}}g^2_{\Lambda\cup\{x\}}$$

$$\rho_2(x,y)=\sum_{\substack{\Lambda;|\Lambda|=n\\ x,y\in\Lambda}}g_\Lambda^2=\sum_{\substack{\Lambda;|\Lambda|=n-2\\ \Lambda\cap\{x,y\}=\phi}}g^2_{\Lambda\cup\{x,y\}}$$

for all $x\neq y$ in $\mathbb{Z}^d$.

Lemma 4.8. *For any site $x\neq y$,*

$$\Big\{\sqrt{\rho_1(y)}-\sqrt{\rho_1(x)}\Big\}^2\le\sum_{\substack{\Lambda;|\Lambda|=n-1\\ \Lambda\cap\{x,y\}=\phi}}\big\{g_{\Lambda\cup\{y\}}-g_{\Lambda\cup\{x\}}\big\}^2$$

$$\Big\{\sqrt{\rho_2(y,z)}-\sqrt{\rho_2(x,z)}\Big\}^2\le\sum_{\substack{\Lambda;|\Lambda|=n-2\\ \Lambda\cap\{z,x,y\}=\phi}}\big\{g_{\Lambda\cup\{y,z\}}-g_{\Lambda\cup\{x,z\}}\big\}^2$$

for all $z\neq x$, y.

Proof. Fix two distinct sites x and y. We may rewrite $\rho_1(x)$ as

$$\rho_1(x) = \sum_{\substack{\Lambda;|\Lambda|=n-1\\ \Lambda\cap\{x,y\}=\phi}} g^2_{\Lambda\cup\{x\}} + \sum_{\substack{\Lambda;|\Lambda|=n-2\\ \Lambda\cap\{x,y\}=\phi}} g^2_{\Lambda\cup\{x,y\}} .$$

To conclude the proof of the first inequality, it remains to rewrite $\rho_1(y)$ in a similar way and recall the Schwarz inequality

$$\left(\left\{\sum_j a_j^2\right\}^{1/2} - \left\{\sum_j b_j^2\right\}^{1/2}\right)^2 \le \sum_j (a_j - b_j)^2 .$$

The proof of the second inequality is similar to the one presented. We leave the details to the reader. □

Lemma 4.8 and the explicit formula (4.1) for the Dirichlet formula give that

$$\sum_{j=1}^d \sum_x \left\{\sqrt{\rho_1(x+e_j)} - \sqrt{\rho_1(x)}\right\}^2 \le K_n(\rho)^{-1} D(g) .$$

By the same reasons,

$$\sum_{j=1}^d \sum_{\substack{x,y\in\mathbb{Z}^d\\ y\neq x,x+e_j}} \left\{\sqrt{\rho_2(x+e_j,y)} - \sqrt{\rho_2(x,y)}\right\}^2$$

$$\le \sum_{j=1}^d \sum_{\substack{x,y\in\mathbb{Z}^d\\ y\neq x,x+e_j}} \sum_{\substack{\Lambda;|\Lambda|=n-1,y\in\Lambda\\ \Lambda\cap\{x,x+e_j\}=\phi}} \{g_{\Lambda\cup\{x+e_j\}} - g_{\Lambda\cup\{x\}}\}^2$$

$$= (n-1)\sum_{j=1}^d \sum_{x\in\mathbb{Z}^d} \sum_{\substack{\Lambda;|\Lambda|=n-1\\ \Lambda\cap\{x,x+e_j\}=\phi}} \{g_{\Lambda\cup\{x+e_j\}} - g_{\Lambda\cup\{x\}}\}^2 = (n-1)K_n(\rho)^{-1}D(g) .$$

We are now in a position to conclude the proof of Theorem 4.7.

Proof of Theorem 4.7. From the definition of the two point functions and the previous estimate, we just have to show that there exists a constant $C > 0$ such that

$$C \sum_{|x-y|=1} \rho_2(x,y) - \sum_{j=1}^d \sum_{\substack{x,y\in\mathbb{Z}^d\\ y\neq x,x+e_j}} \left\{\sqrt{\rho_2(x+e_j,y)} - \sqrt{\rho_2(x,y)}\right\}^2 \le 0$$

for all two point functions. We may rewrite this inequality as

$$\sum_{y}\left\{C\sum_{x;|x-y|=1}\rho_2(x,y)-\sum_{j=1}^{d}\sum_{\substack{x\in\mathbb{Z}^d\\ x\neq y,y-e_j}}\left\{\sqrt{\rho_2(x+e_j,y)}-\sqrt{\rho_2(x,y)}\right\}^2\right\}\leq 0\ .$$

Since the problem is covariant in y, we can assume $y=0$. The problem is thus reduced to show that there exists a constant $C>0$ such that

$$C\sum_{x\in\mathbb{Z}^d}V(x)f(x)^2-\sum_{j=1}^{d}\sum_{x\neq 0,-e_j}\{f(x+e_j)-f(x)\}^2\leq 0 \tag{4.5}$$

for all $L^2(\mathbb{Z}^d)$ functions f. Here $V(x)$ is the finite supported function $V(x)=1\{|x|=1\}$. This estimate follows from a general result based on the Birman–Schwinger kernel.

Consider the Schrödinger operator

$$H=-\Delta-CV$$

where Δ is the Laplace operator in $\mathbb{Z}^d$ with Neumann boundary at origin, V is a nonnegative function with finite support and $C>0$. We claim that in dimension $d\geq 3$, $H\geq 0$ for C small enough. It is enough to show that there exists $C>0$ such that

$$-\Delta+\lambda\geq CV$$

for any constant $\lambda>0$. To keep notation simple, let $V_C=CV$. Multiplying both sides by $V_C^{1/2}[\lambda-\Delta]^{-1}$ from the left and by $V_C^{-1/2}$ from the right, the proof of the last inequality is reduced to show that

$$V_C^{1/2}[\lambda-\Delta]^{-1}V_C^{1/2}\leq 1\ .$$

The kernel of this operator is $K(x,y)=V_C(x)^{1/2}G_\lambda(x,y)V_C(y)^{1/2}$, where $G_\lambda=[\lambda-\Delta]^{-1}$. The Hilbert-Schmidt norm of K is bounded above by

$$\sum_{x,y}K(x,y)^2=C^2\sum_{x,y}V(x)G_\lambda(x,y)^2V(y)\ .$$

As λ decreases to 0 this expression converges to

$$C^2\sum_{x,y}V(x)G(x,y)^2V(y)$$

where, $G(x,y)$ is the Green function of the operator $-\Delta$. The operator $V_C^{1/2}[\lambda-\Delta]^{-1}V_C^{1/2}$ is thus bounded above by 1 for all C small enough because in dimension $d\geq 3$ the Green function G is finite and V has finite support. □

