Probab. Theory Relat. Fields 108, 131–152 (1997)

Probability Theory and Related Fields

Asymptotic behaviour of disconnection and non-intersection exponents

Wendelin Werner

C.N.R.S. Laboratoire de Mathématiques, E.N.S., 45 rue d'Ulm, F-75230 Paris Cedex 05, France, e-mail: wwerner@dmi.ens.fr

Received: 28 February 1996 / In revised form: 3 September 1996

Summary. We study the asymptotic behaviour of disconnection and non-intersection exponents for planar Brownian motion when the number of considered paths tends to infinity. In particular, if η_n (respectively $\xi(n, p)$) denotes the disconnection exponent for n paths (respectively the non-intersection exponent for n paths versus p paths), then we show that $\lim_{n\to\infty} \eta_n/n = \frac{1}{2}$ and that for $a > 0$ and $b > 0$, $\lim_{n\to\infty} \xi([na], [nb])/n = (\sqrt{a} + \sqrt{b})^2/2$.

Mathematics Subject Classification (1991): 60J65

1. Introduction

Several papers have dealt with disconnection and non-intersection exponents for planar Brownian motion in recent years. The disconnection exponent η_n associated to n independent planar Brownian motions $B^1, \ldots, B^n$ (started away from 0) describes the asymptotical decay of the probability

$$P\left(\bigcup_{j=1}^{j=n} B^j[0,t] \text{ does not disconnect 0 from infinity}\right)$$

when $t \to \infty$, which is logarithmically equivalent to $t^{-\eta_n/2}$ (we say that a closed set disconnects 0 from infinity if it contains a closed loop around 0; see Sect. 2 for a rigorous definition of η_n, and e.g. [23, 21] for more details). The exact value of these exponents is not known, but it has been conjectured that they are simple rational numbers and for instance that $\eta_1 = \frac{1}{4}$, $\eta_2 = \frac{2}{3}$ (see for instance Duplantier et al. [10], Puckette–Werner [21]; see also simulations which support these conjectures in [21]). Lawler [14] recently proved that this last conjecture is in fact equivalent to Mandelbrot's conjecture [19] on the Hausdorff dimension of the 'frontier' of a planar Brownian path (Mandelbrot conjectured that this dimension is $\frac{4}{3}$). See [14] and the references therein for more details; see also Burdzy [3], Burdzy–Lawler [5], Burdzy–Werner [7], Lawler [12] and [15] for other related results). It is easy to see that $\eta_n \leqq n/2$ (this corresponds to the fact that if none of the n Brownian motions $B^1, \ldots, B^n$ hits a fixed half-line running from 0 to infinity before time t, then 0 is not disconnected from infinity); see [23, 24] for rigorous bounds for η_n. The first purpose of this paper is to study the asymptotic behaviour of η_n when $n \to \infty$, and to prove the following result that we conjectured in [23].

Proposition 1. *One has*

$$\lim_{n\to\infty} \frac{\eta_n}{n} = \frac{1}{2} . \tag{1}$$

The second result of this paper is the analogous result for non-intersection exponents. Suppose that $k \geqq 2$ is fixed, and define a family of independent planar Brownian motions $(B^{j,l}; j \in \{1, \ldots, k\},\ l \geqq 1)$ started from distinct points. Then, if $a_1, \ldots, a_k$ are positive integers, the exponent $\xi_k(a_1, \ldots, a_k)$ characterizes the asymptotical decay of the probability

$$P\left(\forall (j_1, j_2) \in \{1, \ldots, k\}^2,\ \left(\bigcup_{l=1}^{l=a_{j_1}} B^{j_1,l}[0,t]\right) \cap \left(\bigcup_{l=1}^{l=a_{j_2}} B^{j_2,l}[0,t]\right) = \emptyset \text{ if } j_1 \neq j_2\right)$$

when $t \to \infty$, which is logarithmically equivalent to $t^{-\xi_k(a_1,\ldots,a_k)/2}$ (see e.g. Sect. 3, for a rigorous definition of this exponent). Loosely speaking, this describes the probability that k packs of respectively $a_1, \ldots, a_k$ planar Brownian motions do not intersect each other. Except for the very exceptional case $\xi_2(2,1) = 2$ (see [11]), the exact value of these exponents is not known; it has also been conjectured that they take rational values (see some conjectures in [10, 21]; for simulations see: Duplantier–Kwon [9], Burdzy et al. [6] and Li–Sokal [18]). It has also been conjectured [10, 21] that η_n can be identified with the 'formal' value $\xi_1(n)$ (see Lawler [16] for recent progress on this). For links between these exponents and analogous exponents defined for discrete planar random walks, see Burdzy–Lawler [4], Cranston–Mountford [8], Lawler–Puckette [17] and Lawler [13]).

It is not hard to see (see e.g. in Sect. 3) that

$$\xi_k(a_1, a_2, \ldots, a_k) \leqq \frac{(\sqrt{a_1} + \cdots + \sqrt{a_k})^2}{2}$$

considering the probability that each pack of Brownian motions stays in a well-chosen cone. We are going to show that in fact, if $[u]$ denotes the integer part of the real number u:

Proposition 2. *For all positive real numbers* $\alpha_1, \ldots, \alpha_k$,

$$\lim_{n\to\infty} \frac{\xi_k([\alpha_1 n], \ldots, [\alpha_k n])}{n} = \frac{(\sqrt{\alpha_1} + \cdots + \sqrt{\alpha_k})^2}{2} . \tag{2}$$

It turns out that the proof of (2) is much more technical than that of (1), even if the basic ideas are similar. We make an extensive use of subadditivity properties, combined with Beurling's theorem to obtain (1). For (2), Beurling's theorem has to be replaced by some other estimates that we derive. As in [23, 24], we also use some tools from complex analysis (extremal distance, conformal mappings) that we combine with hitting time properties of Brownian motion.

Notations. In this paper, we will often identify $\mathbb{C}$ with $\mathbb{R}^2$. $\mathscr{D}(z,\alpha)$ (respectively $\mathscr{C}(z,\alpha)$) will denote the closed disc (resp. the circle) centered at z and with radius $\alpha > 0$. We will also write $\mathscr{C}_\alpha$ instead of $\mathscr{C}(0,\alpha)$. The frontier of a set $K \subset \mathbb{C}$ will be denoted ∂K. When $K \subset \mathbb{C}$ is non-empty, we define the sausage $\mathscr{S}_\alpha^K$ of radius α around K as follows:

$$\mathscr{S}_\alpha^K = \bigcup_{z \in K} \mathscr{D}(z,\alpha) .$$

When $a, b \in \mathbb{C}$, then (a,b) will denote the open interval between a and b. When f is a function (or a process), the image of an interval I under f will often be denoted $fI = f(I)$. When $\gamma : I \to \mathbb{C}$ is a continuous path in the plane defined on an open interval I, then we put $\bar{\gamma} = \overline{\gamma I}$.

Suppose Ω is an open subset of $\mathbb{C}$ and that $K \subset \Omega$, $K' \subset \bar{\Omega}$, $K'' \subset \bar{\Omega}$. We say that K disconnects K' from K'' in Ω if any continuous path $\gamma : (0,1) \to \Omega$, such that $\bar{\gamma} \cap K' \neq \emptyset$ and $\bar{\gamma} \cap K'' \neq \emptyset$ does intersect K. When $\Omega = \mathbb{C}$, we will just say that K disconnects K' from K''.

We say that $K \subset \mathbb{C}$ disconnects $K' \subset \mathbb{C}$ from infinity if any unbounded continuous path $\gamma : (0,1) \to \mathbb{C}$ that intersects K' also intersects K.

We say that a set K connects a set K' (respectively a point $z \in \mathbb{C}$) to a set K'', if it contains a continuous path $\gamma : (0,1) \to K$, which intersects both K' (resp. $\{z\}$) and K''.

