Probab. Theory Relat. Fields 108, 171–192 (1997)

Probability Theory and Related Fields

Hyperbolic branching Brownian motion

Steven P. Lalley, Tom Sellke

Department of Statistics, Mathematical Sciences Building, Purdue University, West Lafayette, IN 47907, USA
email: lalley@stat.purdue.edu; tsellke@stat.purdue.edu

Received: 2 November 1995 / In revised form: 22 October 1996

Summary. Hyperbolic branching Brownian motion is a branching diffusion process in which individual particles follow independent Brownian paths in the hyperbolic plane $\mathbb{H}^2$, and undergo binary fission(s) at rate $\lambda > 0$. It is shown that there is a phase transition in λ: For $\lambda \leqq 1/8$ the number of particles in any compact region of $\mathbb{H}^2$ is eventually 0, w.p.1, but for $\lambda > 1/8$ the number of particles in any open set grows to ∞ w.p.1. In the subcritical case ($\lambda \leqq 1/8$) the set Λ of all limit points in $\partial\mathbb{H}^2$ (the boundary circle at ∞) of particle trails is a Cantor set, while in the supercritical case ($\lambda > 1/8$) the set Λ has full Lebesgue measure. For $\lambda \leqq 1/8$ it is shown that w.p.1 the Hausdorff dimension of Λ is $\delta = (1 - \sqrt{1 - 8\lambda})/2$.

Mathematics Subject Classification (1991): 60K35 (primary), 60J80 (secondary)

1 Introduction

Hyperbolic branching Brownian motion is a branching diffusion process in which individual particles execute (independent) Brownian motion(s) in the hyperbolic plane $\mathbb{H}^2$, and undergo binary fission(s) at exponentially distributed random times independent of the motions. The rate λ of fission is assumed to be constant. The model is no different from standard branching Brownian motion (see [1, ch. VI]) *except* that the motion takes place in the hyperbolic plane instead of Euclidean space.

Hyperbolic branching Brownian motion, unlike branching Brownian motion in a Euclidean space, exhibits a phase transition in λ. For $\lambda < 1/8$, the process is *subcritical* in the following sense: with probability 1, for any compact subset K of the hyperbolic plane, the number of particles located in K is eventually 0. For $\lambda > 1/8$ the process is *supercritical*: for each nonempty open set U, the number of particles in U is, w.p.1, eventually positive. At the critical value $\lambda = 1/8$ the process dies out in compact sets. These facts follow from the exponential decay of the heat kernel, which may be written explicitly as

$$p_t(z,z') = \exp\left\{-\frac{t}{4}\right\}\left(\frac{\sqrt{2}}{(4\pi t)^{3/2}}\right)\int_{d(z,z')}^{\infty}\frac{r\exp\left\{-r^2/4t\right\}}{\sqrt{\cosh\, r - \cosh\, d(z,z')}}\,dr\ ,$$

where $d(z,z')$ denotes the hyperbolic distance between z and z': see, e.g., [4, ch. 7]. Elementary arguments (not using the heat kernel) will be given below.

Similar phenomena have been observed in a number of related growth models. Pemantle [10] found that for the contact process on a homogeneous tree, it is possible for the population to grow exponentially (in cardinality) but nevertheless to ultimately die out in every compact region, if the infection rate is below a certain critical value. Results of Benjamini and Peres [3] imply that the same is true of branching random walk on a homogeneous tree or a hyperbolic space. The phenomenon is a manifestation of the exponential growth of volume as a function of radius in the phase space, which forces exponential decay of return probabilities for random walks (and Brownian motions) in these spaces.

The main objective of this paper is to show how the phase transition manifests itself in the behavior of the process "at infinity". The (directed) paths in $\mathbb{H}^2$ traced out by the particles of a branching Brownian motion form a random binary tree $\mathscr{T}$. (More precisely, there is a continuous embedding of the full binary tree into $\mathbb{H}^2$ whose nodes are located at the points of fission and whose branches follow the particles' paths.) In the supercritical regime, there are infinite (directed) paths in this tree that remain forever in bounded regions of $\mathbb{H}^2$ (Corollary 3 below); however, in the critical and subcritical regimes there are no such paths – all infinite directed paths in $\mathscr{T}$ diverge to the boundary circle $\partial\mathbb{H}^2$ at ∞. Define Λ to be the set of all accumulation points of $\mathscr{T}$ in $\partial\mathbb{H}^2$. Observe that, with probability 1, Λ is a nonempty, compact subset of $\partial\mathbb{H}^2$. Our main result is the following.

Theorem 1 *For* $0 < \lambda \leqq 1/8$ *the Hausdorff dimension of* Λ *is, with probability 1, equal to*

$$\delta = \delta(\lambda) = \tfrac{1}{2}(1 - \sqrt{1 - 8\lambda})\ . \tag{1}$$

The upper bound $HD(\Lambda) \leqq \delta$ will be proved in Sect. 6, and the lower bound $HD(\Lambda) \geqq \delta$ in Sect. 7.

Note that it is irrelevant whether $\partial\mathbb{H}^2$ is viewed as the real axis (the half-plane model of $\mathbb{H}^2$) or the unit circle (the disk model) because any (hyperbolic) isometry between the Poincaré half-plane and the Poincaré disk is a linear fractional transformation with a smooth extension to the boundary, and diffeomorphisms preserve Hausdorff dimensions. Observe that as $\lambda \uparrow 1/8$ the Hausdorff dimension increases continuously to 1/2, *not* to 1, as one might at first suspect. In the supercritical regime, the complement of Λ has Lebesgue measure 0 (see Proposition 10 below), so the Hausdorff dimension is discontinuous at the critical value $\lambda = 1/8$. In Sect. 8 we shall give a simple heuristic argument (which we call the "backscattering" principle) to explain why the Hausdorff dimension of the limit set cannot be greater than $\frac{1}{2}$ in the subcritical case. This argument suggests that the phenomenon may occur in a large class of similar growth processes, including branching random walks and contact processes. (See, for instance, Sect. 3.2 of [8], in which it is conjectured that $\rho(\lambda) \leqq 1/\sqrt{d}$ for "weakly supercritical" contact processes on a homogeneous tree of degree $d+1$).

2 Hyperbolic Brownian motion and BBM

2.1 The hyperbolic plane

There are several representations of the hyperbolic plane, the most useful for our purposes being the *Poincaré half-plane* and the *Poincaré disk* representations [2]. In the half-plane model, $\mathbb{H}^2$ is the complex manifold $\{z = x + iy: y > 0\}$ with the Poincaré metric $ds = |dz|/y$. In the disk model, $\mathbb{H}^2$ is the complex manifold $\{z = re^{i\theta}: 0 \leqq r < 1\}$ with the Poincaré metric $ds = 2|dz|/(1-r^2)$. The linear fractional transformation

$$\varphi(z) = \frac{z-i}{z+i} \tag{2}$$

maps the upper half-plane onto the disk and takes the Poincaré metric for the half-plane to the Poincaré metric for the disk. For later reference note that φ maps each *horocyle* $\Gamma_t = \{z: \Im(z) = e^{-t}\}$ to a circle $\varphi(\Gamma_t)$ inside the unit disk tangent to the unit circle at $\varphi(\infty) = 1$. The hyperbolic isometries of the Poincaré half-plane are precisely the linear fractional transformations represented by matrices from the group $PSL(2, \mathbb{R})$; these include

(1) translations $z \to z + r$, $r \in \mathbb{R}$,
(2) homotheties $z \to \alpha z$, $\alpha > 0$,
(3) hyperbolic rotations $z \to \varphi^{-1}(e^{i\theta}\varphi(z))$, and
(4) the inversion $z \to -1/z$.

For any two points z_1, z_2 of $\mathbb{H}^2$ there is an isometry of $\mathbb{H}^2$ taking z_1 to z_2.

Hyperbolic circles are Euclidean circles, and vice versa. To see this, observe that it is enough to consider the hyperbolic circles in the disk model centered at $z = 0$, since hyperbolic isometries, being linear fractional transformations, map Euclidean circles to Euclidean circles. That the hyperbolic circles in the disk model centered at $z = 0$ are also the Euclidean circles centered at $z = 0$ follows from the rotational invariance of the Poincaré metric. Finally, note that a hyperbolic circle of radius r has (hyperbolic) area $4\pi \sinh^2(r/2)$ and perimeter $2\pi \sinh r$ (see [2], Theorem 7.2.2).