We conclude this section with an alternative version of Theorem 4.7 and further estimates on the antisymmetric operator M. Denote by $\mathbb{Z}_*^d$ the lattice $\mathbb{Z}^d$ without the origin and by $\mathscr{P}_n^*$ the subsets of $\mathbb{Z}_*^d$ with cardinality n. For each function $g:\mathscr{P}_n^*\to\mathbb{R}$ in $L^2(\mathscr{P}_n^*)$ ($\sum_{\Lambda\in\mathscr{P}_n^*}g_\Lambda^2<\infty$), denote by ρ_2^* the two point functions:

$$\rho_2^*(x,y) = \sum_{\substack{\Lambda\in\mathscr{P}_n^*;\\ x,y\in\Lambda}} g_\Lambda^2 = \sum_{\substack{\Lambda\in\mathscr{P}_{n-2}^*;\\ \Lambda\cap\{x,y\}=\phi}} g_{\Lambda\cup\{x,y\}}^2$$

for all $x \neq y$ in $\mathbb{Z}_*^d$. Denote furthermore by S^* the generator S with Neumann boundary conditions at the origin: $S^* = \sum_{x,y\in\mathbb{Z}_*^d,\,|x-y|=1} S_{x,y}$ and by D^* the Dirichlet form associated to S^*:

$$D^*(g) = \langle g, (-S^*)g\rangle_\rho = (1/4)K_n(\rho) \sum_{\substack{x,y\in\mathbb{Z}_*^d\\ |x-y|=1}} \sum_{\substack{\Lambda\in\mathscr{P}_{n-1}^*\\ \Lambda\cap\{x,y\}=\phi}} \{g_{\Lambda\cup\{y\}} - g_{\Lambda\cup\{x\}}\}^2 \quad (4.6)$$

for every cylinder function $g = \sum_{\Lambda\in\mathscr{P}_n^*} g_\Lambda \eta_\Lambda$.

The arguments presented in the proof of Theorem 4.7 gives that

Theorem 4.9. *There is a constant $C > 0$ independent of n such that*

$$CK_n(\rho) \sum_{\substack{x,y\in\mathbb{Z}_*^d;\\ |x-y|=1}} \sum_{\substack{\Lambda\in\mathscr{P}_{n-2}^*\\ \Lambda\cap\{x,y\}=\phi}} h_{\Lambda\cup\{x,y\}}^2 \leq (n-1)D^*(h)$$

for every function h in $L^2(\mathscr{P}_n^*)$.

The next result follows straightforwardly from this theorem and Schwarz inequality.

Corollary 4.10. *For every positive integer ℓ, there is a constant $C(\ell)$ depending only on ℓ such that*

$$C(\ell)K_n(\rho) \sum_{\substack{x,y\in\mathbb{Z}_*^d\\ 1\leq|x-y|\leq\ell}} \sum_{\substack{\Lambda\in\mathscr{P}_{n-2}^*\\ \Lambda\cap\{x,y\}=\phi}} h_{\Lambda\cup\{x,y\}}^2 \leq (n-1)D^*(h)$$

for every function h in $L^2(\mathscr{P}_n^*)$.

Recall from (2.6) that for a positive integer ℓ we denote by $W_{n,\ell}$ the subsets of $\mathbb{Z}^d$ in $\mathscr{P}_n$ whose elements are at least at distance ℓ and by $w_\ell = w_{n,\ell}$ the indicator function of this set.

Corollary 4.11. *For $n \geq 2$ and a cylinder function f in $\mathscr{M}_n$,*

$$\|f(1-w_\ell)\|_0^2 \leq C(\ell)n\|f\|_1^2$$

for some constant $C(\ell)$ depending on ℓ only.

Proof. Fix $n \geq 2$ and a cylinder function $f = \sum f_\Lambda \eta_\Lambda$ in $\mathscr{M}_n$. Define a new monomial $F = \sum F_\Lambda \eta_\Lambda$ in $\mathscr{M}_n$ by

$$F_\Lambda = \begin{cases} \sum_{B\sim\Lambda} f_B & \text{if } 0\in\Lambda \\ 0 & \text{otherwise .} \end{cases}$$

With this definition, we may write

$$\|f\|_0^2 = \frac{K_n(\rho)}{n} \sum_{\Lambda \in \mathscr{P}^*_{n-1}} F^2_{\Lambda\cup\{0\}} \text{ and } \|f\|_1^2 = \frac{K_n(\rho)}{4n} \sum_{\substack{\Lambda,\Omega \in \mathscr{P}^*_{n-1} \\ d^*(\Lambda,\Omega)=1}} \{F_{\Lambda\cup\{0\}} - F_{\Omega\cup\{0\}}\}^2 ,$$

where d^* is the natural distance in $\mathscr{P}^*_n$ so that $d^*(\Lambda,\Omega) = 1$ if either $d(\Lambda,\Omega) = 1$ or if there exists x with $|x| = 1$ and $\Lambda = x + \Omega$. In particular, if for a set Λ in $\mathscr{P}^*_{n-1}$ we denote $F_{\Lambda\cup\{0\}}$ by $\tilde{F}_\Lambda$, we have

$$\|f\|_0^2 = \frac{K_n(\rho)}{n} \sum_{\Lambda \in \mathscr{P}^*_{n-1}} \tilde{F}^2_\Lambda \quad \text{and}$$

$$\|f\|_1^2 = \frac{K_n(\rho)}{4n} \sum_{\substack{\Lambda,\Omega \in \mathscr{P}^*_{n-1} \\ d^*(\Lambda,\Omega)=1}} \{\tilde{F}_\Lambda - \tilde{F}_\Omega\}^2 \geq \frac{K_n(\rho)}{4n} \sum_{\substack{x,y \in \mathbb{Z}^d_* \\ |x-y|=1}} \sum_{\substack{\Lambda \in \mathscr{P}^*_{n-2} \\ \Lambda\cap\{x,y\}=\phi}} \{\tilde{F}_{\Lambda\cup\{x\}} - \tilde{F}_{\Lambda\cup\{y\}}\}^2 . \tag{4.7}$$

Fix $n \geq 3$. From the definition of the new monomial F, it follows that $K_n(\rho)^{-1}\|f(1-w_\ell)\|_0^2$ is equal to

$$\frac{1}{n} \sum_{\substack{\Lambda \in \mathscr{P}^*_{n-1}; \\ \Lambda\cup\{0\} \in W^c_{n,\ell}}} \tilde{F}^2_\Lambda . \tag{4.8}$$

