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Strong and conditional invariance principles for samples attracted to stable laws

Raoul LePage[1], Krzysztof Podgórski[2], Michał Ryznar[3]

[1] Department of Probability and Statistics, Michigan State University, East Lansing, MI 48823, USA. E-mail: entropy@msu.edu
[2] Department of Mathematical Sciences, IUPUI, Indianapolis, IN 46202-3216, USA. E-mail: kpodgorski@math.iupui.edu
[3] Institute of Mathematics, Technical University of Wroclaw, 50-370 Wroclaw, POLAND E-mail: ryznar@graf.im.pwr.wroc.p1

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Summary. We prove almost sure convergence of a representation of normalized partial sum processes of a sequence of i.i.d. random variables from the domain of attraction of an α-stable law, $\alpha < 2$. We obtain an explicit form of the limit in terms of the LePage series representation of stable laws. One consequence of these results is a *conditional* invariance principle having applications to option pricing as well as to resampling by signs and permutations.

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1. Introduction

Paper [14] established weak convergence of normalized sums

$$\frac{X_1 + \cdots + X_n}{a_n} - b_n\,, \tag{1}$$

where $(X_i)_{i=1}^n$ is a sequence of i.i.d. random variables from the domain of attraction of an α-stable law, $\alpha \in (0,2)$, utilizing a representation of these sums in terms of order statistics. For the symmetric case this representation can be written in the form

$$\frac{X_1 + \cdots + X_n}{a_n} \stackrel{d}{=} \frac{\delta_1 G^{-1}(\Gamma_1/\Gamma_{n+1}) + \cdots + \delta_n G^{-1}(\Gamma_n/\Gamma_{n+1})}{a_n} \tag{2}$$

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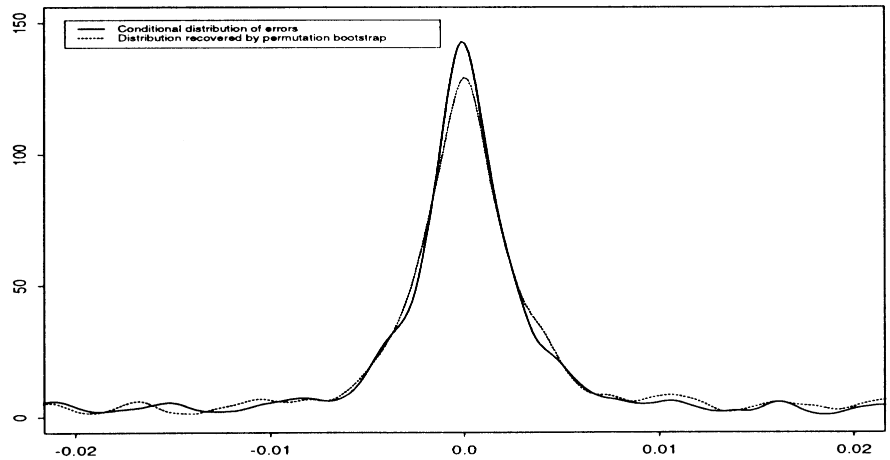

Fig. 1. Conditional distribution of errors versus result of permuting residuals in regression with Cauchy errors

for each $n \in \mathbb{N}$, where $\boldsymbol{\delta} = (\delta_i)_{i=1}^{\infty}$ is a Rademacher sequence, $\boldsymbol{\Gamma} = (\Gamma_i)_{i=1}^{\infty}$ is a sequence of arrival times of a standard Poisson process independent of $\boldsymbol{\delta}$ and $G(x) = P(|(X_1)| \geqq x)$ while $G^{-1}(u) = \inf\{y \geqq 0\colon G(y) \leqq u\}$. There exist natural although not unique extensions of this representation for the non-symmetric case (cf. [14] and also Sect. 3).

The results of [14] claim *convergence in distribution* of such representations. In this paper we show that a use of different techniques and a more suitable choice of representation in the non-symmetric case give *convergence almost surely*. Our approach applies not only for sums of random variables but also to partial sum processes. Namely, corresponding representations in distribution of such processes converge almost surely in $D[0,1]$ with the Skorohod metric. This result can be considered as a strong invariance principle for the stable case. From this we establish a new *conditional* invariance principle for a.s. $D[0,1]$ convergence of the conditional distributions given $\boldsymbol{\Gamma}$. A weak version establishing convergence in probability for the symmetric case was given in [9] by employing martingale methods.

By way of illustrating the potential usefulness of a conditional invariance principle, in Fig. 1 is shown a decidedly non-normal conditional sampling distribution, arising in regression with Cauchy errors. As can be seen in the figure even rather delicate features of this conditional distribution have been recovered by our conditional invariance principle, in this case applied to permutation bootstrap (permuting residuals in regression $Y = X\beta + \varepsilon$ estimates the conditional sampling distribution of the estimation error $\hat{\beta} - \beta$, given the order statistics of the errors (ε_i) actually present in the data, without moment assumptions on (ε_i)). See results in Sect. 4 and [13].

Also in the derivation of the option pricing formula in [16] conditional limits are required. To extend these results to the case of an arbitrary distribution from the stable domain we have used our conditional invariance principle. It is worth emphasizing that the conditional invariance principle stated in Proposition 1 below is not a simple consequence of the a.s. convergence of a version of the partial sum process occurring in the Skorohod representation theorem. This is demonstrated by the examples in Sect. 4.

Our approach to the strong invariance principle, presented here, differs from the one undertaken in a series of papers concluded by [1] (see also references therein). These papers studied the sequence of i.i.d. *symmetric* X_i's from the *normal* α-stable domain, investigating the rate of almost sure approximation of $\sum_{i=1}^n X_i$ by $\sum_{i=1}^n Y_i$, where Y_i's are i.i.d. symmetric α-stable. The relationship between those results and ours is as follows. In this case our strong invariance principle can be viewed as almost sure approximation of $\sum_{i=1}^n X_{i,n}$, where $(X_{i,n})_{i=1}^n$ is a vector of i.i.d. r.v.'s from an α-stable domain, by $\sum_{i=1}^n Y_{i,n}$, where $(Y_{i,n})_{i=1}^n$ is a vector of i.i.d. α-stable r.v.'s. It is clear that ours is a non-sequential form of approximation but, on the other hand, our results hold in the general case of stable domain, not necessarily normal or symmetric. Moreover, through our approach the rate is always at least $o(n^{1/\alpha})$ while this rate cannot always be achieved with the other strong invariance principle (see examples in [1]).

Our methods of proofs are essentially different from those used in [1] and in fact bear more similarity to those applied in [7], where asymptotic behavior of sums of order statistics of i.i.d. variables from stable domain is studied through a Poisson representation of order statistics different from (2). However the results and methods of [7] cannot be directly applied in our proofs as only convergence in L_1 and in probability were considered there and we require methods strong enough to imply almost sure convergence.

The paper is organized in the following way. In Sect. 2 we present lemmas used in the proof of the invariance principle. Different versions of the strong invariance principle for symmetric and non-symmetric cases are proved in Sect. 3, consisting the central part of this paper. In Sect. 4 we prove a conditional invariance principle, applying it to option pricing.

