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Probability Theory and Related Fields

On the smallest maximal increment of partial sums of i.i.d. random variables

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Summary. We study the almost sure limiting behavior of the smallest maximal increment of partial sums of n independent identically distributed random variables for a variety of increment sizes k_n, where k_n is a sequence of integers satisfying $1 \le k_n \le n$, and going to infinity at various rates. Our aim is to obtain universal results on such behavior under little or no assumptions on the underlying distribution function.

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1. Introduction

Let $X, X_1, X_2, \ldots$ be a sequence of independent and identically distributed (i.i.d.) random variables with common nondegenerate distribution function F. For each integer $j \ge 1$, let $S_j = X_1 + \ldots + X_j$ and set $S_0 = 0$. We are interested in the limiting behavior of the *minimum* of the maximal increments of partial sums, where we consider different increment sizes. To be more specific, we shall study the asymptotic behavior of

$$m_n(k_n) = \min_{0 \le i \le n-k_n} \max_{0 \le j \le k_n} |S_{i+j} - S_i - j\, c_n| \big/ a_n, \tag{1.1}$$

for suitable sequences of positive integers $1 \le k_n \le n$, where a_n and c_n are sequences of norming and centering constants depending on the underlying distribution function F and the sequence k_n. Our point of departure is the following result of M. Csörgő and Révész (1981). (For some clarification see Shao (1992) or page 112 of the monograph of Lin and Lu (1992.)

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Let κ_n be a sequence of constants satisfying

(K.1) $1 \le \kappa_n \le n, n \ge 1,$
(K.2) $\kappa_n \nearrow \infty,$ as $n \to \infty,$
(K.3) n/κ_n is nondecreasing,
(K.4) $\log n/\kappa_n \to 0,$ as $n \to \infty,$
(K.5) $b_n := L\Big(nLn\ /\kappa_n\Big)\ /\kappa_n$ is nonincreasing,

where we set $Lx = \log(x \vee e),\ x \ge 0.$

As usual, denote by [x] the integer part of $-\infty < x < \infty$.

Theorem A (Csörgő and Révész (1981)) *Let X be a random variable with* $EX = 0$ *and* $Var(X) = 1$, *and let* $k_n = [\kappa_n]$, *where* κ_n *satisfies (K.1)-(K.5). Set* $a_n = (8\, b_n)^{1/2}\ /\pi$ *and* $c_n = 0$ *in (1.1).*
Then:

(1.2) $$\liminf_{n\to\infty}\ m_n(k_n) = 1 \quad \text{a.s}$$

Moreover, if $L(n/k_n)/LLn \to \infty$, then

(1.3) $$\lim_{n\to\infty}\ m_n(k_n) = 1 \quad \text{a.s}$$

One of the purposes of our paper is to obtain a version of Therore A without any condition at all on the underlying distribution. Among other results we shall obtain the following theorem, which is a consequence of more detailed results stated in the next section.

Theorem B *Let X be a nondegenerate random variable in the Feller class and let* $k_n = [\kappa_n]$, *where* κ_n *satisfies (K.1)-(K.5). There exist sequences of norming and centering constants* a_n *and* c_n *(depending on* F *and* k_n*) such that*

(1.4) $$\liminf_{n\to\infty}\ m_n(k_n) = K_1 \quad \text{a.s}$$

where $K_1 < \infty$ *is a positive constant.*

Moreover, if $L(n/k_n)/LLn \to \infty$, *then*

(1.5) $$\limsup_{n\to\infty}\ m_n(k_n) = K_2 \quad \text{a.s}$$

where $K_2 < \infty$ *is another positive constant.*

The exact form of the norming and centering constants will be given in Sect. 2. Recall that a random variable X is in the *Feller class* if one can find centering constants δ_n and norming constants γ_n so that

$(S_n - \delta_n)/\gamma_n$ is tight with non-degenerate subsequential limits.

The $\liminf$ behavior of $m_n(k_n)$ if k_n is of order $O(\log n)$ is different from the preceding situation. Here it has been shown that one can choose $a_n \equiv 1$ and

$c_n \equiv c = EX$, provided that X has a normal distribution.(Refer to Csörgő and Révész (1981)). Our next result shows, maybe somewhat surprisingly, that such a result holds for random variables with arbitrary distributions (which possibly have no expected value).

Theorem C *Let X be any random variable and assume that $k_n = O(\log n)$ as $n \to \infty$. Then there exists a constant c depending on F and k_n such that*

$$\limsup_{n\to\infty} \min_{0\le i\le n-k_n} \max_{0\le j\le k_n} | S_{i+j} - S_i - j\ c \ | < \infty \text{ a.s.} \tag{1.6}$$

In Sect. 2 we will provide a detailed description of what can happen in this case. It will turn out that in some cases the lim sup in (1.6) can be positive, whereas it can also be equal to zero in other cases. We will also look into the limiting behavior of $m_n(k_n)$ for increment sizes k_n of order $o(\log n)$.

2. Statement of main results

We first introduce some additional notation. Let the quantile function Q be defined as

$$Q(t) = \inf\{x : F(x) \ge t\}, \quad 0 < t < 1. \tag{2.1}$$

For any $0 < s \le 1 - t < 1$ set

$$\bar{\mu}\,(s, 1-t) \ = \ \int_s^{1-t} Q(u)\ du/(1-s-t) \tag{2.2}$$

$$\bar{\tau}^2\,(s, 1-t) \ = \ \int_s^{1-t} Q^2(u)\ du/(1-s-t) - \bar{\mu}^2(s, 1-t), \tag{2.3}$$

and

$$\begin{aligned}\sigma^2\,(s, 1-t) \ &= \ s\ Q^2(s) + t\ Q^2(1-t) + \textstyle\int_s^{1-t} Q^2(u) du \\ &\quad -\{s\ Q(s) + t\ Q(1-t) + \textstyle\int_s^{1-t} Q(u)\ du\}^2\end{aligned} \tag{2.4}$$

The following alternate expression for $\sigma^2(s, 1-t)$ will also come in handy,

$$\sigma^2\,(s, 1-t) = \int_s^{1-t}\int_s^{1-t} (u \wedge v - uv)\ dQ(u)\ dQ(v). \tag{2.5}$$

For any sequence of positive constants satisfying (K.1)-(K.5) and $\gamma_1, \gamma_2 > 0$ such that $(\gamma_1 + \gamma_2)\ b_n < 1$, set

$$\bar{\mu}_n(\gamma_1, \gamma_2) = \bar{\mu}(\gamma_1\ b_n, 1 - \gamma_2\ b_n) \tag{2.6}$$

and

$$\sigma_n^2(\gamma_1, \gamma_2) = \sigma^2(\gamma_1\ b_n, 1 - \gamma_2\ b_n). \tag{2.7}$$

Using (2.5) and recalling (K.5), it is easy to see that $\sigma_n^2(\gamma_1, \gamma_2)$ is a non-decreasing sequence.

