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Euclidean models of first-passage percolation

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Summary. We introduce a new family of first-passage percolation (FPP) models in the context of Poisson-Voronoi tesselations of $\mathbb{R}^d$. Compared to standard FPP on $\mathbb{Z}^d$, these models have some technical complications but also have the advantage of statistical isotropy. We prove two almost sure results: a shape theorem (where isotropy implies an exact Euclidean ball for the asymptotic shape) and nonexistence of certain doubly infinite geodesics (where isotropy yields a stronger result than in standard FPP).

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1. Introduction

Standard first-passage percolation (FPP) was originally formulated by Hammersly and Welsh [HW] as a simplified model of fluid flow in a (random) porous medium. One aspect of the simplification was to use a deterministic lattice (the graph with vertex set $\mathbb{Z}^d$ and all nearest neighbor edges e) with the randomness superimposed by means of i.i.d. non-negative random variables $\tau(e)$ (with common distribution F) representing passage times (of the fluid) through the edges e. The passage time $T(r)$ along a finite nearest neighbor path r is $\sum_{e \in r} \tau(e)$ and the passage time $T(x, y)$ between vertices x

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and y is the infimum of $T(r)$ over paths r connecting x and y. If $F(0) = 0$, then $T(x, y)$ is almost surely (a.s.) a metric on $\mathbb{Z}^d$. If F is continuous then there is a.s. a unique finite geodesic $r^*(x, y)$ connecting x and y such that $T(x, y) = T(r^*(x, y))$.

In this paper we study a family of FPP models, based on homogeneous Poisson processes on $\mathbb{R}^d$, in which, roughly speaking, the randomness affects both the graph and the edge times. We remark that a different class of FPP models, also based on Poisson processes, were introduced earlier by Vahidi-Asl and Wierman [VW1] and other Poisson-based random metrics were studied by Sznitman [S]. The chief advantage of such models is that they have full (statistical) Euclidean invariance and, more particularly, isotropy (i.e., rotational invariance). This will allow us to avoid some serious difficulties encountered in recent work on standard FPP (which we presently briefly review) where the lack of isotropy leads to lattice effects which are hard to control rigorously.

Many of the difficulties in standard FPP revolve around the lack of qualitative information about the asymptotic shape B_0, a deterministic convex subset of $\mathbb{R}^d$ (depending on F) whose existence and significance are supplied by the shape theorem [R, CD, K1]. Roughly speaking, this theorem asserts (under mild conditions on F) that

$$\tilde{B}_t \equiv \{x \in \mathbb{Z}^d : T(0, x) \leq t\} \approx tB_0 \cap \mathbb{Z}^d \quad .$$

More precisely, it asserts that, a.s., for any $\epsilon \in (0, 1)$, $\tilde{B}_t$ is contained in $(1 + \epsilon)tB_0 \cap \mathbb{Z}^d$ and contains $(1 - \epsilon)tB_0 \cap \mathbb{Z}^d$ for all sufficiently large t. Except for certain non-continuous F's (see [DL]), it is a natural conjecture that the boundary of B_0 is smooth and uniformly curved (as defined in [N]), but this has not been proved for *any* F. Sadly, there are a number of interesting results (see [N]) about standard FPP which have thus far only been proved under the *assumption* that B_0 is uniformly curved. The first main result of this paper (see Theorem 1 in the next section) is a shape theorem for our continuum models. Isotropy of course requires the asymptotic shape to be a Euclidean ball, so uniform curvature is assured. We remark that for $d = 2$ such a shape theorem was obtained earlier by Vahidi-Asl and Wierman [VW2] for their continuum FPP models.

The results about standard FPP which assume uniform curvature (and which also require some less objectionable hypotheses on F) concern (either explicitly or implicitly) infinite geodesics, which are semi- or doubly-infinite paths r such that every finite segment of r is the finite geodesic between its endpoints. (We refer to semi-infinite geodesics as uni-geodesics and doubly-infinite geodesics as bi-geodesics.) One such result is that a.s. every uni-geodesic has an asymptotic direction $\hat{x}$ (i.e., if r passes through, in order, the vertices $x_1, x_2, \ldots$ then $\hat{x} \equiv \lim_n x_n/|x_n|$ exists, where $|\cdot|$ is Euclidean length). Another is that, a.s., for every $z \in \mathbb{Z}^d$ and every unit vector $\hat{x} \in \mathbb{R}^d$, there exists at least one such $\hat{x}$-unigeodesic starting from z. A third result concerns the existence of a limit (as $t \to \infty$) of the surface of $\tilde{B}_t$ in specific directions.

There are also results which are weaker than they might otherwise be because of lack of full isotropy or because of lack of uniform curvature. One of these (see [LN] and references cited there) concerns the conjecture, arising from the physics of disordered systems, that (at least for low d) a.s. no bi-geodesic exists. The result of [LN] is a partial confirmation of that conjecture — namely that for $d = 2$ and Lebesgue almost every $\hat{x}$ and $\hat{y}$, there is a.s. no $(\hat{x}, \hat{y})$– or $(\hat{x}, -\hat{x})$–bi-geodesic (defined in the obvious way). We remark that this result has been improved by Zerner [Z], who showed that the conclusions are valid for all but countably many $\hat{x}$'s and $\hat{y}$'s. The second main result of this paper (see Theorem 2 in the next section) is the analogue for our continuum models, but by isotropy the conclusions are valid for *every* $\hat{x}$ and $\hat{y}$.

Other such results concern bounds for the exponents $\chi = \chi(d)$ or $\xi = \xi(d)$ (see [LNP, NP] for the results and precise definitions of these exponents). Roughly speaking, χ and ξ are defined so that the standard deviation of $T(x,y)$ is of order $|x - y|^{\chi}$ while the fluctuations of $r^*(x,y)$ about the straight line from x to y are of order $|x - y|^{\xi}$. In a future paper, we will present such bounds for our continuum models, one of which is $\chi \leq 1/2$, originally proved for standard FPP by Kesten [K2]. This particular bound is not improved from the standard FPP setting because isotropy plays no role in the proof, but this bound is an ingredient for other bounds where isotropy will be a help.

We conclude this section with a brief description of the models studied in the remainder of the paper and some related models. Precise definitions are given in the next section. Basically, our models are defined by taking the (random) complete graph with vertex set

$$Q = \{\text{particle locations of a homogeneous Poisson process on } \mathbb{R}^d\} \ .$$

For each $e = \{x,y\}$ with $x,y \in Q$, define $\tau(e) = |x - y|^{\alpha}$, and then define $T(x,y)$ and $r^*(x,y)$ by minimizing the sum of $\tau(e)$ over "paths" connecting x and y where paths are simply arbitrary finite sequences $(q_1, \ldots, q_k)$ from Q with $q_1 = x$ and $q_k = y$. Note that if $0 \leq \alpha \leq 1$, then $r^*(x,y) = (x,y)$ and the analysis is trivial. If, on the other hand, $\alpha > 1$, then large "leaps" between particles in $r^*(x,y)$ are discouraged. While our results are valid for any $\alpha \geq 2$ (Theorem 1 for any $\alpha > 1$), the model with $\alpha = 2$ is clearly a pleasant choice. It is also clear that our results will be valid for a much larger class of models where $\tau(\{x,y\}) = \phi(|x - y|)$ with other choices of $\phi : \mathbb{R}^+ \to \mathbb{R}^+$. We have not investigated the precise class of ϕ's to which our results apply.