2. Main lemmas

In this paper without further mention $H:(0,1]\mapsto \mathbb{R}$ will always denote a non-increasing function such that for some $\xi>0$ the function $(1/x)^{1/\xi}H(1/x)$ is slowly varying at infinity. Notice that if X is in the domain of an α-stable distribution, then the function $H(x)=(G^{-1}(x))^r$, where $r>0$ and $G(x)=P(|X|\geqq x)$, satisfies this condition with $\xi=\alpha/r$. Usually for such X this property is expressed equivalently in terms of $G(x)$ instead of $H(x)$ (cf. [3], VIII.2, Lemma 1, VIII.8, Lemma 3, XVII.5, Theorem 2). Here and throughout the paper we write $x_n\approx y_n$ if $\lim_{n\to\infty} x_n/y_n=1$. Let $a_n\approx H(1/n)$. Then for every $x>0$:

$$\lim_{n\to\infty} H(x/n)/a_n = x^{-1/\xi}\,. \tag{3}$$

The following lemma is crucial for the results of Sect. 3 where we study almost sure convergence of representations of type (2).

Lemma 1. *Let $\xi, \beta > 0$ and $\xi\beta < 1$. Assume that $V_{n,i}, V_i$, $n \in \mathbb{N}$, $i \leqq n$ are random elements of $D[0,1]$ regarded with the Skorohod metric d such that for each $N \in \mathbb{N}$ and for each $c_{n,i}$, $1 \leqq i \leqq N$, $n \in \mathbb{N}$ satisfying $\lim_{n\to\infty} c_{n,i} = c_i$, we have with probability one in $(D[0,1], d)$:*

$$\lim_{n\to\infty} \sum_{i=1}^{N} c_{n,i} V_{n,i} = \sum_{i=1}^{N} c_i V_i \,. \tag{4}$$

Moreover assume that with probability one

$$M \overset{def}{=} \sup_{k\in\mathbf{N}} \sup_{n\geqq k} \frac{1}{k^\beta} \left\| \sum_{i=1}^{k} V_{n,i} \right\|_\infty < \infty \,, \tag{5}$$

where $\|\cdot\|_\infty$ stands for the uniform norm in $D[0,1]$. Then, assuming independence $(V_{n,i})$ from $\mathbf{\Gamma}$, the series $\sum_{i=1}^\infty V_i/\Gamma_i^{1/\xi}$ is convergent almost surely in $D[0,1]$ both in $\|\cdot\|_\infty$ and in d, and with probability one in $(D[0,1], d)$:

$$\lim_{n\to\infty} \frac{1}{a_n} \sum_{i=1}^{n} V_{n,i} H(\Gamma_i/\Gamma_{n+1}) = \sum_{i=1}^{\infty} V_i/\Gamma_i^{1/\xi} \,. \tag{6}$$

Proof. For $n \in \mathbb{N}$ and $N < n$ let us denote

$$S_N^n = \frac{1}{a_n} \sum_{i=N}^{n} V_{n,i} H(\Gamma_i/\Gamma_{n+1}), \qquad W_n = \sum_{i=1}^{n} V_i/\Gamma_i^{1/\xi} \,.$$

Note that the Law of Large Numbers and (3) imply for each i:

$$\lim_{n\to\infty} \frac{1}{a_n} H(\Gamma_i/\Gamma_{n+1}) = \Gamma_i^{-1/\xi} \quad \text{a.s.} \tag{7}$$

Next, by the triangle inequality, for $N < n$ we have

$$d(S_1^n, W_n) \leqq d(S_1^n - S_N^n, W_{N-1}) + d(S_1^n - S_N^n, S_1^n) + d(W_{N-1}, W_n) \,.$$

Observe that for fixed N, the first term of the right hand side converges to zero by (4), (7) and independence $(V_{n,i})$ from $\mathbf{\Gamma}$. The second term is bounded by $\|S_N^n\|_\infty$. We will show

$$\lim_{N\to\infty} \limsup_{n\to\infty} \|S_N^n\|_\infty = 0 \,. \tag{8}$$

Moreover, as we will see later, this also implies that W_n converges in the uniform norm so in the metric d and then

$$\lim_{N\to\infty} \limsup_{n\to\infty} d(W_{N-1}, W_n) = 0 \,.$$

Now, let us establish (8).

Denoting $T_{n,k} = \sum_{i=1}^k V_{n,i}$ and $\Delta H_{n,k} = H(\Gamma_k/\Gamma_{n+1}) - H(\Gamma_{k+1}/\Gamma_{n+1})$, we have

$$S_N^n = \frac{1}{a_n} \sum_{i=N}^{n-1} T_{n,i} \Delta H_{n,i} + \frac{1}{a_n} T_{n,n} H(\Gamma_n/\Gamma_{n+1}) - \frac{1}{a_n} T_{n,N-1} H(\Gamma_N/\Gamma_{n+1}) \,.$$

Denote the first sum by I_N^n, the rest of the above expression by J_N^n, and by $\widetilde{M}$ the analogue expression to M but with k^β replaced by Γ_k^β. Since H is non-increasing we have

$$\|I_N^n\|_\infty \leqq \frac{\widetilde{M}}{a_n}\Gamma_{n+1}^\beta \sum_{i=N}^{n-1} \frac{\Gamma_i^\beta}{\Gamma_{n+1}^\beta}\Delta H_{n,i} \leqq -\frac{\widetilde{M}}{a_n}\Gamma_{n+1}^\beta \int_{\Gamma_N/\Gamma_{n+1}}^{1} x^\beta \, dH(x)\,.$$

It follows from elementary properties of regularly varying functions (see [3, VIII]) that if $n \to \infty$

$$-\Gamma_{n+1}^\beta \int_{\Gamma_N/\Gamma_{n+1}}^{1} x^\beta \, dH(x) \approx H(\Gamma_N/\Gamma_{n+1})\frac{1}{1-\beta\xi}\Gamma_N^\beta\,.$$

Thus by (7) we have

$$\limsup_{n\to\infty} \|I_N^n\|_\infty \leqq \frac{\widetilde{M}}{1-\beta\xi}\Gamma_N^\beta \lim_{n\to\infty}\frac{1}{a_n}H(\Gamma_N/\Gamma_{n+1}) = \frac{\widetilde{M}}{1-\beta\xi}\Gamma_N^{\beta-1/\xi}\,.$$

Now let us consider J_N^n. We have

$$\begin{aligned}\limsup_{n\to\infty}\|J_N^n\|_\infty &\leqq \limsup_{n\to\infty}\frac{1}{a_n}[\|T_{n,n}\|_\infty H(\Gamma_n/\Gamma_{n+1}) + \|T_{n,N-1}\|_\infty H(\Gamma_N/\Gamma_{n+1})] \\ &\leqq M \lim_{n\to\infty}\frac{n^\beta}{a_n} + \widetilde{M}\Gamma_N^\beta \lim_{n\to\infty}\frac{1}{a_n}H(\Gamma_N/\Gamma_{n+1}) = \widetilde{M}\Gamma_N^{\beta-1/\xi}\,.\end{aligned}$$

Hence

$$\lim_{N\to\infty}\limsup_{n\to\infty}\|S_N^n\|_\infty \leqq \left(\frac{1}{1-\beta\xi}+1\right)\lim_{N\to\infty}\frac{\widetilde{M}}{\Gamma_N^{1/\xi-\beta}} = 0\,.$$

Note that in the proof of (8) independence $(V_{n,i})$ from $\mathbf{\Gamma}$ is irrelevant.