Our first result provides the upper bounds for Theorem B. Note that we do not assume that X is in the Feller class. This assumption is only required for showing that the lim inf in (2.8) and consequently the lim sup in (2.9) are positive.(See Theorem 3 below.) There are cases where these quantities are equal to zero and it would be interesting to know whether and when one can find suitable centering and norming constants leading to positive values for random variables outside the Feller class.

Theorem 1. *Let X be a nondegenerate random variable and let $k_n = [\kappa_n]$, where κ_n satisfies (K.1)-(K.5). For all $\gamma_1, \gamma_2 > 0$ with $\gamma_1 + \gamma_2 < 1$ there exists a constant $0 < K_3 < \infty$ depending only on γ_1, γ_2 such that*

(2.8)
$$\liminf_{n\to\infty} \min_{0 \le i \le n-k_n} b_n^{1/2} \max_{0 \le j \le k_n} \mid S_{i+j} - S_i - j\ \bar{\mu}_n(\gamma_1, \gamma_2) \mid / \sigma_n(\gamma_1, \gamma_2) \le K_3 \text{ a.s.}$$

Moreover, if $L(n/\kappa_n)/LLn \nearrow \infty$ then there exists a constant $0 < K_4 < \infty$ depending only on γ_1, γ_2 such that

(2.9)
$$\limsup_{n\to\infty} \min_{0 \le i \le n-k_n} b_n^{1/2} \max_{0 \le j \le k_n} \mid S_{i+j} - S_i - j\ \bar{\mu}_n(\gamma_1, \gamma_2) \mid / \sigma_n(\gamma_1, \gamma_2) \le K_4 \text{ a.s.}$$

Setting $\kappa_n = n$ in Theorem 1 yields, with a slightly different centering, the upper bound portion of the universal Chung-type law of the iterated logarithm of Einmahl and Mason (1994). Our next result gives Theorem C.

Theorem 2. *Let X be any random variable and assume that $k_n = O(\log n)$. For all $\alpha_1, \alpha_2 > 0$ sufficiently small, so that for all large n, $-c_n \log(1-\alpha) < 1$ with $\alpha = \alpha_1 + \alpha_2$ and $c_n = k_n / \log n$, we have*

$$\limsup_{n\to\infty} \min_{0 \le i \le n-k_n} \max_{0 \le j \le k_n} \mid S_{i+j} - S_i - j\ \bar{\mu}(\alpha_1, 1-\alpha_2) \mid < \infty \text{ a.s.} \tag{2.10}$$

Our next theorem shows, as already indicated, that the lim inf in (2.8) and consequently the lim sup in (2.9) are positive when X is in the Feller class. The case when $k_n = O(\log n)$, however is not so clear cut. (Refer to Sect. 4 for a discussion of the behavior of the limiting constants in this case.)

Theorem 3. *Let X be a nondegenerate random variable in the Feller class and let $k_n = [\kappa_n]$, where κ_n satisfies (K.1)-(K.5). Given $\gamma_1, \gamma_2 > 0$, with $\gamma_1 + \gamma_2 < 1$, there exists a positive constant K_5 depending on F such that*

(2.11)
$$\liminf_{n\to\infty} \min_{0 \le i \le n-k_n} \max_{0 \le j \le k_n} b_n^{1/2} \mid S_{i+j} - S_i - j\ \bar{\mu}_n(\gamma_1, \gamma_2) \mid / \sigma_n(\gamma_1, \gamma_2) \ge K_5 \text{ a.s.}$$

Using the Hewitt-Savage 0-1 law, we readily obtain Theorem B from Theorems 1 and 3.

It is also possible to prove versions of the above results with different centering constants such as $j\mu,\ j \geq 1$ where μ might be the expectation of X provided of course it exists. For more information regarding such questions refer to Sect. 4.

Our final result in this section discloses a somewhat unexpected behavior of the increments of partial sums when $k_n = o\ (\log n)$. This result has been proved for the normal case and the centering $jEX, 1 \leq j \leq n$ by Csörgő and Révész (1981). Note that in the subsequent theorem we have various choices for the centering.

Theorem 4. *Let k_n be any sequence of integers satisfying $1 \leq k_n \leq n$ and $k_n = o\ (\log n)$. For any $0 < t < 1$, there exists a sequence of constants $\epsilon_n \searrow 0$ such that*

$$\lim_{n\to\infty} \min_{0\leq i\leq n-k_n} \max_{0\leq j\leq k_n} \mid S_{i+j} - S_i - j(Q(t) - \epsilon_n) \mid = 0 \text{ a.s.} \tag{2.13}$$

The proofs of Theorem 1-4 will be given in Sect. 3, whereas, as already indicated, Sect. 4 will provide information about possible centerings in (2.12), and the case $k_n = o\ (\log n)$.

We finally mention that results dealing with the *maximum* of the maximal increments can be found in Csörgő and Révész (1981), Lin and Lu (1992), Einmahl and Mason (1996) among other references.

3. Proofs

Without loss of generality we assume throughout that $X_j = Q(U_j), j \geq 1$, where $\{U_j\}$ is a sequence of i.i.d. uniform (0,1) random variables.

Proof of Theorem 1. Our proof will closely parallel the proof of Theorem 1 of Einmahl and Mason (1994). We shall require the following basic facts.

Lemma 1. *For any nondegenerate distribution function F with quantile function Q,*

(a) $s\ Q^2(s) + t\ Q^2(1-t) \leq \sigma^2(s, 1-t)$ *for all* $s, t > 0$ *sufficiently small;*
(b) *Whenever* $E\ X^2 = \infty, \bar{\mu}\ (s, 1-t)/\sigma(s, 1-t) \to 0$ *as* $s, t\ \downarrow 0$.

(See the proof of Lemma 2.1, S. Csörgő, Haeusler and Mason (1988a).)

Set for $0 \leq i \leq n - k_n, n \geq 1$,

$$M_n(i) := \max_{0\leq j\leq k_n} \mid S_{i+j} - S_i - j\ \bar{\mu}_n(\gamma_1, \gamma_2) \mid, \tag{3.1}$$

and $M_n := M_n(0)$. Further let

(3.2) $$\phi_n := n\ (\log n)\ /\kappa_n, n \geq 1, \text{ and}$$

(3.3) $$\psi_n := b_n^{-1/2}\ \sigma_n(\gamma_1, \gamma_2).$$

The next lemma is essential for the proof of Theorem 1.