There are ways to define $\tau(\{x,y\})$ other than as just described which also satisfy (statistical) Euclidean invariance. For example, as in [VW1], one can replace the complete graph on Q by some (Euclidean invariant) random subgraph such as the Voronoi graph whose edge set E' consists of all $\{x,y\}$ such that the (closed) Voronoi region of x (those points of $\mathbb{R}^d$ for which x is a nearest particle in Q) shares a $((d-1)$-dimensional) face with that of y. Then one can take i.i.d. non-negative random variables (independent of the Poisson process), $(\tau(e) : e \in E')$, and proceed as in standard FPP. A particularly esthetic choice here (and non-trivial unlike the analogous choice on

$\mathbb{Z}^d$) is to remove all the randomness from the $\tau(e)$'s (leaving the randomness only in the graph) by taking the $\tau(e)$'s to be constant. For that choice, $T(x,y)$ is simply the usual graph distance on the random Voronoi graph (Q,E'). We remark that one may regard all these types of models as being defined on the complete graph of Q, but with $\tau(e)=\infty$ if $e\notin E'$. We also note that there are interesting recent results about Bernoulli site percolation on the Voronoi graph by Benjamini and Schramm [BS] and by Aizenman [A].

2. Definitions and results

Let Q be a non-empty, locally finite, subset of $\mathbb{R}^d$. We shall refer to Q as a "configuration" and to the elements of Q as "particles". For any $x\in\mathbb{R}^d$, let $q(x)$ denote the particle closest to x in Euclidean distance with any fixed rule for tie-breaking. We note that

$$\mathbb{R}^d=\bigcup_{y\in Q}\{x:q(x)=y\}$$

is the Voronoi tesselation of $\mathbb{R}^d$ with respect to the configuration Q (with the specified rule for allocating boundaries). We refer to these as the induced Voronoi regions. For any $x,y\in\mathbb{R}^d$, a finite sequence of not necessarily distinct particles $(q_1,\ldots,q_k)$ with $k\geq 2$, $q_1=q(x)$, and $q_k=q(y)$ will be referred to as a Q-path from x to y. We will use the notation $\overline{(q_1,\ldots,q_k)}$ to denote the polygonal path of line segments $\overline{q_1q_2}$, $\overline{q_2q_3}$, $\ldots,\overline{q_{k-1}q_k}$. Fix any $\alpha>1$ and let $|\cdot|$ denote Euclidean length. We define the passage time from x to y through Q by

$$T_Q(x,y)=\inf\left\{\sum_{j=1}^{k-1}|q_j-q_{j+1}|^\alpha : k\geq 2 \right.$$
$$\left.\text{and } (q_1,\ldots,q_k) \text{ is a } Q\text{-path from } x \text{ to } y\right\}. \tag{1}$$

Note that if $(q_1,\ldots,q_k)$ is a Q-path from x to y and $(q'_1,\ldots,q'_l)$ is a Q-path from y to z, then $q_k=q'_1$ and $(q_1,\ldots,q_k,q'_2,\ldots,q'_l)$ is a Q-path from x to z. This immediately yields the triangle inequality

$$T_Q(x,z)\leq T_Q(x,y)+T_Q(y,z)\ , \tag{2}$$

which is our motivation for excluding the terms $|q(x)-x|^\alpha$ and $|q(y)-y|^\alpha$ in the definition of $T_Q(x,y)$. Note that x and y belong to the same Voronoi region if and only if $T_Q(x,y)=0$; hence $T_Q(\cdot,\cdot)$ is a pseudometric on $\mathbb{R}^d$ inducing a metric on the Voronoi regions.

We are interested in the situation where the configuration is chosen randomly. In particular, let $(\Omega,\mathscr{F},\mathbb{P})$ be a probability space on which, for each ω, $Q(\omega)$ is an infinite, locally finite subset of $\mathbb{R}^d$ defining a Poisson process with unit density with respect to d-dimensional Lebesgue measure. The specific construction of $(\Omega,\mathscr{F},\mathbb{P})$ is not relevant (although in Section 4 making a specific choice facilitates a proof). In this context $T_{Q(\omega)}(x,y)$ is a family of random variables indexed by x and $y \in \mathbb{R}^d$. To simplify notion, we will drop the subscript $Q(\omega)$ and refer to $T(x,y)$. All 'a.s.' statements are with respect to $\mathbb{P}$.

We observe that, a.s., associated with any two points x and y there is a unique Q-path $r^*(x,y)$ along which $T(x,y)$ is realized. If such a minimizing path did not exist for some configuration Q, one could find Q-paths beginning at $q(x)$ and passing through arbitrarily many particles but with bounded passage time. It would follow that discs of radius ϵ centered at Q's particles would percolate for any choice of $\epsilon > 0$ violating a result of Zuev and Sidorenko [ZS1, ZS2]. Uniqueness follows from the continuous nature of the Poisson process.

Our first result concerns the asymptotic shape of the region of $\mathbb{R}^d$ that can be reached by time t. Specifically, let $\tilde{B}_t = \{x \in \mathbb{R}^d : T(0,x) \leq t\}$ and $B(y,a) = \{x : |x-y| \leq a\}$ where $y \in \mathbb{R}^d$ and $a > 0$. We will show that $\mu = \lim_{|x|\to\infty} T(0,x)/|x|$ exists a.s. and that $0 < \mu < \infty$. (Here μ is nonrandom and depends on d and α.) By a straightforward inversion this will yield:

Theorem 1 (Shape Theorem). *Let* $\alpha > 1$. *Then, a.s.*, for any ϵ *with* $0 < \epsilon < \mu^{-1}$ *we have* $B(0,(\mu^{-1}-\epsilon)t) \subset \tilde{B}_t \subset B(0,(\mu^{-1}+\epsilon)t)$ *for all sufficiently large* t.

We remark that in the Voronoi graph context discussed at the end of Section 1, Vahidi-Asl and Wierman have given optimal conditions on the common distribution of the $\tau(e)$'s to have a shape theorem for $d = 2$ [VW2] (although the precise conditions for their time constant μ to be nonzero have not been determined [VW1]).

Our second result concerns geodesics for the metric induced by T_Q on the Voronoi regions. Identifying each particle q with its own Voronoi region, we define a finite or infinite sequence (q_j) of distinct particles to be a geodesic if, for every $l < k$, the Q-path $(q_l,\ldots,q_k)$ has

$$T_Q(q_l,q_k) = \sum_{j=l}^{k-1} |q_j - q_{j+1}|^\alpha \ .$$

We will call a semi-infinite geodesic $(q_1,q_2,\ldots)$ a uni-geodesic and a doubly infinite geodesic $(\ldots,q_{-1},q_0,q_1,\ldots)$ a bi-geodesic. Our theorem, which supports the conjecture that (at least for $d = 2$) a.s. there are no bi-geodesics, concerns $(\hat{x},\hat{y})$–bi-geodesics. These are bi-geodesics for which $q_j/|q_j| \to \hat{x}$ (resp. $\hat{y}$) as $j \to \infty$ (resp. $-\infty$). The result is an improvement of the analogous result of [LN] for standard FPP on $\mathbb{Z}^2$ because it applies to every deterministic $\hat{x}$ and $\hat{y}$.

Theorem 2. *Let* $d = 2$ *and* $\alpha \geq 2$. *Then for any two unit vectors* $\hat{x}$ *and* $\hat{y}$, a.s. *there is no* $(\hat{x},\hat{y})$*–bi-geodesic.*

The Shape Theorem is proved in Section 3 and Theorem 2 is proved in Section 4.

3. Proof of shape theorem

Lemma 1. *There exist positive constants* c_1, c_2, c_3, *and* κ *depending on* d *and* α *such that*

$$\mathbb{P}\{T(x,y) \geq c_1|x-y|\} \leq c_2 \exp(-c_3|x-y|^{\kappa})$$

for all $x, y \in \mathbb{R}^d$.

