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# Degree theory on Wiener space

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**Summary.** Let $(W, H, \mu)$ be an abstract Wiener space and let $Tw = w + u(w)$, where $u$ is an $H$-valued random variable, be a measurable transformation on $W$. A Sard type lemma and a degree theorem for this setup are presented and applied to derive existence of solutions to elliptic stochastic partial differential equations.

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## 1. Introduction

Let $T$ be a $C^1$ map from $\mathbb{R}^n$ to $\mathbb{R}^n$ and proper (i.e. the inverse image of any compact set is compact). The degree theorem for this map states that for any bounded real valued function $\varphi(x)$, $x \in \mathbb{R}^n$, with compact support

$$\int_{\mathbb{R}^n} J(x)\varphi(Tx)\,dx = q \int_{\mathbb{R}^n} \varphi(x)\,dx\,, \tag{1.1}$$

where $q$, the degree, is an integer and does not depend on $\varphi$, $J(x)$ is the Jacobian of $T$: $\det([\partial T_i(x)]/\partial x_j)_{n\times n}$.

The degree $q$ satisfies

$$q = \sum_{y \in T^{-1}\{x\}} \operatorname{sign}\det\left(\frac{\partial T_i(y)}{\partial y_j}\right)_{n\times n} \tag{1.2}$$

for almost all $x \in \mathbb{R}^n$. The notion of degree was extended in several directions and in particular, applied to establishing the existence of a solution $x$ to equations of the type $f(x) = x$. In 1934 the notion of degree was extended by Leray and Schauder to a class of transformations on Banach space and applied to the proof of existence of solutions to certain partial differential equations cf. [2] or [5] and the references therein.

The possibility of extending the theory of degree to Wiener space was first pointed out by Eels and Elworthy in [7]. In 1986 E. Getzler [10] introduced the notion of degree for the shift transformations $Tw = w + u(w)$, where $u$ is an $H$-valued random variable. The results of [10] were improved by Kusuoka [12] and Üstünel and Zakai [17] and Theorem 5.1 of [18]. The results derived in these papers were extensions of (1.1) and (1.2) under strong integrability assumptions, (cf. Theorems 3.1–3.3 in Sect. 3). In this paper we (a) extend the Leray–Schauder theorem and some results related to the extension of (1.1) and (1.2) to shift transformation on Wiener space under weaker and what we believe to be perhaps more natural assumptions and (b) apply our results to establish the existence of a (not necessarily unique) solution to an elliptic stochastic partial differential equation. Along the way we also derive a general form of Sard's inequality and Sard's lemma on Wiener space which is believed to be of independent interest. In particular it avoids the need to restrict our results to the set on which the Radon–Nikodym derivative is non-zero.

In the next section we present some notation and notions of the Malliavin calculus. In Sect. 3 we restate for later reference a recent version of the change of variables formula, some results on degree theory derived in previous papers. Next we prove, following some results of Kusuoka, a Sard inequality which implies the Sard lemma on the Wiener space and finally a summary of the Leray–Schauder degree and its properties. In Sect. 4 we state and prove the main results of this paper. In Sect. 5 we consider the SPDE

$$-\Delta\xi(x) + g(x,\xi(x)) = \dot{w}(x), \quad x \in D\,, \tag{1.3}$$

$x \in \mathbb{R}^d$, $n = 1, 2$, or 3 and $\xi|_{\partial D} = 0$ and its extension to more general elliptic operators. Existence and uniqueness of solutions to this equation was considered by Buckdahn and Pardoux [3], Dembo and Zeitouni [6] and Mayer-Wolf and Zeitouni [14]. Applying the results of the previous section, the existence of a (possibly nonunique) solution is established under assumptions which extend in certain directions those imposed in the references cited above.

## 2. Notations and preliminaries

$(W, H, \mu)$ denotes an abstract Wiener space, i.e., $H$ is a separable Hilbert space, identified with its continuous dual, $W$ is a Banach space into which $H$ is injected continuously and densely. $\mu$ is the canonical Gaussian measure on $W$ whose reproducing kernel Hilbert space is $H$ and we will call it as the Cameron–Martin space. In the case of classical Wiener space we have $W = C([0,1])$, $H = \{h : [0,1] \to \mathbb{R}\colon h(t) = \int_0^t \dot{h}(s)\,ds,\ \|h\|_H^2 = \int_0^1 |\dot{h}(s)|^2\,ds\}$. Let $X$ be a separable Hilbert space and $a$ be an $X$-valued (smooth) polynomial on $W$:

$$a(w) = \sum_{i=1}^m \eta_i(\langle h_1, w\rangle, \ldots, \langle h_n, w\rangle)\, x_i\,,$$

with $x_i \in X,\ h_i \in W^*$ and $\eta_i \in C_b^\infty(\mathbb{R}^n)$. The Gross–Sobolev derivative of $a$ is defined as

$$\nabla a(w) = \sum_{i=1}^{m} \sum_{j=1}^{n} \partial_j \eta_i(\langle h_1, w\rangle, \ldots, \langle h_n, w\rangle) x_i \otimes h_j \ ,$$

and $\nabla^k a(w)$ is defined recursively. Thanks to the Cameron–Martin theorem, all these operators are closable on all the $L^p$ spaces and the Sobolev spaces $\mathbb{D}_{p,k}(X)$, $p>1,\ k \in \mathbb{N}$, can be defined as the completion of $X$-valued smooth polynomials with respect to the norm:

$$\|a\|_{p,k} = \sum_{i=1}^{k} \|\nabla^i a\|_{L^p(\mu, X \otimes H^{\otimes i})} \ .$$

From the Meyer inequalities, it is known that the $(p,k)$-norm, defined above, is equivalent to the following norm:

$$\|(I+L)^{k/2} a\|_{L^p(\mu;X)} \ ,$$

where $L$ is the Ornstein–Uhlenbeck operator on $W$ (cf. e.g. [16]) and we denote these two norms with the same notation. Since $I+L$ is an invertible operator, we can also define the norms for negative values of $k$ which describe the dual spaces of the positively indexed Sobolev spaces. We denote by $\mathbb{D}(X)$ the intersection of the Sobolev spaces $\{\mathbb{D}_{p,k}(X);\ \ p>1,\ \ k \in Z\}$, equipped with the intersection (i.e., projective limit) topology. The continuous dual of $\mathbb{D}(X)$ is denoted by $\mathbb{D}'(X)$ and in case $X = \mathbb{R}$ we write simply $\mathbb{D}_{p,k}, \mathbb{D}, \mathbb{D}'$ for $\mathbb{D}_{p,k}(\mathbb{R}), \mathbb{D}(\mathbb{R}), \mathbb{D}'(\mathbb{R})$ respectively. Consequently, for any $p>1,\ k \in Z,\ \nabla : \mathbb{D}_{p,k}(X) \mapsto \mathbb{D}_{p,k-1}(X \tilde{\otimes} H)$ continuously, where $X \tilde{\otimes} H$ denotes the completed Hilbert–Schmidt tensor product of $X$ and $H$. Therefore $\delta = \nabla^*$ is a continuous operator from $\mathbb{D}_{p,k}(X \tilde{\otimes} H)$ into $\mathbb{D}_{p,k-1}(X)$ for any $p>1,\ k \in Z$. We call $\delta$ the divergence operator on $W$. Let us recall that, in the case of classical Wiener space, $\delta$ coincides with the Itô stochastic integral on the adapted processes. Recall that, if $F$ is in $\mathbb{D}_{p,1}(H)$ for some $p>1$, then almost surely, $\nabla F$ is an Hilbert–Schmidt operator on $H$ and if $F$ is an $H$-valued polynomial, then $\delta F$ can be written as

$$\delta F = \sum_{i=1}^{\infty} [(F, e_i)_H \delta e_i - (\nabla (F, e_i)_H, e_i)_H] \ ,$$

where $(e_i;\ i \in \mathbb{N})$ is any complete orthonormal basis in $H$.