For each set Λ in $\mathscr{P}_n \cap W^c_{n,\ell}$ there are n distinct sets Ω_i, $1 \leq i \leq n$, equivalent to Λ containing the origin. At least $n-2$ of these sets contain two sites distinct from the origin that are at distance less than ℓ. Since F_{Ω_i} does not depend on i, the previous expression is bounded by

$$\frac{1}{n-2} \sum_{\Lambda \in (W^*_{n-1,\ell})^c} \tilde{F}^2_\Lambda = \frac{1}{n-2} \sum_{\substack{x,y \in \mathbb{Z}^d_* \\ 1 \leq |x-y| \leq \ell}} \sum_{\substack{\Lambda \in \mathscr{P}^*_{n-3} \\ \Lambda\cap\{x,y\}=\phi}} \tilde{F}^2_{\Lambda\cup\{x,y\}}$$

provided $(W^*_{n,\ell})^c$ stands for the set $\mathscr{P}^*_n \cap W^c_{n,\ell}$. By Corollary 4.10, this expression is bounded above by $C(\ell)(n/n-2)K_n(\rho)^{-1}D^*(\tilde{F})$, which in virtue of the definition of $\tilde{F}$, the explicit formula for the Dirichlet form D^* given in (4.6) and equation (4.7), is bounded by $C(\ell)n\|f\|_1^2$.

It remains to consider the case $n = 2$. We have already seen in (4.8) that $K_n(\rho)^{-1}\|f(1-w_\ell)\|_0^2$ is equal to

$$\frac{1}{2} \sum_{1 \leq |x| \leq \ell} \tilde{F}^2_{\{x\}} .$$

By formula (4.5) this expression is bounded above by

$$C(\ell) \sum_{\substack{x,y \in \mathbb{Z}^d_* \\ |x-y|=1}} \{\tilde{F}_{\{x\}} - \tilde{F}_{\{y\}}\}^2$$

for some constant $C(\ell)$ depending only on ℓ. This expression by (4.7) is bounded above by $C(\ell)K_n(\rho)^{-1}\|f\|_1^2$ what concludes the proof of the corollary. □

Lemma 4.12. *There exists a constant $C(d)$ depending only on the dimension such that for every $n \geq 2$ and h in $\mathscr{R}_n$,*

$$\langle\langle h, Mh \rangle\rangle_{-1} \leq \frac{C(d)n^3}{\gamma} \|h\|_1^2 + \gamma \|h\|_{-1}^2$$

for every $\gamma > 0$.

Proof. Denote by $[M, S]$ the commutator of S and M defined by $MS - SM$. A simple computation shows that for any $n \geq 1$ $[M, S]$ is symmetric operator on $\mathscr{R}_n$ and that M and S commute on $W_{n,\ell}$ for $\ell \geq 3$: $MS(w_\ell f) = SM(w_\ell f)$. This follows from the fact that for the generators M and S particles jump at most by one unit. In particular, $[M, S](w_\ell f) = 0$ for all $\ell \geq 3$ and f in $\mathscr{R}_n$.

Consider a function h in the range of S:$h = Sg$ for some g in $\mathscr{R}_n$. By definition of $[M, S]$ and since M is an asymmetric operator,

$$\langle\langle h, Mh \rangle\rangle_{-1} = -(1/2) \langle\langle g, [M, S]g \rangle\rangle_0 \ .$$

By the properties of the commutator mentioned in last paragraph, $\langle\langle g, [M, S]g \rangle\rangle_0$ is equal to

$$\begin{aligned} \langle\langle g, [M, S](1 - w_3)g \rangle\rangle_0 &= \langle\langle [M, S]g, (1 - w_3)g \rangle\rangle_0 \\ &\leq \frac{1}{2\gamma} \|(1 - w_3)[M, S]g\|_0^2 + \frac{\gamma}{2} \|(1 - w_3)g\|_0^2 \end{aligned} \tag{4.9}$$

for every $\gamma > 0$. By Schwarz inequality, Corollary 4.11 and the definition of the commutator $[M, S]$, $\|(1 - w_3)[M, S]g\|_0^2$ is bounded above by

$$2\|(1 - w_3)Mh\|_0^2 + 2\|(1 - w_3)SMg\|_0^2 \leq Cn\Big\{ \|Mh\|_1^2 + \|SMg\|_1^2 \Big\}$$

for some universal constant C because $h = Sg$. By the same reasons $\|(1 - w_3)g\|_0^2$ is bounded above by $Cn\|h\|_{-1}^2$. Therefore, the right hand side of (4.9) is less than or equal to

$$Cn\left\{ \frac{1}{\gamma} \|Mh\|_1^2 + \frac{1}{\gamma} \|SMg\|_1^2 + \frac{\gamma}{2} \|h\|_{-1}^2 \right\} \ .$$

By Lemma 4.3, $\|Mh\|_1^2$ is bounded above by $Cn^2\|h\|_1^2$ for some universal constant C and $\|SMg\|_1^2$ is bounded above by $Cn^2\|Sg\|_1^2 = Cn^2\|h\|_1^2$. We have therefore proved that

$$\langle\langle h, Mh \rangle\rangle_{-1} \leq Cn\left\{ \frac{n^2}{\gamma} \|h\|_1^2 + \frac{\gamma}{2} \|h\|_{-1}^2 \right\}$$

for every $\gamma > 0$ and every h in the range of S. To extend the result to every h in $\mathscr{R}_n$, recall that in virtue of Lemma 2.1, the range of S is dense in $\mathscr{R}_{-1,n}$, $\mathscr{R}_1$ and in $\mathscr{R}_n$. On the other hand, by (4.3), $\langle\langle Mf, Mf \rangle\rangle_{-1}$ is bounded by $\langle\langle f, f \rangle\rangle_0$, what concludes the proof. □

5. Perturbation Theorem

Consider a Hilbert space H, a dense subset U of H and three operators $\mathbb{S}$, $\mathbb{M}$ and $\mathbb{J}$ defined on U and satisfying

(H1) $\mathbb{S}$ is a symmetric nonnegative operator and $\mathbb{M}$, $\mathbb{J}$ are asymmetric.