Proof. By the translation and rotation invariance of the distribution of Q it suffices to show that

$$\mathbb{P}\{T(0,\ell\hat{e}_1) \geq c_1\ell\} \leq c_2 \exp(-c_3\ell^{\kappa}) \ , \tag{3}$$

where $\hat{e}_1 = (1, 0, \ldots, 0) \in \mathbb{R}^d$ and $\ell \geq 0$. Let $\mathscr{C}$ denote the doubly infinite solid cylinder of radius $1/2$ centered about the first coordinate axis. We define the random sequence $(W_n : n \geq 1)$ by

$$W_n = \inf\{w > W_{n-1} : \exists \text{ a particle in the hyperdisc } \mathscr{C} \cap \{x \in \mathbb{R}^d : x_1 = w\}\}$$

where we take $W_0 = 0$. It follows from elementary properties of the Poisson process that the sequence $(W_n : n \geq 1)$ increases (strictly) to infinity and, in fact, is a one dimensional Poisson process. Furthermore there is a unique particle q_n in each hyperdisc $\mathscr{C} \cap \{x \in \mathbb{R}^d : x_1 = W_n\}$. Put $R_n = W_n - W_{n-1}$ and $\tilde{R}_n = 2^{\alpha}(1+R_n)^{\alpha}$ so that R_n and $\tilde{R}_n$ are i.i.d. sequences. With $N = \min\{n : W_n > \ell\}$ we have

$$|q(0) - q_1| \leq |q(0)| + |q_1| \leq 2|q_1| \leq 2(1+R_1) \ ,$$
$$|q_n - q_{n+1}| \leq 1 + R_{n+1} \ ,$$

and, for $N \geq 2$,

$$|q_{N-1} - q(\ell\hat{e}_1)| \leq |q_{N-1} - \ell\hat{e}_1| + |\ell\hat{e}_1 - q(\ell\hat{e}_1)| \leq 2|q_{N-1} - \ell\hat{e}_1| \leq 2(1+R_N) \ ,$$

giving for $N \geq 3$ that

$$T(0,\ell\hat{e}_1) \leq |q(0) - q_1|^{\alpha} + \sum_{n=1}^{N-2} |q_n - q_{n+1}|^{\alpha} + |q_{N-1} - q(\ell\hat{e}_1)|^{\alpha} \leq \sum_{n=1}^{N} \tilde{R}_n \ .$$

It is easy to see that $T(0,\ell\hat{e}_1) \leq \sum_{n \leq N} \tilde{R}_n$ also holds for $N = 1$ and $N = 2$. Therefore, for any $c_1 > 0$ and any choice of m we have

$$\mathbb{P}\{T(0,\ell\hat{e}_1) \geq c_1\ell\} \leq \mathbb{P}\{R_1 + \cdots + R_m \leq \ell\} + \mathbb{P}\{\tilde{R}_1 + \cdots + \tilde{R}_m \geq c_1\ell\} \ . \tag{4}$$

By elementary large deviation theory, for some (large) $b > 0$ and $c_4 > 0$ we have

$$\mathbb{P}\{R_1 + \cdots + R_m < b^{-1}m\} \leq \exp(-c_4 m) \text{ for large } m$$

so, with $b' > b$ and $0 < c_4' < c_4 b'$,

$$\mathbb{P}\{R_1 + \cdots + R_{\lfloor b'\ell \rfloor} < \ell\} \leq \exp(-c_4'\ell) \text{ for large } \ell \ . \tag{5}$$

Also, the $\tilde{R}_n$ are i.i.d. with a sub-exponential tail:

$$\mathbb{P}\{\tilde{R}_n > r\} = \mathbb{P}\left\{R_n > \frac{1}{2}r^{1/\alpha} - 1\right\} = e^v \exp\left(-\frac{v}{2}r^{1/\alpha}\right) \text{ for } r \geq 2^\alpha ,$$

where v is the $d-1$ dimensional volume of a ball with radius $1/2$. Therefore, for some $c_5, c_6 > 0$, we have

$$\mathbb{P}\{\tilde{R}_1 + \cdots + \tilde{R}_m > c_5 m\} \leq \exp\left(-c_6 m^{1/\alpha}\right) \text{ for large } m$$

(see [Na]), and choosing $c_1 > c_5 b'$ and $0 < c'_6 < c_6 {b'}^{1/\alpha}$ yields

$$\mathbb{P}\{\tilde{R}_1 + \cdots + \tilde{R}_{\lfloor b'\ell \rfloor} > c_1 \ell\} \leq \exp\left(-c'_6 \ell^{1/\alpha}\right) \text{ for large } \ell \ . \tag{6}$$

The Lemma follows for $\kappa = 1/\alpha$ by taking $m = \lfloor b'\ell \rfloor$ in (4) and applying (5) and (6). □

Lemma 2. *Suppose* $0 < \gamma < 1$. *Then, almost surely,* $|q(x) - x| \leq |x|^\gamma$ *whenever* $|x|$ *is sufficiently large.*

Proof. It suffices to prove the lemma only for x restricted to $\mathbb{Z}^d$. To see this, for any $x \in \mathbb{R}^d$, let $z \in \mathbb{Z}^d$ be of minimal Euclidean distance from x. Then

$$|q(x) - x| \leq |q(z) - x| \leq |q(z) - z| + |z - x| \leq 2|z|^{\gamma/2} \leq |x|^\gamma$$

whenever $|x|$ is large. The last two inequalities hold for large $|x|$ since Lemma 2 holds (by assumption) for $\gamma/2$ and large $z \in \mathbb{Z}^d$ and since $|z - x| \leq \sqrt{d}/2$. To establish the lemma restricted to $\mathbb{Z}^d$, note that for any x

$$\mathbb{P}\{|q(x) - x| \geq |x|^\gamma\} = \exp\left(-v|x|^{\gamma d}\right)$$

where here v is the d-dimensional volume of $B(0,1)$. Since $\sum_{z \in \mathbb{Z}^d} \exp(-v|z|^{\gamma d}) < \infty$, Lemma 2 follows from an application of the Borel-Cantelli Lemma. □

For $0 \leq m < n$, set $X_{mn} = T(m\hat{e}_1, n\hat{e}_1)$ so, by the triangle inequality (2), $X_{0n} \leq X_{0m} + X_{mn}$. It is easy to verify that this family of random variables satisfies the hypotheses of Liggett's version of Kingman's subadditive ergodic theorem (see [L, Chapter VI]) giving that

$$\mu \equiv \lim_{n\to\infty} X_{0n}/n = \lim_{n\to\infty} \mathbb{E}X_{0n}/n \text{ exists a.s. and in } L^1 \ . \tag{7}$$

Lemma 3. *With* μ *defined above,* $0 < \mu < \infty$.

Proof. The subadditive ergodic theorem gives that $\mu < \mathbb{E}X_{01}$, a quantity that is easily shown to be finite (e.g., by Lemma 1). As for the remaining inequality, we will show that for some positive constants c_7, c_8, c_9 and κ' we have

$$\mathbb{P}\{X_{0n} \leq c_7 n\} \leq c_8 \exp\left(-c_9 n^{\kappa'}\right) \tag{8}$$

yielding that $\mu \geq c_7$.

To see (8), let $(U_n : n \geq 1)$ be a sequence of independent Bernoulli random variables with $\mathbb{P}\{U_n = 1\} = \phi$ and $\mathbb{P}\{U_n = 0\} = 1 - \phi$ and put $V_n = \sum_{i \leq n} U_i$ and let $\theta > 0$. Then

$$\mathbb{P}\{V_n \geq n/(2d)\} \leq e^{-\theta n/(2d)} \mathbb{E} e^{\theta V_n} \leq \left[e^{-\theta/(2d)} \left(1 + \phi \left(e^\theta - 1\right)\right)\right]^n . \tag{9}$$

Clearly for any $\delta > 0$, there are θ, $\phi > 0$ such that $e^{-\theta/(2d)}(1 + \phi(e^\theta - 1)) < \delta$ (choose θ large, then ϕ small).