The *boundary* $\partial\mathbb{H}^2$ of the hyperbolic plane $\mathbb{H}^2$ may be identified with the unit circle $|z| = 1$ (in the disk model) or with $\mathbb{R} \cup \{\infty\}$ (in the half-plane model). The topology of $\partial\mathbb{H}^2$ is that of the unit circle. Convergence of a sequence or a continuous path in $\mathbb{H}^2$ to a point of the boundary $\partial\mathbb{H}^2$

means convergence relative to the usual Euclidean metric on the closed unit disk. (Although it will sometimes be more convenient to use the half-plane representation, it should be understood that in either representation the topology of $\mathbb{H}^2 \cup \partial\mathbb{H}^2$ is that induced by the Euclidean metric on the closed unit disk.)

2.2 Hyperbolic Brownian motion

Brownian motion in the Poincaré half-plane started at $z = i$ may be constructed in two different ways, (1) as a solution to a stochastic differential equation, (2) as a time-change of a two-dimensional (Euclidean) Brownian motion stopped at the x-axis. Brownian motion started at any other point of the Poincaré half-plane may be obtained by isometry: If φ is a hyperbolic isometry and if Z_t is a hyperbolic Brownian motion started at a point z, then $\varphi(Z_t)$ is a hyperbolic Brownian motion started at $\varphi(z)$.

In the first representation, hyperbolic Brownian motion $Z_t = (X_t, Y_t)$ with starting point $z = i$ (written in the usual rectangular coordinates) may be defined as the solution of the stochastic differential equation(s)

$$dY_t = Y_t dY_t^E, \qquad dX_t = Y_t dX_t^E$$

subject to the initial condition $X_0 = 0, Y_0 = 1$, where (X_t^E, Y_t^E) is an ordinary (Euclidean) two-dimensional Brownian motion. This representation, together with Ito's formula, implies

Lemma 1 $\log Y_t$ *is a standard Brownian motion with drift* $-\frac{1}{2}$.

Standard results for one-dimensional Brownian motion with constant drift (see, for instance, [5, ch. 3]) therefore imply

Corollary 1 *Define* $T = T(a) = \inf\{t\colon\ Y_t = e^{-a}\}$ *and* $T = \infty$ *if there is no such* t. *Then for every* $a \neq 0$ *and* $\lambda \leqq 1/8$,

$$Ee^{\lambda T} = \exp\left\{\frac{a}{2}(1 - \sqrt{1 - 8\lambda})\right\} \quad \textit{for } a > 0\ ; \tag{3}$$

$$Ee^{\lambda T}\mathbf{1}\{T < \infty\} = \exp\left\{\frac{a}{2}(1 + \sqrt{1 - 8\lambda})\right\} \quad \textit{for } a < 0\ . \tag{4}$$

For $a \neq 0$ *and* $\lambda > 1/8$ *both expectations are* ∞.

In the half-plane model, hyperbolic Brownian motion $Z_t = (X_t, Y_t)$ started at (0, 1) may also be defined by the requirement that $Z_{\rho(t)}$ be a two-dimensional Euclidean Brownian motion, where

$$t = \int_0^{\rho(t)} Y_{\rho(s)}^{-2}\, ds\ .$$

Note that as $t \to \infty, \rho(t) \to \tau$, where $\tau < \infty$ is the first passage time to the x-axis $y = 0$. Consequently, as $t \to \infty, Z_t$ converges to a unique (random) point Z_∞ of the boundary $\mathbb{R} = \partial\mathbb{H}^2$ of the hyperbolic plane. Similarly, in the disk model hyperbolic Brownian motion $Z_t = (R_t, \Theta_t)$ started at (0, 0) may be defined as a time change of Euclidean Brownian motion, with the time change depending only on the radial process. By the rotational invariance of Euclidean Brownian motion, the limit point Z_∞ is uniformly distributed on the circle $\partial\mathbb{H}^2$.

2.3 Hyperbolic branching Brownian motion

The most natural construction of hyperbolic branching Brownian motion uses a countable collection of independent hyperbolic Brownian motion processes and an independent sequence Υ_t^n of rate λ Poisson processes. The initial particle follows the first Brownian motion, fissioning at the occurrence times of the Poisson process Υ^1. The nth particle, born at a time $t > 0$ and location z, follows the nth Brownian motion, moved (by an isometry) to the starting point z and the starting time t, and fissions at the occurrence times of the Poisson process Υ^n, shifted by t. The number N_t of particles born by time t is a binary fission process. Observe that if an isometry φ of $\mathbb{H}^2$ is applied to the positions of all particles in a branching Brownian motion started at z, the resulting process is a branching Brownian motion started at $\varphi(z)$.

At any time $t \geqq 0$ the state of the branching Brownian motion is determined by the total number $N(t)$ of particles and their current locations $Z_t^1, Z_t^2, \ldots, Z_t^{N(t)}$. For $0 \leqq t \leqq \infty$, define

$$\mathscr{F}_t = \sigma(N(s); Z_s^1, Z_s^2, \ldots, Z_s^{N(s)})_{0 \leqq s \leqq t} \; .$$

Observe that $(\mathscr{F}_t)_{t \geqq 0}$ is a filtration. Consequently, by Lévy's martingale convergence theorem, for every event $A \in \mathscr{F}_\infty$,

$$\mathbf{1}_A = \lim_{t \to \infty} P(A \mid \mathscr{F}_t) \; .$$

3 Phase transition

3.1 The Horocycle GW processes

Fix an integer $n \neq 0$, and consider the following modification of branching Brownian motion started at $z = i$ in the Poincaré half-plane. Let the process evolve in the usual way, but make the horocycle Γ_n an absorbing barrier. Thus, upon reaching Γ_n, a particle will be "frozen" – its motion will cease, and it will undergo no more fission. As $t \to \infty$, more particles will become stuck at Γ_n. Let $\mathscr{P}_n$ denote the point process consisting of the locations (x-coordinates) of *all* stuck particles, and let M_n be the cardinality of $\mathscr{P}_n$ (note that M_n might be ∞).

Observe that a version of hyperbolic branching Brownian motion may be constructed by attaching to each point (x, e^{-n}), where $x \in \mathscr{P}_n$, its own branching Brownian motion (with time adjusted to account for the time the stuck particle at (x, e^{-n}) took to reach its position on Γ_n). These attached branching Brownian motions should be independent of each other and of the pre-Γ_n branching Brownian motion process. That this construction does in fact yield a version of hyperbolic branching Brownian motion is a routine consequence of the strong Markov property.

Proposition 1 *The sequence M_n, $n \geqq 0$, is a Galton–Watson process whose offspring distribution is nondegenerate and has mean*

$$\mu = \exp\left\{\tfrac{1}{2}(1 - \sqrt{1 - 8\lambda})\right\} = e^\delta \tag{5}$$

for $\lambda \leqq 1/8$ and $\mu = \infty$ for $\lambda > 1/8$.

Proof. Consider the particles counted in M_{n+1}. Each particle ξ among these is a descendant of a particle $\zeta \in \mathscr{P}_n$. Now the process consisting of the descendants of ζ born *after* the time of ζ's first visit to Γ_n is itself a branching Brownian motion, as noted above, with initial point located on Γ_n. This process is (in distribution) an isometric replica of the original branching Brownian motion; the isometry (multiplication by e^n followed by a real translation) takes each horocycle Γ_k to the horocycle Γ_{k-n}. Thus, the number M^{ζ} of descendants of ζ counted in M_{n+1} has the same distribution as M_1. Moreover, the post-Γ_n processes engendered by different particles ζ counted in M_n are *independent* branching Brownian motions, so the random variables M^{ζ}, where ζ ranges over the particles counted in M_n, are conditionally independent. It follows that M_n is a GW process. The offspring distribution is clearly nondegenerate since with positive probability the initial particle may reach Γ_1 before fissioning.

That M_1 has the advertised expectation is a routine consequence of Corollary 1 above. □

The same argument proves

Proposition 2 *The sequence* M_n, $n \leqq 0$, *is a Galton–Watson process whose offspring distribution is nondegenerate and has mean*

$$\mu = \exp\left\{-\tfrac{1}{2}(1+\sqrt{1-8\lambda})\right\} = e^{\delta-1} \tag{6}$$

for $\lambda \leqq 1/8$ *and* $\mu = \infty$ *for* $\lambda > 1/8$.

3.2 Critical and subcritical cases

If $\lambda \leqq 1/8$ the Galton–Watson process $\{M_n\}_{n \leqq 0}$ is subcritical. Thus, extinction is certain: with probability 1, for all sufficiently large n, $M_{-n} = 0$. In particular, there is a (random) n_* such that *no* particle of the branching Brownian motion ever reaches the region above the horocycle Γ_{-n_*}.