Let $K$ be a Hilbert–Schmidt operator on $H$ and let $\tilde{K}w$ denote the $H$-valued divergence of $K$. More specifically let $e_i,\ i = 1, 2, \ldots$ be a complete orthonormal base on $H$ and

$$w = \sum \delta e_i \cdot e_i$$

(from a theorem of Ito-Nisio, cf. [11], the above sum converges almost surely and in $L^p$, for any $p>1$, in the norm topology of $W$) then

$$\tilde{K}w =: \delta K = \sum \delta e_i \cdot K e_i \ , \tag{2.1}$$

$\tilde{K}w =: \delta K$ is an $H$-valued random variable and

$$E\tilde{K}_1 w \cdot \tilde{K}_2 w = \sum_i (K_1 e_i,\ K_2 e_i) = (K_1, K_2)_{H-S} .$$

An $X$-valued random variable $F$ is said to be in $\mathbb{D}_{p,1}^{\mathrm{loc}}(X)$ for some $p>1$ if there exists an increasing sequence of measurable subsets $(W_n;\ n\in\mathbb{N})$ of $W$ and $(F_n;\ n\in\mathbb{N}) \subset \mathbb{D}_{p,1}(X)$ such that $\bigcup_n W_n = W$ and $F = F_n$ on $W_n$ almost surely.

Let $K$ be a Hilbert–Schmidt operator on $H$ and denote by $(\lambda_i, i\in\mathbb{N})$ its eigenvalues according to their multiplicity. The Carleman–Fredholm determinant of $I_H + K$ is defined as

$$\det_2(I_H + K) = \prod_{i=1}^{\infty} (1+\lambda_i)e^{-\lambda_i} .$$

Note that $\det_2(I_H + K) \neq 0$ if and only if $(I_H + K)$ is invertible. For $u \in \mathbb{D}_{p,1}^{\mathrm{loc}}(H)$, $\Lambda_u(w)$ will be defined as

$$\Lambda_u(w) = \det_2(I_H + \nabla u(w)) \exp(-\delta u(w) - \tfrac{1}{2}\|u(w)\|_H^2) . \qquad (2.2)$$

We conclude this section with some definitions regarding the behaviour of $u(w+h)$ as a function of $h\in H$.

**Definition 2.1** *Let $u(w)$ be an $H$-valued random variable.*

(a) *$u(w)$ is said to be an $H-C$ map if, for almost all $w\in W, h\mapsto u(w+h)$ is a continuous function of $h\in H$.*
(b) *$u(w)$ is said to be a "compact $H-C$ map" or "$H-C$-compact" if $u$ is $H-C$ and for almost all $w$, $h\mapsto u(w+h)$ is a compact function on $H$. (A map $g: H\to H$ is said to be compact if it maps bounded sets into relatively compact sets).*
(c) *$u(w)$ is said to be $H-C^1$ if it is $H-C$ and for a.a.w. $h\mapsto u(w+h)$ is continuously Frechét differentiable on $H$.*
(d) *$u(w)$ is said to be "locally $H-C^1$" if there exists an almost surely strictly positive random variable $\rho$ such that $h\mapsto u(w+h)$ is $C^1$ on the set $\{h\in H: |h|<\rho(w)\}$.*
(e) *$u(w)$ is said to be "compact $H-C^1$" if it is $H-C^1$ and both $h\mapsto u(w+h)$ and $h\mapsto(\nabla u)(w+h)$ are compact functions on $H$ for almost all $w$.*
(f) *$u(w)$ is said to be "representable by locally $H-C^1$" functions if there exists a sequence of measurable subsets of $W$, say $B_m$, such that $\mu(\bigcup B_m) = 1$ and a sequence of "locally $H-C^1$" $H$-valued random functions $u_m(w)$ such that*

$$\mathbf{1}_{B_m}(w)(u(w) - u_m(w)) = 0 \quad \text{a.s.}$$

*Throughout the paper, $Tw$ will denote*

$$Tw = w + u(w)$$

*where $u(w)$ is an $H$-valued random variable.*

## 3. Some preliminaries and previous results

In this section we, first, state a change of variables formula. Next some previously derived results on the degree theorem are presented and finally, the degree theory of Leray and Schauder is summarized.

The following change of variables formula is proven in Theorem 4.1 of [18]. In fact the first part of it is essential for the proof of the second.

**Theorem 3.1** 1. *Suppose that* $u \in \mathbb{D}_{p,1}(H)$ *for some* $p > 1$. *Assume also that there are two positive constants* $c < 1$ *and* $d$ *with*

$$\|\nabla u\| \leqq c$$

*and*

$$\|\nabla u\|_2 \leqq d\ ,$$

*almost surely, where the first norm is the operator norm and the second is the Hilbert–Schmidt norm on* $H$. *Then the map* $w \mapsto Tw = w + u(w)$ *is almost surely a bijection of* $W$, *moreover, we have*

$$E[F \circ T|\Lambda_u|] = E[F]$$

*for any* $F \in C_b(W)$, *where*

$$\Lambda_u = \det{}_2(I_H + \nabla u)\exp\{-\delta u - \tfrac{1}{2}|u|_H^2\}\ .$$

2. *Let* $u(w) \in \mathbb{D}_{p,1}^{\mathrm{loc}}(H)$ *for some* $p > 1$. *Assume that* $u(w)$ *is representable by locally* $H - C^1$ *functions. Let* $M$ *be the set* $\{w : \det_2(I_H + \nabla u(w)) \neq 0\}$. *Then, for* $Tw = w + u(w)$:

(i) *The cardinality of* $T^{-1}\{w\} \cap M$, *denoted* $N(w, M)$ *is at most countably infinite.*

(ii) *For any positive measurable bounded real random variables* $\rho$ *and* $g$ *and* $\Lambda_u$ *is defined by* (2.1):

$$E[\rho(Tw)g(w)|\Lambda_u(w)|] = E\left\{\rho(w)\sum_{\theta \in T^{-1}\{w\}\cap M} g(\theta)\right\} \tag{3.1}$$

*in the sense that if one side is finite so is the other and equality holds.*

*Remark.* In the sequel, using the Sard lemma we shall prove that one can omit to take into account the set of non-degeneracy $M$ in the above formulas.

The first part of the following result is from [17] and presents an improved version of results of [10] while the second part extends some results of [17].

**Theorem 3.2** (a) *If for some* $\gamma > 0$, $r > (1+\gamma)/\gamma$, $u(w) \in \mathbb{D}_{r,2}(H)$, $\Lambda_u \in L^{1+\gamma}(\mu)$, $\Lambda_u(I_H + \nabla u) \cdot v \in L^{1+\gamma}(\mu, H)$ *for all non random* $v \in H$, *then*

$$E[\varphi(Tw)\Lambda_u(w)] = E[\Lambda_u]E[\varphi] \tag{3.2}$$

*for all bounded and measurable* $\varphi$.

(b) *Suppose that* $u \in \mathbb{D}_{p,1}(H)$ *for some* $p > 1$ *and that*

$$\exp(-\delta u + \tfrac{1}{2}\|\nabla u\|_2^2) \in L^{1+\gamma}(\mu)$$

*for some* $\gamma > 0$, *where* $\|\cdot\|_2$ *denotes the Hilbert–Schmidt norm. Then*

$$E[\phi \circ T\Lambda_u] = E[\phi]$$

*and in particular* $E[\Lambda_u] = 1$.