Now partition $\mathbb{R}^d$ into cubes (overlapping only at their $d-1$ dimensional faces), centered at the points $\epsilon\mathbb{Z}^d$, whose vertices collectively are $(\frac{\epsilon}{2}, \ldots, \frac{\epsilon}{2}) + \epsilon\mathbb{Z}^d$. ($\epsilon$ will be chosen later.) We refer to these as the "ϵ-boxes" and refer to two distinct ϵ-boxes as contiguous if they share a face. Additionally we call any (finite or infinite) sequence of distinct ϵ-boxes an "ϵ-box path" if the first ϵ-box contains the origin and they are sequentially contiguous. The number of ϵ-boxes in an ϵ-box path will be called the length of the path. We note that there are at most $(2d)(2d-1)^{n-2}$ ϵ-box paths of length n, but we will use the cruder bound of $(2d)^n$. We call an ϵ-box occupied if there is at least one particle in it and now let $\phi = \phi_\epsilon$ denote the probability that any given ϵ-box is occupied. Then for any fixed ϵ-box path of length n, the probability that $n/(2d)$ or more are occupied is bounded by the right hand side of (9). Hence if

$$E_n = \{\exists \text{ an } \epsilon\text{-box path of length } n \text{ with } n/(2d) \text{ or more boxes occupied}\} ,$$

then $\mathbb{P}E_n \leq \left[(2d)e^{-\theta/(2d)}(1 + \phi(e^\theta - 1))\right]^n$. Noting that $\phi = \phi_\epsilon \downarrow 0$ as $\epsilon \downarrow 0$ we now choose θ and ϵ so that $(2d)e^{-\theta/(2d)}(1 + \phi(e^\theta - 1)) < e^{-1}$. For this choice of ϵ, $\mathbb{P}E_n \leq e^{-n}$. Furthermore, if

$$\begin{aligned} F_n = \{\exists \text{ an } \epsilon\text{-box path of length } m \geq n \text{ with } m/(2d) \\ \text{or more boxes occupied}\} , \end{aligned}$$

then $\mathbb{P}F_n \leq (1 - e^{-1})^{-1} e^{-n}$.

In establishing Lemma 3, it will be more convenient to consider the "augmented" passage times

$$X^*_{0n} = |q(0)|^\alpha + X_{0n} + |q(n\hat{e}_1) - n\hat{e}_1|^\alpha .$$

We observe that for any $c_7 > 0$,

$$\begin{aligned} \mathbb{P}\{X_{0n} \leq c_7 n\} &\leq \mathbb{P}\left\{X^*_{0n} \leq 3c_7 n\right\} + \mathbb{P}\{|q(0)|^\alpha \geq c_7 n\} + \mathbb{P}\{|q(n\hat{e}_1) - n\hat{e}_1|^\alpha \\ &\qquad \geq c_7 n\} \\ &\leq \mathbb{P}\left\{X^*_{0n} \leq 3c_7 n\right\} + 2\exp\left(-v c_7^{d/\alpha} n^{d/\alpha}\right) \end{aligned} \tag{10}$$

where v is the d-dimensional volume of $B(0,1)$. Presently we bound $\mathbb{P}\{X^*_{0n} \leq 3c_7 n\}$ for an appropriate c_7.

Let

$$r^* = \overline{(0, q_1, \ldots, q_K, n\hat{e}_1)}$$

where $(q_1, \ldots, q_K)$ is the Q-path from 0 to $n\hat{e}_1$ of minimal passage time. Form the corresponding ϵ-box path

$$\beta^* = (\beta_1^*, \ldots, \beta_M^*)$$

as follows: β_1^* is the ϵ-box containing 0; β_{i+1}^* is the ϵ-box that r^* enters when it *last* exits β_i^*. (We note that, a.s., any segment of r^* that passes from one ϵ-box to another passes through the ($d-1$ dimensional) interior of a $d-1$ dimensional face.) Observe that, by this procedure, $n\hat{e}_1 \in \beta_M^*$ and $M \geq m = m(n) = \lceil n\epsilon^{-1} \rceil$. For large n, on the event F_m^c, there are at least $m/(3d)$ indices i with $d|i$, $i+d-1 < m$, and β_j^* unoccupied for $i \leq j \leq i+d-1$.

Note that any straight line segment that passes completely through d sequentially contiguous ϵ-boxes (by passing through the interiors of faces) is at least ϵ in length. This follows from the pigeonhole principle as follows. Each point on the line segment that belongs to a face must have one of its coordinates of the form $\frac{\epsilon}{2} +$ (some integer) $\cdot\, \epsilon$. Since the line segment passes through d sequentially contiguous boxes, there are at least $d+1$ such points on the line segment. Hence there are at least two distinct points x and y on the line segment with the *same* coordinate of the form $\frac{\epsilon}{2} +$ (some integer) $\cdot\, \epsilon$. These two points are at least ϵ apart.

It follows that each portion of r^* passing through one of these unoccupied blocks of boxes contributes at least ϵ^α to the augmented passage time X_{0n}^* and we have, for large n,

$$X_{0n}^* \geq \epsilon^\alpha m/(3d) \geq \epsilon^{\alpha-1} n/(3d) \qquad \text{on } F_m^c \ ,$$

yielding, for large n,

$$\mathbb{P}\left\{X_{0n}^* \leq \frac{1}{3d}\epsilon^{\alpha-1} n\right\} \leq \mathbb{P}F_m \leq \left(1-e^{-1}\right)e^{-m} \leq \left(1-e^{-1}\right)\exp\left(-\epsilon^{-1}n\right) \ . \quad (11)$$

Combining (10) and (11) gives (8) with $c_7 = \epsilon^{\alpha-1}/(9d)$ for appropriate c_8, c_9, and κ'. □

Lemma 4. *Almost surely*, $\lim_{|x|\to\infty} T(0,x)/|x| = \mu$.

Proof. We will show that, a.s., $\limsup_{|x|\to\infty} \big|T(0,x) - \mu|x|\big|/|x| = 0$. Fix ϵ with $0 < \epsilon < 1$ and choose γ such that $0 < \alpha\gamma < 1$ so, additionally, $0 < \gamma < 1$. Pick finitely many unit vectors $\hat{u}_1, \ldots, \hat{u}_m$ such that the region

$$R = \bigcup_{i=1}^{\infty} \bigcup_{j=1}^{m} B\left(i\hat{u}_j, i\epsilon\right)$$

covers all but a bounded subset of $\mathbb{R}^d$. (E.g., choose the vectors so that the $B(\hat{u}_j, \epsilon)$'s cover the unit sphere. Since they also cover everything within some strictly positive distance δ of the unit sphere, R will contain everything beyond some finite distance (e.g., $\lceil 1/(2\delta) \rceil$) from the origin.) We will let u_{ij} denote $i\hat{u}_j$. Since the Poisson process is isotropic, it follows from (7) that, a.s., for sufficiently large i and all $j \leq m$, $|T(0,u_{ij}) - i\mu| < i\epsilon$. Cover each $B(u_{ij}, i\epsilon)$ with $K(i,j)$ balls $B(v_{ijk}, i^\gamma)$ where $\frac{1}{2}i^\gamma \leq |v_{ijk} - u_{ij}| \leq i\epsilon$. For an appropriate $c_{10} > 0$ (depending on ϵ and d) we may take $K(i,j) \leq c_{10} i^{(1-\gamma)d}$.

Now

$$\mathbb{P}\{T(u_{ij}, v_{ijk}) \geq c_1 |u_{ij} - v_{ijk}|\} \leq c_2 \exp\left(-\frac{1}{2^\kappa} c_3 i^{\gamma\kappa}\right)$$

so

$$\mathbb{P}\{T(u_{ij}, v_{ijk}) \geq c_1 \epsilon i\} \leq c_2 \exp\left(-\frac{1}{2^\kappa} c_3 i^{\gamma\kappa}\right)$$

and

$$\mathbb{P}\left[\bigcup_{j=1}^{m} \bigcup_{k=1}^{K(i,j)} \{T(u_{ij}, v_{ijk}) \geq c_1 \epsilon i\}\right] \leq m c_2 c_{10} i^{(1-\gamma)d} \exp\left(-\frac{1}{2^\kappa} c_3 i^{\gamma\kappa}\right) \ . \tag{12}$$

Since the right hand side of (12) is summable over i, we have, a.s., that for large i

$$T(u_{ij}, v_{ijk}) \leq c_1 \epsilon i \text{ whenever } j \leq m \text{ and } k \leq K(i,j) \ . \tag{13}$$