To see that (8) implies convergence of the series in the uniform norm observe that by the lower semicontinuity of $\|\cdot\|_\infty$, (4), (7), and independence $(V_{n,i})$ from $\mathbf{\Gamma}$, for every $k < m$:

$$\left\|\sum_{i=k}^{m} V_i/\Gamma_i^{1/\xi}\right\|_\infty \leqq \limsup_{n\to\infty}\|S_m^n\|_\infty + \limsup_{n\to\infty}\|S_k^n\|_\infty\,.$$

Hence W_n is a Cauchy sequence in the uniform as well as Skorohod topology. □

Remark 1. Condition (4) and independence $(V_{n,i})$ from Γ were used to overcome some difficulties caused by the fact that addition is not continuous in the Skorohod metric. In the scalar version of the result (4) can be replaced by $\lim_{n\to\infty} V_{n,i} = V_i$ and independence $(V_{n,i})$ from Γ is not required.

In Sect. 3 we consider a representation as (2) but for the non-symmetric case. Unfortunately for $1 \leqq \alpha < 2$ Lemma 1 cannot be directly applied to this case. We need also a version with $V_{n,i} = 1$. Such a complementary result is stated below.

Lemma 2. *Let* $1 \leqq \xi < 2$. *Then with probability one*

$$\lim_{n\to\infty} \frac{1}{a_n} \left(\sum_{i=1}^{n} H(\Gamma_i/\Gamma_{n+1}) - n \int_{1/n}^{1} H(x)\,dx \right) = \sum_{i=1}^{\infty} \left(\frac{1}{\Gamma_i^{1/\xi}} - c_i \right) ,$$

where $c_i = \int_i^{i+1} x^{-1/\xi}\,dx$.

Proof. Denote, for $N \leqq n$,

$$S_N^n = \frac{1}{a_n} \sum_{i=N}^{n} H(\Gamma_i/\Gamma_{n+1}), \qquad S_{N,n} = \frac{1}{a_n} \sum_{i=1}^{N} H(\Gamma_i/\Gamma_{n+1}) .$$

First observe that the series $\sum_{i=1}^{\infty}(\Gamma_i^{-1/\xi} - c_i)$ is convergent almost surely. Thus it is enough to show that

$$r_n \overset{\text{def}}{=} \left| S_1^n - \frac{n}{a_n} \int_{1/n}^{1} H(x)\,dx - \sum_{i=1}^{n} (\Gamma_i^{-1/\xi} - c_i) \right|$$

converges almost surely to zero.

For fixed $N < n$ we have

$$r_n \leqq \left| S_{N-1,n} - \frac{n}{a_n} \int_{1/n}^{N/n} H(x)\,dx - \sum_{i=1}^{N-1} (\Gamma_i^{-1/\xi} - c_i) \right| + \left| S_N^n - \frac{n}{a_n} \int_{N/n}^{1} H(x)\,dx \right|$$

$$+ \left| \sum_{i=N}^{n} (\Gamma_i^{-1/\xi} - c_i) \right|$$

with the last term converging to zero.

The first term converges to zero as well since $\lim_{n\to\infty} S_{N-1,n} = \sum_{i=1}^{N-1} \Gamma_i^{-1/\xi}$, and

$$\lim_{n\to\infty} \frac{n}{a_n} \int_{1/n}^{N/n} H(x)\,dx = \lim_{n\to\infty} \frac{1}{a_n} \int_{1}^{N} H(x/n)\,dx = \int_{1}^{N} x^{-1/\xi}\,dx .$$

Thus it is enough to show

$$\lim_{N\to\infty} \limsup_{n\to\infty} \left| S_N^n - \frac{n}{a_n} \int_{N/n}^{1} H(x)\,dx \right| = 0 \quad \text{a.s.} \tag{9}$$

Denote $\alpha_1 = \Gamma_1$ and $\alpha_{i+1} = \Gamma_{i+1} - \Gamma_i$ for $i \in \mathbb{N}$. We have

$$S_N^n = \frac{1}{a_n} \sum_{i=N}^{n} H(\Gamma_i/\Gamma_{n+1})(1 - \alpha_i) + \frac{1}{a_n} \sum_{i=N}^{n} H(\Gamma_i/\Gamma_{n+1})\alpha_i$$

$$= \frac{1}{a_n} \sum_{i=N}^{n} H(\Gamma_i/\Gamma_{n+1})(1 - \alpha_{i+1}) + \frac{1}{a_n} \sum_{i=N}^{n} H(\Gamma_i/\Gamma_{n+1})\alpha_{i+1} .$$

Denoting the first normalized sum by J_N^n and the second one by I_N^n and using the fact that H is nonincreasing we get

$$\frac{\Gamma_{n+1}}{a_n}\int_{\frac{\Gamma_N}{\Gamma_{n+1}}}^{1} H(x)\,dx + J_N^n \leqq S_N^n \leqq \frac{\Gamma_{n+1}}{a_n}\int_{\frac{\Gamma_N}{\Gamma_{n+1}}}^{1} H(x)\,dx + I_N^n + \frac{\alpha_N}{a_n}H(\Gamma_N/\Gamma_{n+1})\,.$$

By (7), we have

$$\lim_{N\to\infty}\lim_{n\to\infty}\frac{\alpha_N}{a_n}H\left(\frac{\Gamma_N}{\Gamma_{n+1}}\right) = \lim_{N\to\infty}\frac{\alpha_N}{\Gamma_N^{1/\xi}} = 0$$

and from the proof of Lemma 1 and Remark 1 (set $V_{n,i} = 1-\alpha_i$ or $V_{n,i} = 1-\alpha_{i+1}$):

$$\lim_{N\to\infty}\limsup_{n\to\infty}|I_N^n| = \lim_{N\to\infty}\limsup_{n\to\infty}|J_N^n| = 0\,.$$

Therefore to prove (9) it is enough to note that

$$\lim_{N\to\infty}\limsup_{n\to\infty}\left|\frac{\Gamma_{n+1}}{a_n}\int_{\Gamma_N/\Gamma_{n+1}}^{1} H(x)\,dx - \frac{n}{a_n}\int_{N/n}^{1} H(x)\,dx\right| = 0 \quad \text{a.s.}\,, \tag{10}$$

which follows from elementary properties of regularly varying functions, and Laws of Iterated Logarithm and Large Numbers applied to Γ_n. □

3. Invariance principle

In the main section we derive various forms of the strong invariance principle. In all of these results we claim almost sure convergence of special representations of sums of random variables in the domain of attraction of a stable law of index $\alpha < 2$. We begin with the general representation of a vector of real i.i.d. random variables expressed through the order statistics of their absolute values.