*Proof.* We shall deal only with the proof of part (b): let $(e_n; n \in \mathbb{N})$ be a complete, orthonormal basis in $H$. Denote by $V_n$ the sigma algebra generated by $\{\delta e_1, \ldots, \delta e_n\}$ and by $\pi_n$ the orthogonal projection of $H$ onto its subspace spanned by $\{e_1, \ldots, e_n\}$. Let $u_n = E[P_{1/n}\pi_n u | V_n]$, where $P_{1/n}$ is the Ornstein–Uhlenbeck semigroup at $t = 1/n$. Then, using the inequality $|\det_2(I_H + A)| \leqq \exp \frac{1}{2}\|A\|_2^2$ for the Hilbert–Schmidt operator $A$, we obtain

$$|\Lambda_{u_n}| \leqq P_{1/n} E[\exp\{-e^{1/n}\delta u + e^{-2/n} 1/2 \|\nabla u\|_2^2\} | V_n]\,,$$

hence there exists some $n_0$ such that for $n \geqq n_0$, we have

$$E[|\Lambda_{u_n}|^{1+\varepsilon}] \leqq E\left[\exp\left\{-(1-\gamma)\delta u + \frac{1+\gamma}{2}\|\nabla u\|_2^2\right\}\right]$$

for some $0 < \varepsilon < \gamma$. Therefore the sequence $(\Lambda_{u_n}; n \in \mathbb{N})$ is uniformly integrable. Now let us replace $u_n$ by $tu_n$. Then from the Jensen inequality $t \mapsto E[\Lambda_{tu_n}]$ is continuous on [0,1] (cf. Lemma 3.1 of [17]) and it is also an integer (this follows from the finite dimensional considerations), hence it is equal to one, which implies also that $E[\Lambda_u] = 1$ by the uniform integrability. The rest is obvious from the part (a) and the uniform integrability. □

**Theorem 3.3** [13] *Suppose that* $u(w)$ *is compact* $H - C^1$ *and* $\nabla u$ *is an* $H - C^1$ *map. Suppose, moreover, that* $u \in \bigcap_{p \in (1,\infty)} \mathbb{D}_{p,2}(H)$ *and for some* $p \in (1, \infty)$ *and* $\varepsilon > 0$,

$$E[\exp p(\tfrac{1}{2}(\varepsilon + \|\nabla u\|_{H-S})^2 - \delta u - \tfrac{1}{2}\|u\|_H^2)] < \infty\,,$$

*then* $E[|\Lambda_u|] < \infty$,

$$E[\varphi \circ T\Lambda_u] = E[\Lambda_u]\, E[\varphi]$$

*and*

$$E[\Lambda_u] = \sum_{\theta \in T^{-1}\{w\}} \operatorname{sign} \det_2(I_H + \nabla u(\theta))\,, \tag{3.3}$$

*almost surely.*

**Theorem 3.4** [18] *If* $u(w)$ *is locally* $H - C^1$ *and if it satisfies the assumptions of part* (a) *of Theorem* 3.2, *then Eq.* (3.3) *holds provided that one replaces in* (3.3) *the set on which the sum is taken by* $T^{-1}\{w\} \cap M$.

*Remark.* As a consequence of Lemma 3.2, we will see that, in fact, one can remove the set $M$, by taking a modification of $u$.

Note the similarity of (3.2) and (3.3) to Eqs. (1.1) and (1.2). The exponential integrability conditions in Theorems 3.2–3.4 are quite difficult to verify in the case of non adapted shifts and consequently the applicability of these results is limited.

In the following pages of this work we shall need Sard's lemma that we prove in this section. First of all we settle the problem of measurability of the forward images of the measurable sets in the following lemma which is of independent interest:

**Lemma 3.1** *Suppose that $u: W \to H$ is a measurable map. Then for any measurable $A \subset W, (I_W + u)(A) = T(A)$ belongs to the universally completed (i.e., the universal) Borel sigma algebra of $W$.*

*Proof.* Let $t: W \times H \to H$ be defined as $t(w,h) = h + u(w+h)$. Then

$$T(A) = \{w \in W \colon t(w, \cdot)^{-1}\{0\} \cap (A - w) \neq \emptyset\} .$$

Let $\Gamma$ be the multifunction with values in the subsets of $H$, defined by

$$\Gamma(w) = t(w, \cdot)^{-1}\{0\} \cap (A - w) .$$

Then

$$T(A) = \{w \in W \colon \Gamma(w) \neq \emptyset\} .$$

Let $G(\Gamma)$ be the graph of $\Gamma$:

$$G(\Gamma) = \{(h, w) \colon h \in \Gamma(w)\} .$$

Note that the projection of $G(\Gamma)$ in $W$ is exactly the set $T(A)$. We have $G(\Gamma) = \{(h,w) \colon t(w,h) = 0, h + w \in A\}$. Since the map $(h,w) \mapsto h + w$ is measurable on $H \times W, G(\Gamma)$ is measurable and $H$ is Suslin, consequently $T(A)$ is measurable with respect to the universally completed Borel sigma algebra of $W$ (cf. [4, p. 75, Theorem III.23]). □

The following result is the infinite dimensional version of the Sard inequality which implies the Sard lemma. In case $u$ is compact $H - C^1$, the validity of the Sard lemma is indicated in [13]. Here we give the proof of the inequality using the technique developed in [12], hence we will not enter too much into technical details.

**Lemma 3.2** *Suppose that $u: W \to H$ is a measurable map in some $\mathbb{D}_{p,1}(H)$ such that there exists a non-negative random variable r, with $\mu(Q) = \mu\{r > 0\} > 0$ and the map $h \mapsto u(w+h)$ is continuously Fréchet differentiable on the random open ball $\{h \in H : |h|_H < r(w)\}$. Then we have, for any $A \in \mathscr{B}(W)$,*

$$\mu(T(A \cap Q)) \leqq \int_{A \cap Q} |\Lambda_u| \, d\mu .$$

*Proof.* Let $(\pi_n;\ n\in\mathbb{N})$ be a sequence of orthogonal projections of $H$ increasing to $I_H$. Define

$$W_{n,m}=\left\{w\in W\colon \|\nabla u(w+h)-\nabla u(w)\|_2<\tfrac{1}{120},\ \text{for all}\ |h|_H<\frac{8}{m}\right\}$$
$$\cap\left\{w\in W\colon |\pi_n^{\perp}u(w)|_H<\frac{1}{120m},\|\pi_n^{\perp}\nabla u(w)\|_2\leqq\tfrac{1}{120},\|\nabla u(w)\|_2\right.$$
$$\left.\leqq m, r(w)>\frac{9}{m}\right\},$$

where $\|\cdot\|_2$ denotes the Hilbert–Schmidt norm. By the $H-C^1$ property, $(W_{n,m};n,m\in\mathbb{N})$ covers almost surely $Q$ (here, if necessary, we add a negligible set to have equality everywhere instead of almost everywhere but we keep the same notation).

Let us denote the set $A\cap W_{n,m}$ by $\Omega$. Let $\rho_\Omega(w)=\inf(|h|_H:h\in(\Omega-w)\cap H)$, note that $|\rho_\Omega(w+h)-\rho_\Omega(w)|\leqq|h|_H$, hence $|\nabla\rho_\Omega(w)|_H\leqq 1$ almost surely (cf., [18]). Define $G(w)=g(w)\pi_n^{\perp}u(w)$, where $g(w)=\phi(m\rho_\Omega(w))$. $\phi$ is chosen as a smooth function from $\mathbb{R}$ to $[0,1]$, it is equal to one on $[-6,6]$ and zero outside the interval $[-7,7]$. Moreover, its derivative is supposed to be bounded by two. Then $\|\nabla G(w)\|_2\leqq 3/10$. To see this, we have

$$\nabla G=\phi'(m\rho_\Omega)m\nabla\rho_\Omega\otimes\pi_n^{\perp}u+g\pi_n^{\perp}\nabla u\,.$$

Since $\nabla$ is a local operator (cf., [15]), on the set $\Omega$, $\nabla\rho_\Omega=0$ almost surely. Hence, for almost all $w\in\Omega$, we have $\nabla G(w)=g(w)\pi_n^{\perp}\nabla u(w)$, which implies that $\|\nabla G(w)\|_2\leqq\frac{1}{120}$ on $\Omega$. For those $w$ who fall outside $\Omega$, the only contribution comes from the ones for which $g(w)\neq 0$. For this condition to be realized, we should have $m\rho_\Omega(w)\leqq 7$. Then, by the very definition of $\rho_\Omega$, for any $\varepsilon>0$, there exist $h\in H$ and $z\in\Omega$, such that $z-w=h$ and $|h|_H<\frac{7}{m}+\varepsilon<\frac{8}{m}$. Hence the norm of the first term at the right hand side of $\nabla G$ can be bounded as

$$m|\phi'(m\rho_\Omega(w))||\nabla\rho_\Omega(w)|_H|\pi_n^{\perp}u(w)|_H\leqq 2m|\pi_n^{\perp}u(w)|_H\,.$$

