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The Burgers equation with a random force and a general model for directed polymers in random environments*

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Summary. The study of the Burgers equation with a random force leads via a Hopf-Cole type transformation to a stochastic heat equation having a white noise with spatial parameters type potential. The latter can be studied by means of a general model of directed polymers in random environments with two point random potentials. These models exhibit a Gaussian behavior at large times and have certain stationary distributions which yield the corresponding results for the above stochastic heat and Burgers equations.

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1 Introduction

The vector Burgers equation

$$\frac{\partial u}{\partial t} + \lambda(u, \nabla)u = v\Delta u + \Phi \ , \tag{1.1}$$

$u = u(x,t) = (u_1(x,t), \ldots, u_d(x,t))$, $x \in \mathbb{R}^d$, with a random force term $\Phi = \Phi(x,t) = (\Phi_1(x,t), \ldots, \Phi_d(x,t))$ was considered recently in a number of both physical (see [KS] and references there) and mathematical (see [Si1, BCJL, HLØUZ1, 2, DDT, DG]) literature. The Burgers and the related scalar Kardar, Parisi, Zhang (KPZ) equation

$$\frac{\partial v}{\partial t} - \frac{\lambda}{2}|\nabla v|^2 = v\Delta v + \frac{\partial}{\partial t}\Psi \ , \tag{1.2}$$

where $u = -\nabla v$ and $\Phi = -\nabla \partial/\partial t\, \Psi$, are among simplest physically interesting nonlinear partial differential equations since by the Hopf-Cole transformation

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$w = \exp(\lambda/2\nu v)$ they can be reduced to the linear heat equation

$$\frac{\partial w}{\partial t} = \nu \Delta w + \frac{\lambda}{2\nu}\left(\frac{\partial}{\partial t}\Psi\right) w \,. \tag{1.3}$$

The interpretation of Eqs. (1.1)–(1.3) depends on a type of the noise term considered. If Φ is the space time white noise then (1.1) should be understood in some generalized sense and [HLØUZ1, 2] interprets the product $(u, \nabla)u$ as a Wick product and the solution is given as a distribution valued process. Physicists usually assume the space–time independency of the noise in the KPZ equation which describes growing surfaces, i.e. that $\partial/\partial t\,\Psi$ is the space–time white noise and, correspondingly, that Φ is its gradient in spatial variables. A mathematical interpretation of (1.1) is not clear then and even its physical consequences lead to certain difficulties (see [KS, p. 517]). In this paper I replace the spatial independence of Ψ by sufficiently weak dependence which enables me to consider a spatially smooth noise and obtain semimartingale solutions of (1.1)–(1.3). Namely, I assume that $\Psi(t) = \Psi(x,t),\ x \in \mathbb{R}^d$ is a temporally and spatially homogeneous C^3-valued Brownian motion with sufficiently fast decreasing spatial covariance function $A(x,y) = A(y-x)$ which means that it is a continuous process with independent increments and values in the space of C^3 functions on $\mathbb{R}^d$ whose distributions are invariant with respect to translations of $\mathbb{R}^d$ (see [Ku2, Ch. 3]). When sufficient smoothness in spatial parameters is assumed the solutions of (1.1) and (1.2) can be represented via the solution of (1.3) by means of the Hopf-Cole transformation as above which can be justified easily using the Ito formula. Furthermore, the solution of (1.3) can be written by means of the Feynman–Kac formula (see [Ku2, p. 308]),

$$\begin{aligned} w(x,t) \;=\; & e^{-\frac{\lambda^2 A(0)t}{8\nu^2}} \int f(y)p(t,x-y)E_y \\ & \times \left(\exp \int_0^t \frac{\lambda}{2\nu}\Psi(W_{t-u},du) \,\Big|\, W_t = x\right) dy\,, \end{aligned} \tag{1.4}$$

where W_u is the Wiener process in $\mathbb{R}^d$ with time scaled by 2ν (i.e. $W_{u/2\nu}$ is the standard Wiener process) which is independent of Ψ, $p(u,z)$ is the transition density of W_u, E_y is the expectation for W_u given $W_0 = y$, and the exponent contains an Ito stochastic integral of the form considered in [Ku2]. The conditional expectation in (1.4) means that I integrate with respect to the distribution of the Brownian bridge from y to x on the interval $[0, 2\nu t]$.

Using the Markov property of W_t on $\mathbb{R}^d$ one can investigate the right hand side of (1.4) by means of a discrete time model which generalizes the set up for a directed polymer in a random environment considered in a number of papers (see [IS, Bo, Si2]). Namely, let $\{F_k(x,y)\},\ k = \dots, -1, 0, 1, \dots$ be a sequence of i.i.d. continuous random fields on $\mathbb{R}^d \times \mathbb{R}^d$, $d \geqq 3$ with distributions depending only on $y - x$ and invariant under the exchange of x and y. Next, consider a random walk $\{W_k,\ k \in \mathbb{Z}\}$ (in particular, W_k may be the above process W_u taken at integer times) with a transition density $p(v,w) = p(w-v)$

and put it into a random environment determined by the potentials F_k, namely, consider another random walk $\{\mathscr{Z}_l\}$ for which probability density of each path $z_m = x, z_{m+1}, \ldots, z_n$ has the (random) weight

$$\prod_{l=m}^{n-1} p(z_{l+1} - z_l) \exp F_l(z_l, z_{l+1}) \ . \tag{1.5}$$

Assuming that the variance of $\exp F_k(x, y)$ is sufficiently small and that the covariances of $\exp F_k(x, y)$ and $\exp F_k(z, v)$ decrease sufficiently fast as the distance between (x, y) and (z, v) grows I shall show that the random walk $\{\mathscr{Z}_l\}$ is asymptotically Gaussian and it has a stationary distribution. This shows that in dimensions bigger than 2 the diffusive behavior in models of directed polymers in random environments is robust in the sense that it occurs for a relatively wide class of models and not only for specific i.i.d. lattice models considered before. In view of (1.4) these results transfer to solutions of (1.3) which yields also certain information about solutions of (1.1) and (1.2), in particular, the existence of a stationary distribution. Actually, I obtain results for general "random positive semigroups" which emerge naturally in representations of solutions of linear second order stochastic partial differential equations even more general than (1.3) (see [KK]). Some other limit theorems related to solutions of (1.3) were derived in [DS]. I note also that there is a somewhat different continuous time and space model considered in [CO] which can be reduced to a discrete time model with quite special two point random potentials whose distributions are invariant only under the action of the diagonal of $\mathbb{Z}^d \times \mathbb{Z}^d$ and not of $\mathbb{R}^d \times \mathbb{R}^d$, as above.

The main part of this paper is concerned with the general model of directed polymers in random environments and in the last section I apply the results to exponential functionals appearing in (1.4). For this I need Ψ to be only space–time continuous Brownian motion. In order to justify the Feynman–Kac formula and the Hopf-Cole transformation one needs some additional smoothness in spatial variables if the equations (1.1)–(1.3) are supposed to be understood in a classical sense. I do not study here specific details of these questions though I believe that the smoothness assumptions considered in [Ku2] can be relaxed in our circumstances. I am grateful to Ya. Sinai who suggested to me that some of the methods from his work [Si2] may be developed to become applicable to the Burgers equation. Actually, this paper is an extension of the approaches from [Bo] and [Si2] which is adequate to deal with the exponential functionals from (1.4) emerging in the Burgers equation related study in view of the Hopf-Cole transformation and the Feynman–Kac formula.