2. Proof of (1)

We first introduce some notation and recall relevant results. Let $B^1, \dots, B^n$, $B^{n+1}, \dots$ denote a sequence of independent planar Brownian motions started from $x_1, \dots, x_n, \dots$ respectively under the probability measure P_X (here $X = (x_j)_{j \geqq 1}$). We will say that $|X| = 1$ if $|x_j| = 1$ for all $j \geqq 1$. We also define the stopping times

$$T_R^j = \inf\{t > 0, |B_t^j| = R\}$$

for $j \geqq 1$ and $R > 1$. Then we put

$$P_R^n = \sup_{|X|=1} P_X \left(\bigcup_{j=1}^{j=n} B^j[0, T_R^j] \text{ does not disconnect 0 from infinity} \right) .$$

The strong Markov property combined with the scaling property of planar Brownian motion shows readily that for all $R, R' > 1$ and $n \geqq 1$,

$$P_{RR'}^n \leqq P_R^n P_{R'}^n . \tag{3}$$

By subadditivity (e.g. Lawler [11, Lemma 5.2.1]) this implies immediately that $\eta_n = \lim_{R\to\infty} -(\ln P_R^n)/\ln R$ exists and that

$$\eta_n = \sup_{R>1} \frac{-\ln P_R^n}{\ln R} . \tag{4}$$

It is also immediate to see that for all $R > 1$ and $n, p \geqq 1$,

$$P_R^{n+p} \leqq P_R^n P_R^p \tag{5}$$

so that $\eta_{n+p} \geqq \eta_n + \eta_p$. Hence, the same subadditivity argument shows that $\eta = \lim_{n\to\infty} \eta_n/n$ exists and that

$$\eta = \sup_{n>0} \frac{\eta_n}{n} ;$$

combined with (4) this shows that

$$\eta = \sup_{n>0} \sup_{R>1} \frac{-\ln P_R^n}{n \ln R} .$$

It is easy to check (see for instance [23]) that $\eta_n \leqq n/2$ (this corresponds to the fact that if none of n planar Brownian motions intersects a fixed half-line then the union of these n paths does not disconnect 0 from infinity), so that $\eta \leqq \frac{1}{2}$. It remains to check the converse inequality.

For all $R > 1$, we define

$$a(R) = \sup_{n>0} \frac{-\ln P_R^n}{n}$$

so that

$$\eta = \sup_{R>1} \frac{a(R)}{\ln R} . \tag{6}$$

Subadditivity (using (5)) shows that

$$a(R) = \lim_{n\to\infty} \frac{-\ln P_R^n}{n} . \tag{7}$$

Furthermore, (3) implies that $a(RR') \geqq a(R) + a(R')$ for all $R, R' > 1$, and therefore (using the same subadditivity argument once more),

$$\eta = \lim_{R\to\infty} \frac{a(R)}{\ln R} . \tag{8}$$

We define the class F of continuous functions $f : [0,1] \to \mathbb{R}^2$ such that

$$f(0) = 0, \ |f(1)| = 1 \quad \text{and} \quad \forall s \in (0,1), \ |f(s)| \in (0,1) .$$

In other words, F is the class of continuous functions joining 0 to the unit circle. We will often identify $f \in F$ with its trace $f[0,1]$. We will use the following easy lemma:

Lemma 1. *For all fixed* $\alpha > 0$ *and* $R > 1$, *there exists* $N = N(\alpha)$ *functions* $f_1, \dots, f_N$ *in* F *such that for all* $f \in F$,

$$f_i \subset \mathscr{G}_\alpha^f, \quad \text{for at least one } i \in \{1, \dots, N\} .$$

Proof of the Lemma. Consider the lattice $G = (\alpha/4)\mathbb{Z}^2$ and the set H of finite injective sequences $S = (s_0, s_2, \dots, s_m) \in G^{m+1}$ (the integer m depends upon S) such that

- $|s_i - s_{i-1}| = \alpha/4$, for all $i \in \{1, \dots m\}$,
- $s_0 = 0$ and $|s_m| \geqq 1$,
- $|s_i| \in (0,1)$ for all $i \in \{1, \dots, m-1\}$.

H is clearly finite. For each $S = (s_0, \dots, s_m) \in H$, consider the function $f_S \in F$ obtained by interpolating linearly the points $f_S(i/m) = s_i$ (for $i = 0, \dots, m-1$) and $f_S(1) = s_{m-1}/|s_{m-1}|$. It is very easy to see that if $f \in F$, then there exists a sequence $S = (s_0, \dots, s_m) \in H$ such that for all $i \in \{0, \dots, m\}$, $s_i \in \mathscr{G}_{\alpha/2}^f$. Hence, $f_S \subset \mathscr{G}_\alpha^f$, which completes the proof of the lemma.

Conclusion of Proof of (1). We fix $\varepsilon > 0$ and we choose $\alpha < \frac{1}{10}$, such that the probability that a planar Brownian motion started from 0 and killed when it first hits the unit circle, does not disconnect the disc $\{z, |z| \leqq \alpha\}$ from infinity, is smaller than ε. Define then N and $f_1, \dots, f_N$ as in Lemma 1.

Let us define, for all $i \in \{1, \dots, N\}$, $j \in \{1, \dots, n\}$ and $r > 1$, the events

$$E_i^j(r) = \{B^j[0, T_r^j] \cap f_i = \emptyset\} ,$$
$$F_i^j(r) = \{B^j[0, T_r^j] \cap f = \emptyset \text{ for some function } f \in F \text{ such that } f_i \subset \mathscr{G}_\alpha^f\} .$$

Note that for all fixed i and r, the events $(F_i^j(r), j \in \{1, \dots, n\})$ are independent. Lemma 2 shows that for all $R > 1$,

$$\begin{aligned}
P_R^n &= \sup_{|X|=1} P_X\left(\exists f \in F, \left(\bigcup_{j=1}^n B^j[0, T_R^j]\right) \cap f = \emptyset\right) \\
&\leqq \sup_{|X|=1} \sum_{i=1}^N P_X\left(\exists f \in F, \left(\bigcup_{j=1}^n B^j[0, T_R^j] \cap f = \emptyset\right) \text{ and } f_i \subset \mathscr{G}_\alpha^f\right) \\
&\leqq \sup_{|X|=1} \sum_{i=1}^N P_X(\forall j \in \{1, \dots, n\},\ \exists f \in F,\ B^j[0, T_R^j] \cap f = \emptyset \text{ and } f_i \subset \mathscr{G}_\alpha^f) \\
&\leqq \sum_{i=1}^N \sup_{|X|=1} P_X\left(\bigcap_{j=1}^n F_i^j(R)\right) \\
&= \sum_{i=1}^N \sup_{|X|=1} P_X(F_i^1(R))^n . \qquad (9)
\end{aligned}$$

We define the stopping times

$$V(i) = \inf\{t > 0,\ B^1_t \in f_i\}\ ,$$
$$V^*(i) = \inf\{t > V(i),\ |B^1_t - B^1_{V(i)}| = 1\}$$

for $i \in \{1, \ldots, N\}$.

Suppose for a moment that $i \in \{1, \ldots, N\}$ is fixed and that $F^1_i(R+1) \setminus E^1_i(R)$ is true; in other words,

$$V(i) < T^1_R \quad \text{and} \quad B^1[0, T^1_{R+1}] \cap f = \emptyset$$

for some $f \in F$ such that $f_i \subset \mathscr{G}^f_\alpha$. Then, as $\mathscr{D}(B^1_{V(i)}, 1) \subset \mathscr{D}(0, R+1)$, one has $V^*(i) < T^1_{R+1}$ and

$$(f \cap B^1[V(i), V^*(i)]) \subset (f \cap B^1[0, T^1_{R+1}]) = \emptyset\ . \tag{10}$$

The definition of $V(i)$ and the fact that $f_i \subset \mathscr{G}^f_\alpha$ ensure that

$$f \cap \mathscr{D}(B^1_{V(i)}, \alpha) \neq \emptyset$$

Hence, f connects $\mathscr{D}(B^1_{V(i)}, \alpha)$ to $\mathscr{C}(B_{V(i)}, 1)$ (because it connects a point in $\mathscr{D}(B^1_{V(i)}, \alpha)$ to $\mathscr{C}_{R+1}$). Combining this with (10) shows that

$$B^1[V(i), V^*(i)] \text{ does not disconnect } \mathscr{D}(B^1_{V(i)}, \alpha) \text{ from infinity}$$

(otherwise it would intersect f). The strong Markov property and the definition of α implies that the probability of this event is smaller than ε. Hence,

$$\sup_{|X|=1} P_X(F^1_i(R+1) \setminus E^1_i(R)) \leqq \varepsilon\ . \tag{11}$$

Combining (9) and (11) shows that

$$P^n_{R+1} \leqq \sum_{i=1}^{N} \sup_{|X|=1} (P_X(E^1_i(R)) + \varepsilon)^n\ .$$

Beurling's theorem (see e.g. [2, Sect. V-4]) shows that for all i,

$$\sup_{|X|=1} P_X(E^1_i(R)) \leqq P_{x_1=1}(B^1[0, T^1_R] \cap (-\infty, 0] = \emptyset)\ .$$

It is easy to see (see e.g. first line of p. 374 in [23]) that this last quantity is smaller than $4R^{-1/2}/\pi$. Hence,

$$P^n_{R+1} \leqq N \left(\frac{4}{\pi\sqrt{R}} + \varepsilon \right)^n$$

and

$$\limsup_{n\to\infty} (P^n_{R+1})^{1/n} \leqq \frac{4}{\pi\sqrt{R}} + \varepsilon\ .$$

As this is true for all $\varepsilon > 0$, and using (7), we get

$$a(R+1) \geqq \ln\left(\frac{\pi\sqrt{R}}{4} \right)\ .$$

Combined with (8), this shows that $\eta \geqq \frac{1}{2}$ and (1) follows.