Since $w=z-h$ and $z\in\Omega$, we have

$$\begin{aligned}|\pi_n^{\perp}u(w)|_H&=|\pi_n^{\perp}u(z-h)|_H\\&\leqq|\pi_n^{\perp}u(z)-\pi_n^{\perp}u(z-h)|_H+|\pi_n^{\perp}u(z)|_H\\&\leqq|\pi_n^{\perp}u(z)-\pi_n^{\perp}u(z-h)|_H+\frac{1}{120m}\,.\end{aligned}$$

Moreover,

$$\begin{aligned}|\pi_n^{\perp}u(z)-\pi_n^{\perp}u(z-h)|_H&\leqq\int_0^1|\nabla_h\pi_n^{\perp}u(z-th)|_H\,dt\\&\leqq\int_0^1[|\nabla_h\pi_n^{\perp}u(z-th)-\nabla_h\pi_n^{\perp}u(z)|_H+|\nabla_h\pi_n^{\perp}u(z)|_H]\,dt\\&\leqq\tfrac{1}{120}|h|_H+\tfrac{1}{120}|h|_H=\tfrac{1}{60}|h|_H\,,\end{aligned}$$

since $|th|_H \leqq |h|_H < 8/m$ and since $z \in \Omega$. Therefore we obtain

$$|\pi_n^\perp u(w)|_H < \frac{1}{60}|h|_H + \frac{1}{120m} \leqq \frac{8}{60m} + \frac{1}{120m} ,$$

consequently

$$m|\phi'(m\rho_\Omega(w))||\nabla\rho_\Omega(w)|_H|\pi_n^\perp u(w)|_H \leqq 2m\frac{17}{120m} = \frac{17}{60} .$$

For the second term at the right hand side of $\nabla G(W)$, we have

$$\begin{aligned}\|g(w)\pi_n^\perp \nabla u(w)\|_2 &\leqq \|\pi_n^\perp \nabla u(z-h)\|_2 \\ &\leqq \|\pi_n^\perp \nabla u(z-h) - \pi_n^\perp \nabla u(z)\|_2 + \|\pi_n^\perp \nabla u(z)\|_2 \\ &\leqq \tfrac{1}{120} + \tfrac{1}{120} = \tfrac{1}{60} .\end{aligned}$$

Therefore, for $w \in \Omega^c$,

$$\|\nabla G(w)\|_2 \leqq \tfrac{17}{60} + \tfrac{1}{60} = \tfrac{3}{10}$$

and for $w \in \Omega$,

$$\|\nabla G(w)\|_2 \leqq \tfrac{1}{120} ,$$

hence

$$\|\nabla G(w)\|_2 \leqq \max\{\tfrac{3}{10}, \tfrac{1}{120}\} = \tfrac{3}{10}$$

for almost all $w \in W$. Consequently $T_G = I_W + G$ is almost surely bijective (cf. [12, 16, 18]). Making exactly the same reasoning, we can see that $\|\nabla(gu)\|_2$ is essentially bounded. Set $E = T_G(\Omega)$, then $E$ is measurable, and if $\rho_E(w) \leqq 3/m$, then $\rho_\Omega(T_G^{-1}(w)) \leqq 5/m$ and $(I_W + \pi_n^\perp u)(T_G^{-1}(w)) = w$ for $\rho_E(w) \leqq 3/m$. Let now $k(w) = \psi(m\rho_E(w))$, $\psi : \mathbb{R} \to [0,1], |\psi'| \leqq 2$, $\psi = 1$ on $[-1,1]$, and zero outside $[-2,2]$. Define $K(w) = k(w)(-w + T_G^{-1}(w))$. Then, as we have done for $\nabla G$ above, we can show easily that $\|\nabla K\|_2 < 1/2$ almost surely.

After all these preparations, define $I_W + S = T \circ T_K$, where $T_K = I_W + K$. We have $\Lambda_S(w) = \Lambda_u(T_K w)\Lambda_K(w)$. Moreover $S(w) = K(w) + u(T_K(w))$. If $\rho_E(w) < 1/m$ (in particular if $w \in E$), from above we have $(I_W + \pi_n^\perp u)(T_G^{-1} w) = w$ and $T_G^{-1}(w) = T_K(w)$, hence

$$\begin{aligned} w &= (I_W + \pi_n^\perp u)(T_K w) \\ &= w + K(w) + \pi_n^\perp u(T_K w)\end{aligned}$$

which gives $K(w) = -\pi_n^\perp u(T_K w)$. Consequently

$$\begin{aligned} S(w) &= K(w) + u(T_K w) \\ &= -\pi_n^\perp u(T_K w) + u(T_K w) \\ &= \pi_n u(T_K w) ,\end{aligned}$$

this means that for $\rho_E(w) < \frac{1}{m}$, $S(w)$ belongs to the finite dimensional space $\pi_n H$. Let $(e_i)$ be a complete orthonormal basis in $H$ corresponding to $(\pi_n)$. Define, for given $w \in W$, $w_2$ as $\sum_{i \leqq n} \delta e_i(w) e_i$ and $w_1 = w - w_2$. Denote respectively by $\mu_n$ and $\mu_n^\perp$ the images of $\mu$ under these two maps. From the Fubini theorem, we have

$$\mu((T_S(E)) = \int_{W_n^\perp \times W_n} \mathbf{1}_{T_S(E)}(w_1 + w_2)\mu_n^\perp(dw_1)\mu_n(dw_2)$$
$$= \int_{W_n^\perp} \mu_n((I_{\mathbb{R}^n} + S(w_1 + \cdot))[(E - w_1) \cap \pi_n H])\mu_n^\perp(dw_1) .$$

Since $w_2 \mapsto S(w_1 + w_2) \in \pi_n H$ for those $w_2$ such that $w_1 + w_2 \in E$, we can use the Sard inequality in finite dimensions for Lipschitz maps (cf. [8], p. 243, Theorem 3.2.3) to obtain

$$\mu(T_S(E)) \leqq \int_{W_n^\perp} \mu_n^\perp(dw_1) \int_{(E-w_1)\cap\pi_n H} |\det(I_{\mathbb{R}^n} + \nabla S(w_1 + w_2))|$$
$$\times \exp[-(S(w_1 + w_2), w_2) + \operatorname{trace}(\nabla S(w_1 + w_2))|_{\pi_n H}$$
$$-1/2|S(w_1 + w_2)|_H^2]\mu_n(dw_2)$$
$$\leqq \int_{W_n^\perp} \mu_n^\perp(dw_1) \int_{(E-w_1)\cap\pi_n H} |\Lambda_S(w_1 + w_2)|\mu_n(dw_2)$$
$$= \int_E |\Lambda_S|\mu(dw)$$
$$= \int_E |\Lambda_u(T_K w)||\Lambda_K(w)|\mu(dw) .$$

Recall that $T_K = T_G^{-1}$ on $E$. $T_G^{-1}$ can be written as $T_G^{-1}(w) = w + \alpha(w)$ for some $\alpha \in \mathbb{D}_{s,1}(H)$ for any $s > 1$ (cf. [16]). From the locality of the operators $\delta$ and $\nabla$ (cf. [15]) $\delta K = \delta\alpha$ and $\nabla K = \nabla\alpha$ almost surely on $E$, hence $\Lambda_K = \Lambda_\alpha$ almost surely on $E$. Moreover, from the Ramer theorem for the contractive case (cf. [18]), we have

$$E[f \circ T_G^{-1}|\Lambda_\alpha|] = E[f] ,$$

for any positive, measurable function $f$ on $W$. Consequently

$$\mu(T_S(E)) \leqq \int_E |\Lambda_u \circ T_K||\Lambda_K|\, d\mu$$
$$= \int_E |\Lambda_u \circ T_G^{-1}||\Lambda_\alpha|\, d\mu$$
$$= \int_W (\mathbf{1}_\Omega|\Lambda_u|) \circ T_G^{-1}|\Lambda_\alpha|\, d\mu$$
$$= \int_W \mathbf{1}_\Omega|\Lambda_u| d\mu .$$