Denote by $\langle \cdot, \cdot \rangle$ the inner product in H, by $D(\sqrt{\mathbb{S}})$ the domain of the symmetric nonnegative operator $\sqrt{\mathbb{S}}$ and by H_1 the closure of $D(\sqrt{\mathbb{S}})$. On H_1 consider the inner product $\langle \cdot, \cdot \rangle_1$ defined by $\langle u, v \rangle_1 = \langle u, \mathbb{S} v \rangle_0$ and denote by H_{-1} the dual of H_1 with respect to $\langle \cdot, \cdot \rangle_0$ so that $\langle u, u \rangle_{-1} = \langle u, \mathbb{S}^{-1} u \rangle := \sup_h \{ 2\langle u, h \rangle - \langle h, h \rangle_1 \}$, where the supremum is carried over all h in H. We assume that

(H2) There is a constant C_1 such that

$$|\langle v, \mathbb{J} u \rangle_0| \le C_1 \|u\|_1 \|v\|_1$$

for all $u, v \in U$.

(H3) Let $\mathbb{B}_\lambda = \mathbb{S} + \lambda \mathbb{M}$. For λ small enough the range of $\mathbb{B}_\lambda$ is dense in H_{-1}: for any $\varepsilon > 0$ and $g \in H_{-1}$ there is $u \in U$ so that

$$\|\mathbb{B}_\lambda u - g\|_{-1} \le \varepsilon \ .$$

(H4) $\mathbb{M}$, $\mathbb{S}$ and $\mathbb{J}$ are bounded operators from H to H_{-1}.

Assumptions (H3) and (H4) guarantee the existence of an inverse $\mathbb{B}_\lambda^{-1}$ from H_{-1} to H. The main theorem of this section states that assumptions (H1)–(H4) guarantee that the range of the operator $\mathbb{S} + \mathbb{J} + \mathbb{M}$ is also dense in H_{-1}. Thus, we may not only extend (H3) to λ large but add a bounded operator in the sense of (H2).

Theorem 5.1. *Assume hypotheses* (*H1*)–(*H4*). *For any g in* H_{-1} *and any* $\varepsilon > 0$ *there exists* $u \in U$ *such that*

$$\|(\mathbb{S} + \mathbb{A})u - g\|_{-1} \le \varepsilon, \ \text{for } \mathbb{A} = \mathbb{M} + \mathbb{J} \tag{5.1}$$

Before proving Theorem 5.1, consider the case where $H = \mathscr{R}_n$, $U = \oplus_{j=1}^n \mathscr{M}_j$, $\mathbb{S} = -S$, $\mathbb{J} = J$, $\mathbb{M} = M$ and $\langle \cdot, \cdot \rangle = \langle\!\langle \cdot, \cdot \rangle\!\rangle_0$. In this context assumption (H2) is satisfied in virtue of Corollary 4.6, assumption (H3) is fulfilled for every $0 \le \lambda \le 1$ by Theorem 2.2 and assumption (H4) follows from Lemma 4.2, Corollary 4.5 (because $\langle\!\langle f, f \rangle\!\rangle_1 \le 2dn \langle\!\langle f, f \rangle\!\rangle_0$ for f in $\mathscr{R}_n$) and the fact that $\langle\!\langle Sf, Sf \rangle\!\rangle_{-1} = \langle\!\langle f, f \rangle\!\rangle_1$. Theorem 2.4 follows therefore from Theorem 5.1 and the results proved in section 2.

We now turn to the proof of Theorem 5.1 and derive some simple bounds of the operators.

Lemma 5.2. *Assume hypotheses* (*H1*), (*H2*) *and* (*H3*). Let $\mathbb{E}$ *be an asymmetric operator defined on U and denote by* C_1 *the constant introduced in assumption* (*H2*). *For any function* $u \in U$ *and any* $\lambda \ge 0$ *we have*

$$\|\mathbb{S}u\|_{-1} \le \|(\mathbb{S} + \lambda \mathbb{E})u\|_{-1} \quad \text{and} \quad \|\mathbb{J}u\|_{-1} \le C_1 \|\mathbb{S}u\|_{-1} \ . \tag{5.2}$$

In particular, setting $\mathbb{E} = \mathbb{M}$, $\|\mathbb{S}u\|_{-1} \le \|\mathbb{B}_\lambda u\|_{-1}$. *Moreover,*

$$\|\mathbb{J}\mathbb{B}_\lambda^{-1}\|_{-1} \le C_1$$

so that $\mathbb{J}\mathbb{B}_\lambda^{-1}$ *defines a bounded operator in* H_{-1}.

Proof. By definition,

$$\|(\mathbb{S} + \lambda\mathbb{E})u\|_{-1}^2 = \|\mathbb{S}u\|_{-1}^2 + 2\lambda\langle u, \mathbb{E}u\rangle_0 + \lambda^2\|\mathbb{E}u\|_{-1}^2 \ge \|\mathbb{S}u\|_{-1}^2$$

because $\mathbb{E}$ is an asymmetric operator by assumption. This proves the first bound of (5.2). On the other hand, by assumption (H2) and the variational formula for the H_{-1} norm, $\|\mathbb{J}u\|_{-1} \le C_1\|\mathbb{S}u\|_{-1}$ for every u in U.

To prove that $\mathbb{J}\mathbb{B}_\lambda^{-1}$ is a bounded operator in H_{-1}, it suffices to show that

$$\|\mathbb{J}\mathbb{B}_\lambda^{-1}v\|_{-1} \le C\|v\|_{-1} \tag{5.3}$$

for a dense subset of functions v in H_{-1}. It follows from the inequalities in (5.2) with $\mathbb{E} = \mathbb{M}$ that $\|\mathbb{J}u\|_{-1} \le C_1\|\mathbb{B}_\lambda u\|_{-1}$ for every u in U. Therefore, (5.3) holds for $v = \mathbb{B}_\lambda u$, u in U. By assumption (H3), the subset $\{\mathbb{B}_\lambda u, u \in U\}$ is dense in H_{-1}. This concludes the proof of the lemma. □

Our goal is to solve (5.1). Note that it is not sufficient to construct the Green function $(\mathbb{S} + \mathbb{A})^{-1}$ as a bounded operator from H_{-1} to H_1. To see this, suppose it is true. By definition,

$$\|(\mathbb{S} + \mathbb{A})(\mathbb{S} + \mathbb{A})^{-1}g - g\|_{-1} = 0$$

We would like to define u to be a local function that approximates $(\mathbb{S} + \mathbb{A})^{-1}g$ in H_1 : $\|u - (\mathbb{S} + \mathbb{A})^{-1}g\|_1 \le \varepsilon$. On the other hand, since $\mathbb{S} + \mathbb{A}$ is not a bounded operator from H_1 to H_{-1}, we can not conclude (5.1).

We start now our route to prove Theorem 5.1. We first show that we can extend property (H3) to large λ.