2 Preliminaries and main results

Let $\{F_k(x, y);\ x, y \in \mathbb{R}^d\},\ k \in \mathbb{Z},\ d \geqq 3$ be a sequence of independent identically distributed $\mathbb{R}$-valued random fields on $\mathbb{R}^d \times \mathbb{R}^d$ with continuous in x, y realizations such that the distribution $Q_{x,y}$ of $F_k(x, y)$ depends only on $y - x$ and $Q_{x,y} = Q_{y,x}$. I can consider $F_k(x, y)$ as random variables on the space Ω of all continuous realizations ω of $\{F_k(x,y)\}$ so that each ω is a function on $\mathbb{Z}\times\mathbb{R}^d\times\mathbb{R}^d$ and $F_k(\omega,x,y)=\omega(k,x,y)$. The distributions of the random fields $\{F_k(x,y)\}$ generate probability measures Q and $Q^{(k)}$ on Ω where $Q=\prod_{k\in\mathbb{Z}}Q^{(k)}$ and $Q^{(k)}$ gives probabilities of events for F_k. I define also the space and time shifts (translations) θ_z, $z\in\mathbb{R}^d$ and η_l, $l\in\mathbb{Z}$ on Ω acting by $\theta_z\omega(k,x,y)=\omega(k,x+z,y+z)$ and $\eta_l\omega(k,x,y)=\omega(k+l,x,y)$, respectively. I assume that the probability measure Q is both θ_z and η_l invariant.

Let $W_0,W_1,W_2,\ldots$ be a Markov chain on $\mathbb{R}^d$ independent of the above random fields and such that

$$P\{W_{n+1}\in\Gamma\mid W_n=x\}=P(x,\Gamma)=\int_\Gamma p(y-x)\,dy \tag{2.1}$$

where the density $p\geqq 0$, $\int p(z)\,dz=1$ satisfies

$$p(z)=p(-z)\quad\text{and}\quad p(z)\leqq C_0e^{-\gamma_0|z|}\,. \tag{2.2}$$

Let $P(l,x,\Gamma)=P\{W_l\in\Gamma\mid W_0=x\}$ be the l-step transition probability and

$$\begin{aligned} p(l,y-x) \;=\; &\int\cdots\int p(z_1-x)p(z_2-z_1)\cdots p(z_{l-1}-z_{l-2})\\ &\times p(y-z_{l-1})\,dz_1\cdots dz_{l-1} \end{aligned} \tag{2.3}$$

be the corresponding density. Observe that in view of (2.1) and (2.3) the increments W_1-W_0, $W_2-W_1,\ldots$ are independent. Next, I shall consider the partition functions

$$\begin{aligned} Z(x,m;y,n) &= p(n-m,y-x)E_x\exp\left(\sum_{l=m}^{n-1}F_l(W_l,W_{l+1})\,\middle|\,W_n=y\right)\\ &= E_xp(y-W_{n-1})\exp\left(\sum_{l=m}^{n-2}F_l(W_l,W_{l+1})\right)\exp F_{n-1}(W_{n-1},y)\,, \end{aligned} \tag{2.4}$$

where E_x is the expectation for $\{W_l\}$ provided $W_0=x$,

$$Z(x,m;n)=\int Z(m,x;n,y)\,dy\quad\text{and}\quad \hat Z(m;y,n)=\int Z(m,x;y,n)\,dx\,. \tag{2.5}$$

Denote $f_k(z,v)=\exp F_k(z,v)$, $\Lambda(z)=E_Qf_0(0,z)$, $\Lambda=\int p(z)\Lambda(z)\,dz$, and $h_k(z,v)=\Lambda^{-1}(f_k(z,v)-\Lambda(v-z))$ where E_Q is the expectation on the probability space (Ω,Q). Clearly,

$$E_Qh_k(z,v)=0\,. \tag{2.6}$$

Assume that for all $v\in\mathbb{R}^d$,

$$q_0(v)\overset{\text{def}}{=}\Lambda^{-1}p(v)\Lambda(v)\leqq C_1e^{-\gamma_1|v|} \tag{2.7}$$

and for all $z,v,w\in\mathbb{R}^d$,

$$\begin{aligned} &p(z)p(w-v)|E_Qh_0(0,z)h_0(v,w)|\\ &\quad\leqq C_1\varepsilon\rho(|v|-2(|z|+|w-v|))\exp(-2\gamma_1(|z|+|w-v|)) \end{aligned} \tag{2.8}$$

for some $1 \leqq C_1 < \infty$, $\gamma_1 > 0$, and a sufficiently small $\varepsilon > 0$, where $\rho(u) = u^{-(d+\delta_0)}, \delta_0 > 0$ for $u \geqq 1$ and $\rho(u) = 1$ for $u < 1$. The last assumption incorporates two conditions. The first one holds true if the correlation coefficient

$$(2.9) \qquad (E_Q h_l^2(0,z))^{-1/2}(E_Q h_l^2(v,w))^{-1/2}|E_Q h_l(0,z)h_l(v,w)|$$

decreases sufficiently fast as $|v| \to \infty$ which follows if one assumes a corresponding ρ-mixing condition (see [Do]). The second condition says that the variance of f_l is sufficiently small.

For all $v \in \mathbb{R}^d$ set,

$$(2.10) \quad q_1(v) = ce^{-\gamma_1|v|} \quad \text{with } c = \left(\int e^{-\gamma_1|v|}\,dv\right)^{-1} = \gamma_1^d((d-1)!s_{d-1})^{-1},$$

where s_{d-1} is the volume of a $(d-1)$-dimensional unit sphere. Let $X_1, X_2, \ldots$ and $Y_1, Y_2, \ldots$ be two independent sequences of i.i.d. random vectors from $\mathbb{R}^d$ constructed on the same probability space with distributions

$$(2.11) \quad P\{X_i \in \Gamma\} = \int_\Gamma q_0(v)\,dv \quad \text{and} \quad P\{Y_i \in \Gamma\} = \int_\Gamma q_1(v)\,dv, \quad \Gamma \subset \mathbb{R}^d .$$

For $r = 0, 1, \ldots, l$ put $S_l^r = \sum_{i=1}^{l-r} X_i + \sum_{j=1}^{r} Y_j$, then

$$(2.12) \qquad P\{S_l^r \in \Gamma\} = \int_\Gamma q_r(l,v)\,dv, \quad \Gamma \subset \mathbb{R}^d$$

where

$$(2.13) \qquad q_r(l,w) = \int \cdots \int \prod_{j=0}^{l-1} q_{i_j}(v_{j+1} - v_j)\,dv_1 \cdots dv_{l-1},$$

$i_j = 0$ or 1, $\sum_{j=0}^{l-1} i_j = r$, $v_0 = 0$, and $v_l = w$. Since the sum S_l^r and so its distribution do not depend on the order of summands the right hand side of (2.13) does not depend also on a particular order of i_j's and it depends only on the number of i_j's equal to 1.

Put

$$(2.14) \qquad \varphi(x,m) = 1 + \sum_{r \geqq 1} \sum_{m \leqq k_1 < k_2 < \cdots < k_r} \int\int q_0(k_1 - m,\ z - x) \times U(k^{(r)}; z, v)\,dz\,dv$$

and

$$(2.15) \qquad \psi(y,n) = 1 + \sum_{r \geqq 1} \sum_{k_1 < k_2 < \cdots < k_r < n} \int\int U(\mathbf{k}^{(r)}; z, v) \times q_0(n - k_r - 1, y - v)\,dz\,dv$$

where for $\mathbf{k}^{(r)} = (k_1, \ldots, k_r)$ with $k_1 < \cdots < k_r$,

$$(2.16)\quad U(\mathbf{k}^{(r)}, z, v) = \int \cdots \int p(\tilde z_1 - z) h_{k_1}(z, \tilde z_1) q_0(k_2 - k_1 - 1, z_2 - \tilde z_1) \times p(\tilde z_2 - z_2) h_{k_2}(z_2, \tilde z_2) \times \cdots \times q_0(k_r - k_{r-1} - 1,\ z_r - \tilde z_{r-1}) \times p(v - z_r) h_{k_r}(z_r, v) \times d\tilde z_1\, dz_2\, d\tilde z_2 \cdots dz_{r-1}\, d\tilde z_{r-1}\, dz_r\ .$$

In the next section I shall prove the following result.