On the other hand $T_S(E) = T \circ T_K(T_G(\Omega)) = T \circ T_G^{-1}(T_G(\Omega))$ and $T_G^{-1} \circ T_G(\Omega) = \Omega$ almost surely. We can add a null set $O$ to have the everywhere equality: $T_K(T_G(\Omega)) = \Omega \cup O$ and we obtain:

$$\mu(T(\Omega)) \leqq \mu(T(\Omega \cup O)) \leqq \int_\Omega |\Lambda_u|\, d\mu\,.$$

Let us now cut and paste the sequence $(\Omega_{n,m})$ to form a partition of $A \cap Q$ (we keep the same notation). Then

$$\begin{aligned}\mu(T(A \cap Q)) &= \mu(T(\cup \Omega_{n,m})) \\ &= \mu(\cup T(\Omega_{n,m})) \\ &\leqq \sum \mu(T(\Omega_{n,m})) \\ &\leqq \sum \int_{\Omega_{n,m}} |\Lambda_u|\, d\mu \\ &= \int_{A\cap Q} |\Lambda_u|\, d\mu\,. \qquad \square\end{aligned}$$

**Theorem 3.5** *Suppose that $u : W \to H$ is as in Lemma* 3.2 *and let $T = I_W + u$. For any positive, bounded, measurable functions $f$ and $g$ on $W$, we have*

$$E[f \circ T \; g\mathbf{1}_Q|\Lambda_u|] = E\left[f \sum_{y \in T^{-1}\{w\} \cap Q} g(y)\right].$$

*where $\Lambda_u = \det_2(I_H + \nabla u)\exp[-\delta u - \frac{1}{2}|u|_H^2]$. Furthermore, if $u$ is locally $H - C^1$, then there exists a modification $u'$ of $u$ (i.e., $u' = u$ almost surely), such that $T' = I_W + u'$ satisfies*

$$E[f \circ T' g|\Lambda_u|] = E\left[f \sum_{y \in T'^{-1}\{w\}} g(y)\right].$$

*If $H + Q \subset Q$, in particular when $u$ is $H - C^1$, then $T'(Q) \subset Q$, hence we can replace $T'$ by the restriction of $T$ to $Q$ and look at $(Q, H, \mu)$ as an abstract Wiener space on which it holds that*

$$\mu(T(A)) \leqq \int_A |\Lambda_u|\, d\mu\,,$$

*for any $A \in \mathscr{B}(Q)$.*

*Proof.* From Theorem 5.2 of [18], we have

$$E[f \circ T \; g\mathbf{1}_Q|\Lambda_u|] = E\left[f \sum_{y \in T^{-1}\{w\} \cap M \cap Q} g(y)\right].$$

Therefore, if $g = g'$ almost surely on $Q$ then

$$\sum_{y \in T^{-1}\{w\} \cap Q \cap M} g(y) = \sum_{y \in T^{-1}\{w\} \cap Q \cap M} g'(y)$$

almost surely. Moreover, we have

$$E\left[f \sum_{y\in T^{-1}\{w\}\cap M^c\cap Q} g(y)\right] = E\left[f\mathbf{1}_{(T(M^c\cap Q))^c} \sum_{y\in T^{-1}\{w\}\cap Q} g(y)\right]$$

and the first part of the corollary follows from Lemma 3.2. For the second part, it suffices to remark that from the local $H-C^1$ property, we have $Q=W$ almost surely. Define $u'(w)$ as $u(w)$ on $Q$ and zero elsewhere, then the corresponding shift $T'$ satisfies the claimed property. □

**Corollary 3.1** *Suppose that $u$ is locally $H-C^1$ and it satisfies the hypothesis of part* (a) *of Theorem* 3.2. *Then the modification $u'$ of $u$ and the corresponding shift $T'$ satisfy the degree identity* (3.3) *of Theorem* 3.3. *Furthermore, if $u$ is $H-C^1$ then* (3.3) *holds also for $u$, i.e.*,

$$E[\Lambda_u] = \sum_{y\in T^{-1}\{w\}} \operatorname{sign} \Lambda_u(y) .$$

We have the following extension of Theorem 8.1 of [12]:

**Lemma 3.3** *Let $u: W\to H$ be as in the Lemma* 3.2. *Suppose that $\nu$ is another probability on $W$ such that $\nu|_{M\cap Q}$ and $\mu$ are mutually singular. Then there exists a universally measurable set $K$ with $\mu(K)=0$ and $T^\star\nu(K) \geqq \nu(Q)$. In other words $\mu$ and $T^\star\tilde{\nu}$ are singular where $\tilde{\nu}$ is defined by*

$$\tilde{\nu}(A) = \frac{\nu(A\cap Q)}{\nu(Q)} .$$

*If $u$ is $H-C^1$ then one can replace $Q$ with $W$.*

*Proof.* By the hypothesis, there exists a measurable set $N\subset M\cap Q$ such that $\mu(N)=0$ and $\nu(N)=\nu(M\cap Q)$. From the Theorem 3.6, we have $\mu(T(N\cup(M^c\cap Q)))=0$ and

$$\begin{aligned} T^\star\nu(T(N\cup(M^c\cap Q))) &\geqq \nu(N\cup(M^c\cap Q)) \\ &= \nu(Q) . \end{aligned}$$

To complete the proof it suffices to take $K=T(N\cup(M^c\cap Q))$. □

Turning to the Leray–Schauder theory cf., e.g., [9] or [5] for a detailed treatment: let $X$ be a Banach space, $D$ a bounded, open subset of $X$ with boundary $\partial D$. Let

$$\psi = I_X + K ,$$

where $K$ is a (not necessarily linear) continuous, compact map on $D$. $\psi$ is called a compact perturbation of the identity. Then there exists a function $\deg(\psi, D, p)$ defined for any $p\in X$, satisfying $p\notin\psi(\partial D)$ which possesses the following properties. Furthermore, the first four properties determines $\deg(\cdot,\cdot,\cdot)$ uniquely.

(a) $\deg(\psi, D, p)$ is integer valued.
(b) $\deg(I_X, D, p) = 1$ for any $p \in D$.
(c) If $D_1, D_2$ are disjoint open subsets of $D$ and $p \notin \psi(D - D_1 \cup D_2)$ then

$$\deg(\psi, D, p) = \deg(\psi, D_1, p) + \deg(\psi, D_2, p)\,.$$

(d) Invariance under homotopy: Let $G : [0,1] \times D \to X$ be a compact map, $y : [0,1] \to X$ continuous and $y(t) \notin (I_X + G(t, \cdot))(\partial D)$ for any $t \in [0,1]$, then $\deg[(I_X + G(t, \cdot)), D, y(t)]$ is independent of $t \in [0,1]$.
(e) $\deg(\psi, D, p) \neq 0$ implies $\psi^{-1}\{p\} \neq \emptyset$.
(f) $\deg(\psi, D, p)$ is *constant* on every connected component of $X - \psi(\partial D)$.
(g) $\deg(I_X + K_1, D, p) = \deg(I_X + K_2, D, p)$ whenever $K_1|_{\partial D} = K_2|_{\partial D}$.
(h) If $\psi$ is one to one and $p \in \psi(D)$, then $d(\psi, D, p) = \pm 1$.

## 4. A degree theorem on Wiener space

**Theorem 4.1** *Let $u(w)$ be an $H$-valued random variable, $Tw = w + u(w)$. Assume that*

(1) *for a.a.w, $h \mapsto u(w+h)$ is a compact map on $H$.*
(2) *for almost all $w$, for any $h_0 \in H$*

$$\sup\{|h| \colon h_0 = h + u(w+h)\} < \infty$$

(*this condition is satisfied if, e.g.,* $\underline{\lim}\ |h_n + u(w+h_n)|_H = \infty$ *whenever* $|h_n| \to \infty$).