For $|x|$ suitably large, we may choose an $i = i(x)$, $j = j(x)$, and $k = k(x)$ so that $x \in B(u_{ij}, i\epsilon) \cap B(v_{ijk}, i^\gamma)$. Note that $i(x) \to \infty$ as $|x| \to \infty$. Put $u = u(x) = u_{ij}$ and $v = v(x) = v_{ijk}$. Then, with $|x|$ large enough to additionally ensure Lemma 2 and (13),

$$\begin{aligned}
\big|T(0,x) - \mu|x|\big| &\leq \big|\mu|x| - \mu i\big| + |\mu i - T(0,u)| + |T(0,u) - T(0,x)| \\
&\leq \mu|x - u| + \epsilon i + T(u,x) \\
&\leq \mu\epsilon i + \epsilon i + T(u,v) + T(v,x) \\
&\leq \mu\epsilon i + \epsilon i + c_1\epsilon i + |q(v) - q(x)|^\alpha \\
&\leq \mu\epsilon i + \epsilon i + c_1\epsilon i + (|q(v) - v| + |v - x| + |q(x) - x|)^\alpha \\
&\leq \mu\epsilon i + \epsilon i + c_1\epsilon i + 3^\alpha|v|^{\gamma\alpha} + 3^\alpha i^{\gamma\alpha} + 3^\alpha|x|^{\gamma\alpha} \ .
\end{aligned} \tag{14}$$

Since $\big||v| - i\big| \leq i\epsilon$, $\big||x| - i\big| \leq i\epsilon$ and $\gamma\alpha < 1$, (14) yields $\limsup_{|x|\to\infty} \big|T(0,x) - \mu|x|\big|/i(x) \leq (\mu + 1 + c_1)\epsilon$ and hence $\limsup_{|x|\to\infty} \big|T(0,x) - \mu|x|\big|/|x| \leq (\mu + 1 + c_1)\epsilon/(1 - \epsilon)$ from which the lemma follows. □

To complete the proof of the Shape Theorem we "invert" Lemma 4. Fix any $\epsilon > 0$ and choose $\tilde{\epsilon} > 0$ so that $(\mu - \tilde{\epsilon})^{-1} \leq \mu^{-1} + \epsilon$ and $\mu^{-1} - \epsilon \leq (\mu + \tilde{\epsilon})^{-1}$. By Lemma 4, a.s. we may choose an $L = L(\omega)$ so that $\big|T(0,x) - \mu|x|\big| \leq \tilde{\epsilon}|x|$ whenever $|x| \geq L$. Hence, whenever $|x| \geq L$,

$$|x| \leq T(0,x)(\mu - \tilde{\epsilon})^{-1} \leq T(0,x)\left(\mu^{-1} + \epsilon\right) \tag{15}$$

and

$$T(0,x)\left(\mu^{-1} - \epsilon\right) \leq T(0,x)(\mu + \tilde{\epsilon})^{-1} \leq |x| \ . \tag{16}$$

It follows from (15) that $\{x : |x| \geq L\} \cap \tilde{B}_t \subset B(0, t(\mu^{-1} + \epsilon))$ and since $\{x : |x| < L\} \subset B(0, t(\mu^{-1} + \epsilon))$ for large t, we have $\tilde{B}_t \subset B(0, t(\mu^{-1} + \epsilon))$ for large t. Also, (16) gives that $\{x : |x| \geq L\} \cap \tilde{B}_t^c \subset B(0, t(\mu^{-1} - \epsilon))^c$ or $B(0, t(\mu^{-1} - \epsilon)) \subset \{x : |x| < L\} \cup \tilde{B}_t$. But, a.s., $\{x : |x| < L\} \subset \tilde{B}_t$ for large t completing the inversion. □

4. Proof of Theorem 2

As in the $\mathbb{Z}^d$ context of [N, LN], a crucial role is played by certain graphs denoted $R(q)$, for each particle q, obtained as the union of all finite geodesics starting from q. More precisely, $R(q)$ is the graph whose vertex set is the set Q of all particles and whose edge set is the set of all $\{q', q''\}$ with $q' \neq q''$ such that there is some finite geodesic $(q = q_0, q_1, \ldots, q_k)$ starting from q with $\{q', q''\} = \{q_{j-1}, q_j\}$ for some $j = 1, \ldots, k$. It is not hard to see that, a.s., $R(q)$ is a spanning tree on the vertex set Q. If one replaces each $\{q', q''\}$ in $R(q)$ by the line segment $\overline{q'q''}$ in $\mathbb{R}^d$, there is the potential complication (for $d = 2$) that two such line segments may cross so that $R(q)$ is not embedded into $\mathbb{R}^d$ in the natural way. The next lemma shows that for $\alpha \geq 2$ this cannot be the case.

Lemma 5. *Suppose $d = 2$ and $\alpha \geq 2$. For a.e. configuration Q, if (q_1, q_2) and (q_1', q_2') are the geodesics from particles q_1 to q_2 and q_1' to q_2', respectively, then either $\overline{q_1 q_2}$ and $\overline{q_1' q_2'}$ are disjoint, or they coincide, or their intersection consists of one point which is an endpoint of both line segments.*

Proof. Since, a.s., there are no four particles a, b, c, $d \in Q$ (at least three of which are distinct) such that $|a - b| = |c - d|$, we may assume that $|q_1 - q_2| \neq |q_1' - q_2'|$. When $\alpha \geq 2$, the direct Q-path (a, b) from particle a to particle b can be a geodesic only if (but not necessarily if) the interior of the disc whose diameter is the line segment $\overline{ab}$ contains no particles. (If it contained such a particle $c \in Q$, then the $\{Q$-path$\}$ (a, c, b) would have smaller passage time. This easily follows from the elementary fact that for c on the boundary of the disc, $|a - c|^2 + |c - b|^2 = |a - b|^2$.) The lemma follows from the following geometric fact: If D and D' are diameters (with unequal length) of discs B and B' such that D and D' intersect at a point that is not an endpoint of (at least one of) D or D', then either the interior of B' contains an endpoint of D or the interior of B contains an endpoint of D'. □

Theorem 2 is a consequence of two other results, which we state as the next two lemmas. We will give the proof of Theorem 2, based on these lemmas, followed by the proofs of the lemmas. Our proof of Theorem 2 and its attendant lemmas parallels [LN, Theorem 2].

Lemma 6. *Let $D_U(\hat{x})$ denote the event that for every particle q there is at most one $\hat{x}$–uni-geodesic $(q, q_1, q_2, \ldots)$ starting at q. For $d = 2$, $\alpha \geq 2$, and any unit vector $\hat{x}$ in $\mathbb{R}^2$, $\mathbb{P}[D_U(\hat{x})] = 1$.*

For a given $\hat{x}$ and any particle q, we denote by $s_q = s_q(\hat{x})$ the a.s. unique (if it exists) $\hat{x}$–uni-geodesic starting at q. For $d = 2$ and $\alpha \geq 2$ it follows from Lemma 6 that if the polygonal paths of s_q and $s_{q'}$ ever meet (which, by Lemma 5, can happen only at a particle location), they must coalesce (i.e., they must be the same path from that particle onward). The next lemma shows that, for $d = 2$ and $\alpha \geq 2$, they must meet (and so coalesce).