3. Proof of (2)

3.1. Preliminaries

We first properly define the exponents $\xi_k(a_1,\dots,a_k)$. We suppose from now on that $k \geqq 2$ is fixed and that

$$X = (x_{j,l})_{j \geqq 1, l \geqq 1}$$

with $x_{j,l} \in \mathbb{C}$ for all j,l. We will say that $|X| = 1$ if $|x_{j,l}| = 1$ for all $j \geqq 1$ and $l \geqq 1$. Define a family $(B^{j,l}, j \geqq 1, l \geqq 1)$ of independent complex Brownian motions started from $B_0 = X$ (i.e. $B_0^{j,l} = x_{j,l}$ for all j,l) under the probability measure P_X. For all $R > 1$, $j \geqq 1$ and $l \geqq 1$, let

$$T_R^{j,l} = \inf\{t > 0,\ |B_t^{j,l}| = R\}\ .$$

Suppose that $a_1,\dots,a_k$ are positive integers. The exponent $\xi_k(a_1,\dots,a_k)$ describes the asymptotic behaviour of the probabilities

$$\begin{aligned} P_R^{a_1,\dots,a_k} = \sup_{|X|=1} P_X(&\forall (j_1,j_2) \in \{1,\dots,k\}^2,\ \forall l_1 \leqq a_{j_1}, \quad \forall l_2 \leqq a_{j_2}, \\ &B^{j_1,l_1}[0,T_R^{j_1,l_1}] \cap B^{j_2,l_2}[0,T_R^{j_2,l_2}] = \emptyset \text{ if } j_1 \neq j_2) \end{aligned}$$

when $R \to \infty$: The strong Markov property and a scaling argument show immediately that for all $R, R' > 1$

$$P_{RR'}^{a_1,\dots,a_k} \leqq P_R^{a_1,\dots,a_k} P_{R'}^{a_1,\dots,a_k}\ , \tag{12}$$

so that the existence of

$$\xi_k(a_1,\dots,a_k) = \lim_{R\to\infty} \frac{-\ln P_R^{a_1,\dots,a_k}}{\ln R}$$

follows from subadditivity. It is also immediate that for all positive integers $a'_1,\dots,a'_k$,

$$P_R^{a_1+a'_1, a_2+a'_2,\dots, a_k+a'_k} \leqq P_R^{a_1,\dots,a_k} P_R^{a'_1,\dots,a'_k}\ .$$

Hence, the same arguments that in Section 2 show that

$$\begin{aligned} \lim_{n\to\infty} \frac{\xi_k(na_1,\dots,na_k)}{n} &= \sup_{R>1} \sup_{n>1} \frac{-\ln P_R^{na_1,\dots,na_k}}{n \ln R} \\ &= \sup_{R>1} \left(\lim_{n\to\infty} \frac{-\ln P_R^{na_1,\dots,na_k}}{n\ln R} \right) \\ &= \lim_{R\to\infty} \left(\lim_{n\to\infty} \frac{-\ln P_R^{na_1,\dots,na_k}}{n \ln R} \right)\ . \end{aligned} \tag{13}$$

3.2. Reduction

We now show that it suffices to prove (2) in the case where $\alpha_1, \dots, \alpha_k$ are positive integers: Assume that (2) is valid for all integer values of $\alpha_1, \dots, \alpha_k$. Suppose that $\alpha_1, \dots, \alpha_k$ are positive real numbers. For all integer $M > 0$, we put (for all $n > 0$) $a(n) = [n/M]$. It is obvious that $\lim_{n\to\infty} n/a(n) = M$ and that

$$[M\alpha_1]a(n) \leqq [n\alpha_1] \leqq [M\alpha_1 + 1](a(n) + 1)\ .$$

Hence,

$$\begin{aligned}
&\frac{1}{2}\left(\sqrt{[M\alpha_1]/M} + \cdots + \sqrt{[M\alpha_k]/M}\right)^2 \\
&\quad = \lim_{a\to\infty} \frac{\xi_k([M\alpha_1]a, \dots, [M\alpha_k]a)}{aM} \\
&\quad \leqq \liminf_{n\to\infty} \frac{\xi_k([n\alpha_1], \dots, [n\alpha_k])}{a(n)M} \\
&\quad \leqq \liminf_{n\to\infty} \frac{\xi_k([n\alpha_1], \dots, [n\alpha_k])}{n} \\
&\quad \leqq \limsup_{n\to\infty} \frac{\xi_k([n\alpha_1], \dots, [n\alpha_k])}{n} \\
&\quad = \limsup_{n\to\infty} \frac{\xi_k([n\alpha_1], \dots, [n\alpha_k])}{Ma(n)} \\
&\quad \leqq \limsup_{a\to\infty} \frac{\xi(([M\alpha_1] + 1)(a + 1), \dots, ([M\alpha_k] + 1)(a + 1))}{aM} \\
&\quad = \frac{1}{2}\left(\sqrt{([M\alpha_1] + 1)/M} + \cdots + \sqrt{([M\alpha_k] + 1)/M}\right)^2\ .
\end{aligned}$$

Letting $M \to \infty$ then shows (2).

3.3. Proof of the upper bound

Define $b_0 = 0$ and for $j \in \{1, \dots k\}$,

$$\begin{aligned}
b_j &= 2\pi \frac{\sum_{u=1}^{j} \sqrt{a_u}}{\sum_{u=1}^{k} \sqrt{a_u}}\ , \\
e_j &= \exp\left(i\left\{\frac{b_j + b_{j-1}}{2}\right\}\right), \\
\Lambda_j &= \{\rho e^{i\varphi};\ \ \rho > 0,\ \ \varphi \in (b_{j-1}, b_j)\}\ .
\end{aligned}$$

Note that the cones Λ_j are disjoint, that $e_j \in \Lambda_u$, that the angle of Λ_j is

$$\varphi_j = \frac{2\pi\sqrt{a_j}}{\sum_{u=1}^{k} \sqrt{a_u}}$$

and that $\bigcup_{j=1}^{k} \bar{\Lambda}_j = \mathbb{C}$. Define $X_0 = (x_{j,l}^0)$ with $x_{j,l}^0 = e_j$, for all $j \geqq 1$ and $l \geqq 1$. Then, estimates for exit probabilities of cones (see, for instance, [23]) show that

$$P_{X_0}(B^{j,1}[0, T_R^{j,1}] \subset \Lambda_j) \geqq \frac{2}{\pi} R^{-\pi/\varphi_j} = \frac{2}{\pi} \exp\left\{ \frac{-\sum_{u=1}^{k} \sqrt{a_u}}{2\sqrt{a_j}} \ln R \right\} .$$

Hence,

$$\begin{aligned} P_R^{a_1,\dots,a_k} &\geqq P_{X_0}(\forall j \in \{1,\dots,k\},\ \forall l \leqq a_j,\ B^{j,l}[0, T^{j,l}] \subset \Lambda_j) \\ &\geqq \prod_{j=1}^{j=k} \prod_{l=1}^{l=a_j} \left(\frac{2}{\pi} \exp\left\{ \frac{-\sum_{u=1}^{k} \sqrt{a_u}}{2\sqrt{a_j}} \ln R \right\} \right) \\ &= \left(\frac{2}{\pi}\right)^{a_1+\cdots+a_k} (\sqrt{R})^{-(\sqrt{a_1}+\cdots+\sqrt{a_k})^2} . \end{aligned}$$

Hence, $\xi_k(a_1,\dots,a_k) \leqq (\sqrt{a_1} + \cdots + \sqrt{a_k})^2/2$ (the method used in [23] shows in fact that this inequality is strict). In particular,

$$\lim_{n\to\infty} \frac{\xi_k(na_1,\dots,na_k)}{n} \leqq \frac{(\sqrt{a_1} + \cdots + \sqrt{a_k})^2}{2} < \infty . \tag{14}$$

3.4. Some applications of conformal invariance

Before deriving the converse inequality, we need to prove some intermediate results: Throughout this subsection, $R > e^{4\pi}$ is fixed. When Ω is an open subset of

$$\mathscr{U} = \{z \in \mathbb{C};\ |z| \in (1, R)\} ,$$

we define

$$\partial\Omega^1 = \partial\Omega \cap \mathscr{C}_1 \quad \text{and} \quad \partial\Omega^R = \partial\Omega \cap \mathscr{C}_R .$$