Theorem 2.1. *Suppose that* (2.7) *and* (2.8) *hold true and* ε *in* (2.8) *is small enough, namely that* $\kappa = C_3\kappa_1 < 1$ *with* κ_1 *from* (3.27) *and* C_3 *from* (3.29). *Then the series* (2.14) *and* (2.15) *converge in* $L^2(\Omega, Q)$. *Furthermore, for all* $x, y \in \mathbb{R}^d$, Q*-almost surely* (*a.s.*),

$$(2.17)\qquad \lim_{n\to\infty} \Lambda^{-(n-m)} Z(x, m; n) = \operatorname*{L.i.m.}_{n\to\infty} \Lambda^{-(n-m)} Z(x, m; n) = \varphi(x, m) > 0\ ,$$

$$(2.18)\qquad \lim_{m\to-\infty} \Lambda^{-(n-m)} \hat Z(m; y, n) = \operatorname*{L.i.m.}_{m\to-\infty} \Lambda^{-(n-m)} \hat Z(m; y, n) = \psi(y, n) > 0\ ,$$

where L.i.m. *denotes the limit in* $L^2(\Omega, Q)$,

$$(2.19)\qquad \varphi(x, m) = \Lambda^{-(n-m)} \int Z(x, m; y, n) \varphi(y, n)\, dy\ ,$$

and

$$(2.20)\qquad \psi(y, n) = \Lambda^{-(n-m)} \int \psi(x, m) Z(x, m; y, n)\, dx$$

for all $x, y \in \mathbb{R}^d$ *and* $n, m \in \mathbb{Z}$ *with* $m < n$. *Finally, there exists* $C > 0$ *such that for all* $x, y \in \mathbb{R}^d$,

$$(2.21)\qquad E_Q(Z(x, m; y, n))^2 \leqq C\Lambda^{2(n-m)} (n-m)^{-d/2} \sum_{r=0}^{n-m} \kappa^r q_r(n-m, y-x)$$

with κ *defined above.*

Observe that $\varphi(x, s)$ and $\psi(x, s)$ have the same distributions (which does not depend on x and s by the stationarity) since they are obtained from each other by the time reversal. In view of (2.19) and (2.20) one can view $\varphi(x, s)$ and $\psi(x, s)$ as (random) densities of backward and forward (random) stationary distributions of the random walk in a random environment. In physical literature (see [KS]) the influence of the random environment (or disorder) is measured by

$$E_Q(Z(x,m;n))^2(E_Q Z(x,m;n))^{-2} = \Lambda^{-2(n-m)} E_Q(Z(x,m;n))^2 \tag{2.22}$$

which by (2.17) converges to $E_Q(\varphi(x,m))^2$, and so, according to physical terminology, this is a weak coupling case.

Let $X_1, X_2, \ldots$ be again i.i.d. random vectors with the distribution given by the first part of (2.11). Since $Q_{x,y} = Q_{y,x}$ then $\Lambda(v) = \Lambda(-v)$, and so

$$EX_i = \Lambda^{-1} \int_\Gamma p(v)\Lambda(v)v\,dv = 0\,. \tag{2.23}$$

Introduce the covariance matrix

$$A = EX_i X_i^* = \Lambda^{-1} \int p(v)\Lambda(v)vv^*\,dv\,, \tag{2.24}$$

where vv^* is the $d \times d$-matrix $(v^{(k)}v^{(l)};\ k,l = 1,\ldots,d)$ if $v = (v^{(1)},\ldots,v^{(d)})$, and assume that the matrix A is positive definite. This will always be the case if $p(Uv) = p(v)$ and $\Lambda(Uv) = \Lambda(v)$ for any orthogonal matrix U (in particular, if the distribution of $W_1 - W_0$ is invariant under rotations and $Q_{x,y}$ depend only on $|x-y|$) since then A will be a scalar multiple of the unit matrix. Set

$$r(l,v) = (2\pi l)^{-d/2}(\det A)^{-1} \exp\left(-\frac{1}{2\sqrt{l}}\langle A^{-1}v, v\rangle\right) \tag{2.25}$$

where $\langle\cdot,\cdot\rangle$ denotes the inner product. Generalizing the arguments from [Bo] I shall derive in Sect. 4 the following limit theorem.

Theorem 2.2. *Suppose again that* ε *in* (2.8) *is small enough. Then for all* $x, y \in \mathbb{R}^d$, $m, n \in \mathbb{Z}$, *and for any continuous function* f *on* $\mathbb{R}^d$ *with at most polynomial growth, i.e.*

$$\lim_{|v|\to\infty} |v|^{-N} f(v) = 0 \tag{2.26}$$

for some $N \in \mathbb{Z}_+$, *one has* Q*-a.s.*,

$$\begin{aligned} &\lim_{n\to\infty} (Z(x,m;n))^{-1} \int Z(x,m;y,n) f\left(\frac{y-x}{\sqrt{n-m}}\right) dy \\ &= \lim_{m\to-\infty} (\hat{Z}(m;y,n))^{-1} \int f\left(\frac{y-x}{\sqrt{n-m}}\right) Z(x,m;y,n)\,dx \\ &= \int f(z) r(1,z)\,dz\,. \end{aligned} \tag{2.27}$$

Furthermore, there exists $\delta_0 > 0$ *such that for any continuous function* f *satisfying*

$$\int |f(v)| e^{-\delta_0 |v|}\,dv < \infty \tag{2.28}$$

one has

$$\begin{aligned} &\operatorname*{L.i.m.}_{n\to\infty} \Lambda^{-(n-m)} \int Z(x,m;y,n) f\left(\frac{y-x}{\sqrt{n-m}}\right) dy \\ &= \varphi(x,m) \int f(z) r(1,z)\,dz \end{aligned} \tag{2.29}$$

and

$$\text{(2.30)}\qquad \underset{m\to-\infty}{\text{L.i.m.}}\, \Lambda^{-(n-m)} \int f\left(\frac{y-x}{\sqrt{n-m}}\right) Z(x,m;y,n)\,dx = \psi(y,n) \int f(z)r(1,z)\,dz\,.$$

It follows by (2.17) *and* (2.18) *that for any continuous function* f *satisfying* (2.28) *the limit* (2.27) *remains true in the sense of convergence in probability in place of* Q*-a.s. convergence. If* f *is continuous and satisfies* (2.26) *then the limits* (2.29) *and* (2.30) *remain true in the sense of* Q*-a.s. convergence.*

Employing the same proof with integrals replaced by sums one obtains the lattice versions of Theorems 2.1 and 2.2. Namely, let $\{F_k(x,y);\ x,y\in\mathbb{Z}^d\},\ k\in\mathbb{Z},\ d\geqq 3$ be a sequence of i.i.d. random fields on $\mathbb{Z}^d\times\mathbb{Z}^d$ with distributions $Q_{x,y}=Q_{y,x}$ depending only on $y-x$ and let $W_0,W_1,\dots$ be a Markov chain on $\mathbb{Z}^d$ independent of these random fields with transition probabilities $P\{W_{n+1}=y|W_n=x\}=p(y-x)$ satisfying (2.2).