Lemma 2. *Given $\gamma_1, \gamma_2 > 0$ with $\gamma_1 + \gamma_2 < 1$, and $0 < \eta < 1$ there exists a constant $C > 0$ depending on γ_1, γ_2 and η only such that for all large n,*

(3.4) $$P(M_n \leq C\psi_n) \geq (\phi_n)^{-\gamma_1-\gamma_2-\eta}.$$

Proof. This proof is a straightforward modification of parts (ii) and (iii) of the proof of Lemma 2 of Einmahl and Mason (1994), and therefore will be omitted. □

We are now ready to finish the proof of Theorem 1. First assume that $k_n/n \to 1$. In this case

$$\min_{0\leq i\leq n-k_n} b_n^{1/2} \max_{0\leq j\leq k_n} \mid S_{i+j} - S_i - j\ \bar{\mu}_n(\gamma_1,\gamma_2) \mid /\sigma_n(\gamma_1,\gamma_2)$$
$$\leq \max_{0\leq j\leq n} b_n^{1/2}\ \mid S_j - j\ \bar{\mu}_n(\gamma_1,\gamma_2) \mid /\sigma_n(\gamma_1,\gamma_2)\ .$$

Noting that Theorem 1 of Einmahl and Mason (1994) is also valid with the centering $\bar{\mu}_n(\gamma_1, \gamma_2)$, we infer that

$$\liminf_{n\to\infty} \max_{0\leq j\leq n} b_n^{1/2}\ \mid S_j - j\ \bar{\mu}_n(\gamma_1,\gamma_2)\ \mid /\sigma_n(\gamma_1,\gamma_2) < \infty \text{ a.s.},$$

which implies (2.8).

Now assume $k_n/n \to \rho < 1$. Set $T_1 = 1$ and define for $n \geq 2$

$$T_{n+1} = \min\{m:\ m - k_m > T_n\}.$$

Applying Lemma 2 with $\eta = (1-\gamma_1-\gamma_2)/2$,we see that for all m large enough

$$\sum_{n=m}^{\infty} P(b_{T_{n+1}}^{1/2}\ M_{T_{n+1}}\ (T_n)/\sigma_{T_{n+1}}(\gamma_1,\gamma_2) \leq C)$$
$$\geq \sum_{n=m}^{\infty} \left(\phi_{T_{n+1}}\right)^{-(1+\gamma_1+\gamma_2)/2},$$

which since $\rho < 1$ is equal to infinity. (This can be seen by using the fact that $T_{n+1} - T_n \sim k_n$ and a standard integral test.) Thus since the $M_{T_{n+1}}(T_n), n \geq 1$, are independent, we have (2.8) by the Borel-Cantelli lemma. This completes the proof of the first part of Theorem 1.

Now assume that

$$L(n/\kappa_n)/LLn \to \infty. \tag{3.5}$$

To finish the proof of Theorem 1 we require two additional lemmas.

Let $k(x), x \geq 1$, be the piecewise linear function satisfying $k(n) = \kappa_n, n \geq 1$, obtained by linear interpolation from the sequence $\kappa_n, n \geq 1$. Recalling assumptions (K.1)-(K.5) it is easy to see that

$$k(x) \text{ and } x/k(x), x \geq 1, \text{ are nondecreasing.} \tag{3.6}$$

Select for any $r \geq 1$, the unique x_r satisfying

$$x_r \ \log(x_r)/k(x_r) = r/(Lr)^2, \tag{3.7}$$

and set $n_r := [x_r] + 1, r \geq 1$.

Lemma 3. *We have*

$$(n_r - n_{r-1})/n_{r-1} = O((Lr)^{-2}) \textit{ as } r \to \infty \tag{3.8}$$

Proof. First observe that by (3.6) and (3.7), $\log(x_r)/r$ is nonincreasing, and, consequently,

$$x_r \leq (x_{r-1})^{r/(r-1)}, r \geq 2. \tag{3.9}$$

Also note that

$$x_r \leq \exp(r/(Lr)^2), r \geq 1, \tag{3.10}$$

and for large r,

$$x_r \geq r/(Lr)^2, \tag{3.11}$$

where we use the facts that $k(x) \leq x$ and $k(x)/\log x \to \infty$, which follow from (K.1) and (K.4), respectively.

Noting that for $r \geq 2$

$$(n_r - n_{r-1})/n_{r-1} \leq (x_r + 1 - x_{r-1})/x_{r-1},$$

which by (3.9) is

$$\leq x_{r-1}^{1/(r-1)} - 1 + 1/x_{r-1},$$

we readily obtain (3.8) from (3.10) and (3.11). □

For any $r \geq 1$ set

$$\Delta(r) = \max_{n_r < n \leq n_{r+1}} b_n^{1/2} \sigma_n^{-1}(\gamma_1,\gamma_2) k_n |\bar{\mu}_n(\gamma_1,\gamma_2) - \bar{\mu}_{n_r}(\gamma_1,\gamma_2)| \tag{3.12}$$

Lemma 4. *With n_r defined as above*

$$\lim_{r\to\infty} \Delta(r) = 0. \tag{3.13}$$

Proof. Using properties (K.1)-(K.5) of the sequence κ_n it is easy to see that for all large r and $n_r \leq n \leq n_{r+1}$

$$\begin{aligned} & k_n\, b_n^{1/2}\, \sigma_n^{-1}(\gamma_1,\gamma_2) \mid \bar{\mu}_n(\gamma_1,\gamma_2) - \bar{\mu}_{n_r}(\gamma_1,\gamma_2) \mid \\ \leq\ & 2\, k_{n_{r+1}}\, b_{n_{r+1}}^{1/2}\, \sigma_n^{-1}(\gamma_1,\gamma_2) \left\{ \int_{\gamma_1 b_n}^{\gamma_1 b_{n_r}} \mid Q(u) \mid du + \int_{1-\gamma_2 b_{n_r}}^{1-\gamma_2 b_n} \mid Q(u) \mid du \right\} \\ +\ & 2\, k_{n_{r+1}}\, b_{n_{r+1}}^{1/2} (b_{n_r} - b_{n_{r+1}}) \mid \int_{\gamma_1 b_n}^{1-\gamma_2 b_n} Q(u)\, du \mid \sigma_n^{-1}(\gamma_1,\gamma_2) \\ =:\ & \delta_{n,1}(r) + \delta_{n,2}(r). \end{aligned}$$

Employing Lemma 1(a), it is readily shown that, uniformly in $n_r \leq n \leq n_{r+1}$, $\delta_{n,1}(r)$ is of order

$$O\left(k_{n_{r+1}}\, b_{n_{r+1}}^{1/2} \left(b_{n_r}^{1/2} - b_{n_{r+1}}^{1/2}\right)\right)$$

which after some calculation turns out to be of order

$$O\left(\log\ \phi_{n_{r+1}} \left(\kappa_{n_{r+1}} - \kappa_{n_r}\right)/\kappa_{n_r}\right).$$

Recalling that κ_n/n is nonincreasing, we find that

$$(\kappa_{n_{r+1}} - \kappa_{n_r})/\kappa_{n_r} \leq (n_{r+1} - n_r)/n_r$$

which in view of Lemma 3 is of order $O\left((Lr)^{-2}\right)$. Finally, noting that by the definition of our subsequence for large r,

$$\log\ \phi_{n_r} \leq \log\ r,$$

we can conclude that

$$\max_{n_r \leq n \leq n_{r+1}} \delta_{n,1}(r) \to 0.$$

By a similar, though somewhat easier argument, we get,

$$\max_{n_r \leq n \leq n_{r+1}} \delta_{n,2}(r) \to 0.$$

Lemma 4 has been proved. □

We are ready to finish the proof of the second part of Theorem 1. Now by Lemma 4 in combination with the fact that (3.5) implies $k_n/n \to 0$, we get for any $C > 0, \varepsilon > 0$ and all large r,

$$\begin{aligned} & P\left(\max_{n_r < n \le n_{r+1}} \ \min_{0 \le i \le n-k_n} \ M_n(i)/\psi_n \ > \ C + \varepsilon\right) \\ \le \ & P\left(\min_{0 \le i \le n_r - k_{n_{r+1}}} \ \max_{0 \le j \le k_{n_{r+1}}} \mid S_{i+j} - S_i - j \ \bar{\mu}_{n_r} \ (\gamma_1, \gamma_2) \mid > C \ \psi_{n_r}\right) \\ \le \ & P\left(\max_{0 \le j \le k_{n_{r+1}}} |S_j - j \ \bar{\mu}_{n_r} \ (\gamma_1, \gamma_2) \mid > C \ \psi_{n_r}\right)^{\frac{n_r}{k_{n_{r+1}}} - 1}. \end{aligned}$$