Let B denote a planar Brownian motion B started from $z \in \mathbb{C}$ under the probability measure P_z, and

$$T = \inf\{t > 0,\ |B_t| = R\} .$$

Throughout this section Ω will denote an open subset of $\mathscr{U}$, which does not disconnect $\mathscr{C}_1$ from $\mathscr{C}_R$. Our aim is to estimate the probability

$$p(\Omega) = \sup_{|z| \leqq 1} P_z(B(0,T) \cap \mathscr{U} \subset \Omega) .$$

Let $(\Omega_i)_{i\in I}$ denote the family of all connected components of Ω (I is at most countable), and define

$$J = \{ j \in I,\ \text{both } \partial\Omega_j^1 \text{ and } \partial\Omega_j^R \text{ contain at least two distinct points}\} .$$

Suppose that $|B_0| \leqq 1$ and put

$$S = \sup\{t < T,\ |B_t| = 1\} .$$

Note that if $B(0,T) \cap \mathscr{U} \subset \Omega$ with $|B_0| \leqq 1$, then

$$B(S,T) \subset \Omega_i \quad \text{for one } i \in I \ . \tag{15}$$

Suppose that $i \notin J$ is fixed; then one of the two following events is almost surely true:

- $B_S \notin \partial\Omega_i^1$ (this happens if $\partial\Omega_i^1$ contains 0 or 1 point).
- $B_T \notin \partial\Omega_i^R$ (this happens if $\partial\Omega_i^R$ contains 0 or 1 point).

Hence, if $|B_0| \leqq 1$ and $B(0,T) \cap \mathscr{U} \subset \Omega$, then (15) is in fact valid for an $i \in J$. In particular,

$$p(\Omega) = 0 \quad \text{if } J = \emptyset \ . \tag{16}$$

Suppose now that $J \neq \emptyset$. For $j \in J$, we define the open set O_j as follows:

$$O_j = O(\Omega_j) = \{z \in \mathscr{U},\ \Omega_j \text{ disconnects } z \text{ from } \mathscr{C}_1 \text{ or from } \mathscr{C}_R \text{ in } \mathscr{U}\} \ .$$

We now start with some elementary remarks:

Fix $j \in J$ for a while. As Ω_j does not disconnect $\mathscr{C}_1$ from $\mathscr{C}_R$, there exists a continuous path γ: $(0,1) \to \mathscr{U}$ such that

$$\bar{\gamma} \cap \mathscr{C}_1 \neq \emptyset, \quad \bar{\gamma} \cap \mathscr{C}_R \neq \emptyset \quad \text{and} \quad \gamma \cap \Omega_j = \emptyset \ .$$

The definition of O_j then readily implies that $\gamma \cap O_j = \emptyset$, and consequently O_j does not disconnect $\mathscr{C}_1$ from $\mathscr{C}_R$ either.

Suppose now that Ω_j does not disconnect the point z from $\mathscr{C}_1$ and from $\mathscr{C}_R$ in $\mathscr{U}$. Then, there exists a continuous path γ: $(0,1) \to \mathscr{U}$ such that

$$\bar{\gamma} \cap \mathscr{C}_1 \neq \emptyset, \quad \bar{\gamma} \cap \mathscr{C}_R \neq \emptyset, \ z \in \gamma \quad \text{and} \quad \gamma \cap \Omega_j = \emptyset \ .$$

Then the definition of O_j implies that $\gamma \cap O_j = \emptyset$, and O_j does not disconnect z from $\mathscr{C}_1$ (and from $\mathscr{C}_R$) in $\mathscr{U}$ either.

Suppose now that O_j is not simply connected. Then for some $z \in \mathscr{U} \backslash O_j$, O_j disconnects z from infinity (and consequently from $\mathscr{C}_R$ or from $\mathscr{C}_1$ in $\mathscr{U}$). The above then implies that Ω_j disconnects z from $\mathscr{C}_R$ or from $\mathscr{C}_1$ in $\mathscr{U}$ too. But the definition of O_j then implies that $z \in O_j$, which contradicts the hypothesis. Hence:

$$O_j \text{ is simply connected for all } j \in J \ .$$

Similarly, it is very easy to check that O_j is an open set, and that both ∂O_j^1 and ∂O_j^R are connected sets, which contain at least two distinct points.

Suppose for a moment that for some fixed $j \neq j'$ in J, $O(\Omega_j) \cap O(\Omega_{j'}) \neq \emptyset$. Then, by definition, Ω_j disconnects a point $z \in O(\Omega_{j'})$ from $\mathscr{C}_1$ or from $\mathscr{C}_R$ in $\mathscr{U}$. But $O(\Omega_{j'})$ is an open connected set in U, and its frontier intersects both $\mathscr{C}_1$ and $\mathscr{C}_R$. Hence,

$$\Omega_j \cap O(\Omega_{j'}) \neq \emptyset \ .$$

The same argument (interchanging j and j') then yields

$$\Omega_j \cap \Omega_{j'} \neq \emptyset \ ,$$

which contradicts the definition of $(\Omega_j)_{j\in I}$. Hence, for all $j \neq j'$ in J,

$$O_j \cap O_{j'} = \emptyset \ .$$

More generally, we also define

$$O(\Omega) = \bigcup_{j \in J} O(\Omega_j) \ .$$

The same arguments than above imply also that if Ω and Ω' are two disjoint open subsets of $\mathscr{U}$, which do not disconnect $\mathscr{C}_1$ from $\mathscr{C}_R$, then

$$O(\Omega) \cap O(\Omega') = \emptyset \ . \tag{17}$$

Define also the measure μ such that

$$d\mu(x,y) = \frac{dx\,dy}{x^2 + y^2} \ .$$

Note that $\mu(\mathscr{U}) = 2\pi \ln R$. We are now ready to state the main result of this subsection.

Lemma 2. *For all open set $\Omega \subset \mathscr{U}$, which does not disconnect $\mathscr{C}_1$ from $\mathscr{C}_R$ and such that $J \neq \emptyset$,*

$$p(\Omega) \leqq K \exp\left(\frac{-\pi(\ln R)^2}{\mu(O(\Omega))} \right) \ ,$$

where the constant $K < \infty$ is independent of Ω and $R > e^{4\pi}$.

Proof of the lemma. For $j \in J$, we define the extremal distance d_j between ∂O_j^1 and ∂O_j^R in O_j (we refer to Ahlfors [1] for definitions and properties of extremal distance). Define also the conformal mapping ϕ_j, which maps O_j onto the rectangle

$$\mathscr{R}_j = (0, 2d_j) \times (-1, 1)$$

and such that $\phi_j(\partial O_j^1) = [-i, i]$ and $\phi_j(\partial O_j^R) = [2d - i, 2d + i]$ (this mapping exists because O_j is simply connected and ∂O_j^1 (and ∂O_j^R) is connected and contains more than two points).

Recall also the following equivalent definition of the extremal distance d_j (see e.g. Ahlfors [1, pp. 51–53]):

$$d_j = \sup_{v \in \mathscr{M}} \frac{(\inf_{\gamma \in \mathscr{L}} \int_\gamma v(z)|dz|)^2}{\int\int_{O_j} v(x,y)^2 dx\,dy} \ , \tag{18}$$

where $\mathscr{M}$ is the set of positive measurable functions in $\bar{O}_j$, and $\mathscr{L}$ the set of all rectifiable arcs in $\bar{O}_j$, which connect ∂O_j^1 and ∂O_j^R. It is easy to see that the 'v-length' of any rectifiable arc $\gamma \in \mathscr{L}$ associated to $v(z) = |z|^{-1}$ is such that

$$\int_\gamma v(z)|dz| = \int_\gamma \frac{|dz|}{|z|} \geqq \int_1^R \frac{d\rho}{\rho} = \ln R \ .$$

Hence (taking $v(z) = |z|^{-1}$ in (18)),

$$d_j \geqq \frac{(\ln R)^2}{\mu(O_j)} \tag{19}$$

and in particular (as $\mu(\mathscr{U}) = 2\pi \ln R$ and $R > e^{4\pi}$),

$$d_j \geqq \frac{(\ln R)^2}{\mu(O(\Omega))} \geqq \frac{(\ln R)^2}{\mu(\mathscr{U})} \geqq \frac{\ln R}{2\pi} \geqq 2 \,. \tag{20}$$