Theorem 2.3. *Suppose that* (2.7) *and* (2.8) *hold true with* ε *small enough. Then Theorems* 2.1 *and* 2.2 *hold true with the corresponding modifications where integrals are replaced by sums.*

Next, I shall consider a general continuous time stochastic process in a random environment, formulate corresponding versions of Theorems 2.1 and 2.2 for it, and, finally, apply these results to solutions of the Eqs. (1.1)–(1.3). Let (Ω,Q) be a probability space generated by a family of $\mathbb{R}_+=(0,\infty)$-valued random fields $\{Z(x,s;y,t);\ x,y\in\mathbb{R}^d\},\ s,t\in\mathbb{R},\ s<t,\ d\geqq 3$ on $\mathbb{R}^d\times\mathbb{R}^d$ with $L^2(\Omega,Q)$-continuous in x,y,s,t realizations. I assume that the distribution of $Z(x,s;y,t)$ depends only on $y-x$ and $t-s$, the distributions of $Z(x,s;y,t)$ and $Z(y,s;x,t)$ are the same, and that the random fields $Z(\cdot,s;\cdot,t)$ and $Z(\cdot,\tilde{s};\cdot,\tilde{t})$ are independent if the open intervals (s,t) and $(\tilde{s},\tilde{t})$ are disjoint. Assume also that for any $s<u<t$,

$$\text{(2.31)}\qquad \int Z(x,s;z,u)Z(z,u;y,t)\,dz = Z(x,s;y,t)\,.$$

Let $p(u,z)=p(u,-z)$ be the transition density of a continuous process W_t with independent increments in $\mathbb{R}^d$, i.e.

$$\text{(2.32)}\qquad \int p(u-s,z-x)p(t-u,y-z)\,dz = p(t-s,y-x)$$

for any $s<u<t$. Then it is known (see [GS, Ch. 4]) that W_t has Gaussian increments, and so $p(u,z)$ is a Gaussian density. Set $\Lambda(u,w)=(p(u,w))^{-1} E_Q Z(0,0;w,u)$ and $\Lambda(u)=\int p(u,w)\Lambda(u,w)\,dw$ then

$$\text{(2.33)}\qquad \int \Lambda(u-s,z-x)p(u-s,z-x)\Lambda(t-u,y-z)p(t-u,y-z)\,dz = \Lambda(t-s,y-x)p(t-s,y-x)$$

and

$$\Lambda(u-s)\Lambda(t-u)=\Lambda(t-s)\,.$$

From the assumptions on the above random fields it follows that $\Lambda(u)$ is continuous in u, and so

(2.34) $$\Lambda(u) = \Lambda^u, \quad u \geqq 0$$

for some constant $\Lambda > 0$. Set $Z(x,s;t) = \int Z(x,s;y,t)\,dy$, $\hat{Z}(s;y,t) = \int Z(x,s;y,t)\,dx, q_0(u,w) = \Lambda^{-u}p(u,w)\Lambda(u,w)$, and

(2.35) $$f(z,s;v,u) = Z(z,s;v,u)(p(u-s,v-z))^{-1}\,,$$
$$h(z,s;v,u) = \Lambda^{-(u-s)}(f(z,s;v,u) - \Lambda(u-s,v-z))\,.$$

In place of (2.7) and (2.8) I assume now that for all $z \in \mathbb{R}^d$, and $1 \leqq u < 2$,

(2.36) $$\Lambda^{-u}p(u,z)\Lambda(u,z) \leqq C_1 e^{-\gamma_1|z|}$$

and

(2.37) $$p(u,z)p(u,w-v)|E_Q h(0,0;z,u)h(v,0;w,u)|$$
$$\leqq C_1\varepsilon\rho(|v| - 2(|z| + |w-v|))\exp(-2\gamma_1(|z| + |w-v|))\,,$$

where $C_1, \gamma_1, \varepsilon$, and ρ are the same as in (2.8). Define also $\varphi(x,s)$ and $\psi(y,s)$ by (2.14)–(2.16) with $m = 0$, $n = 0$, and $p(v) = p(1,v)$ but taking $h^s_{k_i}(z_i,\tilde{z}_i) = h(z_i, s+k_i; \tilde{z}_i, s+k_i+1)$ in place of $h_{k_i}(z_i,\tilde{z}_i)$ in (2.16). The following counterparts of Theorems 2.1 and 2.2 will be proved in Sect. 5.

Theorem 2.4. *Suppose that* (2.36) *and* (2.37) *hold true for* ε *small enough. Then the series for* $\varphi(x,s)$ *and* $\psi(y,t)$ *converge in* $L^2(\Omega,Q)$ *and they are* $L^2(\Omega,Q)$*-continuous in* x,s,y,t. *Furthermore* Q*-a.s.*,

(2.38) $$\underset{t\to\infty}{\text{L.i.m.}}\,\Lambda^{-(t-s)}Z(x,s;t) = \varphi(x,s) > 0\,,$$

(2.39) $$\underset{s\to-\infty}{\text{L.i.m.}}\,\Lambda^{-(t-s)}\hat{Z}(s;y,t) = \psi(y,t) > 0\,,$$

(2.40) $$\varphi(x,s) = \Lambda^{-(t-s)}\int Z(x,s;y,t)\varphi(y,t)\,dy\,,$$

and

(2.41) $$\psi(y,t) = \Lambda^{-(t-s)}\int \psi(x,s)Z(x,s;y,t)\,dx$$

for all $x, y \in \mathbb{R}^d$ *and* $s < t$. *Finally, there exists* $C > 0$ *such that for all* $x,y \in \mathbb{R}^d$ *and* $s,t \in \mathbb{R}$ *satisfying* $t - s \geqq 1$,

(2.42) $$E_Q(Z(x,s;y,t))^2 \leqq C\Lambda^{2(t-s)}(t-s)^{-d/2}\tilde{q}(t-s,y-x)\,,$$

where

(2.43) $$\tilde{q}(u,z) = \sum_{r=0}^{\max(1,[u])}\kappa^r q_{r+1}(\max(1,[u]),z)$$

and $[\,\cdot\,]$ *denotes the integral part.*

Theorem 2.5. *Assume that conditions of Theorem* 2.4 *hold true. Then for all* $x, y \in \mathbb{R}^d$, $s < t$, *and for any continuous function* f *on* $\mathbb{R}^d$ *with at most polynomial growth* Q-*a.s.*,

$$\lim_{t\to\infty} (Z(x,s;t))^{-1} \int Z(x,s;y,t) f\left(\frac{y-x}{\sqrt{t-s}}\right) dy \tag{2.44}$$
$$= \lim_{s\to-\infty} (\hat{Z}(s;y,t))^{-1} \int f\left(\frac{y-x}{\sqrt{t-s}}\right) Z(x,s;y,t)\, dx$$
$$= \int f(z) r(1,z)\, dz$$

where $r(1,z)$ *is given by* (2.25). *Furthermore, there exists* $\delta_0 > 0$ *such that for any continuous function* f *satisfying* (2.28) *one has*

$$\operatorname*{L.i.m.}_{t\to\infty} \Lambda^{-(t-s)} \int Z(x,t;y,s) f\left(\frac{y-x}{\sqrt{t-s}}\right) dy \tag{2.45}$$
$$= \varphi(x,s) \int f(z) r(1,z)\, dz$$

and

$$\operatorname*{L.i.m.}_{s\to-\infty} \Lambda^{-(t-s)} \int f\left(\frac{y-x}{\sqrt{t-s}}\right) Z(x,s;y,t)\, dx \tag{2.46}$$
$$= \psi(y,t) \int f(z) r(1,z)\, dz\, .$$

It follows that for any continuous function f *satisfying* (2.28) *the limit* (2.44) *remains true in the sense of convergence in probability in place of* Q-*a.s. convergence. If* f *is continuous and satisfies* (2.26) *then the limits* (2.45) *and* (2.46) *remain true in the sense of* Q-*a.s. convergence.*

Next, I shall discuss the application of Theorems 2.4 and 2.5 to the Eqs. (1.1)–(1.3). Let $\Psi(t) = \Psi(x,t)$, $x \in \mathbb{R}^d$ be a temporally and spatially homogeneous C^3-valued Brownian motion (see [Ku2]) with the mean $E_Q\Psi(t) = 0$ and let (Ω, Q) be the corresponding probability space. This means that Q is invariant under the space and time shifts θ_x, $x \in \mathbb{R}^d$ and η_t, $t \in \mathbb{R}$, respectively, $\Psi(x,t)$ is continuous in t and C^3-function in x, and for any $0 \leqq t_0 < \cdots < t_l < T, \Psi(t_0)$, $\Psi(t_{i+1}) - \Psi(t_i)$, $i = 0, \ldots, l-1$ are independent random variables with values in the space of C^3 functions on $\mathbb{R}^d$. Note that for any such Brownian motion $\Psi(t)$ its every finite dimensional restriction $(\Psi(x_1,t), \ldots, \Psi(x_n,t))$ is a nd-dimensional Gaussian process with independent increments and zero mean (see, for instance, [GS, Ch. 4]). The existence of such Brownian motion $\Psi(x,t)$ for any smooth spatial covariance function $A(x,y) = A(y-x)$ follows from arguments of Chapters 3 and 4 in [Ku2] (see Theorems 3.2.1, 4.2.5, and 4.2.8 there). Actually, one always has a representation (see [Ku1, Proposition 2.2.8]),

$$\Psi(x,t) = \sum_{i=1}^{\infty} f^{(i)}(x) b_t^{(i)}\, ,$$

where $b_t^{(i)}$ is a sequence of independent one dimensional Brownian motions, $f^{(i)}(x)$ is a sequence of smooth functions, and for each $x \in \mathbb{R}^d$ and