Corollary 2 *Assume that* $\lambda \leqq 1/8$. *Then for every horocycle* Γ_n *the number of particles of the branching Brownian motion* above Γ_n *is eventually* 0, *with probability* 1. *Hence*, *for every compact subset* K *of* $\mathbb{H}^2$, *the number of particles located in* K *is eventually* 0, *with probability* 1.

Proof. For any pair of integers n_1, n_2 there exists $p = p(n_2 - n_1) > 0$ with the following property. For any time $t \geqq 0$, on the event that a particle ζ of the branching Brownian motion is located on or above Γ_{n_1} at time t, the conditional probability that a post-t descendant of ζ will visit Γ_{n_2}, given the history $\mathscr{F}_t$ of the BBM up to time t, is at least p. Consequently, on the event $V(n_1)$ that there are particles on or above the horocycle Γ_{n_1} at arbitrarily large times,

$$\lim_{t\to\infty} P(\Gamma_{n_2} \text{ is never hit} \mid \mathscr{F}_t) \leqq 1 - p < 1 .$$

Since

$$\mathbf{1}\{\Gamma_{n_2} \text{never hit}\} = \lim_{t\to\infty} P(\Gamma_{n_2} \text{ is never hit} \mid \mathscr{F}_t) ,$$

by the martingale convergence theorem, it follows that on the event $V(n_1)$ it is certain that *every* Γ_{n_2} will be visited. But the event that every Γ_{n_2} is visited is the same as the event that $M_n > 0$ for all $n < 0$, and this happens with probability 0, since the GW process $\{M_n\}_{n \leqq 0}$ is subcritical when $\lambda \leqq 1/8$. Thus, for every n,

$$P(V(n)) = 0 .$$

If K is a compact subset of $\mathbb{H}^2$, then for some integer n, K lies entirely above Γ_n. The event that there are particles in K at arbitrarily large times is contained in the event $V(n)$ that there are particles on or above Γ_n at arbitrarily large times; consequently, it has probability 0. □

3.3 Supercritical case

When $\lambda > 1/8$ both of the horocycle GW processes $\{M_n\}_{n \leqq 0}$ and $\{M_n\}_{n \geqq 0}$ are supercritical. Thus, particles reach every horocycle Γ_n, $n < 0$, above i, and those particles all have descendants that eventually return to Γ_0.

Corollary 3 *Assume that* $\lambda > 1/8$. *With probability* 1, *there exist infinite paths in the tree* $\mathscr{T}$ *that remain forever in compact regions of* $\mathbb{H}^2$.

Proof. Consider the modification of the branching Brownian motion in which particles are frozen upon reaching the horocycle Γ_{-1}. By Proposition 2, $EM_{-1} = \infty$. Consequently, for $k \geqq 1$ sufficiently large, the expected number of particles in $\mathscr{P}_{-1}$ whose trajectories did not reach Γ_k before freezing at Γ_{-1} is (strictly) greater than 1.

Let C_1 and C_2 be hyperbolic circles with the same center iy (in the half-plane representation) and with hyperbolic radii $\log y - 1$ and $\log y + k$, repectively. These circles are situated so that C_1 lies above and is tangent to Γ_{-1} and C_2 lies above and is tangent to Γ_k. For large y, these circles closely approximate the horocycles Γ_{-1} and Γ_k to which they are tangent. Consequently, by the result of the preceding paragraph and the Monotone Convergence Theorem, if y is sufficiently large, the expected number of particles in the branching Brownian motion that reach C_1 before C_2 is greater than 1. But branching Brownian motion is invariant (in law) under hyperbolic isometries, in particular hyperbolic rotations centered at iy, so, for branching Brownian motion started at *any* point on the circle C_3 centered at iy with radius $\log y$, the number of particles that reach C_1 before C_2 has the same distribution F as when the branching Brownian motion is started at i. Since the mean of F is greater than 1, any Galton–Watson process with offspring distribution F is supercritical, and so may explode with positive probability.

Such a GW process is embedded in the branching Brownian motion. Let C_a, C_b, and C_c be the circles centered at i with hyperbolic radii $\log y - 1, \log y$, and $\log y + k$, respectively. Follow the initial particle from its initial position i until its first hit of the circle C_b; then track it and all of its subsequent offspring that return to C_a before hitting C_c; after their return(s) to C_a follow these particles (ignoring their offspring) back to C_b; etc. The number K_n of particles in the nth cycle of this process is the nth term of a GW process with offspring distribution F. On the event $K_n \to \infty$ (which has positive probability) there is an infinite path in $\mathscr{T}$ that remains forever inside the circle C_c.

It now follows that with probability 1 there is an infinite path in $\mathscr{T}$ that remains forever in a compact region of $\mathbb{H}^2$. If $K_n \to \infty$ there is certainly such a path. If $K_n \not\to \infty$, start from scratch with the post-C_c branching Brownian motion initiated by the first particle to reach C_c. If *this* process again fails to produce an infinite path that remains bounded, proceed to the next available particle, etc. By the SLLN, one of these processes will produce an infinite bounded path. □

Corollary 4 *Assume that* $\lambda > 1/8$. *Then with probability* 1, *for every nonempty open subset* U *of* $\mathbb{H}^2$, *there are particles in* U *at indefinitely large times.*

Proof. Let B_r be the ball of radius r centered at the initial point i of the B. B. M., and let A_r be the event that there are particles in B_r at *all* times $t > 0$. By the preceding corollary, $P(A_r)$ can be made as close to 1 as desired by making r large.

For any fixed open set U and any fixed $r > 0$, there exists $p > 0$ so that for *any* $z \in B_r$, the probability that a hyperbolic Brownian motion started at z will visit U at time 1 is *at least* p. On the event A_r, there is a particle of the branching Brownian motion inside B_r at every time $n = 0, 1, \ldots$, and conditional on its location z its subsequent movement is a hyperbolic Brownian motion started at z. Consequently, for each $m = 1, 2, \ldots$ and each $n \geqq m$,

$$P(U \text{ visited after time } m \,|\, \mathscr{F}_n) \geqq p\mathbf{1}_{A_r} .$$

(Recall that $\mathscr{F}_t$ is the σ-algebra generated by the history of the branching Brownian motion up to time t.) But by Lévy's martingale convergence theorem,

$$\lim_{n\to\infty} P(U \text{ visited after time } m \,|\, \mathscr{F}_n) = \mathbf{1}\{U \text{ visited after time } m\} ,$$

so $A_r \subset \{U \text{ visited after time } m\}$. Since $P(A_r) \to 1$ as $r \to \infty$, this proves that the event $\{U \text{ visited after time } m\}$ has probability 1 for every m. □

3.4 The Horocycle GW process in the critical case

Proposition 3 *Assume that* $\lambda = 1/8$. *Then* $\lim_{n\to\infty} M_n/e^{n/2} = 0$.

Remarks. (1) This will be of central importance in determining the Hausdorff dimension of Λ in the critical case $\lambda = 1/8$. (2) The critical case differs from the subcritical case in this regard: it can be shown that when $\lambda < 1/8$, $\lim_{n\to\infty} M_n/e^{n\delta} > 0$ with probability 1.