Noting that for any $0 < \tau < 1$ and for all large r, $k_{n_{r+1}} < k_{n_r} + [\tau \kappa_{n_r}]$, we see that this last bound is less than or equal to

$$\begin{aligned} & P\left(\max_{0 \le j \le k_{n_r}} |S_j - j \ \bar{\mu}_{n_r} \ (\gamma_1, \gamma_2)| \ > C_1 \ \psi_{n_r} \text{ or} \right. \\ & \left. \max_{0 \le j \le [\tau \kappa_{n_r}]} |S_{j+k_{n_r}} - S_{k_{n_r}} - j \ \bar{\mu}_{n_r}(\gamma_1, \gamma_2)| > (C_2 + C_3) \ \psi_{n_r}\right)^{\frac{n_r}{(1+\tau)\kappa_{n_r}}} \end{aligned}$$

where $C_1, C_2, C_3 > 0$ are constants satisfying $C_1 + C_2 + C_3 = C$, which will be specified later.

Choose $0 < \tau < 1$ such that $(1 + \tau)(\gamma_1 + \gamma_2) < 1$ and set

$$b_n(\tau) := L(nLn/(\tau \kappa_n))/(\tau \kappa_n),$$

$$\bar{\mu}_n(\tau) := \bar{\mu}(\tau \gamma_1 b_n(\tau), 1 - \tau \gamma_2 b_n(\tau)).$$

Noticing that $\tau b_{n_r}(\tau) > b_{n_r}$, it is routine using Lemma 1(a) to verify that eventually for some $C_3 > 0$,

$$k_{n_r} |\bar{\mu}_{n_r}(\tau) - \bar{\mu}_{n_r}(\gamma_1, \gamma_2)| < C_3 \psi_{n_r}.$$

Further observe that

$$\tau^{-1/2} \psi_n(\tau) := \tau^{-1/2} b_n^{-1/2}(\tau) \sigma(\tau \gamma_1 b_n(\tau), 1 - \tau \gamma_2 b_n(\tau)) \le \psi_n.$$

Therefore for all large r the last probability is less than or equal to

$$\begin{aligned} & P\left(\max_{0 \le j \le k_{n_r}} |S_j - j \ \bar{\mu}_{n_r} \ (\gamma_1, \gamma_2)| \ > C_1 \ \psi_{n_r} \text{ or} \right. \\ & \left. \max_{0 \le j \le [\tau \kappa_{n_r}]} |S_{j+k_{n_r}} - S_{k_{n_r}} - j \ \bar{\mu}_{n_r}(\tau)| > C_2 \tau^{-1/2} \ \psi_{n_r}(\tau)\right)^{\frac{n_r}{(1+\tau)\kappa_{n_r}}}, \end{aligned}$$

which is equal to

$$(1 - p_r p_r(\tau))^{\frac{n_r}{(1+\tau)\kappa_{n_r}}} =: P_r(\tau),$$

where

$$p_r := P\left(\max_{0 \le j \le k_{n_r}} |S_j - j \ \bar{\mu}_{n_r} \ (\gamma_1, \gamma_2) \mid \le C_1 \ \psi_{n_r}\right),$$

and

$$p_r(\tau) := P\left(\max_{0 \le j \le [\tau \kappa_{n_r}]} |S_j - j \ \bar{\mu}_{n_r} \ (\tau) \mid \le C_2 \tau^{-1/2} \psi_{n_r}(\tau)\right).$$

Choosing any $0 < \eta < 1$ and applying Lemma 2, we see that for some $C_1 > 0$ and $C_2 > 0$

$$p_r \geq \phi_{n_r}^{-\gamma_1-\gamma_2-\eta}$$

and

$$p_r(\tau) \geq (\tau^{-1}\phi_{n_r})^{-\tau(\gamma_1+\gamma_2)-\eta}.$$

Hence for all large r

$$P_r(\tau) \leq \Big(1 - \tau^{\eta+\tau(\gamma_1+\gamma_2)}(n_r \log n_r/\kappa_{n_r})^{-(\tau+1)(\gamma_1+\gamma_2)-2\eta}\Big)^{\frac{n_r}{(1+\tau)\kappa_{n_r}}},$$

which is less than or equal to

$$\Big(\exp\Big(-\tau^{\eta+\tau(\gamma_1+\gamma_2)}(n_r \log n_r/\kappa_{n_r})^{1-(\tau+1)(\gamma_1+\gamma_2)-2\eta}(\log n_r)^{-1}\Big)\Big)^{1/(1+\tau)}.$$

Choosing η small enough we see by (3.7) that the last bound is less than or equal to $\exp(-r^{\rho})$ for some $\rho > 0$ and all large r. This implies with this choice of τ, $\sum_{r=1}^{\infty} P_r(\tau) < \infty$. Thus by the Borel-Cantelli lemma we conclude (2.9). □

Proof of Theorem 2 Let $\bar{U}_1, \bar{U}_2, \ldots$ be i.i.d. uniform $(\alpha_1, 1-\alpha_2)$ random variables and set for each $j \geq 1$,

(3.14) $$\bar{S}_j = Q(\bar{U}_1) + \ldots + Q(\bar{U}_j) =: \bar{X}_1 + \ldots + \bar{X}_j.$$

Note that

(3.15) $$E\ \bar{X}_1 = \bar{\mu}(\alpha_1, 1-\alpha_2) =: \bar{\mu} \text{ and } \mathrm{Var}\ (\bar{X}_1) = \bar{\tau}^2(\alpha_1, 1-\alpha_2) =: \bar{\tau}^2.$$

Using the Skorokhod embedding, without loss of generality we can assume that there exists a standard Brownian motion W and a sequence of independent stopping times $\{\nu_i\ :\ i \geq 1\}$ such that for any $j \geq 1$

(3.16) $$\bar{S}_j - j\bar{\mu} = W(\nu_1 + \ldots + \nu_j),$$

(3.17) $$E\ \nu_j = \bar{\tau}^2,$$

and for any integer $r \geq 2$

(3.18) $$E\ \nu_j^r \leq 2\ (8/\pi^2)^{r-1}\ r!\ E\Big(\bar{X}_j - \bar{\mu}\Big)^{2r}.$$

Refer to Theorem A.1 in Hall and Heyde (1980). Set $H := H(\alpha_1, \alpha_2) := Q(1-\alpha_2) - Q(\alpha_1)$. Notice that by the c_r-inequality for $r \geq 2$,

(3.19) $$E\ |\ \nu_1 - \bar{\tau}^2\ |^r \leq 2^{r-1}(E\ \nu_1^r + \bar{\tau}^{2r}),$$

which by (3.17) and (3.18) is

$$\leq \ 2^{r-1}\Big(2\Big(\tfrac{8}{\pi^2}\Big)^{r-1} r!\ \overline{\tau}^2\ H^{2r-2}\ +\ \overline{\tau}^2\ H^{2r-2}\Big)$$
$$\leq \ \tfrac{r!}{2}\ \overline{H}^{r-2}\ \sigma^2,$$

where $\sigma^2 \ := \ (\frac{64}{\pi^2}+1)\ \overline{\tau}^2\ H^2 \ \geq \ \text{Var}\ (\nu_1)$ and $\overline{H} \ := \ \frac{128}{\pi^2} H^2$.