Lemma 5.3. *Let* $\mathbb{E}$ *be an asymmetric operator defined on* U *and satisfying assumption* (*H4*). *Assume that there is* $\lambda_0 > 0$ *such that for any* $g \in H_{-1}$ *and any* $\varepsilon > 0$ *there exists* $u \in U$ *so that*

$$\|(\mathbb{S} + \lambda_0\mathbb{E})u - g\|_{-1} \le \varepsilon\ . \tag{5.4}$$

The statement remains in force for $\lambda = 1$.

Proof. Let $T := \mathbb{S} + \lambda_0\mathbb{E}$. By (5.4) and (H4), we may define the inverse T^{-1} from H_{-1} to H. By the proof of the previous lemma, $\|\mathbb{S}u\|_{-1} \le \|Tu\|_{-1}$ and we may extend $\mathbb{S}T^{-1}$ is a bounded operator on H_{-1} with norm $\|\mathbb{S}\mathbb{T}^{-1}\|_{-1}$ bounded by 1. In particular, for every $0 \le \delta < 1$, we may define the bounded operator $(1 - \delta\mathbb{S}\mathbb{T}^{-1})^{-1}$ by the series $\sum_{n\ge0}(\delta\mathbb{S}T^{-1})^n$.

Fix $\varepsilon > 0$ and g in H_{-1}. Let $h = \lambda_0\big[1 - (1 - \lambda_0)\mathbb{S}\mathbb{T}^{-1}\big]^{-1}g$. Since h belongs to H_{-1}, by assumption (5.4), there exists u in U (or $u \in H \cap H_1$) such that $\|Tu - h\|_{-1} \le \varepsilon$. Since,

$$\mathbb{S} + \mathbb{E} = \lambda_0^{-1}[T - (1-\lambda_0)\mathbb{S}] = \lambda_0^{-1}\big[1 - (1-\lambda_0)\mathbb{S}T^{-1}\big]T \ ,$$

we have that

$$\begin{aligned}\|(\mathbb{S}+\mathbb{E})u - g\|_{-1} &= \lambda_0^{-1}\|\big[1-(1-\lambda_0)\mathbb{S}T^{-1}\big] \\ &\qquad \times \Big(Tu - \lambda_0\big[1-(1-\lambda_0)\mathbb{S}T^{-1}\big]^{-1}g\Big)\|_{-1} \\ &\le C(\lambda_0)\|Tu - h\|_{-1} \le C(\lambda_0)\varepsilon\end{aligned}$$

for some constant $C(\lambda_0)$ that depends only on λ. This proves the lemma. □

Lemma 5.4. *Assume that* $\mathbb{B}$ *and* $\mathbb{E}$ *are operators defined on* U *satisfying* $\|\mathbb{E}\mathbb{B}^{-1}\|_{-1} < 1$. *Suppose that for any* $g \in H_{-1}$ *and any* $\varepsilon > 0$ *there exists* $u \in H$ *so that*

$$\|\mathbb{B}u - g\|_{-1} \le \varepsilon \ . \tag{5.5}$$

The same statement remains in force if $\mathbb{B}$ *is replaced by* $\mathbb{B} + \mathbb{E}$.

Proof. Fix g in H_{-1} and $\varepsilon > 0$. Assumption (5.5) and the proof of the previous lemma show that $[1 + \mathbb{E}\mathbb{B}^{-1}]^{-1}$ is a well defined bounded operator in H_{-1}. Let $h = [1 + \mathbb{E}\mathbb{B}^{-1}]^{-1}g$. By assumption, there exists u in U such that $\|\mathbb{B}u - h\|_{-1} \le \varepsilon$. Since $\mathbb{B} + \mathbb{E} = [1 + \mathbb{E}\mathbb{B}^{-1}]\mathbb{B}$,

$$\begin{aligned}\|(\mathbb{B}+\mathbb{E})u - g\|_{-1} &= \|\big[1+\mathbb{E}\mathbb{B}^{-1}\big]\Big(\mathbb{B}u - \big[1+\mathbb{E}\mathbb{B}^{-1}\big]^{-1}g\Big)\|_{-1} \\ &\le 2\|\mathbb{B}u - h\|_{-1} \le 2\varepsilon \ .\end{aligned}$$

This concludes the proof of the lemma. □

Proof of Theorem 3.1. Let $\mathbb{J}_\lambda = \lambda\mathbb{J}$. By Lemma 5.2 $\mathbb{E} = \mathbb{J}_\lambda$ and $\mathbb{B} = \mathbb{B}_\lambda$ satisfy the assumptions of Lemma 5.4 for λ small enough. In particular, $\mathbb{E} = \mathbb{M} + \mathbb{J}$ satisfy the assumptions of Lemma 5.3. Therefore, for any $g \in H_{-1}$ and any $\varepsilon > 0$ there is a $u \in U$ such that

$$\|(\mathbb{S} + \mathbb{A})u - g\|_{-1} \le \varepsilon \ .$$

what concludes the proof of the theorem. □

The same proof applies to a slightly different set up.

Corollary 5.5. *Instead of* (*H3*) *and* (*H4*), *assume that*

(*H3b*) *Let* $\mathbb{B}_\lambda = \mathbb{S} + \lambda\mathbb{M}$. *Then for* λ *small enough the range of* $\mathbb{B}_\lambda^{-1}$ *restricted to* $U \cap H_1$ is a dense subset of H_{-1}: *for any* $g \in H_{-1}$ *there is* $u \in U \cap H_1$ *so that*

$$\|\mathbb{B}_\lambda u - g\|_{-1} \le \varepsilon \ .$$

(*H4b*) $\mathbb{M}$ *is a bounded operator from* H *to* H_{-1} *and* $\mathbb{S}$ *and* $\mathbb{J}$ *are bounded operators from* H_1 *to* H_{-1}. *Furthermore,* U *is dense in* $H \cap H_1$ *with norm* $\|\cdot\|_0 + \|\cdot\|_1$.