Proof of Proposition 3. Consider the modification of the hyperbolic branching Brownian motion in which the horocycles $\Gamma_{-1}$and Γ_n (for some integer $n \geqq 1$) are absorbing barriers. Thus, the process evolves in the usual way, starting from a single particle located at $z = i$, but particles are instantaneously "frozen" upon reaching either Γ_{-1} or Γ_n. Define $M_{-1,n}$ to be the number of particles in this process that are ultimately frozen on Γ_{-1}. Then as $n \to \infty$, $M_{-1,n} \uparrow M_{-1}$, so by the monotone convergence theorem,

$$EM_{-1,n} \uparrow EM_{-1} = e^{-1/2} .$$

The event $\{\lim_{n\to\infty} M_n/e^{n/2} > 0\}$ has probability 0 or 1. Suppose that it has probability 1. Since $EM_{-1} < 1$, there is positive probability that $M_{-1} = 0$, and hence that $M_{-1,n} = 0$ for all $n \geqq 1$. Consequently, for some $\varepsilon > 0$,

$$P(F_n) > \varepsilon \quad \forall n \geqq 1 ,$$

where

$$F_n = F_n^{\varepsilon} = \{M_{-1,n} = 0 \text{ and } M_n > \varepsilon e^{n/2}\} .$$

Consider the post-$\Gamma_n \cup \Gamma_{-1}$ evolution of the branching Brownian motion. Each particle counted in M_n begets its own branching Brownian motion, started at a point of Γ_n, and these processes are independent (conditional on the locations of the initiating particles on Γ_n). Moreover, each process is an isometric replica (in law) of the original branching Brownian motion started at i. Thus, if the particles of each of these offspring processes are frozen upon reaching Γ_{-1} then for each offspring process the expected number of descendants that are ultimately frozen on Γ_{-1} is $\exp\{-(n+1)/2\}$. Consequently, conditional on the event F_n, the expected total number of particles descendant from particles counted in M_n that are ultimately frozen on Γ_{-1} is greater than $\varepsilon e^{-1/2}$. This implies that

$$E(M_{-1} - M_{-1,n}) > \varepsilon^2 e^{-1/2}$$

for all $n \geqq 1$, which is a contradiction, since $EM_{-1,n} \uparrow EM_{-1}$. $\square$

Remark. A more arduous argument shows that in fact

$$\lim_{n\to\infty} nM_n/e^{n/2}$$

exists and is positive, with probability 1. The proof uses the fact that the martingale

$$Z(t) = -\sum_{i=1}^{N(t)} Y_i(t)^{\frac{1}{2}} \log Y_i(t)$$

has a finite, strictly positive limit (by Theorem 1 of [7]). Here $Y_i(t)$ denotes the y-coordinate of the ith particle in existence at time t.

4 Diameter of the limit set

Recall that $\mathscr{T}$ is the random binary tree embedded in $\mathbb{H}^2$ whose nodes are located at the points of fission of the branching Brownian motion and whose (directed) edges follow the paths of particles between successive fissions. Define Λ to be the set of all points in $\partial\mathbb{H}^2 = \mathbb{R} \cup \{\infty\}$ that are accumulation points of $\mathscr{T}$, and define

$$D = \text{diameter}(\Lambda) .$$

(The diameter refers to the usual metric on $\mathbb{R}$.) The following estimate on the tail of D will be our primary tool for establishing the topological and dimensional properties of Λ in the subcritical and critical regimes.

Proposition 4 *For each* $\lambda \leqq 1/8$ *there exists a constant* $0 < C < \infty$ *such that for all sufficiently large* t,

$$P\{D \geqq t\} \leqq C t^{\delta - 1} .$$

Proof. Recall that D is the diameter of Λ as measured using the usual Euclidean distance on $\partial \mathbb{H}^2 = \mathbb{R}$. On the event $D \geqq t$ it must be the case that some point of Λ has absolute value at least $t/2$; consequently, it is enough to prove that for a suitable constant $C < \infty$,

$$P(\Lambda \nsubseteq [-t, t]) \leqq C t^{\delta - 1} \tag{7}$$

for all sufficiently large t.

Recall that branching Brownian motion in the Poincaré disk may be obtained from branching Brownian motion in the half-plane by applying the isometry φ defined by (2). The mapping φ takes $\mathbb{R}$ onto the circle $|z| = 1$, with $\varphi(\infty) = 1, \varphi(i) = 0, \varphi(0) = -1$; the complement $[-t, t]^c$ of $[-t, t]$ is mapped onto an open arc of the unit circle centered at 1 with arc-length $\sim 4/t$ (as $t \to \infty$). Consequently, the probability in (7) is the same as the probability that the limit set Λ_D of branching Brownian motion in the Poincaré disk intersects an open arc A_t of arclength $\sim 4/t$ centered at $z = 1$.

Let C_1, C_2, C_3 be Euclidean circles of radius $2/t$ interior and tangent to the unit circle at $1, z_1, z_2$, respectively, where z_1, z_2 are the endpoints of the arc A_t. Any continuous path from $z = 0$ that enters a neighborhood of the arc A_t must first intersect at least one of the circles C_i. Thus, $\Lambda_D \cap A_t \neq \emptyset$ only if a particle of the branching Brownian motion hits one of the circles C_i. Because branching Brownian motion in the disk is rotationally invariant (in law), the probability that a particle hits C_i is the same as the probability that a particle hits C_1; hence, the probability that a particle hits one of the circles C_i is no larger than 3 times the probability that a particle hits C_1.

Now return to the Poincaré half-plane via the mapping φ^{-1}: this maps the circle C_1 onto the horocycle

$$\Gamma = \{z = x + iy \mid y = t/2 - 1\} .$$

Consequently, the preceding paragraph implies that

$$P(\Lambda \nsubseteq [-t, t]) \leqq 3P\{\text{a particle hits } \Gamma\} .$$

The latter probability may be bounded with the aid of the upper horocycle GW process $\{M_n\}_{n \leqq 0}$ introduced in Sect. 3. A particle of the branching Brownian motion reaches the horocycle $\Gamma_n = \{y = e^{-n}\}$ iff $M_n > 0$. By Proposition 2, $\{M_n\}_{n \leqq 0}$ is a subcritical Galton–Watson process with mean offspring number $\mu = \exp\{\delta - 1\}$. Thus, by the Markov inequality, $P\{M_n > 0\} \leqq EM_n = \mu^{|n|}$.

The estimate (7) follows easily from this: for $n = -[\log(t/2-1)]$,

$$P\{\Lambda \nsubseteq [-t,t]\} \leqq 3P\{M_n > 0\} \leqq 3e^{1-\delta}\left(\frac{t}{2}-1\right)^{\delta-1}. \qquad \square$$

Remark. Similar arguments give an inequality in the reverse direction: there exists a constant $C' < \infty$ such that $P\{D > t\} \geqq C' t^{\delta-1}$.

Corollary 5 *If* $\lambda < 1/8$ *then* $ED^{\delta+\varepsilon} < \infty$ *for sufficiently small* $\varepsilon > 0$.

Proof. Since $\delta < 1/2$ whenever $\lambda < 1/8$, this follows immediately from the tail probability estimate of Proposition 4. $\square$

5 Topological consequences

5.1 Critical and subcritical cases ($\lambda \leqq 1/8$)

Assume that $\lambda \leqq 1/8$.

Proposition 5 *With probability* 1, Λ *is a totally disconnected closed subset of* $\partial\mathbb{H}^2$ *with Lebesgue measure* 0.

Proof. That Λ is closed follows from its definition as the set of accumulation points of $\mathscr{T}$ in $\partial\mathbb{H}^2$. Proposition 4 implies that w.p.1, the point $\infty \in \partial\mathbb{H}^2$ is not an element of Λ. Now recall that branching Brownian motion is invariant (in law) under hyperbolic rotations about the initial point. Since the point at ∞ may be mapped to any other point of $\partial\mathbb{H}^2$ by such a rotation, it follows that for every fixed $\xi \in \partial\mathbb{H}^2$,

$$\xi \notin \Lambda \text{ a.s.}$$

Fubini's theorem therefore implies that, w.p.1, Λ has Lebesgue measure 0. Moreover, for any countable dense subset D of $\partial\mathbb{H}^2$, $D \cap \Lambda = \emptyset$ almost surely, so Λ is totally disconnected. $\square$

Define a path in $\mathscr{T}$ to be a continuous function $\gamma : \mathbb{R}_+ \to \mathscr{T}$ such that the arcs of γ between successive nodes follow the Brownian trajectories *at their natural speed*. It follows from Corollary 2 that, under the assumption $\lambda \leqq 1/8$, infinite paths γ in $\mathscr{T}$ must eventually exit any compact subset of $\mathbb{H}^2$ and therefore converge (as $t \to \infty$) to $\partial\mathbb{H}^2$.

Proposition 6 *With probability* 1, *every infinite continuous path* $\gamma(t)$ *in the tree* $\mathscr{T}$ *converges to a* unique *point of* $\partial\mathbb{H}^2$.

Proof. Since w.p.1 every continuous path in $\mathscr{T}$ eventually exits every compact subset of $\mathbb{H}^2$, by Corollary 2, it suffices to show that with probability 1 no continuous path in $\mathscr{T}$ has more than one accumulation point in $\partial\mathbb{H}^2$. Suppose that γ is a continuous path in $\mathbb{H}^2$ that eventually exits every compact subset of $\mathbb{H}^2$ and has two distinct accumulation points ξ_1, ξ_2 in $\partial\mathbb{H}^2$. Then γ accumulates at every point on one of the two arcs of $\partial\mathbb{H}^2$ connecting ξ_1 and ξ_2. But this contradicts Proposition 5, which asserts that w.p.1 Λ is totally disconnected. $\square$

Proposition 7 *With probability* 1, *for every* $\xi \in \Lambda$ *there is an infinite path* γ *in the tree* $\mathscr{T}$ *that converges to* ξ.