Lemma 7. *For $d = 2$, $\alpha \geq 2$, and any unit vector $\hat{x} \in \mathbb{R}^2$, there is zero probability that there are any two disjoint $\hat{x}$–uni-geodesics.*

Proof of Theorem 2. By Lemma 6, we may assume that $\hat{x} \neq \hat{y}$. If there were two distinct $(\hat{x}, \hat{y})$–bi-geodesics, then two applications of Lemma 7 would show that they meet at two particles q and q' while being distinct in between. This would violate the a.s. uniqueness of the (finite) geodesic between q and q'. Hence there is a.s. at most one $(\hat{x}, \hat{y})$–bi-geodesic. Let A be the event that there is exactly one $(\hat{x}, \hat{y})$–bi-geodesic; we must show that $\mathbb{P}A = 0$. For $L > 0$ and $z \in \mathbb{R}^2$, let $A(z, L)$ be the event that there is exactly one $(\hat{x}, \hat{y})$–bi-geodesic and it passes through a particle $q \in z + [-L, L]^2$. Now choose $\hat{w} \neq \hat{x}$ or $\hat{y}$. By translation invariance, $\mathbb{P}[A(k\hat{w}, L)] = \mathbb{P}[A(0, L)]$ and, by ergodicity,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} I[A(k\hat{w}, L)] = \mathbb{P}[A(0, L)] \text{ a.s.} \tag{17}$$

By the choice of $\hat{w}$, any $(\hat{x}, \hat{y})$–bi-geodesics can touch particles in at most finitely many of the $k\hat{w} + [-L, L]^2$ yielding that $\sum_k I[A(k\hat{w}, L)] < \infty$ a.s. and, in conjunction with (17), that $\mathbb{P}[A(0, L)] = 0$. But $A(0, L) \uparrow A$ as $L \uparrow \infty$ giving that $\mathbb{P}A = 0$. □

Proof of Lemma 6. Let $\tilde{e} = (q, q')$ be an ordered pair of particles. The spanning tree $R(q)$ contains one or more (infinite) uni-geodesics starting from q. If one or more of these begins with $\tilde{e}$, then we will define a particular one, denoted $r^+(\tilde{e})$; otherwise $r^+(\tilde{e})$ will be undefined. The uni-geodesic $r^+(\tilde{e}) = (q_1 = q, q_2 = q', q_3, \ldots)$ is the one obtained by a "counterclockwise search algorithm" within $R(q)$. That is, if $(q_1, q_2, \ldots, q_k, \bar{q}_j)$ are all the possible initial segments of the uni-geodesics which begin with $(q_1, \ldots, q_k)$, then $q_{k+1} = \bar{q}_{j^*}$ where j^* is chosen to maximize the angle (in $(-\pi, \pi)$) from $q_k - q_{k-1}$ to $\bar{q}_j - q_k$.

If there are two distinct $\hat{x}$–uni-geodesics r_1 and r_2 starting from some particle q_0, they must bifurcate at some particle q, going respectively to $q^{(1)}$ and $q^{(2)}$ in their next steps. After q, the polygonal paths of r_1 and r_2 never touch (by Lemma 5), and any uni-geodesic (starting from q_0) caught "between" them must be an $\hat{x}$–uni-geodesic as well. Depending on whether r_1 is asymptotically counterclockwise to r_2 or vice-versa, either $r^+((q, q^{(2)}))$ or $r^+((q, q^{(1)}))$ will be such an $\hat{x}$–uni-geodesic. We conclude that $D_U(\hat{x})$ occurs unless the event $G(\hat{x})$, that for some $\tilde{e}$, $r^+(\tilde{e})$ is defined and is an $\hat{x}$–uni-geodesic, occurs. Now, a.s., only countably many $\hat{x}$'s have the property that some $r^+(\tilde{e})$ is defined and is an $\hat{x}$–uni-geodesic. Denoting the uniform measure on the $\hat{x}$'s by $d\hat{x}$, we have, by this fact and Fubini's Theorem, that

$$\begin{aligned} 1 &\geq \int \mathbb{P}[D_U(\hat{x})] d\hat{x} \\ &\geq 1 - \int \mathbb{P}[G(\hat{x})] d\hat{x} \\ &= 1 - \int \left[\int I[G(\hat{x})] d\hat{x} \right] d\mathbb{P} = 1 - \int 0 \, d\mathbb{P} = 1 \ . \end{aligned}$$

This proves that $\mathbb{P}[D_U(\hat{x})]$ (which is ≤ 1 for every $\hat{x}$) must equal 1 for Lebesgue-a.e. $\hat{x}$. But by isotropy, $\mathbb{P}[D_U(\hat{x})]$ is independent of $\hat{x}$ and so equals 1 for every $\hat{x}$, as desired. □

Proof of Lemma 7. For the given $\hat{x}$, we let $S = S(\hat{x})$ denote the union, over all particles q, of $s_q = s_q(\hat{x})$, the (a.s. unique, if it exists) $\hat{x}$–uni-geodesic starting from q. More precisely, S is the graph whose vertices are all the particles and whose edges are those $\{q', q''\}$ belonging to some $\hat{x}$–uni-geodesic. Since s_q and $s_{q'}$ must coalesce if they ever meet, it follows that S either has no edges or else is a forest consisting of $N \geq 1$ distinct infinite trees (plus, perhaps, some isolated vertices). Lemma 7 is equivalent to the claim that, for $d = 2$ and $\alpha \geq 2$, $\mathbb{P}\{N \geq 2\} = 0$. The proof has three parts with a structure parallel to the proof in [BK] for uniqueness of infinite clusters in Bernoulli percolation. For the remainder of this section we work with the canonical realization of our underlying Poisson process, in which each ω is a locally finite subset of $\mathbb{R}^2$ and $Q(\omega) = \omega$.

Part 1. $\mathbb{P}\{N \geq 2\} > 0 \Longrightarrow \mathbb{P}\{N \geq 3\} > 0$. We may and will assume (without loss of generality by isotropy) that $\hat{x} = \hat{e}_1 = (1, 0)$. Then any $\hat{x}$–uni-geodesic is eventually to the right of any vertical line. Translation invariance and standard arguments show that if $\mathbb{P}\{N \geq 2\} > 0$, then, for some $\delta > 0$ and some $x = (x_1, x_2)$ and $x' = (x'_1, x'_2)$ with $x_1,\ x'_1 < -\delta$, $\mathbb{P}[A_\delta(x, x')] > 0$, where $A_\delta(x^{(1)}, x^{(2)}, \ldots, x^{(m)})$ denotes the event that: there is for $1 \leq j \leq m$ a unique particle $q^{(j)} = q(x^{(j)})$ in the disc $B(x^{(j)}, \delta)$; there is a unique $\hat{x}$–uni-geodesic $s^{(j)} = s_{q^{(j)}}$ starting from each $q^{(j)}$; every particle touched by each $s^{(j)}$ after $q^{(j)}$ has strictly positive first coordinate; and the $s^{(j)}$'s are all disjoint.

Since the line segment connecting the first two particles of $s_{q(x)}$ and the corresponding segment for $s_{q(x')}$ each cross the y-axis somewhere with y-coordinates we denote $W(x)$ and $W(x')$, we have also that $\mathbb{P}[A_\delta(x, x'; \eta, w, w')] > 0$ for some $\eta > 0$ and real $w,\ w'$ with $w' - w > 2\eta$, where $A_\delta(x, x'; \eta, w, w')$ (which we denote A^0_δ) is the intersection of $A_\delta(x, x')$ with the events that $|W(x) - w| < \eta$ and $|W(x') - w'| < \eta$. (We note that to take $w' > w$ in this definition, an interchange of x and x' may be needed.) Note that on A^0_δ, since $W(x') > W(x)$, the polygonal path of $s_{q(x')}$ after crossing the y-axis is always "above" that of $s_{q(x)}$. Choose $h > w' - w + 2\eta$ (e.g., $h = 2(w' - w - 2\eta)$) and now consider the translates of A^0_δ by nh (in the y-direction), for integers n:

$$A^n_\delta = A_\delta(x + (0, nh), x' + (0, nh); \eta, w + nh, w' + nh) \ . \tag{18}$$

By translation invariance, $\mathbb{P}[A^n_\delta] = \mathbb{P}[A^0_\delta] > 0$ for all n, which implies that for some $n_1 < n_2$, $\mathbb{P}[A^{n_1}_\delta \cap A^{n_2}_\delta] > 0$. We set $x^{(1)} = x + (0, n_1 h)$, $x^{(2)} = x' + (0, n_1 h)$, $x^{(3)} = x' + (0, n_2 h)$, and $x^{(4)} = x + (0, n_2 h)$. The choice of h ensures that on $A^{n_1}_\delta \cap A^{n_2}_\delta$, $W(x^{(1)}) < W(x^{(2)}) < W(x^{(4)}) < W(x^{(3)})$ and thus that the uni-geodesic $s^{(2)}$ must be disjoint from $s^{(3)}$ since, if they met they would coalesce and $s^{(4)}$, trapped between them, would be forced to intersect $s^{(3)}$ which would contradict the definition of $A^{n_2}_\delta$. Thus, even though $s^{(2)}$ and $s^{(4)}$ may coalesce (so that $A_\delta(x^{(1)}, x^{(2)}, x^{(3)}, x^{(4)})$ may not occur), $s^{(1)}$, $s^{(2)}$, and $s^{(3)}$ remain distinct and so $A_\delta^{n_1} \cap A_\delta^{n_2}$ is contained in the event $A_\delta(x^{(1)}, x^{(2)}, x^{(3)})$, which consequently has strictly positive probability.