*Let $D_n$ denote an increasing sequence of bounded, open subsets of $H$, $D_n \nearrow H$ as $n \to \infty$. Then, as $n$ goes to $+\infty$*

$$\deg(T(w+h) - w, D_n, h_0) \to q \tag{4.1}$$

*almost surely, where $q$ is a non random constant. If $q \neq 0$, then $\{v \colon T(v) = w\}$ is a.s. non empty and the equation for $v$: $v + u(v) = w$ has a measurable solution $v(w)$.*
*If (2) is replaced by (2)′:*

(2)′ *for a.a. $w$, $|u(w+h)|_H = o(|h|_H)$ as $|h|_H \to \infty$, then (2) is also satisfied and $q = 1$.*

*Proof.* Note first that if $v_1(w)$, $v_2(w)$ are two measurable solutions to $w = T(v_i) = v_i + u(v_i)$, $i = 1, 2$ on $(W, H, \mu)$ then $v_1 - v_2 \in H$ and in order to consider solutions to $w = v + u(v)$ we have to consider solutions $h_0 \in H$ to the equation $0 = h + u(w+h)$ and then $v = w + h_0$. Setting $g(h) = h + u(w+h)$ and fixing $w$, by the Leray–Schauder theorem and our assumptions $\deg(g, D_k, p)$ is a well defined integer for every $p \in H$. Moreover by assumption (2), for $p$ fixed and some $k_0$ which may depend on $w$, there is no solution to $p = g(h)$ on $\partial D_k$ for all $k \geqq k_0$. Consequently, by property (c), we may define $\deg(g, H, p)$ as

$$\deg(g, D_k, p) \underset{k \to \infty}{\longrightarrow} \deg(g, H, p) \quad \text{a.s.} \tag{4.2}$$

Again by assumption (2) and property (d) with $G$ independent of $t$, the limit is independent of $p$. Moreover, for any $h_1, h_2, h_3 \in H$

$$\begin{aligned}\lim_{k\to\infty} \deg(g+h_1, D_k+h_3, p+h_2) &= \lim_{k\to\infty} \deg(g, D_k+h_3, p+h_2-h_1) \\ &= \lim_{k\to\infty} \deg(g, D_k+h_3, p) \\ &= \lim_{k\to\infty} \deg(g, D_k, p)\,. \end{aligned} \tag{4.3}$$

Let $d_g(w)$ denote this limit and note that $d_g(w) = d_g(w+h)$. Consequently let $A = \{w\colon d_g(w) \in [a,b]\}$ note that $A$ is shift invariant ($A = A+H$) and consequently the probability of $A$ is zero or one. Therefore $d_g(w)$ is a.s. a (deterministic) constant, say $d_g(w) = q$ and by property ($e$) of the Leray–Schauder degree, if $q \neq 0$ then $\{v\colon Tv = w\}$ is non-empty.

Assuming now that $(2)'$ is satisfied, set $g^\rho(h) = h + \rho u(w+h)$, $0 \leqq \rho \leqq 1$, then by the invariance of the degree under homotopy, $q = d(g) = d(g^\rho) = d(g^0) = 1$.

Finally, to show the existence of a measurable solution $v(w)$ to the equation $w = v + u(v)$, let $(W, \mathscr{B})$ be a measurable space and $X$ a topological space. Let $F(w)$, $w \in W$, take values in the class of non-empty subsets of $X$. The Kuratowski and Ryll-Nardzewski theorem (cf. e.g. [4]) states that if $X$ is Polish, for all $w \in W$, $F(w)$ is closed and for all open set $U$ in $X$,

$$\{w\colon F(w) \cap U \neq \phi\} \in \mathscr{B}\,, \tag{4.4}$$

then there exists a measurable selection i.e., there exists a measurable function $f(w)$, from $(W, \mathscr{B})$ to $X$ such that $f(w) \in F(w)$ for all $w \in W$. In our case $X = H$, $Tw = w + u(w)$ and $F(w) = \{h\colon h + u(w+h) = 0\}$. Evidently,

$$F(w) \cap U = \{h \in U\colon h + u(w+h) = 0\}\,.$$

Since $H$ is separable and $U$ is open, there exists a sequence $(h_i;\, i \geqq 1)$ which is dense in $U$. Let $U_n = \{h_1, h_2, \ldots, h_n\}$. Set

$$\rho_n(w) = \inf_{\substack{m \leqq n \\ h_m \in U_n}} |h_m + u(w+h_m)|_H\,. \tag{4.5}$$

Then $\rho_n(w)$ is a random variable, $\rho_n(w) \searrow \rho_\infty(w)$ hence $\rho_\infty(w)$ is also a random variable and $\{w\colon \rho_\infty(w) = 0\}$ is a measurable set.

Since $u(w+h)$ is continuous in $h$, it follows that

$$\{w\colon \rho_\infty(w) = 0\} = \{w\colon F(w) \cap U \neq \phi\}\,, \tag{4.6}$$

therefore $F$ is a measurable multivalued map. This proves the existence of a measurable selection and completes the proof of Theorem 4.1. □

The following extends Theorem 8.2 of [12]:

**Theorem 4.2** *Assume that* $u\colon W \to H$ *is* $H-C$*-compact and locally* $H-C^1$ *with* $Q + H \subset Q$ (*cf. Theorem* 3.5). *Suppose also that* $\sup(|h|_H\colon h + u(w+h)$

$=h_0)<\infty$ *almost surely for any given* $h_0 \in H$ *and that the degree of* $T=I_W+u$ *is nonzero. Let* $M=\{w\in W\colon \det_2(I_H+\nabla u(w))\neq 0\}$. *Then*

1. $\mu \approx T^*(\mu|_M)$ *with*

$$\frac{d\mu}{dT^\star \mu|_M}(w)=\left[\sum_{y\in T^{-1}\{w\}\cap Q}\frac{1}{|\Lambda_u(y)|}\right]^{-1} .$$

2. *Let* $S$ *be the right inverse of* $T$ *whose existence is proved in Theorem* 4.1. *Then* $S^*\mu \ll \mu|_M$. *Moreover we have the following Girsanov-type identity*:

$$E[f\circ T\,\mathbf{1}_{S(W)}|\Lambda_u|]=E[f]\,,$$

*for any* $f\in C_b(W)$. *This identity, combined with the absolute continuity of* $S^*\mu$ *implies that*

$$\frac{dS^*\mu}{d\mu|_M}(w)=|\Lambda_u(w)|\mathbf{1}_{S(W)}(w)\,.$$

*Proof.* Define $T'$ as $T$ on $Q$ and as being equal to $I_W$ otherwise. Note that $T=T'$ outside a slim set. We have, from Theorem 3.5, for any $A\in\mathscr{B}(W)$,

$$\mu(T'(A))\leqq\int_A|\Lambda_u|\,d\mu\,.$$

Let $S$ be the right inverse of $T$, since $Q+H\subset Q$, $S$ is also the right inverse of $T'$. We have

$$\begin{aligned}\mu(A)\ &=\ \mu(T'(S(A)))\\ &\leqq \int_{S(A)}|\Lambda_u|\,d\mu\\ &\leqq \int \mathbf{1}_A\circ T|\Lambda_\mu|\,d\mu\,,\end{aligned}$$

hence we have $\mu\ll T^*(\mu|_M)$. We know already that $T^*\mu|_M\ll\mu$ (cf. [12, 18]), hence the first part of (1) follows. The expression for the density is an immediate consequence of the equivalence of these two measures and of the following formula which is proven in [18, Theorem 5.2], combined with Theorem 3.5 (Theorem 3.5 permits us to get rid of the dependence on the set of non-degeneracy of $T$):

$$\frac{dT^*(\mu|_M)}{d\mu}(w)=\sum_{y\in T^{-1}\{w\}\cap Q}\frac{1}{|\Lambda_u(y)|}\ .$$

To prove the second part, since, $S^{-1}(A)\subset T'(A)$, we have

$$\mu(S^{-1}(A))\leqq\int_A|\Lambda_u|\,d\mu\,,$$

hence $S^*\mu\ll\mu|_M$.