Proof. If $\xi \in \Lambda$ then there is a sequence z_n of points in (the image of) $\mathscr{T}$ that converge to ξ. For each n there is a *finite* path γ_n in $\mathscr{T}$ that terminates at z_n. Let D_m be the hyperbolic disk of radius m centered at the initial point of the branching Brownian motion. Since $z_n \to \partial \mathbb{H}^2$, for every m all but finitely many of the paths γ_n must exit D_m a first time. But there are only finitely many finite paths in $\mathscr{T}$ that stay in D_m and terminate on the boundary of D_m, in view of Corollary 2; consequently, for each m there is a finite path β_m in $\mathscr{T}$ that stays in D_m and terminates on the boundary of D_m such that infinitely many of the paths γ_n begin with the segment β_m. Moreover, these may be chosen so that for each m, β_{m+1} is an extension of β_m. Thus, there is an *infinite* path in $\mathscr{T}$ that extends all of the finite paths β_m. By Proposition 6, β converges to a unique point $\zeta \in \Lambda$.

That $\zeta = \xi$ follows from Proposition 5. If ζ were distinct from ξ, then for every neighborhood U of ζ and every neighborhood V of ξ there would be a path in $\mathscr{T}$ beginning in U and ending in V (just take the appropriate terminal segments of the paths γ_n). These paths would have to accumulate on one of the boundary arcs connecting ζ and ξ. But this is impossible, because Λ is totally disconnected. □

Proposition 8 *With probability* 1, Λ *is a* perfect *set, i.e., every point* $\xi \in \Lambda$ *is an accumulation point of* $\Lambda - \{\xi\}$.

Proof. Suppose that $\xi \in \Lambda$ is an isolated point of Λ. Then there is an infinite path γ in $\mathscr{T}$ that converges to ξ. There are infinitely many points of fission along this path, because w.p.1 there are no particles that fail to fission in an infinite time interval.

From every fission emerge two particles, each of which initiates its own branching Brownian motion from the point of fission, independent of the other. The particles themselves follow (conditionally) independent Brownian paths to (random) points of $\partial \mathbb{H}^2$, and the distribution(s) of these exit points are absolutely continuous. Let $\zeta_1, \zeta_2, \ldots$ be the termination points of the trajectories of particles $1, 2, \ldots$. Then w.p.1, $\zeta_n \neq \zeta_m$ for all $n \neq m$, $\zeta, \xi \in \partial \mathbb{H}^2$.

Now take a sequence of fissions along γ leading to ξ, and let ξ_n be the termination points of the trajectories of the particles born at these fission points. By the previous paragraph, the points ξ_n are distinct, and they are certainly elements of Λ. The points ξ_n must converge to ξ, because otherwise there would be a sequence of trajectories in $\mathscr{T}$ accumulating along a nonempty open arc of $\partial \mathbb{H}^2$, contradicting Proposition 5. □

5.2 Supercritical case ($\lambda > 1/8$)

Assume now that $\lambda > 1/8$. Then the GW process $\{M_n\}_{n \leqq 0}$ is supercritical, with probability of extinction 0. Consequently, every horocycle is visited by particles of the branching Brownian motion. It follows that ∞ is a cluster point of the tree $\mathscr{T}$. The following is a stronger assertion.

Proposition 9 *With probability* 1 *there is a path in* $\mathscr{T}$ *that converges to* ∞.

Proof. Recall that M_{-1} is the number of particles "frozen" upon reaching Γ_{-1}. Since $EM_{-1}=\infty$, the monotone convergence theorem implies that there exists an integer $k \geqq 1$ such that $EM_{-1}^{(k)}>1$, where $M_{-1}^{(k)}$ is the number of particles counted in M_{-1} that reach Γ_{-1} without first having visited Γ_k. Consequently, a GW process with offspring distribution $\mathscr{L}(M_{-1}^{(k)})$ is supercritical. Such a process is contained in $\{M_n\}_{n \leqq 0}$: beginning with M_{-1}, throw away all particles *and their descendants* that visit Γ_k before being frozen at Γ_{-1}; then at each subsequent Γ_{-n}, throw away all particles (not already thrown away) and their descendants that visit Γ_{-n+k} before being frozen at Γ_{-n} (and after being "unfrozen" from Γ_{-n+1}).

The GW process so constructed is supercritical, but may reach extinction. However, each of the particles in the branching Brownian motion begets its own branching Brownian motion, and embedded in each of these is a copy of the GW process built above. By the ergodic theorem, at least one of these will explode. Now any path γ that follows a sequence of nodes corresponding to the particles in such an exploding GW process must converge to ∞, because after reaching each horocycle Γ_n, $n<0$, it never again drops below Γ_{n+k}. □

Proposition 10 *Define* Λ_0 *to be the set of points in* $\partial\mathbb{H}^2$ *to which paths in* $\mathscr{T}$ *converge. With probability* 1, *the complement of* Λ_0 *in* $\partial\mathbb{H}^2$ *has Lebesgue measure* 0.

Proof. Here it is convenient to work in the disk model. Since hyperbolic branching Brownian motion is rotationally invariant (in law), Proposition 9 implies that for every $\xi \in \partial\mathbb{H}^2$, with probability one, there is a path in $\mathscr{T}$ that converges to ξ. Fubini's theorem therefore implies that with probability 1 the set Λ_0 has full Lebesgue measure. □

Corollary 6 *With probability* 1, $\Lambda = \partial\mathbb{H}^2$.

Proof. Clearly, $\Lambda_0 \subset \Lambda$. By the preceding proposition, with probability 1, the complement of Λ_0 is a set of Lebesgue measure 0; hence, with probability 1, the complement of Λ has Lebesgue measure 0. Since Λ is necessarily closed, it follows that with probability 1 its complement is empty. □

6 Hausdorff dimension: the upper bound

Assume that $\lambda \leqq 1/8$. To show that the Hausdorff dimension of Λ is no larger than the constant δ defined by (1) it suffices to exhibit, for each small $\varepsilon>0$ and each $n \geqq 1$, a covering $\mathscr{C}_n$ of Λ by arcs J_{nk} such that w.p.1

$$\sum_{k=1}^{M_n} |J_{nk}|^{\delta+\varepsilon} \longrightarrow 0 \tag{8}$$

as $n \to \infty$. (For any arc J, $|J|$ is its length.) It is sufficient to show convergence in probability, because this implies the existence of an almost surely convergent subsequence.

6.1 Coverings of Λ

We will work in the Poincaré half-plane model. Fix an integer $n \geqq 1$ and consider again the modification of the branching Brownian motion process in which particles are "frozen" upon reaching Γ_n (Sect. 3). Recall that to construct a version of the entire branching Brownian motion process (with no freezing), one attaches to each particle in $\mathscr{P}_n$ its own branching Brownian motion, each independent of the rest and of the pre-Γ_n process. For each of these attached branching Brownian motions there will be a limit set Λ_{nk}, and clearly

$$\Lambda = \bigcup_{k=1}^{M_n} \Lambda_{nk} .$$

Conditional on M_n, the random sets $\Lambda_{nk},\ 1 \leqq k \leqq M_n$, are independent and identically distributed. Moreover, conditional on M_n, the random sets Λ_{nk} are independent scaled replicas (in distribution) of Λ, with scale factor e^{-n}. (This is because the branching Brownian motion engendered by a particle in $\mathscr{P}_n$ is an isometric replica of the whole branching Brownian motion, via an isometry gotten by composing translations with the homothety $z \to e^n z$.)

Define J_{nk} to be the smallest closed interval in $\partial \mathbb{H}^2 = \mathbb{R}$ that contains Λ_{nk}. Then for each integer $n \geqq 1$, the collection

$$\mathscr{C}_n = \{J_{nk}\}_{1 \leqq k \leqq M_n}$$

is a covering of Λ by intervals. We will prove that for every sufficiently small $\varepsilon > 0$, (8) holds for this sequence of coverings.