Thus by the Bernstein inequality as given in Exercise 14, page 111, Chow and Teicher (1988),

$$\text{(3.20)} \qquad P\ (\nu_1+\ldots+\nu_j - j\ \overline{\tau}^2\ \geq\ x)\ \leq\ \exp\ \Big(-\ x^2/(2j\ \sigma^2+2\overline{H}x)\Big)$$

Applying (3.20) with $x = j\ \overline{\tau}^2$, we get

$$\text{(3.21)} \qquad P\ (\nu_1+\ldots+\nu_j - j\ \overline{\tau}^2\ \geq\ j\ \overline{\tau}^2)\ \leq\ \exp\ (-\ d\ j\ \overline{\tau}^2/H^2),$$

where $0 < d < 1$ is a constant independent of α_1, α_2 and Q.

To complete the proof of Theorem 2, we require the following lemma.

Lemma 5 *There exists a constant $0 < \overline{d} < 1$ independent of α_1, α_2 and Q such that for all k sufficiently large and $0 < x \leq 1$,*

$$\text{(3.22)} \qquad P\ \Big\{\max_{0\leq j\leq k}\ |\ \overline{S}_j - j\ \overline{\mu}\ |\ \leq 2H/\sqrt{dx}\Big\} \geq \tfrac{1}{2}\ \exp\ (-\overline{d}x\ k)$$

Proof. We shall assume that $H > 0$, otherwise (3.22) is trivial. Applying (3.21), we see that the probability in (3.22) is greater than or equal to

(3.23)

$$P\ \Big\{\max_{0\leq s\leq 2k\overline{\tau}^2}\ |\ W(s)\ |\ \leq 2H/\sqrt{dx}\Big\}$$
$$-\ P\Big\{\textstyle\sum_{j=1}^{k}\ (\nu_j - \overline{\tau}^2)\ \geq k\ \overline{\tau}^2\Big\}\ =:\ P_k(1) - P_k(2)$$

Using relation (2.1) of Jain and Pruitt (1975), we get for all k large enough,

$$\text{(3.24)} \qquad P_k(1) \geq \frac{8}{3\pi}\ \exp\ (-x\pi^2\ kd\ \overline{\tau}^2/16\ H^2)$$

and using (3.21) we have

$$\text{(3.25)} \qquad P_k(2) \leq\ \exp\ (-dk\ \overline{\tau}^2/H^2)$$

Setting $\overline{d} =\ \pi^2 d/16$ and noting that $\overline{\tau}^2/H^2 \leq 1$ completes the proof. □

To finish the proof of Theorem 2, set

$$\text{(3.26)} \qquad m_n = \min_{0\leq i\leq n-k_n}\ \max_{0\leq j\leq k_n}\ |S_{i+j} - S_i - j\ \overline{\mu}|,$$

and assume without loss of generality that $k_n \to \infty$.

Observe that for any $0 < x \leq 1$

$$P_n := P(m_n > 2H/\sqrt{dx}) \leq \left(P(\max_{1\leq j\leq k_n} |S_j - j\ \bar{\mu}\ | > 2H/\sqrt{dx})\right)^{\frac{n}{k_n}-1} \tag{3.27}$$

Now since conditioned on $U_1, \ldots, U_{k_n} \in (\alpha_1, 1-\alpha_2)$, $\{X_j\}_{j=1}^{k_n} =_d \{\bar{X}_j\}_{j=1}^{k_n}$,

$$P\left(\max_{1\leq j\leq k_n} |S_j - j\ \bar{\mu}\ | \leq 2H/\sqrt{dx}\right) \geq P\left(\max_{1\leq j\leq k_n} |\bar{S}_j - j\ \bar{\mu}\ | \leq 2H/\sqrt{dx}\right) \exp\left(k_n \log(1-\alpha)\right), \tag{3.28}$$

where $\alpha = \alpha_1 + \alpha_2$. This last lower bound is by Lemma 5 for all n large enough greater than or equal to

$$\frac{1}{2} \exp\left(-k_n(x\bar{d} - \log(1-\alpha))\right), \tag{3.29}$$

Therefore for all n sufficiently large

$$P_n \leq \left(1 - \frac{1}{2} \exp(-k_n(x\bar{d} - \log(1-\alpha)))\right)^{\frac{n}{k_n}-1}, \tag{3.30}$$

Now by our choice of $\alpha = \alpha_1 + \alpha_2$ and selecting $x > 0$ sufficiently small, using $k_n = O(\log n)$, we get from (3.26) for some $\delta > 0$,

$$P_n \leq \exp(-n^{\delta}). \tag{3.31}$$

Hence $\sum_{n=1}^{\infty} P_n < \infty$, which by the Borel-Cantelli lemma implies (2.10). □

We have all the tools now to prove Theorem 4.

Proof of Theorem 4 Choose any $t \in (0,1)$ and for $0 < \eta < t$ set $\bar{\mu}(\eta) = \bar{\mu}(t-\eta, t)$, $\bar{\tau}^2(\eta) = \bar{\tau}^2(t-\eta, t)$ and $H(\eta) = Q(t) - Q(t-\eta)$.

Further, set

$$m_n(\eta) = \min_{0\leq i\leq n-k_n} \max_{0\leq j\leq k_n} |S_{i+j} - S_i - j\ \bar{\mu}(\eta)|, \tag{3.32}$$

and assume without loss of generality that $k_n \to \infty$.