For the second part of the proof of Lemma 7, we let $F_{M,K}$ denote the event that some tree in S touches a particle in the rectangle $R_{M,K} = [0, M] \times [-K, K]$ but touches no particle in $\{(x_1, x_2) \in \mathbb{R}^2 : x_1 \le M\} \backslash R_{M,K}$.

Part 2: $\mathbb{P}\{N \ge 3\} > 0 \Longrightarrow \mathbb{P}[F_{M,K}] > 0$ *for some* $M > 0$ *and* $K > 0$. Starting from $\mathbb{P}\{N \ge 3\} > 0$ or from $\mathbb{P}[A_\delta(x^{(1)}, x^{(2)}, x^{(3)})] > 0$, it follows easily that for some $\delta > 0$, some $x^{(1)}$, $x^{(2)}$, $x^{(3)}$, with $x_1^{(j)} < -\delta$ for each j, and some $\bar{x}^{(1)}$, $\bar{x}^{(2)}$, $\bar{x}^{(3)}$ with $\bar{x}_1^{(j)} > \delta$ for each j, we have $\mathbb{P}[A'_\delta] > \delta$, where $A'_\delta = A'_\delta(x^{(1)}, x^{(2)}, x^{(3)}; \bar{x}^{(1)}, \bar{x}^{(2)}, \bar{x}^{(3)})$ is the event that: each disc $B(x^{(j)}, \delta)$ and $B(\bar{x}^{(j)}, \delta)$ contains a unique particle $q^{(j)}$ or $\bar{q}^{(j)}$; there is a unique $\hat{x}$–uni-geodesic $s^{(j)}$ starting from each $q^{(j)}$ whose second particle is $\bar{q}^{(j)}$; every particle in $s^{(j)}$ after $q^{(j)}$ has strictly positive first coordinate; and the three $s^{(j)}$'s are disjoint. By relabelling, we may assume that $W(x^{(1)}) < W(x^{(2)}) < W(x^{(3)})$ so that $s^{(2)}$ is the "middle" geodesic, "trapped" between the other two.

Choose $c_{11} > \max(\bar{x}_1^{(1)}, \bar{x}_1^{(2)}, \bar{x}_1^{(3)}) + \delta$ and consider the annulus of width c_{11} about $R_M^- = [-2M, 0] \times [-M, M]$. Specifically, set

$$\mathscr{A}_M = \mathscr{A}_M^E \cup \mathscr{A}_M^N \cup \mathscr{A}_M^W \cup \mathscr{A}_M^S \ ,$$

where

$$\begin{aligned} \mathscr{A}_M^E &= [0, c_{11}] \times [-M - c_{11}, M + c_{11}], \\ \mathscr{A}_M^N &= [-2M - c_{11}, c_{11}] \times [M, M + c_{11}], \\ \mathscr{A}_M^W &= [-2M - c_{11}, -2M] \times [-M - c_{11}, M + c_{11}], \text{ and} \\ \mathscr{A}_M^S &= [-2M - c_{11}, c_{11}] \times [-M - c_{11}, -M] \ . \end{aligned}$$

Note that, for large M, $\mathscr{A}_M$ contains $\bar{x}^{(1)}$, $\bar{x}^{(2)}$, and $\bar{x}^{(3)}$ and the associated particles $\bar{q}^{(j)}$. Subdivide each of the four parts of $\mathscr{A}_M$ into rectangles of size c_{11} by $2c_{11}^{-1} \ln M$ (for $\mathscr{A}_E$ and $\mathscr{A}_W$) and $2c_{11}^{-1} \ln M$ by c_{11} (for $\mathscr{A}_N$ and $\mathscr{A}_S$). (If these rectangles don't fit evenly, make the final rectangle slightly larger than the rest – but no more than twice as large.) Then each rectangle has probability less than $e^{-2\ln M} = M^{-2}$ of containing no particles. Since there are $O(M/\ln M) = o(M^2)$ rectangles, elementary considerations show that $\mathbb{P}[B_M^\sharp] \to 1$ where $B_M^\sharp$ is the event that every rectangle in $\mathscr{A}_M$'s partition contains at least one particle. We then have, using the fact that $\hat{x} = \hat{e}_1$, for sufficiently large M and then for K sufficiently large (depending on M), that $\mathbb{P}[A'_\delta \cap B_M^\sharp \cap \tilde{C}_{M,K}] > 0$ where $\tilde{C}_{M,K}$ is the event that the polygonal path of each $s^{(j)}$ is disjoint from $\{(x_1, x_2) : 0 \le x_1 \le M; |x_2| \ge K\}$.

Let

$$\Theta_M = R_M^- \backslash \left(B\left(x^{(1)}, \delta\right) \cup B\left(x^{(3)}, \delta\right) \right)$$

and, for each $\omega \in A'_\delta \cap B_M^\sharp \cap \tilde{C}_{M,K}$, consider the modified particle configuration $\omega \backslash \Theta_M$ in which all particles in Θ_M are deleted. By Lemma 8 below, there is a (measurable) event A_M^* such that

$$A_M^* \subset \left\{\omega\backslash\Theta_M : \omega \in A'_\delta \cap B_M^\sharp \cap \tilde{C}_{M,K}\right\} \text{ and } \mathbb{P}\left[A_M^*\right] > 0 \ . \qquad (19)$$

We make several observations about $\omega^* = \omega\backslash\Theta_M \in A_M^*$. Since the removal of a particle does *not* affect the minimizing character of any (finite or infinite) geodesic which touched only the non-removed particles, it follows that the outer geodesics $s^{(1)}$ and $s^{(3)}$ (for ω) remain geodesics for ω^*. The geodesic $\bar{s}^{(2)}$ (which is $s^{(2)}$ without its first particle $q^{(2)}$) is also a geodesic for ω^* for the same reason. The crucial point is the claim for ω^* that for sufficiently large M and then sufficiently large K, no particle q in $\{(x_1,x_2) \in \mathbb{R}^2 : x_1 \leq M\}\backslash R_{M,K}$ has an $\hat{x}$–uni-geodesic s_q which touches the middle geodesic $\bar{s}^{(2)}$. According to this claim, for large M and then large K, $A_M^* \subset F_{M,K}$ and so (19) implies $\mathbb{P}[F_{M,K}] > 0$ as desired.

To justify the above claim, consider an s_q starting from such a q. By the definition of $\tilde{C}_{M,K}$ and the fact that the polygonal path of s_q cannot cross that of $s^{(1)}$ or $s^{(3)}$ to reach $\bar{s}^{(2)}$, it must be that for s_q to reach $\bar{s}^{(2)}$, there is a particle q' in the left half-plane $\{(x_1,x_2) : x_1 < 0\}$ and a particle q'' in the region of the right half-plane lying between the polygonal paths of $s^{(1)}$ and $s^{(3)}$ with $\{q',q''\}$ an edge in s_q. We note that (a.s.) q' cannot be either $q^{(1)}$ or $q^{(3)}$ (the only particles in R_M^-) by Lemma 6 and the definition of A'_δ. Hence $q' \in \{(x_1,x_2) : x_1 < 0\}\backslash R_M^-$. We will complete the justification of the claim by showing that, for large M, such an edge $\{q',q''\}$ cannot be part of any geodesic since there is a Q-path from q' to q'' with passage time strictly less than $|q'-q''|^\alpha$.