Hence for all $j \in J$, the set

$$L_j = \phi_j^{-1}(1 - i, 1 + i)$$

is not empty. We put $L = \bigcup_{j \in J} L_j$ and we define the sequences of stopping times (U_n) and (V_n) as follows: $V_0 = 0$ and for all $n \geqq 1$,

$$U_n = \inf\{t \geqq V_{n-1},\ B_t \in L\} \,,$$

$$V_n = \inf\{t \geqq U_n,\ B_t \notin O(\Omega)\} \,.$$

We also define the random number

$$N = \sup\{n \geqq 0,\ V_n \leqq T\} \,.$$

It is easy to see that if $B(0, T) \subset \Omega$ then $N < \infty$ almost surely. (15) shows that if $B(0, T) \subset \Omega$, then $N \geqq 1$ and that $V_N = T$. Note also that if $N > j \geqq 1$, then, almost surely $|B_{V_j}| = 1$. Then, using the strong Markov property, one gets

$$\begin{aligned} p(\Omega) &\leqq \sup_{|z|=1} \sum_{n=1}^{\infty} P_z(N = n,\ V_n = T \text{ and } |B_{V_j}| = 1 \text{ for all } j = 1, \ldots, n-1) \\ &\leqq \sum_{n=1}^{\infty} \left(\sup_{z \in L} P_z(|B_{V_1}| = 1) \right)^{n-1} \sup_{z \in L} P_z(V_1 = T) \,. \end{aligned} \tag{21}$$

But for all $j \in J$,

$$\begin{aligned} \sup_{z \in L_j} P_z(|B_{V_1}| = 1) &\leqq \sup_{z \in L_j} P_{\phi_j(z)}(B \text{ exits } \mathscr{R}_j \text{ through } [-i, i]) \\ &\leqq \sup_{z \in (1-i, 1+i)} P_z(\Re(B) \text{ hits } 0 \text{ before } |\Im(B)| \text{ hits } 1) \,, \end{aligned}$$

where the first inequality is a consequence of conformal invariance of planar Brownian motion (here under the mapping ϕ_j). The strong Markov property shows readily that this last supremum is obtained when $z = 1$, and therefore, there exists $\gamma < 1$ (γ is independent from R and Ω), such that

$$\sup_{z \in L} P_z(|B_{V_1}| = 1) < \gamma \,. \tag{22}$$

On the other hand,

$$\begin{aligned}
\sup_{z\in L} P_z(V_1 = T) &\leqq \sup_{j\in J}\sup_{z\in L_j} P_z(B \text{ exits } O_j \text{ through } \partial O_j^R) \\
&\leqq \sup_{j\in J}\sup_{z\in(1-i,1+i)} P_z(B \text{ exits } \mathscr{R}_j \text{ through } [2d-i, 2d+i]) \\
&\leqq \sup_{j\in J}\sup_{z\in(1-i,1+i)} P_z(\Re(B) \text{ hits } 2d_j \text{ before } |\Im(B)| \text{ hits } 1) \\
&= \sup_{z\in(1-i,1+i)} P_z(\Re(B) \text{ hits } 2d \text{ before } |\Im(B)| \text{ hits } 1)\,,
\end{aligned}$$

where $d = \inf_{j\in J} d_j$. Again, the strong Markov property shows that this last supremum is obtained when $z = 1$ (i.e. $\Im(B_0) = 0$). Eventually,

$$\sup_{z\in L} P_z(V_1 = T) \leqq P(\sigma_{2d-1} < \tilde{\sigma}_1)\,, \tag{23}$$

where σ_{2d-1} denotes the hitting time of $2d-1$ by the linear Brownian motion $\Re(B)-1$ started from 0, and where $\tilde{\sigma}_1$ denotes the hitting time of 1 by the reflected linear Brownian motion $|\Im(B)|$ started from 0.

But we know (see e.g. [23], or Revuz–Yor [22, II.(3.7)]) that

$$E\left(\exp\left(\frac{-\lambda^2\sigma_{2d-1}}{2}\right)\right) = \exp(-\lambda(2d-1))\,. \tag{24}$$

On the other hand, an easy consequence of the decomposition of the Dirichlet Laplacian in an interval shows (see e.g. Port–Stone [20, p. 52]), that, for all $x > 0$,

$$P(\tilde{\sigma}_1 > x) \leqq \frac{4}{\pi}\exp\left(\frac{-\pi^2 x}{8}\right). \tag{25}$$

Hence, using the independence between $\Re(B)$ and $\Im(B)$, (23)–(25), we get

$$\begin{aligned}
\sup_{z\in L} P_z(V_1 = T) &\leqq \frac{4}{\pi}E\left(\exp\left(\frac{-\pi^2\sigma_{2d-1}}{8}\right)\right) \\
&\leqq \frac{4}{\pi}\exp\left(\frac{-\pi(2d-1)}{2}\right) \\
&\leqq \frac{4e^{\pi/2}}{\pi}\exp(-\pi d)\,.
\end{aligned} \tag{26}$$

Finally, putting the pieces together using (21), (22) and (26), we get

$$p(\Omega) \leqq \frac{4e^{\pi/2}}{\pi}\sum_{n\geqq 1}\gamma^{n-1}e^{-\pi d} = \frac{4e^{\pi/2}}{\pi(1-\gamma)}e^{-\pi d}\,.$$

(19) shows that

$$d \geqq \frac{(\ln R)^2}{\mu(O(\Omega))}\,.$$

Hence,

$$p(\Omega) \leqq \frac{4e^{\pi/2}}{\pi(1-\gamma)} \exp\left(\frac{-\pi(\ln R)^2}{\mu(O(\Omega))}\right), \tag{27}$$

and the proof of the lemma is complete.

3.5. Spiders and colourings

We start with some informal considerations: The main idea of the proof of the upper bound for (2) is similar to that of the proof of (1). This time, the set F of functions that can be avoided by a family of paths that do not disconnect the origin has to be replaced by 'colourings'. More precisely, if

$$B^{j_1,l_1}[0, T_R^{j_1,l_1}] \cap B^{j_2,l_2}[0, T_R^{j_2,l_2}] = \emptyset$$

for all $j_1 \neq j_2 \in \{1,\dots,k\}$ and $l_1 \leqq a_{j_1}$, $l_2 \leqq a_{j_2}$, then, for some disjoint open sets $\Lambda_1,\dots,\Lambda_k$ in $\mathscr{D}(0,R)$,

$$B^{j,l}[0, T_R^{j,l}] \subset \Lambda_j \ ,$$

for all $j \in \{1,\dots,k\}$ and $l \leqq a_j$. Loosely speaking, we want to construct a deterministic finite family of such partitions of $\mathscr{D}(0,R)$, which do cover almost (i.e. except for a set of sufficiently small probability) all possibilities. However, a major additional difficulty, compared with the proof of (1), is that the 'optimal' deterministic partition is not exactly the partition of $\mathscr{D}(0,R)$ into cones, that we constructed in Sect. 3.3 (in Sect. 2, we used Beurling's Theorem, i.e. the fact that the 'optimal' path in F, was the straight line). But, using Sect. 3.4, we will loosely speaking see that it is not too far of being optimal, and this is sufficient for our purpose.

We start with some new definitions (colourings etc...): In the sequel $k \geqq 2$ is fixed. Suppose $R > 1$ is fixed. A *leg* is a continuous function $f : [0,1] \to \mathbb{C}$ such that

$$|f(0)| = 1, \qquad |f(1)| = R \quad \text{and} \quad \forall s \in (0,1),\ f(s) \in \mathscr{U} \ .$$

We will sometimes identify f with its trace $f[0,1]$. An *l-legged spider* is a family $f = (f_1,\dots,f_l)$ of legs such that for all $1 \leqq i < j \leqq l$,

$$f_i \cap f_j = \emptyset \ ,$$

i.e. any two legs are disjoint. We will also often identify the spider with its trace ($f = \bigcup_{i=1}^{l} f_i[0,1]$)

An l-legged spider divides $\mathscr{U}$ into l parts (the connected components of $\mathscr{U} \backslash f$), which we denote $C_1^f,\dots,C_l^f$ (in cyclic order). Suppose now that f is an l-legged spider and that $k \leqq l$. We say that the function $c : \mathscr{U} \to \{1,\dots,k\}$ is a *spider-k-colouring* of $\mathscr{U}$ associated to f if:

- For all $i \in \{1,\dots,l\}$, c is constant on $C_i^f : c(C_i^f) = \{c_i\}$.
- For all $j \in \{1,\dots,k\}$, there exists $i \in \{1,\dots,l\}$ such that $c_i = j$.

- For all $i \in \{1, \ldots, l-1\}$, $c_i \neq c_{i+1}$ and $c_1 \neq c_l$ (i.e. two adjacent connected components do not have the same 'colour').