Let

$$P_n(\eta) = P(m_n(\eta) > 2H(\eta)/\sqrt{d}). \tag{3.33}$$

We get from (3.30) above with $x = 1$ and $1 - \alpha = \eta$ for all large n

$$P_n(\eta) \leq \left(1 - \frac{1}{2} \exp(-k_n(\bar{d} - \log(\eta)))\right)^{\frac{n}{k_n}-1}. \tag{3.34}$$

This last bound in conjunction with $k_n = o(\log n)$ gives

(3.35) $$\sum_{n=1}^{\infty} P_n(\eta) < \infty.$$

Further, it is routine using (3.35) to select a sequence $\eta_n \geq 0$ decreasing to 0 such that

(3.36) $$\sum_{n=1}^{\infty} P_n(\eta_n) < \infty.$$

Therefore by the Borel-Cantelli lemma

(3.37) $$P\left(m_n(\eta_n) > 2H(\eta_n)/\sqrt{d}\ i.o.\right) = 0.$$

Now by left continuity of $Q,\ H(\eta_n) \to 0$, thus

(3.38) $$\lim_{n\to\infty} m_n(\eta_n) = 0 \quad \text{a.s.}$$

Setting $\varepsilon_n = Q(t) - \overline{\mu}(\eta_n) \geq 0$ completes the proof of (2.13). □

Proof of Theorem 3 We first collect a number of auxiliary results.

Lemma 6 *We have for any nondegenerate random variable X,*

$$\limsup_{s,t\to 0} \overline{\tau}^2(s, 1-t)/\sigma^2(s, 1-t) \leq 1.$$

When $EX^2 < \infty$, Lemma 6 is trivial, and when $EX^2 = \infty$ it follows from (2.3) and (2.4) via Lemma 1(b).

Lemma 7 *Let X be a nondegenerate random variable in the Feller class, and let $0 < \lambda_1, \lambda_2 < 1$ be constants. Then we have for some positive constant $K_6 = K_6(\lambda_1, \lambda_2)$,*

$$\limsup_{s,t\to 0} \sigma^2(\lambda_1\ s, 1-\lambda_2\ t)/\sigma^2(s, 1-t) \leq K_6.$$

See relation (1.42) of Csörgő, Haeusler and Mason (1988b).

Lemma 8 *Let X be a nondegenerate random variable in the Feller class. There exists a positive constant K_7 (depending on the distribution of X) such that for large m,*

$$\sup_{\beta} P\{|S_m - \beta| \leq K_7\sqrt{m}\ \sigma(1/m)\} \leq e^{-5}$$

where $\sigma^2(s) := \sigma^2(s, 1-s), 0 < s < 1/2.$

Lemma 8 is an immediate consequence of Lemma 5, Einmahl and Mason (1994). Arguing as in the proof of (3.8) of the same paper, we can infer from Lemma 8,

Lemma 9 *Let X be a nondegenerate random variable in the Feller class, and let*

$\{\beta_{n,k} : 1 \le k \le k_n, n \ge 1\}$ *be an array of real numbers. Then we have for large* n,

$$P\{\max_{1\le k\le k_n} |S_k - \beta_{n,k}| \le \frac{1}{2}K_7 \sqrt{k_n/\log \phi_n}\, \sigma_n\} \le \phi_n^{-5/2},$$

where $\sigma_n^2 := \sigma^2(b_n, 1-b_n), n \ge 2$, *and* ϕ_n *is defined as in (3.2).*

The next lemma is due to Shao (1992). It can also be found on page 113 of the monograph of Lin and Lu (1992).

Lemma 10 *Let* $Z_1, Z_2, \ldots$ *be independent random variables satisfying for some* $0 < \alpha < 1$ *and* $\varepsilon > 0$,

$$P\left\{\max_{1\le i\le L} \left|\sum_{k=1}^{i} Z_k\right| \ge \varepsilon x\right\} \le \alpha. \tag{3.39}$$

Then we have:

$$P\left\{\min_{0\le i\le L} \max_{1\le j\le M} \left|\sum_{k=i+1}^{i+j} Z_k\right| \le x\right\} \le (1-\alpha)^{-1} P\left\{\max_{1\le j\le M} \left|\sum_{k=1}^{j} Z_k\right| \le x(1+\varepsilon)\right\}.$$

We are now ready to prove Theorem 3. Let $\{n_r : r \ge 1\}$ be the same subsequence as in the proof of the second part of Theorem 2.1 (see relation (3.7)). Let $\overline{\psi}_n := \sqrt{k_n/\log \phi_n}\, \sigma_n,\ n \ge 1$. In view of Lemma 7 it suffices to prove that with probability 1,

$$\liminf_{n\to\infty} \min_{0\le i\le n-k_n} \max_{0\le j\le k_n} |S_{i+j} - S_i - j\,\overline{\mu}_n(\gamma_1,\gamma_2)| / \overline{\psi}_n > 0. \tag{3.40}$$

Noting that for $n_r \le n \le n_{r+1}$,

$$\begin{aligned}
&\min_{0\le i\le n-k_n} \max_{0\le j\le k_n} |S_{i+j} - S_i - j\,\overline{\mu}_n(\gamma_1,\gamma_2)| / \overline{\psi}_n \\
&\ge \min_{0\le i\le n_{r+1}-k_{n_r}} \max_{0\le j\le k_{n_r}} |S_{i+j} - S_i - j\,\overline{\mu}_{n_r}(\gamma_1,\gamma_2)| / \overline{\psi}_{n_{r+1}} \\
&\quad - \max_{n_r<n\le n_{r+1}} \left\{k_n|\overline{\mu}_n(\gamma_1,\gamma_2) - \overline{\mu}_{n_r}(\gamma_1,\gamma_2)| / \overline{\psi}_n\right\} \\
&=: \widetilde{M}(r) - \Delta'(r),
\end{aligned}$$

it is obviously enough to show,

$$\Delta'(r) \to 0 \text{ as } r \to \infty, \tag{3.41}$$

and for a suitable positive constant K_8,

$$\liminf_{r\to\infty} \widetilde{M}(r) \ge K_8 \text{ a.s.} \tag{3.42}$$

Statement (3.41) follows by combining Lemma 4 and Lemma 7.

Further note that by Lemma 7 in conjunction with Lemma 3 we have,

$$\limsup_{r\to\infty} \overline{\psi}_{n_{r+1}} / \overline{\psi}_{n_r} < \infty, \tag{3.43}$$

and we can reduce the proof of (3.42) to showing for some positive constant K_9,

$$\liminf_{r\to\infty} M(r) \ \geq \ K_9 \text{ a.s.} \tag{3.44}$$

where $M(r) \ := \ \widetilde{M}(r)\ \overline{\psi}_{n_{r+1}} \ / \ \overline{\psi}_{n_r}\ ,\ r \geq 1$.

We need an additional lemma. To simplify our notation, we set for $0 < \varepsilon < 1$, $m_r(\varepsilon) \ := [\varepsilon^2\ k_{n_r} \ / \ \log \ \phi_{n_r}]$.

Lemma 11 *We have for* $0 < \varepsilon < 1/2$ *and large* r

$$P\ \left\{\max_{1\leq j\leq m_r(\varepsilon)} |S_j - j\ \bar{\mu}_{n_r}\ (\gamma_1, \gamma_2)\ | \ \geq \ K_{10}\ \varepsilon\ \overline{\psi}_{n_r}\right\} \ \leq \frac{1}{2},$$

where $K_{10} > 0$ *is a constant independent of* ε.