Then, for any g in H_{-1} and any $\varepsilon > 0$ there exists $u \in U$ such that

$$\|(\mathbb{S} + \mathbb{A})u - g\|_{-1} \leq \varepsilon, \text{ for } \mathbb{A} = \mathbb{M} + \mathbb{J}$$

6. The Green–Kubo formula

Recall the nearest-neighbor asymmetric simple exclusion process is a Markov process on $\{0,1\}^{\mathbb{Z}^d}$ whose generator L acts on cylinder functions as

$$(Lf)(\eta) = \sum_{j=1}^{d} \sum_{\substack{x \in \mathbb{Z}^d \\ e, |e|=1}} p(e)\eta_x[1 - \eta_{x+e}][f(\eta^{x,x+e}) - f(\eta)] \ .$$

where $p(e)$ is the jump rate from x to $x+e$. For configuration η and a density ρ, denote respectively by $\mathbb{P}_\eta$ and $\mathbb{P}_\rho$ the probability on the path space $D([0,T], \{0,1\}^{\mathbb{Z}^d})$ corresponding to the Markov process with generator L starting from η, ν_ρ. Expectations with respect to P_η or P_ρ are respectively denoted by $\mathbb{E}_\eta$ and $\mathbb{E}_\rho$. Thus $\mathbb{E}_\rho[\{\eta_t(x) - \eta_0(x)\}\eta_0(0)]$ stands for the time dependent correlation functions of a general driven diffusive system in equilibrium with density ρ. Suppose these correlation functions behave like (non centered) Gaussian. Then one obtains the diffusion coefficient (the bulk diffusion coefficient) by the following limit

$$D^{(1)}_{i,j}(\rho) = \lim_{t\to\infty} \frac{1}{t}\frac{1}{2\chi}\left\{\sum_{x\in\mathbb{Z}^d} x_i x_j \mathbb{E}_\rho[\{\eta_t(x) - \eta_0(x)\}\eta_0(0)] - \chi(v_i t)(v_j t)\right\} \tag{6.1}$$

where v in $\mathbb{R}^d$ is the velocity defined by

$$vt = \frac{1}{\chi}\sum_{x\in\mathbb{Z}^d} x\mathbb{E}_\rho[\{\eta_t(x) - \eta_0(x)\}\eta_0(0)] \tag{6.2}$$

and χ the static compressibility which for simple exclusion processes is equal to $\chi(\rho) = \rho(1-\rho)$. Here we have followed the convention of [LOY2] to denote the diffusion coefficient obtained in (6.1) as the first definition adn thus the superscript 1. The velocity can be explicitly computed (cf. [S]):

$$v = (1 - 2\rho)\sum_{e:|e|=1} p(e)e \ . \tag{6.3}$$

Another definition of the diffusion coefficient is through the linear response theory. To fix ideas, consider the nearest neighbor simple exclusion process. Denote the instantenuous currents (that is difference between the rate at which a particle jumps from x to $x + e_i$ and the rate at which a particle jumps from $x + e_i$ to x) by $W_{x,x+e_i}$:

$$W_{x,x+e_i} = p(e_i)\eta(x)[1 - \eta(x+e_i)] - p(-e_i)\eta(x+e_i)[1 - \eta(x)] \tag{6.4}$$

so that

$$L\eta(0) = \sum_i \left\{ W_{-e_i,0} - W_{0,e_i} \right\} \ .$$

Let $w_i(\rho, \eta) = w_i(\eta)$, $1 \le i \le d$, denote the normalized current in the i-th direction :

$$w_i(\eta) = W_{0,e_i} - \langle W_{0,e_i} \rangle_\rho - \frac{d}{d\theta} \langle W_{0,e_i} \rangle_\theta \Big|_{\theta=\rho} (\eta(0) - \rho) \ . \tag{6.5}$$

Similarly, we can define the currents $W^*_{x,x+e_i}$ of the reversed process characterized by the generator L^* which is the formal adjoint of L with respect to ν_ρ, or the generator of the reversed dynamics. The current $W^*_{x,x+e_i}$ is given explicitly by

$$W^*_{x,x+e_i} = p(-e_i)\eta(x)[1 - \eta(x + e_i)] - \check{p}(e_i)\eta(x + e_i)[1 - \eta(x)] \ .$$

Similarly $w^*_i(\rho, \eta) = w^*_i(\eta)$ is defined by

$$w^*_i = W^*_{0,e_i} - \langle W^*_{0,e_i} \rangle_\rho - \frac{d}{d\theta} \langle W^*_{0,e_i} \rangle_\theta \Big|_{\theta=\rho} (\eta(0) - \rho) \tag{6.6}$$

and a simple computation shows that

$$\begin{aligned} w_i &= [\rho - p(-e_i)]\nabla_{e_i}\eta(0) - [p(e_i) - p(-e_i)][\eta(0) - \rho][\eta(e_i) - \rho] \ , \\ w^*_i &= [\rho - p(e_i)]\nabla_{e_i}\eta(0) + [p(e_i) - p(-e_i)][\eta(0) - \rho][\eta(e_i) - \rho] \ . \end{aligned} \tag{6.7}$$

The second definition of diffusion coefficient according to the convention of [LOY2], $D^{(2)}(\rho) = (D^{(2)}_{i,j}(\rho))_{1 \le i,j \le d}$, obtained through the linear response theory is given by the Green–Kubo formula as [ELS]:

$$\begin{aligned} D^{(2)}_{i,j}(\rho) = \frac{1}{\chi(\rho)} \Bigg\{ &- \frac{1}{2} \delta_{i,j} \langle [\eta(e_i) - \eta(0)] W_{0,e_i} \rangle_\rho \\ &- \int_0^\infty dt \sum_x \left\langle w_i(\eta); e^{tL^*} \tau_x w^*_j(\eta) \right\rangle_\rho \Bigg\} \ . \end{aligned} \tag{6.8}$$

In this formula and below $\delta_{i,j}$ (or $\delta_{x,y}$) stands for the delta of Kroenecker and is equal to 1 if $i = j$ and 0 otherwise. Moreover, e^{tL} (e^{tL^*}) represents the semigroup of the Markov process with generator L (L^*).

The static term of the Green–Kubo formula is easy to compute. It is equal to $(1/2)\delta_{i,j}\chi$ so that

$$\begin{aligned} D^{(2)}_{i,j} - (1/2)\delta_{i,j} &= -\frac{1}{\chi} \int_0^\infty dt \sum_x \left\langle w_i; e^{tL^*} \tau_x w^*_j \right\rangle_\rho \\ &= -\frac{1}{\chi} \int_0^\infty dt \sum_x \left\langle w^*_j; e^{tL} \tau_x w_i \right\rangle_\rho \ . \end{aligned} \tag{6.9}$$

The purpose of this section is to show that the bulk diffusion coefficient $D^{(1)}$ is equal to the symmetric part of the Green–Kubo coefficient : $D^{(1)} = (D^{(2)})^s$. In [LOY2] we proved that the bulk diffusion coefficient is such that

$$D_{i,j}^{(1)} - (1/2)\delta_{i,j} = \lim_{t\to\infty} \frac{1}{2t\chi}\Big\{ - \int_0^t ds \int_0^s dr \sum_x \Big\langle w_i ; e^{rL^*} \tau_x w_j^* \Big\rangle_\rho - \int_0^t ds \int_0^s dr \sum_x \Big\langle w_j ; e^{rL^*} \tau_x w_i^* \Big\rangle_\rho \Big\} \cdot \tag{6.10}$$