Since the segment from q' to q'' cannot cross the polygonal path of $s^{(1)}$ or $s^{(3)}$, it must cross the vertical axis between $W(x^{(1)})$ and $W(x^{(3)})$ and hence between $-c_{12}$ and c_{12} for some c_{12} that does not depend on M, e.g.,

$$c_{12} = \max\left(\left|x_2^{(1)}\right|, \left|\bar{x}_2^{(1)}\right|, \left|x_2^{(3)}\right|, \left|\bar{x}_2^{(3)}\right|\right) + \delta \ .$$

Now construct a new Q-path $(q_1 = q', q_2, \ldots, q_{k-1}, q_k = q'')$ where $q_2, \ldots, q_{k-1}$ are in successive rectangles of $\mathscr{A}_M$'s partition with q_2 in the first rectangle crossed by the segment from q' to q'' and q_{k-1} in the last such rectangle. (We are using here the fact that ω^* is still in $B_M^\sharp$ since in obtaining ω^* from ω, only particles in Θ_M (but not in $\mathscr{A}_M$) were deleted.) It follows that, for some c_{13}, c_{14}, c_{15}, and large M,

$$|q'-q''| \geq |q'-q_2| + |q_2 - q_{k-1}| + |q_{k-1} - q''| - c_{13}\ln M,$$

while

$$|q_2 - q_{k-1}| \geq c_{14}M$$

and

$$\sum_{j=2}^{k-2} \left|q_j - q_{j+1}\right|^\alpha \leq \ c_{15}(\ln M)^\alpha M/\ln M.$$

Thus, for some c_{16} and large M,

$$\begin{aligned}|q'-q''|^\alpha &\geq (|q'-q_2|+|q_{k-1}-q''|+c_{16}M)^\alpha \\ &\geq |q'-q_2|^\alpha+|q_{k-1}-q''|^\alpha+c_{16}^\alpha M^\alpha \\ &> |q'-q_2|^\alpha+|q_{k-1}-q''|^\alpha+c_{15}(\ln M)^{\alpha-1}M \\ &\geq \sum_{j=1}^{k-1}|q_j-q_{j+1}|^\alpha. \end{aligned} \tag{20}$$

The second inequality above is valid for any $\alpha \geq 1$ and we have $\alpha \geq 2$. The strict inequality (20) contradicts the assertion that $\{q', q''\}$ is an edge in s_q.

We complete Part 2 by proving:

Lemma 8. *With the definitions made above, there exists an* $A_M^* \in \mathscr{F}$ *satisfying* (19).

Proof. We let Ω_1 denote all finite subsets of Θ_M, $\mathscr{F}_1$ denote the sigma field generated by events of the form $\{\omega_1 \in \Omega_1 : \omega_1 \cap B = \emptyset\}$ where B ranges over all discs contained in Ω_1, and $\mathbb{P}_1$ denote the probability measure on $(\Omega_1, \mathscr{F}_1)$ such that the outcomes are a Poisson process on Θ_M with unit density with respect to Lebesgue measure on Θ_M. Also, let Ω_2 denote all infinite, locally finite, subsets of Θ_M^c, and define $\mathscr{F}_2$ and $\mathbb{P}_2$ analogously to $\mathscr{F}_1$ and $\mathbb{P}_1$. We identify $(\Omega, \mathscr{F}, \mathbb{P})$ with the product space $(\Omega_1 \times \Omega_2, \mathscr{F}_1 \times \mathscr{F}_2, \mathbb{P}_1 \times \mathbb{P}_2)$ by the measure-preserving bijection $\omega \mapsto (\omega_1, \omega_2) = (\omega \cap \Theta_M, \omega \cap \Theta_M^c)$ and observe that the inverse of this mapping is given by $(\omega_1, \omega_2) \mapsto \omega_1 \cup \omega_2$. Letting A denote the event $A'_\delta \cap B_M^\sharp \cap \tilde{C}_{M,K}$ and using Fubini's Theorem gives that

$$0 < \mathbb{P}[A] = \int_{\Omega_1} \int_{\Omega_2} I_A(\omega_1 \cup \omega_2) d\mathbb{P}_2 d\mathbb{P}_1$$

and hence for some ω_1^*, $I_A(\omega_1^* \cup \omega_2)$ is an $\mathscr{F}_2$-measurable function and

$$0 < \int_{\Omega_2} I_A(\omega_1^* \cup \omega_2) d\mathbb{P}_2 = \mathbb{P}_2[A^*]$$

where $A^* = \{\omega_2 : \omega_1^* \cup \omega_2 \in A\} \in \mathscr{F}_2$. Since $A^* \in \mathscr{F}_2$, we also have that $A^* \subset \Omega$ and $A^* \in \mathscr{F}$ when each $\omega_2 \in A^*$ is viewed as a subset of $\mathbb{R}^2$ (i.e., as an element of Ω). Furthermore, viewing A^* as an element of both $\mathscr{F}$ and $\mathscr{F}_2$, we have

$$\mathbb{P}[A^*] \;=\; \mathbb{P}_1\{\omega_1 = \emptyset\}\mathbb{P}_2[A^*] > \exp(-4M^2)\mathbb{P}_2[A^*] > 0.$$

Finally, when viewed as an element of $\mathscr{F}$, $A^* \subset \{\omega \backslash \Theta_M : \omega \in A\}$. □

Part 3: $\mathbb{P}[F_{M,K}] > 0$ *leads to a contradiction*. Since each tree in S is infinite, it follows that for any positive integer j, $F \equiv F_{M,K}$ is the increasing limit as $l' \to \infty$ (with l' assuming integer values) of $F(j, l')$, the event that some tree in S touches a particle in $R_{M,K}$, no particles in $\{x \leq M\} \backslash R_{M,K}$, and at least j particles in $R_{l'M,l'K}$. Thus $\rho \equiv \mathbb{P}[F] > 0$ implies that for large l', $\mathbb{P}[F(j, l')] \geq \frac{1}{2}\rho > 0$.

Consider a rectangular array of translates $R^u_{M,K}$ of $R_{M,K}$ which intersect only on their boundaries and are indexed by $u \in \mathbb{Z}^2$ (in a natural way). Consider also the corresponding translated events $F^u(j, l')$. If $F^u(j, l')$ and $F^v(j, l')$ (with $u \neq v$) both occur, then the corresponding trees in S are easily seen to be disjoint. For $l = 2l'$, let n_l denote the number of $R^u_{M,K}$'s in $R_{lM,lK}$ such that the corresponding translate of $R_{l'M,l'K}$ is also in $R_{lM,lK}$ and let $N_l = N_l(j)$ denote the (random) number of the corresponding $F^u(j, l')$'s that occur. Note that all the corresponding trees touch at least j particles in $R_{lM,lK}$. Now $n_l \geq \frac{1}{2} l^2$ so, by translation invariance,

$$\mathbb{E}[N_l(j)] = \mathbb{P}[F(j, l')] n_l \geq \frac{1}{2} \rho n_l \geq \frac{1}{4} \rho l^2. \tag{21}$$

On the other hand by the properties of the corresponding $N_l(j)$ trees in S, we have that $\tilde{N}_l$, the total number of all particles in $R_{lM,lK}$, satisfies $\tilde{N}_l \geq j N_l(j)$ so that, for all l,

$$j\mathbb{E}[N_l(j)] \leq \mathbb{E}[\tilde{N}_l] = MKl^2. \tag{22}$$

By choosing j so large that $\frac{1}{4} j\rho > MK$, we obtain a contradiction between (21) and (22), completing the proof of Part 3.

Combining Parts 1, 2, and 3 we see that $\mathbb{P}\{N \geq 2\} > 0$ has been ruled out (for $d = 2$ and $\alpha \geq 2$) which, as noted earlier, is equivalent to Lemma 7. □

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