From Theorem 3.5, we have, for any $f\in C_b^+(W)$,

$$E[f\circ T\mathbf{1}_{S(W)}|\Lambda_u|]=E\left[f\sum_{y\in T^{-1}\{w\}\cap Q}\mathbf{1}_{S(W)}(y)\right]\,.$$

It is easy to see that

$$\sum_{y\in T^{-1}\{w\}\cap Q} \mathbf{1}_{S(W)}(y)=\mathbf{1}_Q(Sw)=1$$

$\mu$-almost surely by $S^*\mu\ll\mu$. Therefore we obtain

$$E[f\circ T\mathbf{1}_{S(W)}|\Lambda_u|]=E[f]\ .$$

To calculate the Radon–Nikodym density, we have, from Theorem 3.5,

$$\begin{aligned}E[f\mathbf{1}_{S(W)}|\Lambda_u|] &= E\left[\sum_{y\in T^{-1}\{w\}\cap Q} f(y)\mathbf{1}_{S(W)}(y)\right]\\ &= E\left[\sum_{y\in T^{-1}\{w\}\cap Q} f(Sw)\mathbf{1}_{S(W)}(y)\right]\\ &= E\left[f(Sw)\sum_{y\in T^{-1}\{w\}\cap Q} \mathbf{1}_{S(W)}(y)\right]\\ &= E[f(Sw)\mathbf{1}_Q(Sw)]\\ &= E[f(Sw)]\ ,\end{aligned}$$

which completes the proof. □

*Remark.* If we suppose that $u$ is $H-C^1$ then the hypothesis $Q+H\subset Q$ is automatically satisfied. Moreover, if we make the aesthetical convention that $Q=W$, then we can replace everywhere above the set $Q$ with $W$.

In the sequel we shall use the notations $N^+(w), N^-(w), N(w)$ defined as following

$$N^+(w)=\text{The cardinality of } \{v\colon\ Tv=w \text{ and } \det{}_2(I_H+\nabla u(v))>0\}\ ,$$

$$N^-(w)=\text{The cardinality of } \{v\colon\ Tv=w \text{ and } \det{}_2(I_H+\nabla u(v))<0\}\ ,$$

$$N(w)=N^+(w)+N^-(w)\ .$$

In the following theorem the convention $\text{sign}(0)=0$ is used:

**Theorem 4.3** *Assume that conditions* (1), (2) *of Theorem* 4.1 *are satisfied. Further assume that $u$ is locally $H-C^1$. Then for any $f\in C_b^+(W)$*

$$\lim_{k\to\infty} E[(f\mathbf{1}_{N<k})\circ T\Lambda_u]=qE[\mathbf{1}_{N<\infty}f] \tag{4.7}$$

*and if $\mu\{w\colon N(w)<\infty\}\neq 0$ then for almost all $w$ in $\{w\colon N(w)<\infty\}$*

$$q=N^+(w)-N^-(w)\ ,$$

*namely,*

$$q=\sum_{\theta\in T^{-1}\{w\}} \text{sign}\,\Lambda_u(\theta)\ . \tag{4.8}$$

*If moreover, for some positive, bounded, measurable function $f$ on $W$, we have $E[f\circ T|\Lambda_u|]=E[fN]<\infty$, then*

$$E[f\circ T\Lambda_u]=qE[f]\,.$$

*Proof.* From Theorem 3.1, we have

$$E[(f\mathbf{1}_{\{N\leqq k\}})\circ T|\Lambda_u|]=E[f\mathbf{1}_{\{N\leqq k\}}N]<\infty\,,$$

moreover, replacing $g$ by $\operatorname{sign}\Lambda_u$ there, we obtain

$$E[(f\mathbf{1}_{\{N\leqq k\}})\circ T\Lambda_u]=E[f\mathbf{1}_{\{N\leqq k\}}(N^+-N^-)]\,.$$

From Theorem 4.1 and the definition of the Leray–Schauder degree, with the notations of the proof of Theorem 4.1, we have

$$q_k=\sum_{h\in D_k,\,h+u(w+h)=0}\operatorname{sign}\det{}_2(I_H+\nabla u(w+h))\,,$$

and $q_k\to q$ stationarily, hence we should have $q=N^+-N^-$, which is almost surely a constant. Consequently

$$\begin{aligned}E[(f\mathbf{1}_{\{N\leqq k\}})\circ T\Lambda_u]&=E[f\mathbf{1}_{\{N\leqq k\}}(N^+-N^-)]\\&=qE[f\mathbf{1}_{\{N\leqq k\}}]\,.\end{aligned}$$

If $q=0$, there is nothing to prove, if not, dividing both sides of the above equation by $q$ we obtain positive functionals, hence we can pass to monotone limit which proves the first claim. From the argument above, the second claim is obvious. To prove the last claim, remark that, from the hypothesis $E[fN]<\infty$, hence $f$ should be zero almost surely on the set $\{N=\infty\}$. Therefore,

$$\begin{aligned}E[f(N^+-N^-)]&=E[f(N^+-N^-)\mathbf{1}\{N<\infty\}]\\&=qE[f\mathbf{1}_{\{N<\infty\}}]\\&=qE[f]\,.\qquad\square\end{aligned}$$

## 5. An application

We start with the following corollary to Theorem 4.1.

**Corollary 5.1** *Let $K$ be a linear Hilbert–Schmidt operator on $H$ and $G(h)$ a continuous function from $H$ to itself such that $|G(h)|_H=o(|h|_H)$ as $|h|_H\to\infty$. Then the equation for $y$:*

$$y+KG(y)=\tilde{K}w,\tag{5.1}$$

*where $\tilde{K}$ is as defined by Eq. (2.1), possesses a measurable $H$-valued solution. If we suppose further that $G$ is Fréchet differentiable, then the law of $y(w)$ is absolutely continuous with respect to the law of $\tilde{K}(w)$ and we have*

$$E[F(y(w))]=E[F(\tilde{K}w)|\Lambda_K|]\,,$$

*where*

$$\Lambda_K(w) = \det_2(I_H + KDG(\tilde{K}w)) \exp -\delta(G \circ \tilde{K}) - 1/2|G \circ \tilde{K}|_H^2 .$$

*Proof.* Consider the shift:

$$Tw = w + G(\tilde{K}w) . \tag{5.2}$$

Since $K$ is compact, $G(\tilde{K}w)$ satisfies the requirements on $u(w)$ in Theorem 4.1. Therefore the equation $Tv = w$ has a measurable solution, say $v = T^{-1}w$. Replacing $w$ with $T^{-1}w$ in (5.2) and operating with $\tilde{K}$ yields

$$\begin{aligned}\tilde{K}w &= \tilde{K}(T^{-1}w + G(\tilde{K}T^{-1}w)) \\ &= \tilde{K}T^{-1}w + KG(\tilde{K}T^{-1}w)\end{aligned}$$

comparing with (5.1) yields that $y = \tilde{K}T^{-1}w$ solves (5.1). The expression for the law of $y(w)$ follows from Theorem 4.2. □

Let $\mathscr{D}$ be a bounded domain in $\mathbb{R}^d$, $\dot{w}$ a white noise on $\mathbb{R}^d$ and $H_o$ will denote the Hilbert space of real valued functions $L^2(\mathscr{D})$. Let $g(x,r)$, $x \in \mathscr{D}$, be real valued and such that for any $f \in H_o$, $g(x, f(x))$, $x \in \mathbb{R}^d$ is a continuous and bounded transformation from $H_o$ to itself satisfying the assumption on $G$ imposed in Corollary 5.1 with $H$ replaced by $H_o$. We want to consider the stochastic partial differential equation

$$-\Delta\xi(x) + g(x, \xi(x)) = \dot{w}, \quad x \in \mathscr{D} , \tag{5.3}$$

where $\Delta$ is the Laplace operator on $\mathscr{D}$ with the boundary condition

$$\xi|_{\partial\mathscr{D}} = 0 . \tag{5.4}$$

We restrict $d$ to be $1, 2$ or $3$ since, as shown by Buckdahn and Pardoux [3], $K = (-\Delta)^{-1}$ subject to the boundary condition (5.4) is a strictly positive Hilbert–Schmidt kernel and this is needed later. Eqs. (5.3), (5.4) can, therefore, be written as

$$\xi(x) + Kg(x, \xi(x)) = K\dot{w} . \tag{5.5}$$

Let $\mathbb{R}_x^d$ denote $\{y\colon\ y_i \leqq x_i,\ \forall i \leqq d\}$. Define the Cameron–Martin space $H$ induced by $H_o$ as follows:

$$\eta(x) = \int_{\mathbb{R}_x^d \cap \mathscr{D}} \xi(y)\, dy , \qquad (\eta_1, \eta_2)_H = (\xi_1, \xi_2)_{H_o} . \tag{5.6}$$

Set

$$(G_1(\eta))(x) = \int_{\mathbb{R}_x^d \cap \mathscr{D}} g(y, \xi(y))\, dy , \tag{5.7}$$

and

$$(K_1\eta)(x) = \int_{\mathbb{R}_x^d \cap \mathscr{D}} \int_{\mathscr{D}} K(y_1, y_2)\xi(y_2)\, dy_2\, dy_1 . \tag{5.8}$$

Then Eq. (5.5) is equivalent to:

$$\eta + K_1 G_1(\eta) = \tilde{K}_1 w \tag{5.9}$$

and the existence of a measurable solution to (5.9) follows from Corollary 5.1.