We say that the function $c : \mathscr{U} \to \{1, \ldots, k\}$ is a *k-colouring* of $\mathscr{U}$ if there exists a family $(A_1, \ldots, A_k)$ of open (possibly empty) subsets of $\mathscr{U}$, such that

- $\bigcup_{i=1}^{i=k} \bar{A}_i = \bar{\mathscr{U}}$.
- For all $j \in \{1, \ldots, k\}$ such that $A_j \neq \emptyset$, $c(A_j) = \{j\}$.
- For all $j \in \{1, \ldots, k\}$, A_j is equal to the interior of $\bar{A}_j$.

Note that for a k-colouring c, the choice of the sets $A_1, \ldots, A_k$ is unique (A_j is the interior of $c^{-1}(\{j\})$). We will sometimes write $A_1(c), \ldots, A_k(c)$. The *frontier* ∂c *of the k-colouring* c is the set of points in $\mathscr{U}$ at which c is not continuous. In other words

$$\partial c = \mathscr{U} \cap \left(\bigcup_{j=1}^{j=k} \partial A_j \right) .$$

The set of all k-colourings of $\mathscr{U}$ will be denoted Γ_k. Note that any spider-k-colouring is a k-colouring and that its frontier is the trace in $\mathscr{U}$ of a k-legged spider.

We can now state the following lemma.

Lemma 3. *For all* $\alpha > 0$, *there exists a* (*finite*) *family of k-colourings* $c^1, \ldots, c^N$ *of* $\mathscr{U}$ *such that for all k-colouring c of* $\mathscr{U}$, *there exists an* $i \in \{1, \ldots, N\}$ *such that*

$$\partial c^i \subset \mathscr{G}_\alpha^{\partial c}$$

and

$$\forall z \in \mathscr{U} \backslash \mathscr{G}_\alpha^{\partial c}, \quad c^i(z) = c(z) .$$

Proof. We consider the lattice $G = (\alpha/4)\mathbb{Z}^2$ and the 'projection' $\Pi : \mathscr{U} \to G$ defined as follows, for all $z = (x, y) \in \mathscr{U}$:

$$\Pi(x, y) = \left(\frac{\alpha}{4} \left[\frac{4x}{\alpha} \right], \frac{\alpha}{4} \left[\frac{4y}{\alpha} \right] \right) \in G .$$

Note that $|\Pi(z) - z| \leqq \alpha/2$ for all $z \in \mathscr{U}$. The set $\Pi(\mathscr{U})$ is finite, and for all $g \in \Pi(\mathscr{U})$, we choose $g^* \in \mathscr{U}$ such that $\Pi(g^*) = g$. We now define

$$\Xi = \{L = (L(g))_{g \in \Pi(\mathscr{U})},\ L(g) \in \{1, \ldots, k\}\} = \{1, \ldots, k\}^{\Pi(\mathscr{U})} .$$

For all $L \in \Xi$, we define the k-colouring c^L as follows:

$$c^L(z) = L(\Pi(z)), \quad z \in \mathscr{U} .$$

$(c^L, L \in \Xi)$ is a finite family of k-colourings of $\mathscr{U}$, and it is easy to see that if c_0 is a k-colouring of $\mathscr{U}$ and if we define $L(c_0) \in \Xi$ by

$$L(c_0)(g) = c_0(g^*)$$

then

$$\partial c^{L(c_0)} \subset \mathscr{G}_\alpha^{\partial c_0}$$

and

$$\forall z \in \mathscr{U} \backslash \mathscr{G}_\alpha^{\partial c_0}, \quad c^{L(c_0)}(z) = c_0(z) ,$$

and the lemma is proved.

Let us now state the following result:

Lemma 4. *For all* $c \in \Gamma_k$, *and for* K *defined as in Lemma* 2,

$$\prod_{j=1}^{k} p(A_j(c))^{a_j} \leqq K^{a_1+\cdots+a_k} \exp\left(\frac{-(\sqrt{a_1} + \cdots + \sqrt{a_k})^2 \ln R}{2} \right) . \tag{28}$$

Proof. Suppose for a moment that $A_i(c)$ disconnects $\mathscr{C}_1$ from $\mathscr{C}_R$ for some $i \in \{1, \ldots, k\}$. Then, it is easy to see that for $i' \in \{1, \ldots, k\} \backslash \{i\}$, $p(A_{i'}(c)) = 0$, because $A_i(c)$ and $A_{i'}(c)$ are disjoint. As $k \geqq 2$, this shows that

$$\prod_{j=1}^{k} p(A_j(c))^{a_j} = 0 .$$

We now assume that $A_i(c)$ does not disconnect $\mathscr{C}_1$ from $\mathscr{C}_R$ for all $i \in \{1, \ldots, k\}$. We can also assume that $O(A_j(c)) \neq \emptyset$ for all $j \in \{1, \ldots, k\}$, as otherwise (cf. (16)), $p(A_j(c)) = 0$. As $(A_1(c), \ldots, A_k(c))$ are disjoint, Eq. (17) implies that $(O(A_1(c)), \ldots, O(A_n(c)))$ are also disjoint open subsets of $\mathscr{U}$. Hence

$$\sum_{i=1}^{i=k} \mu(O(A_i(c))) \leqq \mu(\mathscr{U}) = 2\,\pi \ln R . \tag{29}$$

Using Lemma 2, we get

$$\prod_{j=1}^{j=k} p(A_j(c))^{a_j} \leqq K^{a_1+\cdots+a_k} \exp\left(-\pi(\ln R)^2 \sum_{j=1}^{j=k} \frac{a_j}{\mu(O(A_j(c)))} \right) .$$

The Cauchy–Schwarz inequality and (29) show that

$$\begin{aligned} \sum_{j=1}^{j=k} \frac{a_j}{\mu(O(A_j(c)))} &\geqq \left(\sum_{j=1}^{j=k} \frac{a_j}{\mu(O(A_j(c)))} \right) \frac{(\sum_{j=1}^{j=k} \mu(O(A_j(c))))}{2\pi \ln R} \\ &\geqq \frac{1}{2\pi \ln R} \left(\sum_{j=1}^{j=k} \sqrt{a_j} \right)^2 \end{aligned}$$

and the lemma follows.

3.6. Choice of α

Suppose that $R > 2$ and define the set

$$\mathscr{U}_2 = \{ z \in \mathbb{C}, \ |z| \in (2, R) \} .$$

Define a planar Brownian motion B started from $z \in \mathbb{C}$ under the probability measure P_z and the hitting times $T_r = \inf\{t > 0,\ |B_t| = r\}$, for $r \geqq 0$. Suppose that $z \in \mathscr{U}_2$ and define the conditional probability

$$P_z^*(\ \cdot\ ) = P_z(\ \cdot\ | T_R < T_1)\ .$$

The skew-product decomposition of B (see below) shows immediately that for all $z \in \mathscr{U}_2$,

$$P_z(T_R < T_1) = \frac{\ln |z|}{\ln R} \geqq \frac{\ln 2}{\ln R}\ .$$

Suppose now that ε is fixed, and chose $\alpha = \alpha(R, \varepsilon)$ in such a way that

$$P_0(B[0, T_{1/4}] \text{ does not disconnect } \mathscr{D}(0, \alpha) \text{ from infinity}) \leqq \frac{\varepsilon \ln 2}{\ln R}\ .$$

Then, for any $z \in \mathscr{U}_2$, if $T_{1/4}^z = \inf\{t > 0,\ |B_t - z| = 1/4\}$,

$$P_z^*(B[0, T_{1/4}^z] \text{ does not disconnect } \mathscr{D}(z, \alpha) \text{ from } \mathscr{D}(z, 1/4)) \leqq \varepsilon\ . \tag{30}$$

Suppose now that $R > 4$ is fixed and let

$$S = \sup\{t \leqq T_R,\ |B_t| = 1\}\ ,$$
$$S' = \inf\{t > S,\ |B_t| = 2\}\ .$$

Recall the skew-product decomposition of B (see e.g. Revuz–Yor [22, Ch. V, Theorem (2.11)]):

$$B_t = \exp(X_{A_t} + iY_{A_t})\ ,$$

where $(X + iY)$ is a complex Brownian motion, and where

$$A_t = \int_0^t |B_s|^{-2} ds\ .$$

We define the times

$$\tau = A_{T_R} = \inf\{t > 0,\ X_t = \ln R\}\ ,$$
$$\sigma = A_S = \sup\{t < \tau,\ X_t = 0\}\ ,$$
$$\sigma' = A_{S'} = \inf\{t > \sigma,\ X_t = \ln 2\}\ .$$