Let X be a real random variable and let

$$G_+(x) = P(X \geqq x | X \geqq 0), \qquad G_-(x) = P(-X > x | -X > 0)\,,$$

$$p_0 = P(X \geqq 0), \qquad q_0 = P(-X > 0)\,.$$

Here $0/0$ is treated as zero. Let $(\delta_i)_{i=1}^n$ be a vector of independent possibly *non-symmetric* random signs such that $P(\delta_i = +) = p_0,\ P(\delta_i = -) = q_0$ and $(U_i)_{i=1}^n$ be a vector of independent random variables with the uniform distribution on $[0,1]$. As usual $U_{(j)}$ denotes here the j-th order statistic of $(U_i)_{i=1}^n$ and π denote a uniformly distributed random permutation applied to the coordinates of n-space. We assume also that π, $(\delta_i)_{i=1}^n$ and $(U_i)_{i=1}^n$ are mutually independent. The elementary proof of the following representation is omitted.

Lemma 3. *Let $(X_i)_{i=1}^n$ be a vector of i.i.d. random variables with the distribution of X. Then*

$$(X_1,\ldots,X_n) \stackrel{d}{=} \pi(\delta_1 G_{\delta_1}^{-1}(U_{(1)}),\ldots,\delta_n G_{\delta_n}^{-1}(U_{(n)})) \,.$$

For X_i's i.i.d. symmetric random variables with $G(x)=P(|X_1|>x)$ from our representation

$$X_1+\cdots+X_n \stackrel{d}{=} \delta_1 G^{-1}(\Gamma_1/\Gamma_{n+1})+\cdots+\delta_n G^{-1}(\Gamma_n/\Gamma_{n+1}) \,,$$

which is a simple consequence of the fact that $(\Gamma_1/\Gamma_{n+1},\ldots,\Gamma_n/\Gamma_{n+1})$ has the same distribution as $(U_{(1)},\ldots,U_{(n)})$. If we assume that $x^\alpha G(x)$, $0<\alpha<2$, is slowly varying at infinity, i.e. when the distribution of X_1 belongs to the symmetric α-stable domain, then as an immediate consequence of Lemma 1 we derive the following result.

Theorem 1. *If $a_n = G^{-1}(1/n)$, then*

$$\lim_{n\to\infty} \frac{\delta_1 G^{-1}(\Gamma_1/\Gamma_{n+1})+\cdots+\delta_n G^{-1}(\Gamma_n/\Gamma_{n+1})}{a_n} = \sum_{n=1}^{\infty} \frac{\delta_i}{\Gamma_i^{1/\alpha}} \quad a.s.$$

Proof. Take $V_{n,i}=\delta_i$ and $H(x)=G^{-1}(x)$ and then apply Lemma 1. □

The rest of this section is mostly devoted to various extensions of this result. In Theorem 2 and Corollary 1 we formulate the invariance principle for partial sums processes in the general and symmetric case respectively, where we establish almost sure convergence in the Skorohod metric of a proper representation in $D[0,1]$ of the partial sums process

$$\frac{1}{a_n}\sum_{i=1}^{[nt]} X_i, \quad t\in[0,1] \,.$$

Here and in what follows $a_n = G^{-1}(1/n)$. It is well known that such a process after appropriate centering converges weakly in $D[0,1]$ to an α-stable process with independent increments (cf. [5, IX.6, Theorem 2]). Our goal is to prove almost sure convergence for a special representation of such a process. In order to state an appropriate representation of the partial sum process we have to introduce the following notation and definitions.

Let $\mathbf{u}=(u_i)_{i=1}^\infty$ by any sequence of numbers belonging to $(0,1)$. For $n\in\mathbb{N}$ and $t\in[0,1]$ define

$$I_1^n(\mathbf{u},t) = \mathbf{1}_{\{s\in[0,1]:\, u_1 \leqq [ns]/n\}}(t) \,,$$

$$I_i^n(\mathbf{u},t) = \mathbf{1}_{\{s\in[0,1]:\, u_i \leqq ([ns]-\Sigma_{r=1}^{i-1} I_r^n(u,s))/(n+1-i)\}}(t), \quad 1<i\leqq n \,.$$

The following list of properties of $(I_i^n(\mathbf{u},\cdot))_{i=1}^n$ is verified in [10, 9]. For fixed $t\in[0,1]$ a sequence $(I_i^n(\mathbf{u},t))_{i=1}^n$ consists of exactly $k=[nt]$ ones and can be interpreted as a combination of k-elements from n-elements. On the other hand, with fixed n and $\mathbf{u}$, the number of ones increase with t from 0 to n and the sequence of positions of jumps in the sequence $(I_i^n(\mathbf{u},\cdot))_{i=1}^n$ is a permutation of elements of $\{1,\dots,n\}$. We refer to them as the combinations and the permutation chosen by $\mathbf{u}$. Moreover $I_i^n(\mathbf{u},t)$ as a function of t changes its value from zero to one at exactly one of the points k/n, say J_i^n, being right continuous at this point, i.e.

$$I_i^n(\mathbf{u},t)=\mathbf{1}_{[J_i^n,1]}(t)=\mathbf{1}_{[0,t]}(J_i^n)\ .$$

If $\mathbf{U}=(U_i)_{i=1}^\infty$ is a sequence of i.i.d. random variables with the uniform distribution on $(0,1)$, then for fixed $t\in[k/n,(k+1)/n)$, $k\in\{0,\dots,n\}$, a random combination chosen by $\mathbf{U}$ has a uniform distribution over all possible $\binom{n}{k}$ combinations. Also the positions of jumps of $(I_i^n(\cdot))_{i=1}^n \stackrel{\text{def}}{=} (I_i^n(\mathbf{U},\cdot))_{i=1}^n$ i.e. the permutation chosen by $\mathbf{U}$ will be uniformly distributed over all $n!$ possible permutations.

According to Lemma 3 and assuming mutual independence of $\boldsymbol{\Gamma},\mathbf{U}$ and $\boldsymbol{\delta}$ we get the following representation of the partial sum process

$$\frac{1}{a_n}\sum_{i=1}^{[n\,\cdot\,]} X_i \stackrel{d}{=} \frac{1}{a_n}\sum_{i=1}^{n}\delta_i G_{\delta_i}^{-1}(\Gamma_i/\Gamma_{n+1})I_i^n(\cdot)\ , \tag{11}$$

where the equality is understood in $D[0,1]$. From now on the right hand side of (11) will be denoted by S_n. For a real continuous function v on $[0,1]$ we can write

$$\int_0^1 v\,dS_n=\frac{1}{a_n}\sum_{i=1}^{n} v(J_i^n)\delta_i G_{\delta_i}^{-1}(\Gamma_i/\Gamma_{n+1})\ .$$

Assume now that the distribution of X_i is in the domain of attraction of an arbitrary stable law with $\alpha<2$ and $G(x)=P(|X_1|\geqq x)$. Then the following limits are well defined (see [3, XVII.5 Theorem 2])

$$p=\lim_{x\to\infty}\frac{P(X\geqq x)}{G(x)},\qquad q=\lim_{x\to\infty}\frac{P(-X>x)}{G(x)}\ .$$

This implies that the normalizing sequences (in the sense given by (3)) for G_+^{-1} and G_-^{-1} have the form $a_n(p/p_0)^{1/\alpha}$ and $a_n(q/q_0)^{1/\alpha}$ respectively.