Proof. Recall that $X_j = Q(U_j)$, where U_j is uniform $(0,1)$, $j \geq 1$. Let $V_1(r), V_2(r), \ldots$ be independent uniform $(\gamma_1\ b_{n_r},\ 1 - \gamma_2\ b_{n_r})$ random variables. Using a simple conditioning argument, it is easy to see that the above probability is less than or equal to

$$m_r(\varepsilon) b_{n_r}\ +\ P\ \Big\{\max\nolimits_{1\leq j\leq m_r(\varepsilon)}\ \big|\ \sum\nolimits_{i=1}^{j}\ \{Q(V_i(r))\ - \bar{\mu}_{n_r}(\gamma_1, \gamma_2)\}\ \big| \\ \geq\ K_{10}\ \varepsilon\ \overline{\psi}_{n_r}\Big\},$$

which by definition of b_n and Kolmogorov's maximal inequality is

$$\leq\ \varepsilon^2 + K_{10}^{-2}\ \text{Var}\ (Q(V_1(r)))\ /\ \sigma_{n_r}^2\ \ \leq \frac{1}{2}$$

provided we have chosen K_{10} large enough so that for large r,

$$\text{Var}\ (Q(V_1(r)))\ /\ \sigma_{n_r}^2\ \ \leq K_{10}^2/4.$$

The existence of such a constant follows from Lemmas 6 and 7. □

Setting $\ell(\varepsilon, r) := [n_{r+1}/m_r(\varepsilon)] + 1$, we readily obtain that for $0 < \varepsilon < 1/2$,

$$P\ \{M(r)\ \leq\ \big(K_7/2 - K_{10}\ \varepsilon\big)\} \\ \leq \sum\nolimits_{\ell=1}^{\ell(\varepsilon,r)}\ P\Big\{\min\nolimits_{(\ell-1)m_r(\varepsilon)\leq i\leq \ell m_r(\varepsilon)}\ \ \max\nolimits_{0\leq j\leq k_{n_r}}\ |S_{i+j} - S_i - j\ \bar{\mu}_{n_r}(\gamma_1, \gamma_2)| \\ \leq\ \big(K_7/2 - K_{10}\ \varepsilon\big)\ \overline{\psi}_{n_r}\Big\}$$

which in turn by Lemma 10 and 11 is

$$\leq\ 2\ \ell(\varepsilon, r)\ P\Big\{\max_{0\leq j\leq k_{n_r}}\ |S_{i+j} - S_i - j\ \bar{\mu}_{n_r}(\gamma_1, \gamma_2)| \leq K_7 \overline{\psi}_{n_r}/2\Big\}.$$

Recalling Lemma 9, we can further conclude that this is

$$\leq 2\ \varepsilon^{-2}\ (n_{r+1}/k_{n_r})\ \log\ \phi_{n_r}\ \phi_{n_r}^{-5/2},$$

which in turn by Lemma 3 is for large r,

$$\leq 3\ \varepsilon^{-2}\ \log\ \phi_{n_r}\ \phi_{n_r}^{-3/2},$$

From the definition of the subsequence $\{n_r : r \geq 1\}$ it follows that

$$\phi_{n_r}\ \sim\ r/(Lr)^2 \text{ as } r \to \infty, \tag{3.45}$$

and we can infer that for $0 < \varepsilon < 1/2$,

$$\sum_r P\left\{M(r) \leq K_7/2 - \varepsilon K_{10}\right\} < \infty. \tag{3.46}$$

Since ε can be made arbitrarily small, this implies, via the Borel-Cantelli lemma, statement (3.44) with $K_9 = K_7/2$, and Theorem 3 has been proved. □

Remark Given the general formulation of Lemma 9, one might ask whether it is not possible to prove Theorem 3 for more general centering sequences. An inspection of the proof shows that one can replace the centerings $\{j\ \bar{\mu}_n(\gamma_1, \gamma_2)\ :\ 1 \leq j\ \leq k_n\}$ by
$\{j\ \beta_n\ :\ 1 \leq j\ \leq k_n\}$ whenever the following two conditions are satisfied,

$$|\beta_n - \bar{\mu}_n(\gamma_1, \gamma_2)|\ =\ o\ (\sqrt{b_n}\ \sigma_n) \text{ as } n \to \infty, \tag{3.47}$$

and

$$\max_{n_r \leq n \leq n_{r+1}} \left\{\sqrt{k_n\ \log\ \phi_n}\ |\ \beta_n - \beta_{n_r}|/\sigma_n\right\} \to 0 \text{ as } r \to \infty, \tag{3.48}$$

where $\{n_r\}$ is the subsequence defined in (3.7). The latter condition might be somewhat difficult to verify in general, but it is trivial in one case of particular interest, namely if $\beta_n \equiv \beta$ is constant.

4. Discussion and further results

Our Theorem 2 is related to the following conjecture of Csörgő and Révész (1981). For some other results in this direction see also Csáki and Földes(1984).

Conjecture Let $X, X_1, X_2, \ldots,$ be i.i.d. random variables with $EX = 0$ and Var $(X) = 1$. Then

$$\lim_{n\to\infty}\ \min_{0 \leq i \leq n-k_n}\ \max_{0 \leq j \leq k_n} |S_{i+j} - S_i| = r(c),\ \text{ a.s.}, \tag{4.1}$$

where $k_n = [c\ \log n]$ and $r(c)$ is a function which uniquely defines the distribution function F of X.

Our Theorem 2 says that whenever $k_n = [c\ \log n]$, for some $c > 0$, with probability 1, (4.2) $$\limsup_{n\to\infty} \min_{0\le i\le n-k_n} \max_{0\le j\le k_n} |S_{i+j} - S_i - j\ \bar{\mu}(\alpha_1, 1-\alpha_2)| < \infty$$

as long as $\alpha_1, \alpha_2 > 0$ are chosen so that $-c\ \log(1-\alpha) < 1$ where $\alpha = \alpha_1 + \alpha_2$. In light of the Csörgő and Révész conjecture it would be interesting to know whether lim sup can be replaced by limit in (4.2), and, further, assuming it exists, whether as a function of c it determines F.

When $k_n/\log n \to \infty$ and F is in the Feller class our Theorem 3 states that the lim inf in (2.11) is strictly positive. We shall show that this need not be the case in (2.10) when $k_n = [c\ \log n]$ for some $c > 0$. In fact, we shall provide an example where the lim sup in (4.2) is 0. Moreover, we shall also give an example for which

(4.3) $$\liminf_{n\to\infty} \min_{0\le i\le n-k_n} \max_{0\le j\le k_n} |S_{i+j} - S_i - j\ \bar{\mu}(\alpha_1, 1-\alpha_2)| > 0,$$

where $k_n = [c\ \log n]$ and $-c\ \log(1-\alpha) < 1$ with $\alpha = \alpha_1 + \alpha_2$. In order to construct our first example we shall require the following auxiliary result.