Williams' decomposition of the Brownian path (see Revuz–Yor [22, Ch. VII, Theorem (4.9)]) shows that $(\beta_t) = (X_{\sigma+t}, t \in [0, \tau - \sigma])$ is a three-dimensional Bessel process stopped at its hitting time of $\ln R$, which is independent of $(X_t,\ t \leqq \sigma)$. As $\sigma' - \sigma$ is a stopping time for β, it follows that $(X_t,\ t \in [0, \sigma'])$ and $(X_t,\ t \in [\sigma', \tau])$ are independent. Moreover (using standard properties of the three-dimensional Bessel processes), the law of $(X_t,\ t \in [\sigma', \tau])$ is identical to that of linear Brownian motion started from $\ln 2$ and conditioned on hitting $\ln R$ before 0. In other words (and using the independence between X and Y), if $z = 2$, then $B[0, T]$ under the probability measure P_z^* is identical in law to

$B[S', T]$, conditioned on $\{B_{S'} = z\}$. In particular, if Ω is a fixed open subset of $\mathscr{U}_2$, then:

$$\sup_{|z|=1} P_z(B[S', T] \cap \mathscr{U}_2 \subset \Omega) \leqq \sup_{|z|=2} P_z^*(B[0, T_R] \cap \mathscr{U}_2 \subset \Omega)$$
$$\leqq \frac{\ln R}{\ln 2} \sup_{|z|=2} P_z(B[0, T_R] \cap \mathscr{U}_2 \subset \Omega) \,. \qquad (31)$$

3.7. Conclusion of the proof

We are now ready to proceed to the proof of (2). Suppose for a while that $R > 2e^{4\pi}$ and ε are fixed. Define α as in the previous section, and then N, $c^1, \ldots, c^N$ as in Lemma 3. Suppose that $|X| = 1$ and that $(B_0) = X$ (i.e. all Brownian motions are started on the unit circle $\{z; |z| = 1\}$. Define the times (for all j, l and $R > 1$)

$$S^{j,l} = \sup\{t < T_R^{j,l},\ |B_t^{j,l}| = 1\} \,,$$
$$S_2^{j,l} = \inf\{t > S^{j,l},\ |B_t^{j,l}| = 2\} \,,$$
$$T_R^{j,l*} = \inf\{t > T_R^{j,l},\ |B_t^{j,l} - B_{T_R^{j,l}}^{j,l}| > 1/2\} \,.$$

Suppose now that for all $1 \leqq j_1 < j_2 \leqq k$, $l_1 \leqq na_{j_1}$, $l_2 \leqq na_{j_2}$,

$$B^{j_1,l_1}[0, T_{R+1}^{j_1,l_1}] \cap B^{j_2,l_2}[0, T_{R+1}^{j_2,l_2}] = \emptyset \,.$$

Then, there exists a spider k-colouring c of $\mathscr{U}$ (associated to a spider f) such that, for all $1 \leqq j \leqq k$ and for all $1 \leqq l \leqq na_j$,

$$c(B^{j,l}(S^{j,l}, T_R^{j,l})) = \{j\} \,.$$

Furthermore, it is easy to see that it is possible to choose c in such a way that

$$B^{j,l}(S^{j,l}, T_R^{j,l*}) \cap \partial c = \emptyset \,.$$

Lemma 3 ensures that for at least one $i \in \{1, \ldots, N\}$,

$$c^i = c \quad \text{on } \mathscr{U} \backslash \mathscr{G}_\alpha^f \quad \text{and} \quad \partial c^i \subset \mathscr{G}_\alpha^f \,.$$

Define for $i \in \{1, \ldots, N\}$, the set

$$\Gamma_k^i = \{c \in \Gamma_k,\ c \text{ is a spider } k\text{-colouring, } c^i = c \text{ on } \mathscr{U} \backslash \mathscr{G}_\alpha^{\partial c} \text{ and } \partial c^i \subset \mathscr{G}_\alpha^{\partial c}\}$$

and the events (for all $j \in \{1, \ldots, k\}$ and $l \geqq 1$)

$$F_i^{j,l} = \{\exists c \in \Gamma_k^i,\ c(B^{j,l}(S^{j,l}, T_R^{j,l})) = \{j\} \text{ and } \partial c \cap B^{j,l}(S^{j,l}, T_R^{j,l*}) = \emptyset\} \,.$$

Note that for a fixed i, the sets $(F_i^{j,l},\ l \geqq 1,\ j \in \{1, \ldots, k\})$ are independent. The above implies that

$$P_{R+1}^{na_1, \ldots, na_k} \leqq \sup_{|X|=1} P_X \left(\bigcup_{i=1}^N \left(\bigcap_{j=1}^k \bigcap_{l=1}^{na_j} F_i^{j,l} \right) \right) ,$$

and consequently,

$$P_{R+1}^{na_1,\dots,na_k} \leqq \sup_{|X|=1} \left(\sum_{i=1}^{N} P_X \left(\bigcap_{j=1}^{k} \bigcap_{l=1}^{na_j} F_i^{j,l} \right) \right)$$
$$\leqq \sum_{i=1}^{N} \left(\prod_{j=1}^{k} \sup_{|X|=1} P_X(F_i^{j,1})^{na_j} \right) . \tag{32}$$

Define the times:

$$W_i^{j,1} = \inf\{t > S^{j,1},\ B_t^{j,1} \in \mathscr{U}_2 \text{ and } c^i(B_t^{j,1}) \neq j\} ,$$
$$W_i^{j,1*} = \inf\{t > W_i^{j,1};\ |B_t^{j,1} - B_{W_i^{j,1}}^{j,1}| = 1/4\} .$$

Suppose now for a moment that $F_i^{j,l}$ is true and that

$$\{W_i^{j,1} < T_R\} .$$

Define the spider k-colouring c as in the definition of $F_i^{j,l}$. Then for a sequence $h_n \to 0+$,

$$c^i(B_{W_i^{j,1}+h_n}^{j,1}) \neq j = c(B_{W_i^{j,1}+h_n}^{j,1}) .$$

As $c^i = c$ on $\mathscr{U} \backslash \mathscr{G}_\alpha^{\partial c}$, this implies that

$$B_{W_i^{j,1}}^{j,1} \in \mathscr{G}_\alpha^{\partial c} ,$$

and hence, there exists $z_0 \in \partial c$, such that

$$|z_0 - B_{W_i^{j,1}}^{j,1}| \leqq \alpha .$$

The definitions of $W_i^{j,1*}$ and $T_R^{j,1*}$, and the fact that $W_i^{j,1} < T_R$ imply that

$$W_i^{j,1*} < T_R^{j,1*} .$$

Combining this with our choice of c shows that

$$(B^{j,1}[W_i^{j,1}, W_i^{j,1*}] \cap \partial c) \subset (B^{j,1}(S^{j,1}, T_R^{j,1*}) \cap \partial c) = \emptyset .$$

But, as c is a spider k-colouring of $\mathscr{U}$ and $z_0 \in \partial c$, ∂c connects z_0 to the circles $\mathscr{C}_1$ and $\mathscr{C}_R$. As $R > 4$, this implies in particular that ∂c connects z_0 to the circle $\mathscr{C}(z_0, 1/2)$. Hence,

$$B^{j,1}[W_i^{j,1}, W_i^{j,1*}] \text{ does not disconnect } \mathscr{D}(B_{W_i^{j,1}}^{j,1}, \alpha) \text{ from } \mathscr{C}(B_{W_i^{j,1}}^{j,1}, \tfrac{1}{4}) .$$

The choice of α ensures that the probability of this event is smaller than ε (it is easy to check that $W_i^{j,1} - S^{j,1}$ is a stopping time for the Markov process $(B_{S^{j,1}+t}^{j,1},\ t \geqq 0)$). Finally, we get

$$\sup_{|X|=1} P_X(F_i^{j,1}) \leqq \varepsilon + \sup_{|X|=1} P_X(T_R^{j,1} < W_i^{j,1})$$

and (using (32)),

$$P_{R+1}^{na_1,\dots,na_k} \leqq \sum_{i=1}^{N} \prod_{j=1}^{k} \left(\varepsilon + \sup_{|X|=1} P_X(T_R^{j,1} < W_i^{j,1}) \right)^{na_j} .$$

Note that if $T_R^{j,1} < W_i^{j,1}$ then

$$c^i(B^{j,1}(S_2^{j,1}, T_R^{j,1}) \cap \mathscr{U}_2) = \{j\} .$$

Consequently,

$$\limsup_{n\to\infty} (P_{R+1}^{na_1,\dots,na_k})^{1/n} \leqq \sup_{c\in\Gamma_k} \prod_{j=1}^{k} \left(\sup_{|X|=1} P_X(c(B^{j,1}(S_2^{j,1}, T_R^{j,1})) = \{j\}) + \varepsilon \right)^{a_j} .$$