$t \geqq 0$ this series converges in $L^2(\Omega, Q)$. Then it follows also that $A(y-x) = \sum_i f^{(i)}(x) f^{(i)}(y)$. Usually, one considers a Brownian motion $\Psi(x,t)$ for non-negative t only, but taking $\Psi_s(x,t) = \Psi(x, t-s)$ I can employ Brownian motions starting at any time $s \in \mathbb{R}$.

The process $\Psi(t)$ is a martingale with spatial parameters and stochastic integrals with respect to such martingales are studied in [Ku2]. The stochastic heat equation (1.3) should be considered in the integral form

$$w(x,t) = w(x,s) + v\int_s^t \Delta w(x,u)\,du + \frac{\lambda}{2v}\int_s^t w(x,u)\Psi(x,du) \tag{2.47}$$

where the last term is an Ito stochastic integral (see Remark 2.8). The solution of (2.47) with the condition $w(x,s) = f(x)$ can be written via a Feynman–Kac type formula in the form (see [Ku2, p. 308]),

$$w(x,t) = w_f(x,s;t) = \int f(y) Z(y,s;x,t)\,dy \tag{2.48}$$

with the "stochastic heat kernel" given by

$$Z(y,s;x,t) = e^{-\frac{\lambda^2 A(0)(t-s)}{8v^2}} p(t-s, x-y) \times E_y\left(\exp\int_s^t \frac{\lambda}{2v}\Psi(W_{t-u}, du)\,\middle|\, W_{t-s} = x\right), \tag{2.49}$$

where, W_u is the $2v$-time scaled Wiener process independent of Ψ, E_y is the expectation for W_u given $W_0 = y$, and

$$p(u,z) = (4\pi v u)^{-d/2} \exp\left(-\frac{|z|^2}{4vu}\right). \tag{2.50}$$

Observe that stochastic integrals depend only on increments of $\Psi(\cdot, u)$, and so the distribution of any $Z(x,s;y,t)$ defined by $\Psi_r(x,u) = \Psi(x, u-r)$ will be the same for all $r \leqq s$. Letting $r \to -\infty$ I obtain a probability measure on the space of functions of x,s,y,t with $s < t$ which determines the distributions of $Z(x,s;y,t)$ for all real $s < t$, and so I may speak about these random fields for all real $s < t$ as required for the application of Theorems 2.4 and 2.5. Of course, this means also that I consider the solutions of (2.47) in the weak sense, i.e. only their distributions and not their specific representation make any sense. It is clear that (2.31)–(2.34) hold true and by the definition of the Ito stochastic integral, $Z(x,s;z,t)$ and $Z(v,\tilde{s};w,\tilde{t})$ are independent if the open intervals (s,t) and $(\tilde{s},\tilde{t})$ are disjoint. Employing the generalized Ito formula from [Ku2] I shall show in Sect. 5 that the condition (2.36) is automatically satisfied in our circumstances. On the other hand, in order to satisfy (2.37) one has to assume that the random field Ψ has sufficiently small and fast decreasing spatial covariances. Namely, suppose that for some constants $C > 0$ and a sufficiently small $\varepsilon > 0$,

$$\sup_{x : |x| \geqq r} A(x) \leqq C\varepsilon\rho(r) \quad \text{for all } r \geqq 0 \tag{2.51}$$

where ρ is the same as in (2.37). Observe, that since Ψ is a Gaussian random field then (2.51) implies the ergodicity of the spatial shift θ_x, $x \in \mathbb{R}^d$ (see Sects. 1 and 2 of Ch. 2 in [Te]).

Theorem 2.6. *If* (2.51) *holds true with* ε *small enough then the conditions of Theorems* 2.4 *and* 2.5 *are satisfied with* $Z(x,s;y,t)$ *defined by* (2.49), *and so these theorems remain true for the stochastic heat kernel describing the evolution of solutions of the Eq.* (2.47).

If w is the solution of (2.47) then a direct computation shows (cf. Theorem 4.1 in [HLØUZ2]) that $v = \frac{2\nu}{\lambda} \log w$ and $u = -\nabla v$ are solutions of the Eqs. (1.2) and (1.1), respectively, considered in the integral form similar to (2.47). This computation made in the Stratonovich form (and passing back to the Ito differentials afterwards) is essentially the same as in the deterministic case and its justification follows from Sects. 6.1 and 6.2 in [Ku2]. I shall show in Sect. 5 that the function $f(y) = \psi(y,s)$ can be taken as an initial condition in (2.47) and that $\Lambda = 1$ in our circumstances. Then by (2.48) and (2.41), $w(x,s;t) = \psi(x,t)$, and so $w(x,s;t)$ has the same distribution on (Ω, Q) for all $t \geqq s$. Since the corresponding solution of (1.2) has the form

$$v(x,s;t) = \frac{2\nu}{\lambda} \log w(x,s;t) = \frac{2\nu}{\lambda} \log \psi(x,t) \tag{2.52}$$

then the distribution of $v(x,s;t)$ on (Ω, Q) is the same for all $t > s$. Finally, the solution of (1.1),

$$u(x,s;t) = -\nabla v(x,s;t) = -2\nu\lambda^{-1} \nabla \log \psi(x,t) \tag{2.53}$$

has the same distribution on (Ω, Q) for all $t > s$. Thus (2.40) and (2.41) yield backward and forward stationary distributions for solutions of (1.1)–(1.3). Moreover, these distributions are attracting for natural classes of solutions. Namely, the following result (which will be proved in Sects. 5 and 6) holds true.

Theorem 2.7. *The distributions of* $\psi(x,t)$, *of* $\frac{2\nu}{\lambda} \log \psi(x,t)$, *and of* $-2\nu\lambda^{-1}\nabla \log \psi(x,t)$ *are stationary in time distributions* (*invariant measures*) *for the Eqs.* (1.3), (1.2), *and* (1.1), *respectively. Furthermore, suppose that the initial* (*at time* s) *condition* $f = f(y)$, $y \in \mathbb{R}^d$ *in* (2.47) *is a* $L^2(\Omega, Q)$ *stationary under the action of* $\mathbb{R}^d$ *random field independent of the Brownian motion with spatial parameters* $\Psi(x,t)$ *for all* $t \geqq s$. *Then the solution of* (1.3) (*i.e. of* (2.47)) *given by* (2.48) *satisfies*

$$\lim_{t\to\infty} Q\{|w_f(x,s;t) - \psi(x,t) E_Q f(0)| > \delta\} = 0, \quad \forall \delta > 0 \,. \tag{2.54}$$

The corresponding results hold true for solutions of (1.2) *and* (1.1).