Recall the inner product $\langle\!\langle \cdot, \cdot \rangle\!\rangle_{\rho,0}$ from (1.2) and define the norm:

$$\|f\|_{\rho,0}^2 = \langle\!\langle f, f \rangle\!\rangle_{\rho,0} \ . \tag{6.11}$$

Fix a unit vector $\xi \in \mathbb{Z}^d$. We can rewrite (6.10) as

$$\xi \cdot D^{(1)}\xi - (1/2) = \lim_{t\to\infty} \frac{1}{t\chi}\Big\{ - \int_0^t ds \int_0^s dr \langle\!\langle e^{rL} w_\xi ; w_\xi^* \rangle\!\rangle_{\rho,0} \Big\} \ .$$

where $w_\xi = \xi \cdot w$. Since for the inner product $\langle\!\langle \cdot, \cdot \rangle\!\rangle_{\rho,0}$, the gradients are equivalent to 0, we have that $w_j^* = -w_j$ for this inner product. Hence the last term is equal to

$$\begin{aligned} \xi \cdot D^{(1)}\xi - (1/2) &= \lim_{t\to\infty} \frac{1}{t\chi} \int_0^t ds \int_0^s dr \langle\!\langle e^{rL} w_\xi ; w_\xi \rangle\!\rangle_{\rho,0} \\ &= \lim_{t\to\infty} \frac{1}{\chi} \Big\| t^{-1/2} \int_0^t ds \ w_\xi(\eta(s)) \Big\|_{\rho,0}^2 \ . \end{aligned}$$

We shall prove that the time correlations of the current decay fast enough so that

$$\lim_{t\to\infty} \Big\| t^{-1/2} \int_0^t ds \ w_\xi(\eta(s)) \Big\|_{\rho,0}^2 = \int_0^\infty ds \langle\!\langle e^{sL} w_\xi ; w_\xi \rangle\!\rangle_{\rho,0} \tag{6.12}$$

so that

$$\xi \cdot D^{(1)}\xi - (1/2) = \frac{1}{\chi} \int_0^\infty ds \langle\!\langle e^{sL} w_\xi ; w_\xi \rangle\!\rangle_{\rho,0} \ .$$

This is exactly the symmetrization of the Green–Kubo formula (6.8).

We start with a general result on Markov processes.

Lemma 6.1. *Suppose μ is an invariant measure of a Markov process with generator L. Then, for every function w in $H_{-1}(\mu)$,*

$$\limsup_{t\to\infty} \mathbb{E}_\mu \left[\left(t^{-1/2} \int_0^t w(x(s)) \ ds \right)^2 \right] \le 6 \langle w, (-L_s)^{-1} w \rangle := 6 \|w\|_{-1}^2 \ . \tag{6.13}$$

In this formula $\langle \cdot, \cdot \rangle$ stands for the inner product in $L^2(\mu)$ and L_s for the symmetric part of the generator L.

Proof. Fix a function w in H_{-1}. Consider the resolvent equation

$$(\lambda - L) u_\lambda = w \tag{6.14}$$

By Ito's formula

$$u_\lambda(t) = u_\lambda(0) + \int_0^t Lu_\lambda(x(s))\, ds + M_t \ ,$$

where M_t is a martingale satisfying

$$\mathbb{E}_\mu[M_t^2] = t\langle u_\lambda, (-L_s)u_\lambda\rangle := t\|u_\lambda\|_1^2 \ .$$

In particular, by the resolvent equation, the left hand side of (6.13) is bounded above by

$$8t^{-1}\|u_\lambda\|_0^2 + 4\|u_\lambda\|_1^2 + 4\lambda^2\mathbb{E}_\mu\left[\left(t^{-1/2}\int_0^t u_\lambda(x(s))\, ds\right)^2\right] \ .$$

By Schwarz inequality, the last term is bounded by

$$\lambda^2\mathbb{E}_\mu\left[\int_0^t u_\lambda(x(s))^2\, ds\right] = \lambda^2 t\|u_\lambda\|_0^2 \ .$$

Set $\lambda = t^{-1}$. We have thus proved that

$$\mathbb{E}_\mu\left[\left(t^{-1/2}\int_0^t w(x(s))\, ds\right)^2\right] \le 12\lambda\|u_\lambda\|_0^2 + 4\|u_\lambda\|_1^2 \ .$$

To conclude the proof of the lemma, it remains to estimate the L^2 and the H_1 norms of u_λ in terms of the H_{-1} norm. Multiplying the resolvent equation by u_λ and taking the expectation with respect to μ, we have that

$$\lambda\|u_\lambda\|_0^2 + \|u_\lambda\|_1^2 = \langle u_\lambda, w\rangle \le (1/2)\|u_\lambda\|_1^2 + (1/2)\|w\|_{-1}^2 \ .$$

Hence,

$$\lambda\|u_\lambda\|_0^2 + (1/2)\|u_\lambda\|_1^2 \le (1/2)\|w\|_{-1}^2 \ ,$$

what concludes the proof. □

Notice that Lemma 6.1 remains in force if the left hand side of (6.13) is replaced by the expression $\|t^{-1/2}\int_0^t ds\, w_\xi(\eta(s))\|_{\rho,0}^2$ and the inner product on the right hand side is replaced by $\langle\!\langle w_\xi, (-L_s)^{-1}w_\xi\rangle\!\rangle_{\rho,0}$. We have thus proved that

$$\limsup_{t\to\infty}\left\|t^{-1/2}\int_0^t ds\, w_\xi(\eta(s))\right\|_{\rho,0}^2 \le \langle\!\langle w_\xi, (-L_s)^{-1}w_\xi\rangle\!\rangle_{\rho,0} \ .$$

It is proved in [EMY1] that $\langle\!\langle w_\xi, (-L_s)^{-1}w_\xi\rangle\!\rangle_{\rho,0}$ is finite. Therefore, as $t\uparrow\infty$ a limit up to a subsequence of the left hand side of (6.12) exists. The subtle point is to prove that this limit is indeed given by the right hand side of (6.12). The following lemma gives sufficient conditions for the convergence.