6.2 Subcritical case

Assume now that $\lambda < 1/8$. In this case, by Corollary 5,

$$ED^{\delta+\varepsilon} < \infty$$

for some $\varepsilon > 0$. Define $D_{nk} = e^n |J_{nk}|$ to be the scaled diameter of the interval J_{nk}; then, conditional on M_n, the random variables D_{nk} are independent and identically distributed, with the same distribution as D. Proposition 1 implies that $EM_n = e^{n\delta}$. Consequently, for each $n \geqq 1$,

$$E\left(\sum_{k=1}^{M_n} |J_{nk}|^{\delta+\varepsilon}\right) = e^{-n\varepsilon} ED^{\delta+\varepsilon} . \tag{9}$$

As $n \to \infty$, this converges to zero. It follows that (8) holds in probability.

This argument may be modified to yield a sharper result:

Proposition 11 *If* $\lambda < 1/8$ *then with probability* 1,

$$\mathscr{H}^\delta(\Lambda) < \infty .$$

Proof. First, observe that (9) implies that

$$\max_{1 \leqq k \leqq M_n} |J_{nk}| \longrightarrow 0$$

in probability. Since this maximum is nonincreasing in n, it must in fact converge to 0 with probability 1. Now by Corollary 5, $ED^\delta < \infty$, and by the same argument as used in proving (9),

$$E\left(\sum_{k=1}^{M_n} |J_{nk}|^\delta\right) = ED^\delta .$$

Thus, by Fatou's Lemma,

$$E\left(\liminf_{n\to\infty} \sum_{k=1}^{M_n} |J_{nk}|^\delta\right) \leqq ED^\delta ;$$

in particular, the lim inf is almost surely finite. Since $\max_k |J_{nk}| \to 0$ as $n \to \infty$, this proves, by definition of the outer δ-dimensional Hausdorff measure, that $\mathscr{H}^\delta(\Lambda) < \infty$ almost surely. □

6.3 Critical Case

Assume that $\lambda = 1/8$. Then $\delta = 1 - \delta = \frac{1}{2}$, so the argument used in the subcritical case breaks down. The tail probability estimate given in Proposition 4 must now be used in a more delicate manner.

Lemma 2 *There is a constant $C < \infty$ such that for every $\varepsilon > 0$ and $n \geqq 1$,*

$$E\{e^{-n}D\mathbf{1}\{e^{-n}D \leqq 1\}\}^{\frac{1}{2}+\varepsilon} \leqq C\left(\frac{1+2\varepsilon}{2\varepsilon}\right)\exp\left\{-\frac{n}{2}\right\} + \exp\left\{-\frac{n}{2} - n\varepsilon\right\} .$$

Proof. Let $C < \infty$ be the constant in Proposition 4. Then

$$E\{e^{-n}D\mathbf{1}\{e^{-n}D \leqq 1\}\}^{\frac{1}{2}+\varepsilon} \leqq \exp\left\{-\frac{n}{2} - n\varepsilon\right\} \int_0^{e^{n/2+n\varepsilon}} P\{D^{\frac{1}{2}+\varepsilon} > t\}\, dt$$

$$\leqq \exp\left\{-\frac{n}{2} - n\varepsilon\right\} \int_1^{e^{n/2+n\varepsilon}} Ct^{-\frac{1}{1+2\varepsilon}}\, dt + \exp\left\{-\frac{n}{2} - n\varepsilon\right\}$$

$$\leqq C\left(\frac{1+2\varepsilon}{2\varepsilon}\right)\exp\left\{-\frac{n}{2}\right\} + \exp\left\{-\frac{n}{2} - n\varepsilon\right\} . \quad \square$$

To prove that (8) holds in probability it suffices to prove

Lemma 3 *For every $\varepsilon > 0$ and $\rho > 0$,*

$$\lim_{n\to\infty} P\left\{\sum_{k=1}^{M_n} |J_{nk}|^{\frac{1}{2}+\varepsilon} > \rho\right\} = 0 . \tag{10}$$

Proof. For any $\alpha > 0$,

$$P\left\{\sum_{k=1}^{M_n} |J_{nk}|^{\frac{1}{2}+\varepsilon} > \rho\right\} \leqq P\{M_n > \alpha e^{n/2}\}$$

$$+ P\left(\max_k |J_{nk}| \geqq 1 \mid M_n \leqq \alpha e^{n/2}\right)$$

$$+ P\left(\sum_k |J_{nk}|^{\frac{1}{2}+\varepsilon} \mathbf{1}\{|J_{nk}| < 1\} > \rho \mid M_n \leqq \alpha e^{n/2}\right) .$$

Proposition 3 implies that for every $\alpha > 0$,

$$\lim_{n\to\infty} P\{M_n > \alpha e^{n/2}\} = 0 .$$

Conditional on M_n, the random variables $D_{nk} = e^n|J_{nk}|$ are independent and identically distributed, with the same distribution as D, so Proposition 4 implies that

$$P\left(\max_k |J_{nk}| \geqq 1 \mid M_n \leqq \alpha e^{n/2}\right) \leqq \alpha e^{n/2} P\{D \geqq e^n\} \leqq C\alpha ,$$

which is small when $\alpha > 0$ is small. Finally, by the Markov inequality and Lemma 2,

$$P\left(\sum_{k=1}^{M_n} |J_{nk}|^{\frac{1}{2}+\varepsilon}\mathbf{1}\{|J_{nk}| < 1\} > \rho \mid M_n \leqq \alpha e^{n/2}\right)$$

$$\leqq \rho^{-1} E\left(\sum_k |J_{nk}|^{\frac{1}{2}+\varepsilon}\mathbf{1}\{|J_{nk}| < 1\} \mid M_n \leqq \alpha e^{n/2}\right)$$

$$\leqq \rho^{-1}\alpha e^{n/2} E(e^{-n}D\mathbf{1}\{e^{-n}D \leqq 1\})^{\frac{1}{2}+\varepsilon}$$

$$\leqq \rho^{-1}\alpha\left(C\left(\frac{1+2\varepsilon}{2\varepsilon}\right) + e^{-n\varepsilon}\right) ,$$

which is also small when $\alpha > 0$ is small. □

7 Hausdorff dimension: the lower bound

7.1 Frostman's Lemma

Assume that $\lambda \leqq 1/8$. To show that the Hausdorff dimension of Λ is no smaller than the constant δ defined by (1), we will use the following variation of a well known criterion due to Frostman.

Lemma 4 (Frostman) *Let A be a compact subset of a Euclidean space. If there exists a probability measure ν with support contained in A such that for ν-a.e. x,*

$$I_t(\nu, x) = \int |x - y|^{-t}\, d\nu(y) < \infty$$

then the Hausdorff dimension of A is at least t.

Proof. The usual Frostman Lemma states that if there is a probability measure v_* such that

$$I_t(v_*) = \int I_t(v_*, x)\, dv_*(x) < \infty$$

then the Hausdorff dimension of A is at least t: see, e.g., [6], Corollary 6.6. Suppose, then, that v is a probability measure for which $I_t(v, x) < \infty$ for v-a.e. x. Choose $a < \infty$ so large that $v(B_a) > 0$, where $B_a = \{x : I_t(v, x) < a\}$, and define a probability measure v_* by $v_*(A) = v(A \cap B_a)/v(B_a)$. Then

$$\int\int |x-y|^{-t}\, dv_*(x) dv_*(y) \leqq \frac{1}{v(B_a)^2} \int_{B_a}\int_{B_a} |x-y|^{-t}\, dv(x) dv(y) \leqq \frac{a}{v(B_a)^2}\ .$$

□

7.2 *Construction of measures* v_m *on* Λ

The use of Frostman's Lemma requires suitable probability measures on the set Λ. We will define a sequence of probability measures v_m on Λ, each the exit measure of a certain random walk on the tree of the branching Brownian motion. For each measure v_m we will then estimate the energies $I_{\delta-\varepsilon}(v_m, x)$ in Sect. 7.3 below.

It is most convenient to work in the half-plane representation. Recall that $\mathscr{P}_n$ is the set of (x-coordinates of) particles "frozen" on the horocycle Γ_n in the modification of the branching Brownian motion introduced in Sect. 3. Fix $m \geqq 1$, and fix a realization of the entire branching Brownian motion. Conditional on this realization, construct random variables $X_1, X_2, \ldots$ as follows. First, choose an element X_1 of $\mathscr{P}_m$ at random. Conditional on X_1, choose an element X_2 at random from among those elements of $\mathscr{P}_{2m}$ representing particles descendant from the particle at (X_1, e^{-m}). Conditional on $X_1, X_2, \ldots, X_k$, choose an element X_{k+1} at random from among those elements of $\mathscr{P}_{km+m}$ representing particles descendant from the particle at (X_k, e^{-km}).