Remark 2.8. Let $w(x,t) = w_f(x,s;t) = \Lambda^{-(t-s)} \int f(y) Z(y,s;x,t)\,dy$ and $v(x,s) = v_g(x,s;t) = \Lambda^{-(t-s)} \int Z(x,s;y,t) g(y)\,dy$ where $Z(\cdot,\cdot;\cdot,\cdot)$ is as in Theorem 2.1

or as in Theorem 2.4 and f and g are $L^2(\Omega, Q)$ stationary under the $\mathbb{R}^d$-action random fields independent of the σ-algebras $\mathscr{F}_s^\infty$ and $\mathscr{F}_{-\infty}^t$, respectively (where $\mathscr{F}_s^t$ is the σ-algebra generated by all $Z(y,\alpha;z,\beta)$ with $s \leqq \alpha \leqq \beta \leqq t$). Assume ergodicity of the $\mathbb{R}^d$-action on (Ω, Q) (which holds automatically in the framework of Theorems 2.6 and 2.7) then in the same way as in Theorem 2.7 it follows using Theorem 6.1 that for any $\delta > 0$,

$$\text{(2.55)} \qquad \lim_{t\to\infty} Q\{|w(x,t) - \psi(x,t)E_Q f(0)| > \delta\} + \lim_{s\to-\infty} Q\{|v(x,s) - \varphi(x,s)E_Q g(0)| > \delta\} = 0\,.$$

Remark 2.9. Observe that if (2.47) (and so, (1.1)–(1.3)) is considered in the Stratonovich form then the solution still can be represented by (2.49) but the first factor in the right hand side of (2.49) should be omitted. Then Theorems 2.4 and 2.5 can be applied again with $\Lambda = \exp(\lambda^2 A(0)/8v^2)$ and all arguments remain valid. In particular, the stationary distribution of the Burgers equation is given by (2.53) but the solution of (2.47) at time t with the initial condition $f(y) = \psi(y,s)$ at time s will be given now by $\psi(x,t)\Lambda^{t-s}$.

Remark 2.10. In the proofs I do not use in a substantial way temporal homogeneity of the corresponding random fields. Some versions of the main results can be obtained without this condition, as well, using the corresponding versions of local limit theorems for sums of non identically distributed independent random vectors and assuming uniform in time bounds in (2.7) and (2.8) (in (2.36) and (2.37) in the continuous time case). I can do without temporal homogeneity of the Markov chain W_k (of the process W_t, in the continuous time case), as well.

Remark 2.11. Making easy modifications in the proof I can derive certain versions of Theorems 2.2, 2.3, and 2.5 without assuming that $p(u,z) = p(u,-z)$ and that the distributions of $Z(x,s;y,t)$ and of $Z(y,s;x,t)$ are the same. In this case $b = u^{-1}\int vq_0(u,v)dv$ may be $\neq 0$ and in Theorems 2.2 and 2.5 one should replace $y-x$ by $y-x-b(n-m)$ and by $y-x-b(t-s)$, respectively.

3 $\mathbf{L^2(\Omega, Q)}$-estimates

In this section I shall prove Theorem 2.1. Writing $f_k(W_i, W_{i+1}) = \Lambda(h_k(W_i, W_{i+1}) + \Lambda^{-1}\Lambda(W_{i+1} - W_i))$ I obtain from (2.4) and (2.16) that

$$\text{(3.1)} \qquad Z(x,m;y,n) = \Lambda^{n-m}(q_0(n-m, y-x) + \tilde{Z}(x,m;y,n))\,,$$

where

$$\text{(3.2)} \qquad \tilde{Z}(x,m;y,n) = \sum_{r\geqq 1}\ \sum_{m\leqq k_1 < k_2 < \cdots < k_r < n} V(\mathbf{k}^{(r)};x,m;y,n)$$

and for $\mathbf{k}^{(r)} = (k_1,\ldots,k_r)$, $k_1 < k_2 < \cdots < k_r$,

$$\text{(3.3)} \qquad V(\mathbf{k}^{(r)};x,m;y,n) = \int\int q_0(k_1 - m, z-x) \times U(\mathbf{k}^{(r)};z,v)q_0(n-k_r-1, y-v)\,dz\,dv\,.$$

Observe that from the local limit theorems (see [Pe, Sect. 7.2]) together with (2.7) it follows that for some $C_2 > 0$,

$$|q_0(l,v) - r(l,v)| \leqq \frac{C_2}{l^{\frac{d}{2}+1}} \quad \text{for all } v \in \mathbb{R}^d \text{ and } l \geqq 1\,. \tag{3.4}$$

In particular, if C_2 is chosen large enough then

$$q_0(l,v) \leqq C_2 l^{-\frac{d}{2}} \quad \text{for all } v \in \mathbb{R}^d \text{ and } l \geqq 1 \tag{3.5}$$

and

$$q_0(l,v) = r(l,v)(1 + l^{-1/2} c_R(l,v))\,, \tag{3.6}$$

provided $|v| \leqq R l^{1/2}$ where

$$\sup_{l,v} c_R(l,v) \stackrel{\text{def}}{=} c_R < \infty\,. \tag{3.7}$$

Writing each product of integrals $U(\mathbf{k}^{(r)}; v, \tilde{v}) U(\mathbf{1}^{(s)}; w, \tilde{w})$ as one multiple integral and applying the conditional expectations $E_Q(\,\cdot|\mathscr{F}_j)$ with respect to the σ-algebras $\mathscr{F}^j_{-\infty}$ generated by all random variables $F_i(x,y)$ with $i < j$ I derive from (2.16) and the independence of the fields F_i, $i \in \mathbb{Z}$ that the sums in (2.14) and (2.15) consist of uncorrelated random variables. Thus the series in (2.14) converge in $L^2(\Omega, Q)$ if

$$\sum_{r \geqq 1} \sum_{m \leqq k_1 < \cdots < k_r} W(\mathbf{k}^{(k)}; x, m) < \infty \tag{3.8}$$

where

$$W(\mathbf{k}^{(k)}; x, m) = E_Q\left(\iint q_0(k_1 - m, z - x) U(\mathbf{k}^{(r)}; z, v)\, dz\, dv\right)^2 \tag{3.9}$$

and similarly for the series in (2.15). It is easy to see that

$$\begin{aligned} W(\mathbf{k}^{(r)}; x, m) \\ &= \int \cdots \int \mathscr{W}(\mathbf{k}^{(r)}; x, m; \mathbf{z}^{(r)}, \tilde{\mathbf{z}}^{(r)}, \mathbf{v}^{(r)}, \tilde{\mathbf{v}}^{(r)})\, d\mathbf{z}^{(r)}\, d\tilde{\mathbf{z}}^{(r)} d\mathbf{v}^{(r)}\, d\tilde{\mathbf{v}}^{(r)} \end{aligned} \tag{3.10}$$

where for $\mathbf{z}^{(r)} = (z_1, \ldots, z_r), \tilde{\mathbf{z}}^{(r)} = (\tilde{z}_1, \ldots, \tilde{z}_r), \mathbf{v}^{(r)} = (v_1, \ldots, v_r), \tilde{\mathbf{v}}^{(r)} = (\tilde{v}_1, \ldots, \tilde{v}_r)$,

$$\begin{aligned} &\mathscr{W}(\mathbf{k}^{(r)}; x, m; \mathbf{z}^{(r)}, \tilde{\mathbf{z}}^{(r)}, \mathbf{v}^{(r)}, \tilde{\mathbf{v}}^{(r)}) \\ &= q_0(k_1 - m, z_1 - x) q_0(k_1 - m, v_1 - x)(E_Q h_{k_1}(z_1, \tilde{z}_1) h_{k_1}(v_1, \tilde{v}_1)) \\ &\quad \times p(\tilde{z}_1 - z_1)\, p(\tilde{v}_1 - v_1) \cdots q_0(k_r - k_{r-1} - 1, z_r - \tilde{z}_{r-1}) \\ &\quad \times q_0(k_r - k_{r-1} - 1, v_r - \tilde{v}_{r-1}) \\ &\quad \times (E_Q h_{k_r}(z_r, \tilde{z}_r) h_{k_r}(v_r, \tilde{v}_r))\, p(\tilde{z}_r - z_r)\; p(\tilde{v}_r - v_r)\,. \end{aligned} \tag{3.11}$$