Let us define

$$z_i=(p/p_0)^{1/\alpha}\frac{\delta_i+1}{2}+(q/q_0)^{1/\alpha}\frac{\delta_i-1}{2}\ .$$

Then $P(z_i=(p/p_0)^{1/\alpha})=p_0$ and $P(z_i=-(q/q_0)^{1/\alpha})=q_0$ and

$$E(z_i^+)^\alpha=p,\qquad E(z_i^-)^\alpha=q\ . \tag{12}$$

Set $C_i=Ez_i\int_i^{i+1}x^{-1/\alpha}\,dx$ and $b_n=E(z_i\int_{1/n}^1 G_{\delta_i}(x)\,dx)n/a_n$.

The next result is the general version of the strong invariance principle.

Theorem 2. *With the above notation for any* $\alpha \in (0,2)$ *we have*

$$\lim_{n\to\infty}(S_n(t)-b_nt)=\sum_{i=1}^{\infty}\left(\frac{z_i}{\Gamma_i^{1/\alpha}}\mathbf{1}_{[U_i,1]}(t)-C_it\right)\quad a.s.\,,$$

in the Skorohod metric in $D[0,1]$. *Moreover for a continuous real function* v *of bounded variation on* $[0,1]$

$$\lim_{n\to\infty}\left(\int_0^1 v\,dS_n-b_nEv(U)\right)=\sum_{i=1}^{\infty}\left(\frac{z_iv(U_i)}{\Gamma_i^{1/\alpha}}-C_iEv(U_i)\right)\quad a.s.$$

In the proof of the above theorem we will use the following result which is proven right after the proof of Theorem 2.

Lemma 4. *Let* $\mathbf{r}=(r_i)_{i=1}^{\infty}$ *be a sequence of i.i.d. zero mean random variables independent of* $\mathbf{U}=(U_i)_{i=1}^{\infty}$ *and such that* $|r_i|\leqq a<\infty$ *a.s. Then for every* $\beta>1/2$ *we have*

$$\sup_{k\in\mathbb{N}}\sup_{k\leqq n}\frac{1}{k^{\beta}}\left\|\sum_{i=1}^{k}(I_i^n(t)-t)\right\|_{\infty}<\infty\,, \tag{13}$$

$$\sup_{k\in\mathbb{N}}\sup_{k\leqq n}\frac{1}{k^{\beta}}\left\|\sum_{i=1}^{k}r_iI_i^n(t)\right\|_{\infty}<\infty\,. \tag{14}$$

Proof of Theorem 2. Let us define

$$R_{1,n}(t)=\frac{1}{a_n}\sum_{i=1}^{n}\left[G_+^{-1}(\Gamma_i/\Gamma_{n+1})\left(\frac{\delta_i+1}{2}I_i^n(t)-p_0t\right)\right.$$
$$\left.+G_-^{-1}(\Gamma_i/\Gamma_{n+1})\left(\frac{\delta_i-1}{2}I_i^n(t)+q_0t\right)\right]$$

and

$$R_{2,n}=\left[\frac{p_0}{a_n}\sum_{i=1}^{n}G_+^{-1}(\Gamma_i/\Gamma_{n+1})-\frac{q_0}{a_n}\sum_{i=1}^{n}G_-^{-1}(\Gamma_i/\Gamma_{n+1})-b_n\right]\,.$$

Then to prove the first part of the theorem it is enough to show

$$\lim_{n\to\infty}R_{1,n}(t)=\sum_{i=1}^{\infty}\left(\frac{z_i}{\Gamma_i^{1/\alpha}}\mathbf{1}_{[U_i,1]}(t)-\frac{C_i}{c_i}\frac{t}{\Gamma_i^{1/\alpha}}\right)\,, \tag{15}$$

where convergence is understood in $(D[0,1],d)$, and

$$\lim_{n\to\infty}R_{2,n}=\sum_{i=1}^{\infty}\left(\frac{1}{c_i\Gamma_i^{1/\alpha}}-1\right)C_i\,. \tag{16}$$

Indeed, the last statement implies that $tR_{2,n}$ converges in the uniform norm in $D[0,1]$ and consequently this and (15) establish the convergence of R_1+R_2 in the Skorohod metric. Note that in general addition is not continuous in

$(D[0,1],d)$ but we can use here the following property. If $h_n \to h$ in $(D[0,1],d)$ and $g_n \to g$ in $\|\cdot\|_\infty$ then $h_n + g_n \to h + g$ in $(D[0,1],d)$.

To prove (15) we use Lemma 1 with $H = G_+^{-1}$, $\tilde{H} = G_-^{-1}$ with the corresponding normalizing sequences $(p/p_0)^{1/\alpha}a_n$, $(q/q_0)^{1/\alpha}a_n$ and

$$V_{n,i}(t) = \frac{\delta_i + 1}{2} I_i^n(t) - p_0 t = \left(\frac{\delta_i + 1}{2} - p_0\right) I_i^n(t) + p_0(I_i^n(t) - t) ,$$

$$\tilde{V}_{n,i}(t) = \frac{\delta_i - 1}{2} I_i^n(t) - q_0 t = \left(\frac{\delta_i - 1}{2} + q_0\right) I_i^n(t) - q_0(I_i^n(t) - t) .$$

Note that $V_{n,i}$ and $\tilde{V}_{n,i}$ satisfy assumptions of Lemma 1, i.e. first (4) since $\lim_{n\to\infty} J_i^n = u_i$ (see [9]) and then (5) by Lemma 4. Moreover,

$$\lim_{n\to\infty} V_{n,i}(t) = \frac{\delta_i + 1}{2} \mathbf{1}_{[U_i,1]}(t) - p_0 t$$

and

$$\lim_{n\to\infty} \tilde{V}_{n,i}(t) = \frac{\delta_i - 1}{2} \mathbf{1}_{[U_i,1]}(t) - q_0 t ,$$

where convergence holds a.s. in $(D[0,1],d)$.

Equality (16) follows easily by a double application of Lemma 2 to functions $H = G_+$ and $H = G_-$.