As this is true for all $\varepsilon > 0$,

$$\limsup_{n\to\infty} (P_{R+1}^{na_1,\dots,na_k})^{1/n} \leqq \sup_{c\in\Gamma_k} \prod_{j=1}^{k} \sup_{|X|=1} P_X(B^{j,1}(S_2^{j,1}, T_R^{j,1}) \cap \mathscr{U}_2 \subset A_j(c))^{a_j} .$$

(31) implies that

$$\limsup_{n\to\infty} (P_{R+1}^{na_1,\dots,na_k})^{1/n}$$
$$\leqq \left(\frac{\ln R}{\ln 2} \right)^{a_1+\cdots+a_k} \sup_{c\in\Gamma_k} \prod_{j=1}^{k} \sup_{|X|=2} P_X(B^{j,1}(0, T_R^{j,1}) \cap \mathscr{U}_2 \subset A_j(c))^{a_j}.$$

Note that the restriction of c to $\mathscr{U}_2$ is a k-colouring of $\mathscr{U}_2$ (with obvious definitions). Using the scaling property and Lemma 4 shows eventually that

$$\limsup_{n\to\infty} (P_{R+1}^{na_1,\dots,na_k})^{1/n}$$
$$\leqq \left(\frac{K \ln R}{\ln 2} \right)^{a_1+\cdots+a_k} \exp\left(-\frac{(\sqrt{a_1} + \cdots + \sqrt{a_k})^2}{2} \ln(R/2) \right) .$$

Combining this with (13) and (14) completes the proof of (2).

4. Remarks

Remark 1. Proposition 2 shows in particular that

$$\lim_{n\to\infty} \frac{\xi_2(n, 2n)}{n} = \frac{3}{2} + \sqrt{2} ,$$

which is an irrational number. This implies that $\xi_2(n, p)$ (as a function of n, p) cannot be the ratio of two polynoms (in n and p) with integer coefficients (in other words, $\xi_2(n, p) \notin \mathbb{Z}(n, p)$).

Remark 2. Note that in particular, for $a_1 = \cdots = a_k = 1$,

$$\lim_{n\to\infty} \frac{\xi_k(n,\ldots,n)}{n} = \frac{k^2}{2} ,$$

which is consistent with the following conjecture made in [21]:

$$\xi_k(n,\ldots,n) = \frac{n(((n+1)k)^2 - 1)}{2(n+1)(n+2)} .$$

Remark 3. Note also that if $0 < \alpha < \alpha' \leqq 1/2$ then,

$$\begin{aligned} \lim_{n\to\infty} \frac{\xi_2([\alpha n],[(1-\alpha)n])}{n} &= \frac{1+2\sqrt{\alpha(1-\alpha)}}{2} \\ &\leqq \frac{1+2\sqrt{\alpha'(1-\alpha')}}{2} \\ &= \lim_{n\to\infty} \frac{\xi_2([\alpha' n],[(1-\alpha')n])}{n} , \end{aligned}$$

which does not contradict the conjecture (6) in [23] (i.e. $\xi_2(N-n,n) < \xi_2(N-n',n')$ as soon as $1 \leqq n < n' \leqq N/2$).

Remark 4. We believe that the following result is true: $(\eta_n,\ n \geqq 1)$ is a *convex* function of n (i.e. $\eta_{n+1} - \eta_n$ is increasing), in which case (1) would imply that

$$\lim_{n\to\infty} (\eta_{n+1} - \eta_n) = \tfrac{1}{2} . \tag{33}$$

Similarly, it is likely that for fixed positive integers $a_1,\ldots,a_k$, the difference

$$\xi_k((n+1)a_1,\ldots,(n+1)a_k) - \xi_k(na_1,\ldots,na_k)$$

is an increasing function of n, and that consequently

$$\lim_{n\to\infty} (\xi_k((n+1)a_1,\ldots,(n+1)a_k) - \xi_k(na_1,\ldots,na_k)) = \frac{(\sqrt{a_1} + \cdots + \sqrt{a_k})^2}{2} . \tag{34}$$

Note that on the other hand, for fixed positive integers $n_1,\ldots,n_u$ $(u \geqq 1)$, the function

$$\xi_{u+1}(n_1,\ldots,n_u,an)$$

is a *concave* function of n (this is a straightforward application of the Cauchy–Schwarz inequality). In fact, Greg Lawler [16] proved that this function is strictly concave.

On the other hand, one also has, for $u \geqq 1$, $v \geqq 1$, (this can be derived, using (2) for the lower bound, and a similar argument than in Sect. 3.3 for the upper bound) that

$$\lim_{n\to\infty} \frac{\xi_{u+v}(n_1,\ldots,n_u,a_1 n,\ldots,a_v n)}{n} = \frac{(\sqrt{a_1} + \cdots + \sqrt{a_v}\,)^2}{2} . \tag{35}$$

Acknowledgements. I thank Emily Puckette, Gérard Ben Arous, Philippe Biane, Chris Burdzy and Greg Lawler for stimulating (sometimes electronic) discussions on these topics.

References

1. Ahlfors, L.V.: Conformal invariants, Topics in geometric function theory. New York: Mc-Graw-Hill, 1973
2. Bass, R.F.: Probabilistic techniques in analysis. New York: Springer, 1995
3. Burdzy, K.: Labyrinth dimension of Brownian trace, Probability and Mathematical Statistics **15**, 165–193 (1995)
4. Burdzy, K., Lawler, G.F.: Non-intersection exponents for Brownian paths, Part I. Existence and an invariance principle, Probab. Theory Rel. Fields **84**, 393–410 (1990)
5. Burdzy, K., Lawler, G.F.: Non-intersection exponents for random walks and Brownian motion. Part II: Estimates and applications to a random fractal. Ann. Probab. **18**, 981–1009 (1990)
6. Burdzy, K., Lawler, G., Polaski, T.: On the critical exponent for random walk intersections. J. Statist. Phys. **56**, 1–12 (1989)
7. Burdzy, K., Werner, W.: No triple point of planar Brownian motion is accessible. Ann. Probab. **24**, 125–147 (1996)
8. Cranston, M., Mountford, T.: An extension of a result by Burdzy and Lawler. Probab. Theory Rel. Fields **89**, 487–502 (1991)
9. Duplantier, B., Kwon, K.-H.: Conformal invariance and intersections of random walks. Phys. Rev. Lett. **61**, 2514–2517 (1988)
10. Duplantier, B., Lawler, G.F., Le Gall, J.-F., Lyons, T.J.: The geometry of the Brownian curve, in: Probabilités et Analyse stochastique. Tables rondes de St-Chéron Janvier 1992, Bull. Sc. Math. **117**, 91–106 (1993)
11. Lawler, G.F.: Intersections of random walks. Boston: Birkhäuser, 1991
12. Lawler, G.F.: Nonintersecting planar Brownian motions. Math. Physics Electronic J. **1**, paper no. 4, 1995
13. Lawler, G.F.: Cut times for simple random walks. Electronic J. Probab. **1** (13) (1996)
14. Lawler, G.F.: The dimension of the frontier of planar Brownian motion. Electron. Comm. Probab. **1** (5) (1996)
15. Lawler, G.F.: Hausdorff dimension of cut points for Brownian motion. Electronic J. Probab. **1** (2) (1996)
16. Lawler, G.F. Strict concavity for the intersection exponent for Brownian motion in two and three dimensions, preprint.
17. Lawler, G.F., Puckette, E.E.: The disconnection exponent for simple random walk. Israel J. Math. (to appear) (1996)
18. Li, B., Sokal, A.D.: High-precision Monte-Carlo test of the conformal-invariance predictions for two-dimensional mutually avoiding walks. J. Statist. Phys. **61**, 723–748 (1990)
19. Mandelbrot, B.B.: The fractal geomerty of nature. New York: Freeman (1982)
20. Port, S.C., Stone, C.J.: Brownian motion and classical potential theory. New York: Academic Press, 1987
21. Puckette, E.E., Werner, W.: Monte-Carlo tests and conjectures for disconnection exponents, Electron. Comm. Probab. **1** (6) (1996)
22. Revuz, D., Yor, M.: Continuous martingales and Brownian motion. Berlin: Springer, 1991
23. Werner, W.: On Brownian disconnection exponents. Bernoulli **1**, 371–380 (1995)
24. Werner, W.: Bounds for disconnection exponents. Electron. Comm. Probab. **1** (4) (1996)