Lemma 5 *With probability* 1, $\lim_{k\to\infty} X_k = \xi$ *exists, and* $(\xi, 0) \in \Lambda$.

Proof. By construction, for each k the particle at location (X_k, e^{-km}) is a (post-Γ_{km-m}) descendant of the particle at (X_{k-1}, e^{-km+m}). Thus, for each k there is a finite path γ_k in $\mathscr{T}$ terminating at (X_k, e^{-km}), and for each k the path γ_{k+1} is an extension of γ_k. The paths γ_k, $k < \infty$, may be knitted together to form an *infinite* path γ in $\mathscr{T}$. By Proposition 6, γ converges to a unique point $(\xi, 0) \in \Lambda$. Since the points (X_k, e^{-km}) converge to $\partial\mathbb{H}^2$ and are all on γ, it must be that $X_m \to \xi$. □

Define v_m to be the conditional distribution of ξ given the realization of the branching Brownian motion. Call the sequence X_k a $\mathscr{P}_m$ random walk on the tree $\mathscr{T}$, and ξ the *exit point* of the random walk.

Note. The random measure v_m is a function of the Galton–Watson tree, which in turn is dependent on the particle histories, which also determine distances in the limit set Λ. Thus, our computation of the energy $I_{\delta-\varepsilon}(v_m, x)$ in Sect. 7.3

below is somewhat more complicated than might at first be expected (e.g., compare with the superficially similar computation of the Hausdorff dimension of harmonic measure in [9]).

Observe that the construction just outlined does not require that the initial point of the branching Brownian motion be at $z = i$: If $\mathscr{T}'$ is the tree of a BBM started at *any* point $z = x + ie^{-nm}$ of a horocycle Γ_{nm}, $n = 0, 1, 2, \ldots$, a $\mathscr{P}_m$-random walk on $\mathscr{T}'$ will converge w.p.1 to an exit point $(\xi, 0) \in \Lambda$. Note, however, that the unconditional distribution of ξ depends on the initial point z. In fact, if ξ and ξ' are the exit points of $\mathscr{P}_m$-random walks on the trees of branching Brownian motions started at $z = i$ and $z = x + ie^{-nm}$, respectively, then ξ' has the same distribution as $e^{-nm}\xi + x$.

Lemma 6 *The* unconditional *distribution of* ξ *has a bounded density with respect to Lebesgue measure on* $\mathbb{R} = \partial\mathbb{H}^2$ (*in the upper half-plane representation of* $\mathbb{H}$).

Proof. Assume that the branching Brownian motion is defined on a probability space sufficiently large to accomodate an independent random variable U with density $2x$ for $0 < x < 1$. Let C_r and B_r be the hyperbolic circle and disk, respectively, of radius r centered at i. Consider two modifications of branching Brownian motion started at i: in the first, particles are "frozen" upon reaching the circle C_1; in the second, particles are "frozen" upon reaching the circle C_U. Since $U \leqq 1$, the mean number κ of particles created in the first modification is no smaller than the mean number of particles created in the second modification. Let $\mathscr{Q}$ be the point process consisting of the positions on C_U of all particles created in the second modification. Then the intensity measure of $\mathscr{Q}$ has a density relative to Lebesgue measure on the ball B_1 that is bounded by κ – this follows from the rotational symmetry of B.B.M. and the choice of density of U.

Now consider the modified branching Brownian motion in $\mathbb{H}^2$ defined in Sect. 3, in which particles are frozen upon reaching the horocycle Γ_m. A version of this process may be obtained from the second modification of the previous paragraph by attaching to each particle in $\mathscr{Q}$ its own branching Brownian motion (independent of the others, and of the pre-C_U process), with time shifted to account for the time it took to reach C_U, and freezing particles when they hit Γ_m. Consider the intensity measure μ_m of the point process $\mathscr{P}_m$ of particles frozen at Γ_m: Since each of these particles is an offspring of one of the particles in $\mathscr{Q}$,

$$\mu_m = \int_{B_1} \mu_m^z \, dI_{\mathscr{Q}}(z) \leqq \kappa \int_{B_1} \mu_m^z \, dz \ ,$$

where dz denotes Lebesgue measure on the ball B_1, $I_{\mathscr{Q}}$ the intensity measure of $\mathscr{Q}$, and μ_m^z the intensity measure of the point process of frozen particles in Γ_m descended from a single ancestor located originally at z. Observe that (a) if $z = x + iy$ and $z' = x' + iy$ then $\mu_m^{z'}$ is the $x' - x$ translate of μ_m^z; and (b) the total mass of μ_m^z is bounded above by $E|\mathscr{P}_{m+1}| = e^{m\delta+\delta}$, for every $z \in B_1$. It therefore follows by an easy argument from the integral representation of

μ_m above that μ_m has a bounded density $h(x) = d\mu_m/dx$ relative to Lebesgue measure on the horocycle Γ_m.

Finally, consider the exit random variable ξ for a $\mathscr{P}_m$-random walk X_k. The first step X_1 is to a point of $\mathscr{P}_m$, chosen at random; conditional on $X_1 = x$, the distribution of ξ is the same as the unconditional distribution of $e^{-m}(\xi - x)$. Thus, the distribution of ξ may be bounded above by summing the conditional distributions over all $x \in \mathscr{P}_m$, then integrating against the distribution of $\mathscr{P}_m$. Consequently, for any Borel subset B of $\mathbb{R}$,

$$P\{\xi \in B\} \leqq \int P\{e^{-m}(\xi - x) \in B\} h(x)\, dx \ .$$

This shows that ξ has a density bounded by $\max_x h(x)$. □

Corollary 7 *Let ξ, ξ' be the exit points of* independent *$\mathscr{P}_m$-random walks $(X_k)_{k \geqq 0}$ and $(X'_k)_{k \geqq 0}$ on the (same) tree $\mathscr{T}$ of a branching Brownian motion started at $z = i$. For $i = 1, 2, \ldots$, let $N_i = N_i^m$ denote the number of descendants in $\mathscr{P}_{mi}$ of the particle with x-coordinate X_{i-1} in $\mathscr{P}_{mi-m}$, and define $\mathscr{G}$ to be the σ-algebra generated by the random variables X_k and N_i. For $k \geqq 0$, define events*

$$A_k = \{X_k = X'_k\} \ ,$$
$$B_k = A_k \cap A_{k+1}^c \ .$$

Then there exists a constant $C < \infty$ such that for all $k \geqq 0$ and all $\varepsilon > 0$,

$$P(|\xi - \xi'| < \varepsilon \,|\, \mathscr{G} \vee B_k) \leqq Ce^{km}\varepsilon \ .$$

Here $\mathscr{G} \vee B_k$ denotes the smallest σ-algebra containing $\mathscr{G}$ and B_k.

Proof. On the event B_k the random walks X_j, X'_j follow the same path through $\mathscr{T}$ for the first k steps, then proceed through the (different) subtrees attached at the distinct points $X_{k+1}, X'_{k+1} \in \mathscr{P}_{km+m}$. Now the event B_k and the random variables X_{k+1}, X'_{k+1} depend only on the pre-$\mathscr{P}_{km+m}$ section of $\mathscr{T}$, and conditional on the pre-$\mathscr{P}_{km+m}$ process, the subtrees attached at *distinct* points x, x' of $\mathscr{P}_{km+m}$ are conditionally independent B.B.M. trees. Therefore, conditional on $\mathscr{G} \vee B_k$, the exit random variable ξ', scaled by e^{km+m}, has the same law as (a translate of) the exit random variable for a branching Brownian motion started at i. The result now follows from the preceding lemma. □

7.3 Energy of ν_m

By Frostman's Lemma, to prove that the Hausdorff dimension of Λ is almost surely at least δ it suffices to prove

Lemma 7 *For every $\varepsilon > 0$ sufficiently small, there exists an integer $m \geqq 1$ such that with probability* 1, *for ν_m-a.e. x,*

$$I_{\delta-\varepsilon}(\nu_m, x) < \infty.$$

Proof. Let $\mathscr{F}_\infty$ be the σ-algebra generated by the entire history of the branching Brownian motion, and let ξ and ξ' be conditionally independent given

$\mathscr{F}_\infty$, each with conditional distribution ν_m. It suffices to show that for any $\varepsilon \in (0, \delta/3)$, if m is sufficiently large then

$$E(|\xi - \xi'|^{-\delta+3\varepsilon} \mid \mathscr{F}_\infty \vee \xi) < \infty \text{ a.s.} \tag{11}$$