It follows from (2.8) that

$$(3.12)\quad \beta_l(z,u;v,w) \stackrel{\text{def}}{=} p(u-z)p(w-v)|E_Q h_l(z,u)h_l(v,w)|$$
$$\leqq C_1 K\varepsilon \exp(-\gamma_1(|u-z|+|w-v|))\min(1,|v-z|^{-(d+\delta_0)}),$$

where

$$K = \sup_{z,u,v,w}(e^{-\gamma_1(|u-z|+|w-v|)}(|v-z|^{d+\delta_0}+1)\rho(|v-z|-2|u-z|-2|w-v|))$$
$$\leqq \sup_{z,u,v,w}(e^{-\gamma_1(|u-z|+|w-v|)}(2|u-z|+2|w-v|+1)^{d+\delta_0}$$
$$+(|v-z|^{d+\delta_0}+1)\min(1,(\tfrac{1}{2}|v-z|)^{-(d+\delta_0)})) \leqq \infty .$$

Therefore, integrating first the exponent in (3.12) in u and w, then estimating $q_0(j,z-\tilde{z})$ by (3.5), and, finally, integrating the remaining part in the right hand side of (3.12) in z I derive for any $j \geqq 1$ that

$$(3.13)\quad \iiiint q_0(j,z-\tilde{z})q_0(j,v-\tilde{v})\beta_l(z,u;v,w)\,dz\,dv\,du\,dw$$
$$\leqq C_1C_2K\varepsilon j^{-d/2}\left(\int e^{-\gamma_1|x|}\,dx\right)^2\int \min(1,|y|^{-(d+\delta_0)})\,dy$$
$$\leqq C_1C_2K\varepsilon j^{-d/2}c^{-2}(b_d+s_{d-1}\delta_0^{-1})$$

where b_d is the volume of the d-dimensional unit ball, s_{d-1} is the volume of the $(d-1)$-dimensional unit sphere, and c is the same as in (2.10). Finally, I obtain by (3.9)–(3.13) that

$$(3.14)\quad \sum_{m\leqq k_1<\cdots<k_r} W(\mathbf{k}^{(r)};x,m) \leqq \kappa_0^r$$

where $\kappa_0 = C_1C_2K\varepsilon c^{-2}d(d-2)^{-1}(b_d+s_{d-1}\delta_0^{-1})$. Thus if ε in (2.8) is chosen so small that $\kappa_0 < 1$ then (3.8) holds true.

It remains to establish (2.17)–(2.21). Since the series in (2.14) converges in $L^2(\Omega,Q)$ it is clear from (3.2) that

$$(3.15)\quad \operatorname*{L.i.m.}_{n\to\infty} \int \tilde{Z}(x,m;y,n)\,dy = \varphi(x,m)-1$$

which together with (3.1) gives the $L^2(\Omega,Q)$-limit in (2.17), whilst the corresponding limit in (2.18) follows similarly. In order to derive the Q-a.s. limits in (2.17) and (2.18) I observe that $\Lambda^{-(n-m)}Z(x,m;n)$ is a martingale in n and $\Lambda^{-(n-m)}\hat{Z}(m;y,n)$ is a backward martingale in m. Indeed, $E_Q Z(x,k;n) = \Lambda^{n-k}$ for any $k<n$, and so

$$\Lambda^{-(n-m)}E_Q(Z(x,m;n)|\mathscr{F}^k_{-\infty})$$
$$= \Lambda^{-(n-m)}E_Q\left(\int Z(x,m;y,k)Z(y,k;n)\,dy|\mathscr{F}^k_{-\infty}\right)$$
$$= \Lambda^{-(n-m)}\int Z(x,m;y,k)E_Q Z(y,k;n)\,dy = \Lambda^{-(k-m)}Z(x,m;k).$$

Thus by the martingale convergence theorem the Q-a.s. limit (2.17), and similarly for (2.18), follows.

Next, I derive the Q-a.s. positivity of φ and ψ. It suffices to deal only with $\varphi(x,m)$ since the proof for $\psi(y,n)$ is the same using the limit in (2.18) in place of (2.17). Actually, $\psi(y,n)$ can be obtained by the time reversal from $\varphi(y,n)$, and so they have the same distribution. Since

$$E_Q\varphi(x,m)=1 \tag{3.16}$$

then

$$E_Q(\varphi(x,m))^2=1+E_Q(\varphi(x,m)-1)^2>0\,. \tag{3.17}$$

By (2.4) and (2.5),

$$Z(x,m;n)=\int p(z-x)e^{F_m(z,z)}Z(z,m+1;n)\,dz\,. \tag{3.18}$$

Dividing this by $\Lambda^{(n-m)}$ and applying (2.17) I obtain that for any $z\in\mathbb{R}^d$ and $l\in\mathbb{Z}$, Q-a.s.,

$$\varphi(z,l)=\Lambda^{-1}\int p(v-z)e^{F_l(z,v)}\varphi(v,l+1)\,dv\,. \tag{3.19}$$

Also by (2.17) for any $v\in\mathbb{R}^d$ and $l\in\mathbb{Z}$, Q-a.s.,

$$\varphi(v,l)\geqq 0\,. \tag{3.20}$$

For $l=1,2,\ldots$ set

$$L_{x,l}=\{z:p(l,z-x)>0\}\quad\text{and}\quad \Xi_l=\left\{\omega\in\Omega:\int_{L_{x,l}}\varphi(\omega,z,m+l)\,dz=0\right\}\,.$$

Let Ξ be the subset of $\omega\in\Omega$ such that $\varphi(x,m)=\varphi(\omega,x,m)=0$. From (3.19) and (3.20) it follows that $\Xi=\Xi_l$ for any $l=1,2,\ldots$. Since $\varphi(z,m+l)$ depends only on the random fields $\{F_{m+l},F_{m+l+1},\ldots\}$ then Ξ is the tail event of the sequence of independent random fields F_k, $k\in\mathbb{Z}$, and so by the Kolmogorov 0–1 law $Q(\Xi)=0$ or $=1$. This together with (3.17) yields that $Q(\Xi)=0$ completing the proof of (2.17).

Next, for $k>n$ set $\delta(y,n;k)=\Lambda^{-(k-n)}Z(y,n;k)-\varphi(y,n)$ then

$$I_k\overset{\text{def}}{=}\underset{k\to\infty}{\text{L.i.m.}}\int\Lambda^{-(n-m)}Z(x,m;y,n)\delta(y,n;k)\,dy=0\,. \tag{3.21}$$

Indeed, by the Cauchy–Schwartz inequality, by the independence of $Z(x,m;n)$ $Z(x,m;y,n)$ and $\delta(y,n;k)$, and by the invariance of Q with respect to space and time shifts I obtain from (2.17) that

$$\begin{aligned}E_Q(I_k)^2 &\leqq E_Q\int\frac{Z(x,m;y,n)}{Z(x,m;n)}(\Lambda^{-(n-m)}Z(x,m;n)\delta(y,n;k))^2\,dy\\ &=\int\Lambda^{-2(n-m)}E_Q\,(Z(x,m;n)Z(x,m;y,n))\,E_Q\delta^2(y,n;k)\,dy\\ &=E_Q\delta^2(0,0;k-n)E_Q(\Lambda^{-(n-m)}Z(x,m;n))^2\to 0\\ &\quad\text{in } L^2(\Omega,Q)\text{ as } k\to\infty\,.\end{aligned} \tag{3.22}$$

Now by (2.17) and (3.21),

$$\int \Lambda^{-(n-m)} Z(x,m;\,y,n)\varphi(y,n)\,dy \tag{3.23}$$
$$= \int \Lambda^{-(n-m)} Z(x,m;\,y,n) \operatorname*{L.i.m.}_{k\to\infty} \frac{Z(y,n;k)}{\Lambda^{(k-n)}}\,dy$$
$$= \operatorname*{L.i.m.}_{k\to\infty} \Lambda^{-(k-m)} \int Z(x,m;\,y,n)Z(y,n;k)\,dy$$
$$= \operatorname*{L.i.m.}_{k\to\infty} \Lambda^{-(k-m)} Z(x,m;k) = \varphi(x,m)\,,$$

and (2.20) follows similarly.