In order to extend the application of the degree theorem to more general elliptic stochastic partial differential equations, we first reformulate Corollary 5.1 as follows. Let $\mathscr{D}$ denote a bounded domain in $\mathbb{R}^d$. For a multi-index $\alpha$, $D^\alpha$ denotes the corresponding partial differentiation. For $\xi \in C^m(\mathscr{D}), \|\xi\|_{m,2}$ denotes the norm

$$\|\xi\|_{m,2} = \left( \sum_{0 \leqq \alpha \leqq m} \|D^\alpha \xi\|_2^2 \right)^{1/2} .$$

We define now, $H_o^m$ and $\mathring{H}_o^m$ as follows. Set $H_o^m$ to be the completion of $\{\xi \in C^m(\mathscr{D})\}$ with respect to the $\| \cdot \|_{m,2}$ norm and $\mathring{H}_o^m$ the completion on the elements of $C^m(\mathscr{D})$ with *compact support* in $\mathscr{D}$. Let $H^m$ and $\mathring{H}^m$ denote the Cameron–Martin subspaces associated with $H_o^m$ and $\mathring{H}_0^m$ respectively,

$$H^m = \{\eta(x),\ x \in \mathbb{R}^d \colon \eta(x) = \int_{\mathscr{D} \cap \mathbb{R}_x^d} \xi(y)\,dy,\ \xi \in H_o^m\} \tag{5.10}$$

with $\|\eta\|_{H^m} = \|\xi\|_{m,2}; \mathring{H}^m$ is defined similarly by (5.10) but with $H_o^m$ replaced by $\mathring{H}_o^m$.

The superscript $m$ will be dropped whenever $m = 0$.

**Corollary 5.2** *Let $K_1$ be a bounded linear operator from $H$ to $H^m, m \geqq 1$ and assume that embedding of $H^m$ into $H$ is Hilbert–Schmidt, the operator from $H$ to $H$ induced by $K_1$ will also be denoted $K_1$. Further assume that $G(z)$ is a continuous function from $H^m$ to $H$ and that $|G(z)|_H = o(|z|_{H^m})$ as $|z|_{H^m} \to \infty$. Then the equation*

$$y + K_1 G(y) = \tilde{K}_1 w . \tag{5.11}$$

*possesses an $H^m$-valued solution.*

*Proof.* Note that $\tilde{K}_1 w$ as defined by Eq. (2.1) is an $H^m$-valued random variable. The rest of the proof is the same as that of Corollary 5.1 and therefore omitted. □

Consider, now, the elliptic differential operator on $\mathscr{D}$:

$$P = \sum_{0 \leqq |\alpha| \leqq 2m} \beta_\alpha(x) D^\alpha, \quad x \in \mathscr{D}$$

also consider $g(x, u, \ldots \partial^\beta u \ldots)$ where $g$ is a function of $x \in \mathscr{D}$ and $u$ and its partial derivatives up to the order $\delta$. We want to consider the SPDE

$$Pu + g(\cdot, u, \ldots \partial^\beta u \ldots) = \dot{w} \tag{5.12}$$

subject to zero Dirichlet Boundary Conditions. For this purpose we list the following restrictions. $\mathscr{D}$ will be assumed to be a bounded open set.

a) Assume that the coefficients $\beta_\alpha(x)$ are in $C^{|\alpha| - m}$ for $m < |\alpha| \leqq 2m$. Then (5.12) can be written as

$$Pu = \sum_{0 \leqq |\rho|, |\sigma| \leqq m} (-1)^{|\rho|} D^\rho a^{\rho\sigma} D^\sigma u . \tag{5.13}$$

Further assume that $P$ is uniformly strongly elliptic on $\mathscr{D}$: $\forall x \in \mathscr{D}$

$$\sum_{|\rho|=|\sigma|=m} x^\rho a^{\rho\sigma} x^\sigma \geqq E_0 \cdot |x|^{2m}, \quad E_0 > 0 \tag{5.14}$$

b) $a^{\rho\sigma}(x)$ are uniformly continuous on $\mathscr{D}$ for $|\sigma| = |\rho| = m$.
c) $a^{\rho\sigma}(x)$ is bounded and measurable for $|\alpha| + |\beta| \leqq 2m$.
d) For $u, v \in C_0^\infty(\mathscr{D})$, set

$$B(v,u) = (v, Pu)$$

where $(g,h) = \int_{\mathscr{D}} g(x)h(x)\,dx$, and assume that for all $\varphi \in \mathring{H}_o^m$

$$B(\varphi, \varphi) \geqq C\|\varphi\|_{m,2}^2 .$$

e) The function $g(x, u, \ldots, \partial^\beta u, \ldots), |\beta| \leqq m-1$, $u \in H_o^m$, takes values in $H_o$ and

$$\|g(z)\|_{H_o} = o(\|z\|_{H_o^m}) .$$

**Theorem 5.1** *Let $\mathscr{D}$ be a bounded domain and $P$ and $g$ satisfy assumptions* a)–e). *Assume then $\partial\mathscr{D}$ possesses the finite cone property (there exists a finite cone $C$ such that every $x \in \mathscr{D}$ is the vertex of a finite cone $C_x$ congruent to $C$) and $m > d/2$. Then there exists an $\mathring{H}_o^m$-valued r.v. which solves*

$$(Pu)(x) + g(x, u, \ldots, \partial^\beta u \cdots) = \dot{w}, \quad x \in \mathscr{D} . \tag{5.15}$$

*Proof.* Under the above assumptions $K = P^{-1}$ is a bounded linear transformation from $H_0$ to $\mathring{H}_o^m$ (cf. [2, Theorem 8.2, p. 101]). Moreover, since $\mathscr{D}$ was assumed to possess the finite cone property, it follows from Maurin's theorem ([1, Theorem 6.53]), that the embedding from $H^m$ to $H$ is Hilbert–Schmidt. Consequently $K$ when considered as a transformation from $H$ to $H$ is the product of a bounded operator and an H–S operator. Consequently $K$ is Hilbert–Schmidt. Eq. (5.15) together with the homogeneous Dirichlet boundary condition is equivalent to $u + K \circ g = K\dot{w}$, or by Eq. (5.6)–(5.8), (5.10) it is equivalent to

$$\eta + K_1 G_1(\eta) = \tilde{K}_1 w$$

and the result follows from Corollary 5.2. □

The restriction $m > d/2$ of Theorem 5.1, can be improved if some conditions of smoothness are imposed on $\partial\mathscr{D}$ and on the coefficients $a^{\rho\sigma}$:

**Theorem 5.2** *Assume that conditions* a), c) *and* d) *are satisfied. Assume also that condition* (e) *is satisfied with m and replaced by 2m. Further assume*
f) *$\mathscr{D}$ is of class $C^{2m}$ (in the sense defined in* [2, *p.* 128]);
g) *the coefficients $a^{\rho\sigma}$ are bounded and measurable on $\mathscr{D}$ and $a^{\rho\sigma} \in C^{|\alpha|}$ for all $|\alpha| > 0$.*

*Then, for all $2m > d/2$, there exists* an *$\mathring{H}_o^{2m}$-valued random variable which solves Eq.* (5.9).

*Proof.* Applying Theorems 8.2 and 9.8 of [2], it follows that $P^{-1}$ is a bounded linear transformation from $H_o$ to $\mathring{H}_o^{2m}$. The rest is the same as in the proof of Theorem 5.1. □

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