Let $X_1, X_2, \dots$ and $X_1', X_2', \dots$ be conditionally independent $\mathscr{P}_m$-random walks on $\mathscr{T}$. Define $\mathscr{H}$ to be the σ-algebra generated by the branching Brownian motion and the random walk $(X_k)_{k \geqq 1}$, and define $\mathscr{G}$ to be the σ-algebra generated by the random walk $(X_k)_{k \geqq 1}$ and the random variables $(N_i)_{i \geqq 1}$ defined in Corollary 7. Note that $\mathscr{F}_\infty \subset \mathscr{H}$ and $\mathscr{G} \subset \mathscr{H}$, and that $\mathscr{H} = \mathscr{F}_\infty \vee \xi$ (because for any given value of ξ there is only one path through $\mathscr{T}$ that converges to ξ, so the steps of the random walk X_k are determined by ξ). Let the events A_k, B_k be as in Corollary 7; then

$$P(A_k \mid \mathscr{H}) = 1 \Big/ \prod_{i=1}^{k} N_i = P(A_k \mid \mathscr{G}) \ ,$$

the last because the random variables N_i are measurable relative to $\mathscr{G}$. Unconditionally, $(N_i)_{i \geqq 1}$ are independent and identically distributed, each with the same distribution as M_m. By the SLLN,

$$\lim_{k\to\infty} \left(\prod_{i=1}^{k} N_i \right)^{1/k} = \exp\{E \log M_m\}$$

almost surely, and by Lemma 8 below,

$$\lim_{m\to\infty} m^{-1} E \log M_m = \delta.$$

Consequently, for each $\varepsilon > 0$ there exists m and a finite, nonnegative, $\mathscr{G}$-measurable random variable $\kappa = \kappa_\varepsilon$ such that for every $k \geqq 1$, with probability 1,

$$P(A_k \mid \mathscr{G}) \leqq \kappa \exp\{-km(\delta - \varepsilon)\}.$$

Now let $\Delta_n = \{|\xi - \xi'| < e^{-nm}\}$. By Corollary 7, for any $n \geqq 1$,

$$\begin{aligned} P(\Delta_n \mid \mathscr{G}) &= \sum_{k=0}^{\infty} E(P(\Delta_n \mid \mathscr{G} \vee B_k) \mathbf{1}_{B_k} \mid \mathscr{G}) \\ &\leqq \sum_{k=0}^{n-1} E(P(\Delta_n \mid \mathscr{G} \vee B_k) \mathbf{1}_{A_k} \mid \mathscr{G}) + P(A_n | \mathscr{G}) \\ &\leqq \sum_{k=0}^{n} C e^{-(n-k)m} P(A_k \mid \mathscr{G}) \\ &\leqq \sum_{k=0}^{n} (C e^{-(n-k)m})(\kappa e^{-km(\delta-\varepsilon)}) \\ &\leqq C' \kappa e^{-nm(\delta - 2\varepsilon)} \ . \end{aligned}$$

Thus, for all $\varepsilon > 0$,

$$E(|\xi - \xi'|^{-\delta+3\varepsilon} \mid \mathscr{G}) < \infty \ .$$

Since $\mathscr{G} \subset \mathscr{H} = \mathscr{F}_\infty \vee \xi$, (11) follows. □

Lemma 8 *Let Z_n be a Galton–Watson process whose offspring distribution has mean $e^{\delta} > 1$ and is supported by $\{1, 2, \ldots\}$ (i.e., there is no extinction). Then*

$$\lim_{n\to\infty} n^{-1} E \log Z_n = \delta \ .$$

Proof. By Jensen's inequality, $E \log Z_n \leqq n\delta$ for every n. Thus, it suffices to show that $\liminf n^{-1} E \log Z_n \geqq \delta$. Routine arguments using the generating function φ of the offspring distribution (or alternatively the Seneta–Heyde theorem) show that for any $\varepsilon > 0$,

$$Z_n / e^{n\delta - n\varepsilon} \to \infty$$

in probability. Consequently, for all sufficiently large n,

$$n^{-1} E \log Z_n \geqq \delta - \varepsilon \ . \qquad \square$$

8 The backscattering principle

That the Hausdorff dimension of the limit set Λ increases to $\frac{1}{2}$, and not 1, as $\lambda \uparrow \frac{1}{8}$ is rather striking, especially in view of a recent conjecture of Liggett [8] concerning "weakly supercritical" contact processes on homogeneous trees. Liggett's conjecture[1] states (in equivalent form) that the limit set of such a contact process cannot have Hausdorff dimension larger than $\frac{1}{2}$ the Hausdorff dimension of the boundary of the tree. In this section, we give a simple heuristic argument that explains why the limit set of a subcritical hyperbolic branching Brownian motion cannot have Hausdorff dimension larger than $\frac{1}{2}$. It is possible that this argument may be adapted to other growth processes on state spaces with exponential volume growth.

Consider subcritical branching Brownian motion ($\lambda \leqq 1/8$) in the hyperbolic plane $\mathbb{H}^2$, viewed as the Poincaré disk. Assume, as usual, that the process is initiated by a single particle located at 0. Suppose that particles are "frozen" upon reaching the circle C_n of (hyperbolic) radius n centered at 0; then eventually all particles will be frozen. Let N_n be the total number of frozen particles on C_n. Observe that the original branching Brownian motion (with no freezing) may be obtained by attaching conditionally independent branching Brownian motions to each of the N_n frozen particles on C_n.

Lemma 9 $\lim_{n\to\infty} \frac{1}{n} \log N_n = \delta$ *a.s.*

This lemma may be deduced from Propositions 1 and 2 by trapping a long arc of the circle C_n between the horocycles Γ_n and $\Gamma_{(1-\varepsilon)n}$.

Lemma 9 implies that, for every $\varepsilon > 0$ and every arc A of C_n with (hyperbolic) length 1, the expected number of frozen particles in A is at least $\exp n(\delta - \varepsilon - 1)$, provided n is sufficiently large, because the branching Brownian motion is radially symmetric, and because the hyperbolic arclength

[1] This conjecture has now been proved by the authors.

of C_n is $\sim e^n$. Now consider the post-C_n branching Brownian motions initiated from the frozen particles in C_n; each of these is, in law, an isometric replica of the original B.B.M. Suppose that each of these post-C_n B.B.M.s is run until "freezing" at a circle of hyperbolic radius n centered at the initial point $z \in C_n$; then for each the expected number of descendants located at positions at distance $\leqq 1$ from 0 is at least $\exp n(\delta - \varepsilon - 1)$, by the argument above. But there are N_n such processes, and, by Lemma 9, $N_n \geqq \exp n(\delta - \varepsilon)$ for all sufficiently large n. Thus, the expected number of particle trails that reach C_n and then return to the ball of radius 1 centered at 0 is, for large n, at least

$$\exp n(2\delta - 2\varepsilon - 1) .$$

If it were the case that $\delta > \frac{1}{2}$, so that, for sufficiently small $\varepsilon > 0$, $\delta - \varepsilon > \frac{1}{2}$, then one should expect large numbers of particle trails returning to a neighborhood of the origin after reaching C_n. This would contradict the "subcriticality" of the process.

Acknowledgements. We would like to thank the referee for pointing out an error in our original proof of Proposition 3. This work was supported by NSF grant DMS-9307855.

References

[1] Athreya, K., Ney, P.: Branching Processes. Berlin Heidelberg New York: Springer, 1972
[2] Beardon, A.: The Geometry of Discrete Groups. Berlin Heidelberg New York: Springer, 1985
[3] Benjamini, I., Peres, Y.: Markov chains indexed by trees. Ann. Probab. **22**, 219–243 (1994)
[4] Buser, P.: Geometry and Spectra of Compact Riemann Surfaces. Basel: Birkhäuser, 1992
[5] Chhikara, R.S., Folks, J.L.: The Inverse Gaussian Distribution. New York: Marcel Dekker, 1989
[6] Falconer, K.J.: The Geometry of Fractal Sets. Cambridge: Cambridge Univ. Press, 1985
[7] Lalley, S.P., Sellke, T.: A conditional limit theorem for the frontier of a branching Brownian motion. Ann. Probab. **15**, 1052–1061 (1987)
[8] Liggett, T.: Stochastic models of interacting systems. Wald Memorial Lectures, 1996
[9] Lyons, R.: Random walks and percolation on trees. Ann. Probab. **18**, 931–958 (1990)
[10] Pemantle, R.: The contact process on trees. Ann. Probab. **20**, 2089–2116 (1992)