To prove the second statement we use partial integration to get

$$\int_0^1 v d(S_n(t) - b_n t) = (S_n(1) - b_n)v(1) - S_n(0)v(0) - \int_0^1 (S_n(t) - b_n t)\, dv .$$

Now by the first part of the theorem the integrand converges in $D[0,1]$ to $L(t)$ defined as the right hand side limiting process from the first part of the theorem. Since dv has no atoms (v is continuous) then $S_n(t) - b_n t$ is convergent to $L(t)$ a.e. with respect to dv and by the Dominated Convergence Theorem we obtain

$$\lim_{n\to\infty} \int_0^1 v d(S_n(t) - b_n t) = L(1)v(1) - \int_0^1 L(t)\, dv . \tag{17}$$

Next since $L(t)$ as a series of functions is convergent with probability one a.e. with respect to dv we have by the Dominated Convergence Theorem

$$\begin{aligned}\int_0^1 L(t) dv &= \lim_{n\to\infty} \sum_{i=1}^n \int_0^1 \left(\frac{z_i}{\Gamma_i^{1/\alpha}} \mathbf{1}_{[U_i,1]}(t) - C_i t\right) dv \\ &= \lim_{n\to\infty} \sum_{i=1}^n \left(z_i \frac{v(1) - v(U_i)}{\Gamma_i^{1/\alpha}} + C_i \left(\int_0^1 v(t)\, dt - v(1)\right)\right) \\ &= v(1)L(1) - \sum_{i=1}^\infty \left(\frac{z_i v(U_i)}{\Gamma_i^{1/\alpha}} - C_i E v(U_i)\right) . \qquad \square\end{aligned}$$

Proof of Lemma 4. Let $T_{n,k}(t) = \sum_{i=1}^{k} (I_i^n(t) - t)$ for $1 \leqq k \leqq n$. Let us fix k and let $U_{(1)} \leqq U_{(2)} \leqq \cdots \leqq U_{(k)}$ be the order statistics of $U_1, U_2, \ldots, U_k$. We define

$$A = \{U_{(2)} - U_{(1)} > k^{-4}, U_{(3)} - U_{(2)} > k^{-4}, \ldots, U_{(k)} - U_{(k-1)} > k^{-4}\} .$$

Let $n_0 = 2k^5, J_i^n$ be the jump of I_i^n and σ be a permutation of $\{1, 2, \ldots, k\}$ such that $U_{(i)} = U_{\sigma(i)}$. From the definition of I_i^n we have for $n \geqq n_0$ that

$$|J_{\sigma(i)}^n - U_{(i)}| \leqq \frac{\sigma(i)}{n} \leqq \frac{k}{n_0} \leqq \frac{1}{2k^4} .$$

Hence on the event A we have

$$U_{(i-1)} < J_{\sigma(i)} < U_{(i+1)}, \quad i = 1, \ldots, k , \tag{18}$$

where we set $U_{(0)} = 0$ and $U_{(k+1)} = 1$. Moreover $J_{\sigma(i)}^n$ is increasing in $i, 1 \leqq i \leqq k$. Since $\sum_{i=1}^{k} I_i^n(t) = \max\{i \leqq k\colon J_{\sigma(i)}^n \leqq t\}$ it follows from (18) that on A for $n \geqq n_0$

$$\left\| \sum_{i=1}^{k} (I_i^n(t) - \mathbf{1}_{[U_i,1]}(t)) \right\|_\infty \leqq 1 . \tag{19}$$

Next, by the classical estimate for the uniform empirical process (see [2]) we have

$$P\left(\left\| \sum_{i=1}^{k} \mathbf{1}_{[U_i,1]}(t) - t \right\|_\infty \geqq x \right) \leqq 58 \exp(-2x^2/k)$$

and by (19) we conclude that for $x \geqq 0$ we have

$$P\left(\sup_{n \geqq n_0} \|T_{n,k}(t)\|_\infty \geqq x + 1, A \right) \leqq 58 \exp(-2x^2/k) . \tag{20}$$

Now we will deal with $T_{n,k}$ for $n \leqq n_0$. First observe that $\sum_{i=1}^{k} I_i^n(t)$ has a hypergeometric distribution with parameters $i, [nt], n$. By the result of Hoeffding (see [6]) for $x \geqq 0$ we have

$$P\left(\left| \sum_{i=1}^{k} (I_i^n(t) - [nt]/n) \right| \geqq x \right) \leqq 2 \exp(-2x^2/k) .$$

Since $I_i^n(t)$ have jumps only at points of the form $j/n,\ 0 \leqq j \leqq n$ we obtain

$$P\left(\left\| \sum_{i=1}^{k} (I_i^n(t) - [nt]/n) \right\|_\infty \geqq x \right) = P\left(\max_{1 \leqq j \leqq n} \left| \sum_{i=1}^{k} (I_i^n(j/n) - j/n) \right| \geqq x \right)$$
$$\leqq 2n \exp(-2x^2/k) .$$

This leads to

$$P\left(\sup_{n \leqq n_0} \|T_{n,k}\| \geqq x + 1 \right) \leqq \sum_{n=1}^{n_0} P(\|T_{n,k}\| \geqq x + 1) \leqq 2n_0^2 \exp(-2x^2/k) . \tag{21}$$

Next, notice that

$$P(A^c) \leqq \binom{k}{2} P(|U_1 - U_2| \leqq k^{-4}) \leqq \text{const}/k^2 \tag{22}$$

and this last inequality together with (20) and (21) yields (set $x = k^\beta$ and recall that $n_0 = 2k^5$)

$$P\left(\sup_{n \in \mathbb{N}} \|T_{n,k}\|_\infty > k^\beta + 1\right) \leqq \text{const}/k^2 + (8k^{10} + 58)\exp(-2k^{2\beta-1}) .$$

Application of the Borel–Cantelli Lemma concludes the proof of (13).

To prove (14) denote $\tilde{T}_{n,k} = \sum_{i=1}^k r_i I_i^n$ for $n \geqq n_0$ and observe that on the even A we have

$$\|T_{n,k}\|_\infty = \left\|\sum_{i=1}^k r_{\sigma(i)} I_{\sigma(i)}^n(t)\right\|_\infty = \sup_{n \leqq k}\left|\sum_{i=1}^n r_{\sigma(i)}\right|$$

since the jumps $J_{\sigma(i)}^n$, $i = 1, \dots, k$, form an increasing sequence. Now using independence of $\mathbf{r} = (r_i)_{i=1}^\infty$ and $\mathbf{U}$, exchangeability of $(r_i)_{i=1}^n$ and finally Hoeffding's bound for $\sup_{n \leqq k} |\sum_{i=1}^n r_i|$ (see [6]) we obtain

$$P\left(\sup_{n \geqq n_0} \|\tilde{T}_{n,k}\|_\infty \geqq y, A\right) \leqq P\left(\sup_{n \leqq k}\left|\sum_{i=1}^n r_i\right| \geqq y\right) \leqq 2\exp(-y^2/(2a^2k)) . \tag{23}$$

Next observe that for a fixed t again by Hoeffding's inequality applied conditionally we have

$$P\left(\left|\sum_{i=1}^k I_i^n(t) r_i\right| \geqq y\right) \leqq 2\exp(-y^2/(2a^2k)) .$$

Having this inequality we can repeat all steps in the proof of the first part leading to the inequality (21). Then making some minor adjustments we obtain

$$P\left(\sup_{n \leqq n_0} \|\tilde{T}_{n,k}\|_\infty \geqq y\right) \leqq 2n_0 \exp(-y^2/(2a^2k)) .$$

The last inequality together with (22) and (23) provide us with all we need to conclude the second part of the lemma in a similar fashion as we did the first one. □

Remark 2. If $Ev(U) = \int_0^1 v(t)\,dt = 0$, then by the above result

$$\lim_{n \to \infty} \int_0^1 v\,dS_n = \sum_{i=1}^\infty \frac{z_i v(U_i)}{\Gamma_i^{1/\alpha}} .$$

Remark 3. For $\alpha < 1$ the constants b_n, C_i can be set zero. In this case we can use directly Lemma 1 since $k^{-1}\|\sum_{i=1}^k I_i^n\|_\infty \leqq 1$.