Let $U_1, U_2, \ldots,$ be a sequence of uniform $(0,1)$ random variables and J be any measurable subset of $(0,1)$, where $1-\alpha$ denotes the Lebesque measure of J. Choose any $c > 0$ and let k_n be a sequence of integers satisfying $1 \le k_n \le n$ and $k_n = [c\ \log n]$ for large n. Let

(4.4) $$A_{j,n} = \{U_{j+1}, \ldots, U_{j+k_n} \in J\}, j = 0, \ldots, n-k_n,$$

and

(4.5) $$A_n = \bigcup_{j=0}^{n-k_n} A_{j,n}.$$

Proposition *For any $c > 0$*

(4.6) $$P\ (A_n \text{ eventually}) = 1 \text{ or } 0$$

according as $-c\ \log(1-\alpha) < 1$ or $-c\ \log(1-\alpha) > 1$.

Proof. First assume that $-c\ \log(1-\alpha) < 1$. In this case, note that

(4.7) $$P\ (A_n^c) \ \le\ \left(P(A_{0,n}^c)\right)^{\frac{n}{k_n}-1},$$

which for all large enough n is less than or equal to

(4.8) $$2\ \left(1-(1-\alpha)^{k_n}\right)^{n/k_n}.$$

This last expression is in turn less than or equal to

(4.9) $$2\ \left(1-(1-\alpha)^{\bar{c}\ \log n}\right)^{n/k_n} = 2\left(1-\exp(-\bar{c}\ \log(1-\alpha)\ \log\ n)\right)^{n/k_n}$$

for any $\bar{c} > c$ and all large n. Now since we can choose $\bar{c} > c$ so that $\bar{c}\ \log(1-\alpha) > -1$, (4.9) is less than or equal to $\exp(-n^{\delta})$ for some $\delta > 0$. From this we get that

$$\sum_{n=1}^{\infty} P(A_n^c) < \infty, \tag{4.10}$$

which by an application of the Borel-Cantelli lemma implies $P(A_n$ eventually$) = 1$.

Now assume $-c\ \log(1-\alpha) > 1$. Set for $r = 0, 1, 2, \ldots$, and $i = 0, 1, 2, \ldots$

$$B_{i,r} = \left\{U_{i+1}, \ldots, U_{i+k_{2^r}} \in J\right\} \tag{4.11}$$

and

$$B_r = \bigcup_{i=0}^{2^{r+1}} B_{i,r} \tag{4.12}$$

Notice that $A_n \subset B_r, 2^r < n \le 2^{r+1}$ and

$$P(B_r) \le 2^{r+2} \exp\left(k_{2^r}\ \log(1-\alpha)\right) \le 4 \cdot 2^{-\delta r}, \tag{4.13}$$

for some $\delta > 0$. Since this last bound is summable, we infer from the Borel-Cantelli lemma that $P(B_r\ i.o.) = 0$, which implies $P(A_n$ eventually$) = 0$.

Example 1 Choose any Q that is constantly equal to 1 on $[\alpha_1, 1-\alpha_2]$, where $\alpha_1, \alpha_2 > 0$ satisfy $-c\ \log(1-\alpha) < 1$. Observe that this makes $\bar{\mu}(\alpha_1, \alpha_2) = 1$. Applying our Proposition we conclude that with probability 1 for each sufficiently large n there exists an $0 \le i \le n - k_n$ such that $S_{i+j} - S_i = j$ for all $1 \le j \le k_n = [c\ \log n]$. This, of course, implies that the lim sup in (4.2) is equal to 0 almost surely for all F having a Q with this property and with $\alpha_1, \alpha_2 > 0$ chosen in this way.

Our next example shows that the lim inf in (4.3) can be positive.

Example 2 For any choice of $0 < \alpha < 1$ such that $-c\ \log(1-\alpha) < 1$, define

$$Q(u) = \begin{cases} 1 \text{ for } \frac{1}{2} < u < 1 \\ 0 \text{ for } 0 < u \le 1/2. \end{cases} \tag{4.14}$$

Notice that $\bar{\mu}(\alpha/2, 1-\alpha/2) = 1/2$. Hence for any sequence of integers ℓ_n satisfying $1 \le \ell_n \le n$ and all $0 \le i \le n - \ell_n$

$$|S_{i+1} - S_i - \bar{\mu}(\alpha/2, 1-\alpha/2)| = |S_{i+1} - S_i - 1/2| = 1/2. \tag{4.15}$$

This implies that

(4.16) $$\min_{0 \le i \le n-\ell_n} \max_{0 \le j \le \ell_n} |S_{i+j} - S_i - j\ \bar{\mu}(\alpha/2, 1-\alpha/2)\ | \ge 1/2;$$

which, in particular, says that the lim inf in (4.3) is greater than or equal to $1/2$.

We finally would like to make some comments on the problem of what are alternative choices for the centering constants $\{j\ \bar{\mu}_n(\gamma_1, \gamma_2)\ :\ 1 \le j \le k_n\}$ used in Theorem 1. Since this problem has been extensively studied in connection with the Chung-type LIL proved in Einmahl and Mason (1994), it will be enough to only state the results without giving detailed proofs.

We first note that using the method employed in the proof of Theorem 3 of the aforementioned paper along with the remark at the end of the proof of Theorem 3 of the present paper, one can obtain the following refinement of Theorem 1 for random variables in the Feller class.

Theorem 5 *Let X be a nondegenerate random variable in the Feller class, and let $\{k_n\}$ be as in Theorem 1. Assume that we have for suitable constants $\gamma_1, \gamma_2 > 0$ with $\gamma_1 + \gamma_2 < 1$,*

(4.17) $$|\beta_n - \bar{\mu}_n(\gamma_1, \gamma_2)| = o\ (\sqrt{b_n}\ \sigma_n) \text{ as } n \to \infty,$$

and

(4.18) $$\max_{n_r \le n \le n_{r+1}} \left\{ \sqrt{k_n\ \log\ \phi_n}\ |\beta_n - \beta_{n_r}|/\sigma_n \right\} \to 0 \text{ as } r \to \infty,$$

where $\{n_r\}$ is the subsequence defined by (3.7).

Then we have, with probability 1,

(4.19) $$\liminf_{n \to \infty} \min_{0 \le i \le n-k_n} \max_{0 \le j \le k_n} |S_{i+j} - S_i - j\ \beta_n|\ /\ \sqrt{k_n/\log\ \phi_n}\ \sigma_n = K_{11},$$

where $K_{11} > 0$ is a finite constant.

Moreover, if

(4.20) $$\log\ (n/\kappa_n)/LLn \to \infty \text{ as } n \to \infty,$$

we also have with probability 1,

(4.21) $$\limsup_{n \to \infty} \min_{0 \le i \le n-k_n} \max_{0 \le j \le k_n} |S_{i+j} - S_i - j\ \beta_n|\ /\ \sqrt{k_n/\log\ \phi_n}\ \sigma_n = K_{12},$$

where K_{12} is a finite constant.

As in Sect. 5, Einmahl and Mason (1994), we can infer from Theorem 5 among other results that if X is a random variable in the domain attraction of a stable law of index $\alpha \in (0, 2], \alpha \neq 1$, which is not completely asymmetric, both (4.19) and (4.21) hold true with $\beta_n \equiv \mu_\alpha$, where $\mu_\alpha = 0$ if $0 < \alpha < 1$ and $\mu_\alpha = EX$ if $1 < \alpha < 2$.

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