Lemma 6.2. *Suppose that equation* (*6.14*) *can be solved for each* $\lambda > 0$ *and that*

$$\lim_{\lambda\to 0}\lambda\|u_\lambda\|_0^2 = 0 \ . \tag{6.15}$$

Then,

$$\lim_{t\to\infty} \mathbb{E}_\mu \left[\left(t^{-1/2} \int_0^t w(x(s))\, ds \right)^2 \right] = \lim_{\lambda\to 0} \langle w, (\lambda - L)^{-1} w\rangle := \langle w, (-L)^{-1} w\rangle \ .$$

Suppose, on the other hand, that we are able to solve the equation $-Lu = w$ *in* H_{-1} *in the sense that for any* $\varepsilon > 0$ *there is a bounded local function* u_ε *such that*

$$\|Lu_\varepsilon + w\|_{-1}^2 \le \varepsilon. \tag{6.16}$$

Then,

$$\lim_{\varepsilon\to 0} \|u_\varepsilon\|_1^2 = B$$

exists for some constant B and

$$\lim_{t\to\infty} \mathbb{E}_\mu \left[\left(t^{-1/2} \int_0^t w(x(s))\, ds \right)^2 \right] = B := \langle w, (-L)^{-1} w\rangle \ .$$

Proof. The proof is similar to the proof of the previous lemma. For a fixed $\lambda > 0$, consider the resolvent equation (6.14). Replacing w by its value in terms of u_λ and expanding the suqare, we obtain that for any $\gamma > 0$,

$$\mathbb{E}_\mu \left[\left(t^{-1/2} \int_0^t w(x(s))\, ds \right)^2 \right] = (1 + O(\gamma))\|u_\lambda\|_1^2 + O(1+\gamma^{-1})\lambda\|u_\lambda\|_0^2 \ .$$

Here we applied Scwharz inequality to estimate the cross terms. By assumption, the last term vanishes for any γ fixed. On the other hand,

$$\lambda\|u_\lambda\|_0^2 + \|u_\lambda\|_1^2 = \langle (\lambda - L)^{-1} w, (\lambda - L_s)(\lambda - L)^{-1} w\rangle \ .$$

By definition, we can replace the operator L_s by L because it appears in a quadratic form. Hence,

$$\lambda\|u_\lambda\|_0^2 + \|u_\lambda\|_1^2 = \langle (\lambda - L)^{-1} w, w\rangle \ .$$

This proves the first statement of the lemma.

Assume now that (6.16) holds. For each $\varepsilon > 0$, let $h_\varepsilon = w + Lu_\varepsilon$. Then,

$$\|u_\varepsilon\|_1^2 = \langle u_\varepsilon, (-L)u_\varepsilon\rangle = \langle u_\varepsilon, w\rangle - \langle u_\varepsilon, h_\varepsilon\rangle$$

The second term on the right hand side is bounded by $\|u_\varepsilon\|_1 \|h_\varepsilon\|_{-1}$, while the first one is bounded by $\|u_\varepsilon\|_1 \|w\|_{-1}$. In particular, by assumption (6.16),

$$\|u_\varepsilon\|_1 \le C_\varepsilon + \|w\|_{-1} \ , \tag{6.17}$$

where C_ε is such that $\lim_{\varepsilon\to 0} C_\varepsilon = 0$.

We can now take a weakly convergent subsequence, still denoted by u_ε, that converges to some u in H_1. Since $Lu_\varepsilon = h_\varepsilon + w$, h_ε converges to 0 in H_{-1} and $\|u_\varepsilon\|_1$ is a bounded sequence,

$$\lim_{\varepsilon\to 0}\langle u_\varepsilon, (-L)u_\varepsilon\rangle = \lim_{\varepsilon\to 0}\langle u_\varepsilon, w\rangle = \langle u, w\rangle \ . \tag{6.18}$$

Using (6.16) again, we obtain that

$$\langle u, w\rangle = \lim_{\varepsilon\to 0}\langle u, Lu_\varepsilon\rangle = \lim_{\varepsilon\to 0}\langle L^* u, u_\varepsilon\rangle \ .$$

Since $L^*u = 2L_s u - Lu = 2L_s u - w$, from the a priori estimate (6.17) we deduce that L^*u belongs to H_{-1}. Since u_ε converges weakly to u in H_1, we have thus proved that

$$\langle u, w\rangle = \lim_{\varepsilon\to 0}\langle L^* u, u_\varepsilon\rangle = \langle L^* u, u\rangle = \|u\|_1^2 \ . \tag{6.19}$$

It follows from identities (6.18) and (6.19) that

$$\lim_{\varepsilon\to 0}\|u_\varepsilon\|_1^2 = \|u\|_1^2 \ .$$

Since u_ε converges weakly to u in H_1, we conclude that u_ε converges strongly to u H_1. Moreover, the limit is unique.

By the previous lemma and by assumption (6.16), for each fixed $\varepsilon > 0$,

$$\mathbb{E}_\mu\left[\left(t^{-1/2}\int_0^t w(x(s)) + Lu_\varepsilon(x(s))\,ds\right)^2\right] \le 6\|w + Lu_\varepsilon\|_{-1}^2 \le 6\varepsilon \tag{6.20}$$

On the other hand, for any bounded local function u_ε,

$$u_\varepsilon(t) = u_\varepsilon(0) + \int_0^t Lu_\varepsilon(x(s))\,ds + M_t \ ,$$

where M_t is a martingale such that

$$\mathbb{E}_\mu[M_t^2] = \langle u_\varepsilon, (-L_s)u_\varepsilon\rangle = \|u_\lambda\|_1^2 \ .$$

Since u_ε is a bounded local function,

$$t^{-1}\mathbb{E}_\mu[u_\varepsilon(t)^2] = t^{-1}E_\mu[u_\varepsilon(0)^2]$$

that vanishes as $t \uparrow \infty$. Hence

$$\lim_{t\to\infty}\mathbb{E}_\mu\left[\left(t^{-1/2}\int_0^t Lu_\varepsilon(x(s))\,ds\right)^2\right] = \|u_\varepsilon\|_1^2 \ .$$

This limit together with (6.20) concludes the proof of the lemma. □

This lemma holds in our setting with the norm defined according to (6.11). The right hand side of (6.12) is equal to $\lim_{\lambda\to 0}\langle\langle \mathrm{w}, (\lambda - \mathrm{L})^{-1}\mathrm{w}\rangle\rangle = \langle\langle w, (-L)^{-1}w\rangle\rangle$, which, up to the change of the norm, appears at the right hand side of the equations after (6.15) and (6.16). Hence we only have to prove (6.15) or (6.16) (with the correct norm) in order to prove (6.12). The condition (6.15) is suitable for reversible models and is used extensively in [KV] for proving central limit theorem of tagged particles. In nonreversible setting, we have to prove (6.16). But this is the main context Theorem 1.1.

This concludes the proof of (6.12) and the identification of $D^{(1)}$ and $D^{(2)}$ is completed.

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