As an immediate consequence we obtain the invariance principle for the symmetric case which can be considered as an extension of the results given in [9].

Corollary 1. *With the above notation in the symmetric case* ($p_0 = q_0 = 1/2$) *we have with probability one in* $(D[0,1], d)$:

$$\lim_{n\to\infty} \frac{1}{a_n} \sum_{i=1}^{n} \delta_i G^{-1}(\Gamma_i/\Gamma_{n+1}) I_i^n(\,\cdot\,) = \sum_{i=1}^{\infty} \frac{\delta_i}{\Gamma_i^{1/\alpha}} \mathbf{1}_{[U_i,1]} .$$

Next we deal with the partial sum process of absolute values of the sequence $(X_i)_{i=1}^{\infty}$. Analogously as before we have the following representation in $D[0,1]$

$$\frac{1}{a_n^r} \sum_{i=1}^{[n\,\cdot\,]} |X_i|^r \overset{d}{=} \frac{1}{a_n^r} \sum_{i=1}^{n} [G^{-1}(\Gamma_i/\Gamma_{n+1})]^r I_i^n(\,\cdot\,) .$$

The following simple consequence of Lemma 1 establishes a.s. limit of the above representation.

Theorem 3. *For* $r > \alpha$ *with probability one in* $(D[0,1], d)$:

$$\lim_{n\to\infty} \frac{1}{a_n^r} \sum_{i=1}^{n} [G^{-1}(\Gamma_i/\Gamma_{n+1})]^r I_i^n(\,\cdot\,) = \sum_{i=1}^{\infty} \frac{1}{\Gamma_i^{r/\alpha}} \mathbf{1}_{[U_i,1]}(\,\cdot\,) .$$

Proof. Apply Lemma 1 with $V_{n,j} = I_j^n(\,\cdot\,)$. Let $\xi = \alpha/r$, $\beta = 1$ and $H(x) = [G^{-1}(x)]^r$. Then since $\|I_j^n(\,\cdot\,)\|_\infty \leqq 1$ the condition (5) is satisfied. □

4. Conditional invariance principle and its applications

As an important consequence of Theorem 2 we have the following *conditional invariance principle*.

Proposition 1. *Let us denote by* $W_\alpha(t)$ *the limiting process from Theorem* 2, *and by* $\mathscr{J}_n$, $\mathscr{J}$ *the* σ*-fields generated by the values of the jumps of* $S_n(t)$ *and* $W_\alpha(t)$, *respectively. Almost surely*

$$\lim_{n\to\infty} \mathscr{L}(S_n(t) - b_n t | \mathscr{J}_n) = \mathscr{L}(W_\alpha(t) | \mathscr{J}) ,$$

where the convergence is meant as the weak convergence in $D[0,1]$. *Here the symbol* $\mathscr{L}(X|\mathscr{F})$ *stands for the conditional distribution of* X *with respect to the* σ*-field* $\mathscr{F}$.

Proof. We may assume that the underlying probability space can be written in the form $\Omega = \Omega_1 \times \Omega_2$, $P = P_1 \times P_2$, where the sequences (U_i) for (Γ_i, δ_i) are defined on Ω_1 and Ω_2, respectively. By Theorem 2 and the Fubini theorem, for almost all $\omega_2 \in \Omega_2$

$$\lim_{n\to\infty} (S_n(t)(\,\cdot\,, w_2) - b_n t) = W_\alpha(t)(\,\cdot\,, \omega_2)$$

P_1-a.s. Now, we can observe that

$$\mathscr{L}(S_n(t) - b_n t \mid \mathscr{J}_n) = \mathscr{L}(S_n(t) - b_n t \mid (\Gamma_i, \delta_i))$$

and the result follows by the above stated convergence. □

As mentioned in the Introduction, the above conditional principle is not a simple consequence of the a.s. convergence of any version of the partial sum process and so cannot be obtained by appealing to the Skorohod representation theorem. This is in spite of the fact that one can show that ordered jumps and their locations are continuous on $D[0,1]$. The following simple example shows that a.s. convergence of two sequences of random variables does not imply that their conditional distributions converge to the conditional distribution of the limits.

Example 1. Let X be a nondegenerated random variable, and let $X_n = X$, $Y_n = X/n$. We have that $\lim_{n\to\infty}(X_n, Y_n) = (X, 0)$ a.s. Next observe that $\mathscr{L}(X_n \mid Y_n) = \delta_{\{X\}}$ while $\mathscr{L}(X \mid 0) = \mathscr{L}(X)$. Hence

$$\lim_{n\to\infty} \mathscr{L}(X_n \mid Y_n) \neq \mathscr{L}(X \mid 0) .$$

Moreover, the *conditional invariance principle* cannot be obtained by the following, apparently reasonable, modification of our construction. Our original representation of a sample $(X_i)_{i=1}^n$ relied on generating the sequence of independent signs, and then (conditional on their values) generating absolute values (see Lemma 3). The example below shows that reversing the order of conditioning can lead to a version of the partial sum process for which the conditional invariance principles do not hold.

Example 2. Consider the following two subsets of $[0,1]$:

$$A_+ = \bigcup_{n=1}^{\infty} \left(\frac{1}{2n+1}, \frac{1}{2n} \right], \quad A_- = [0,1] \backslash A_+ .$$

Then for the function $v = I_{A_+} - I_{A_-}$ the random variable $X = v(U)U^{-1}$, where U is uniform on $[0,1]$, is from the 1-stable domain of attraction.

Now, suppose that we reverse the order of conditioning in the construction of the partial sum process. That is, contrary to our previous approach, we generate absolute values first and then, conditional on their values, we choose the signs according to the conditional distribution $sign(X)\big||X|$. The resulting version of the partial sum process assumes the following form

$$\tilde{S}_n(t) = \frac{1}{n} \sum_{i=1}^{n} v(\Gamma_i/\Gamma_{n+1})(\Gamma_i/\Gamma_{n+1})^{-1} I_{[J_i^n,1]}(t) .$$

We will show that both $\mathscr{L}(\tilde{S}_n(t) - b_n t \mid \mathscr{J}_n)$ and $\tilde{S}_n(t) - b_n t$ cannot converge almost surely. Since any function $x(t) \in D[0,1]$ has a finite number of jumps exceeding $\varepsilon > 0$ in absolute value, we can order the jumps of $x(t)$

according to their absolute values. Let $T(x(t))$ denote the first jump of $x(t)$ with respect to this ordering. Next, observe that

$$T(\tilde{S}_n(t) - b_n t) = \frac{\Gamma_{n+1}}{n\Gamma_1} v(\Gamma_1/\Gamma_{n+1}) .$$

Thus if $\mathscr{L}(\tilde{S}_n(t) - b_n t \mid \mathscr{J}_n)$ and $\tilde{S}_n(t) - b_n t$ were convergent a.s., then $\lim_{n\to\infty} v(\Gamma_1/\Gamma_{n+1})$ would exist a.s. since T is a continuous functional on $D[0,1]$.

For any $A \in (0,1)$ let

$$N(A) = \sum_{n=1}^{\infty} I_{\{\Gamma_1/\Gamma_{n+1} \in A\}} .$$

Note that $\Gamma_{n+1} = \Gamma_1 + \tilde{\Gamma}_n$ with $\tilde{\Gamma}_n = \Gamma_{n+1} - \Gamma_1$ being consecutive arrival times of a Poisson process with rate 1 independent of Γ_1. Conditionally on Γ_1, $N(A)$ is a Poisson point process with the intensity measure $\lambda_{\Gamma_1}(t_1, t_2] = \Gamma_1(1/t_1 - 1/t_2)$, $0 < t_1 < t_2 < 1$. Since $\lambda_{\Gamma_1}(A_+) = \lambda_{\Gamma_1}(A_-) = \infty$ we have $N(A_+) = N(A_-) = \infty$ a.s. which proves that i.o. $v(\Gamma_1/\Gamma_{n+1}) = 1$ and i.o. $v(\Gamma_1/\Gamma_{n+1}) = -1$ a.s.

Application: We extend the option pricing formula from [16], which was assumed a binomial model of stock price movements with condition that increments of logarithms of prices follow a symmetric Pareto distribution. For detailed discussion of the model as well as a survey of results related to stock price processes see [8, 16]. The aim of this section is to derive the option pricing formula but with more general assumptions about increments. Namely, we consider logarithm-price increments to be properly normalized i.i.d. random variables from the domain of attraction of a stable law.

Now, let us recall the model studied in [16]. Without losing generality we can assume that the time of expiration of the call option equals to one. Next, let us assume that there are n movements of the stock price until the expiration time. The consecutive price movements at moment k/n are determined by

$$S_k = S_0 \prod_{i=1}^{k} U_i^{r_i} D_i^{1-r_i}, \quad 1 \leqq k \leqq n$$

where r_i's are i.i.d. r.v.'s with $P(r_i = 0) = 1/2$, $P(r_i = 1) = 1/2$, independent of positive random variables U_i and D_i of the form

$$U_i = \exp(\sigma|X_i|/a_n), \qquad D_i = \exp(-\sigma|X_i|/a_n) ,$$

where X_i are i.i.d. from the domain of a symmetric α-stable law, $\sigma > 0$, and the normalizing constants a_n are defined as before ([16] studied the case $G(x) = \min(1, x^{-\alpha})$ and $a_n = n^{1/\alpha}$). Thus

$$S_k = S_0 \exp\left(\frac{\sigma}{a_n} \sum_{i=1}^{k} (2r_i - 1)|X_i|\right) .$$

The random *riskless interest rate* at the ith period R_i is the average of the price up and down movement, i.e. $R_i = (D_i + U_i)/2$.

Now, we can write down the option pricing formula from [16]

$$C_n = E\left[\frac{(S_n - K)_+}{R_1 \dots R_n}\right],$$

where K is a striking price at the expiration time (after n movements). Denoting $|\mathbf{X}_n| = (|X_i|)_{i=1}^n$ we can write

$$R_1 \dots R_n = \frac{1}{2^n} \prod_{i=1}^n (\exp(-\sigma|X_i|/a_n) + \exp(\sigma|X_i|/a_n)) = E\left(\exp\left(\frac{\sigma}{a_n}\sum_{i=1}^n X_i\right)\middle| |\mathbf{X}_n|\right).$$

Thus

$$C_n = E\,\frac{[S_0 \exp(\sigma/a_n \sum_{i=1}^n X_i) - K]_+}{E(\exp(\sigma/a_n \sum_{i=1}^n X_i)\big||\mathbf{X}_n|)} \tag{24}$$

As the next step we want to derive a limiting pricing formula taking $n \to \infty$. In order to do that we will have to change the form of the expression under expected value without changing its distribution. The order statistics of $|\mathbf{X}_n|$ have the same distribution as $(G^{-1}(\Gamma_i/\Gamma_{n+1}))_{i=1}^n$ and the sums in (24) are not sensitive to permutations of $|\mathbf{X}_n|$. Thus,

$$C_n = E\,\frac{E([S_0 \exp(\sigma \sum_{i=1}^n \delta_i G^{-1}(\Gamma_i/\Gamma_{n+1}))/a_n) - K]_+ \big| \mathbf{\Gamma})}{E(\exp(\sum_{i=1}^n \delta_i G^{-1}(\Gamma_i/\Gamma_{n+1}))/a_n) \big| \mathbf{\Gamma})} \tag{25}$$

Theorem 4. *With the above notation we have*

$$\lim_{n\to\infty} C_n = C = E\,\frac{[S_0 \exp(\sigma \sum_{i=1}^\infty \delta_i/\Gamma_i^{1/\alpha}) - K]_+}{E(\exp(\sigma \sum_{i=1}^\infty \delta_i/\Gamma_i^{1/\alpha})\big|\mathbf{\Gamma})}.$$

Proof. Note that the ratio under the expectation in (25) is bounded by S_0. Thus by the Dominated Convergence Theorem it is enough to show that for a fixed value of $\mathbf{\Gamma}$ the sequences in the numerator and denominator of (25) are convergent. Of course, it is enough to show that for the latter sequence.

The convergence is not a direct consequence of Proposition 1 since the exponent function is not bounded. However note that $\sum_{i=1}^n [G^{-1}(\Gamma_i/\Gamma_{n+1})]^2/a_n^2$ is convergent a.s. which together with the inequality

$$P(W_n > x) \leqq 2\exp\left(-(\sigma x)^2/2\sum_{i=1}^n [G^{-1}(\Gamma_i/\Gamma_{n+1})]^2/a_n^2\right)$$

implies uniform integrability of $\exp(W_n)$, where $W_n = \frac{\sigma}{a_n}\sum_{i=1}^n \delta_i G^{-1}(\Gamma_i/\Gamma_{n+1})$. Consequently,

$$\lim_{n\to\infty} E\left(\exp\left(\frac{\sigma}{a_n}\sum_{i=1}^n \delta_i G^{-1}(\Gamma_i/\Gamma_{n+1})\right)\middle|\mathbf{\Gamma}\right) = E\left(\exp\left(\sigma\sum_{i=1}^n \delta_i/\Gamma_i^{1/\alpha}\right)\middle|\mathbf{\Gamma}\right). \qquad \square$$

Remark 4. The result establishes existence of a unique price independent of a distribution of logarithm of price increments from the domain of a symmetric α stable distribution, where $\alpha \in (0,2)$ is fixed. The formula poses interesting practical difficulties in computing the conditional expectation $E(\exp(\sigma \sum_{i=1}^{\infty} \delta_i/\Gamma_i^{1/\alpha})|\mathbf{\Gamma}) = \prod_{i=1}^{\infty} \cosh(\sigma\Gamma_i^{-1/\alpha})$, which we do not address here.

An application of the conditional invariance principle to resampling by signs and permutations in multiple linear regression is given in [13].

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