Next, in the same way as above

$$E_Q(\tilde{Z}(x,m;\,y,n))^2 = \sum_{r=1}^{n-m} \sum_{m \leqq k_1 < \cdots < k_r < n} W(\mathbf{k}^{(r)};x,m;\,y,n) \tag{3.24}$$

where

$$W(\mathbf{k}^{(r)};x,m;\,y,n) = E_Q(V(\mathbf{k}^{(r)};x,m;\,y,n))^2 \tag{3.25}$$
$$= \int\cdots\int \mathscr{W}(\mathbf{k}^{(r)};x,m;\mathbf{z}^{(r)},\tilde{\mathbf{z}}^{(r)},\mathbf{v}^{(r)},\tilde{\mathbf{v}}^{(r)})$$
$$\times q(n-k_r,\,y-\tilde{z}_r)q_0(n-k_r,\,y-\tilde{v}_r)$$
$$\times d\mathbf{z}^{(r)}\,d\tilde{\mathbf{z}}^{(r)}\,d\mathbf{v}^{(r)}\,d\tilde{\mathbf{v}}^{(r)}$$

with V and $\mathscr{W}$ given by (3.3) and (3.11), respectively. From (2.10), (2.13), (3.5), and (3.12) I derive similarly to (3.13) that for any $a = 0,1,2,\ldots$ and $i,j \geqq 1$,

$$\int\!\!\int\!\!\int\!\!\int q_0(i,u-z)q_0(i,v-z')\beta_l(u,\tilde{u};v,\tilde{v})q_a(j,w-\tilde{u}) \tag{3.26}$$
$$\times q_a(j,w'-\tilde{v})\,du\,dv\,d\tilde{u}\,d\tilde{v}$$
$$\leqq C_1C_2^2K\varepsilon(\max(1,i))^{-d/2}(\max(1,j))^{-d/2}\int\!\!\int\!\!\int\!\!\int q_0(i,u-z)$$
$$\times e^{-\gamma_1|\tilde{u}-u|}e^{-\gamma_1|\tilde{v}-v|}\min(1,|v-z|^{-(d+\delta_0)})q_a(j,w-\tilde{u})\,du\,dv\,d\tilde{u}\,d\tilde{v}$$
$$\leqq C_1C_2^2Kc^{-2}\varepsilon(b_d+s_{d-1}\delta_0^{-1})(\max(1,i))^{-d/2}$$
$$\times(\max(1,j))^{-d/2}q_{a+1}(i+j+1,w-z)\,,$$

where c is the same as in (2.10). From (3.25) and (3.26) it follows that

$$\sum_{r=1}^{n-m} \sum_{m \leqq k_1 < \cdots < k_r < n} W(\mathbf{k}^{(r)};x,m;\,y,n) \tag{3.27}$$
$$\leqq \sum_{r=1}^{n-m} \kappa_1^r q_r(n-m,\,y-x)\alpha_r(m,n)\,,$$

where $\alpha_r(m,n) = \sum_{m \leqq k_1 < \cdots < k_r < n} \alpha(\mathbf{k}^{(r)},m,n)$, $\kappa_1 = C_2\kappa_0$, and

$$\alpha(\mathbf{k}^{(r)},m,n) = (\max(1,k_1-m))^{-d/2} \tag{3.28}$$
$$\times\prod_{i=1}^{r}(\max(1,k_{i+1}-k_i-1))^{-d/2}$$

with $k_{r+1} = n$. I claim that

$$\alpha_r(m,n) \leqq C_3^r (n-m)^{-d/2} \tag{3.29}$$

for some constant $C_3 > 0$ independent of $\mathbf{k}^{(r)}, m$, and n. Indeed, for $r = 1$ and $n - m > 1$,

$$\begin{aligned}(3.30)\quad & \sum_{m \leqq k < n} (\max(1, k-m))^{-d/2} (\max(1, n-k-1))^{-d/2} \\ & = (n-m-1)^{-d/2} \left(2 + \sum_{1 \leqq l \leqq n-m-2} (l^{-1} + (n-m-1-l)^{-1})^{d/2}\right) \\ & \leqq (n-m-1)^{-d/2} \left(2 + 2^d \sum_{l \geqq 1} l^{-d/2}\right) \\ & \leqq 2^{d/2} (n-m)^{-d/2} (2 + 2^d d(d-2)^{-1})\ .\end{aligned}$$

If (3.29) holds true for some $r \geqq 1$ then for $r \leqq \frac{1}{2}(m-n)$,

$$\begin{aligned}(3.31)\quad \alpha_{r+1}(m,n) & \leqq C_3^r \sum_{m \leqq k \leqq n-r} (\max(1, k-m))^{-d/2} (\max(1, n-k-1))^{-d/2} \\ & \leqq C_3^r (n-m)^{-d/2} (1 - r(n-m)^{-1})^{-d/2} (2 + 2^d d(d-2)^{-1})\end{aligned}$$

and (3.29) follows for such r by induction. For $r > \frac{1}{2}(m-n)$ I estimate $\alpha_r(m,n)$ just by $(2 + \sum_{l \geqq 1} l^{-d/2})^{r+1} \leqq ((2 + d(d-2))^{r+1}$ and so (3.29) follows for C_3 large enough. Assuming that ε in (2.8) is sufficiently small so that $\kappa = C_3 \kappa_1 < 1$, I obtain (2.21) from (3.1), (3.24), and (3.27)–(3.29).

4 The limit theorem

In this section I shall prove Theorem 2.2 employing a generalization of martingale arguments from [Bo] together with estimates of the previous section. First, I shall prove (2.27) for any f having the form $f(v) = \prod_{j=1}^d v_j^{k_j}$ where $v = (v_1, \ldots, v_d) \in \mathbb{R}^d$ and $k = (k_1, \ldots, k_d) \in \mathbb{Z}^d, k_j \geqq 0\ \forall j = 1, \ldots, d$. For any such v, k and each $t \geqq 0$ set

$$\begin{aligned}(4.1)\qquad W_k(t,x) & = \frac{\partial^{\|k\|}}{\partial \alpha_1^{k_1} \cdots \partial \alpha_d^{k_d}} \exp((\alpha, x) - t \log \rho(\alpha))|_{\alpha=0} \\ & = \sum A_k(i_1, \ldots, i_d, j) x_1^{i_1} \cdots x_d^{i_d} t^j\end{aligned}$$

where $\|k\| = k_1 + \cdots + k_d \geqq 1$,

$$\rho(\alpha) = \int q_0(x) \exp(\alpha, x)\, dx \tag{4.2}$$

and $(\alpha, x) = \sum_{j=1}^d \alpha_j x_j$ for $\alpha, x \in \mathbb{R}^d$. In view of (2.7), $\rho(\alpha)$ is finite and infinitely differentiable in α provided $|\alpha| < \gamma_1$, and so the polynomials W_k are

well defined by (4.1). It is easy to see that the coefficients A_k satisfy the following properties (see [Bo]):
(i) If $i_1 + \cdots + i_d + 2j > \|k\|$, then $A_k(i_1, \ldots, i_d, j) = 0$;
(ii) The coefficients A_k with $\|k\| = i_1 + \cdots + i_d + 2j$ are determined uniquely by the second derivatives of log ρ at 0;
(iii) If $i_1 + \cdots + i_d = \|k\|$, then $A_k(i_1, \ldots, i_d, 0) = \delta_{i_1k_1}\delta_{i_2k_2} \cdots \delta_{i_dk_d}$ where δ_{ij} is the Kronecker symbol.

Observe that

$$(4.3) \qquad \int q_0(y-z)W_k(n-m, y-x)\,dy = W_k(n-m-1, z-x)\ .$$

Indeed, by (4.2),

$$(4.4) \qquad \begin{aligned} &\int q_0(y-z)\exp((\alpha, y-x) - (n-m)\log\rho(\alpha))\,dy \\ &\quad = \exp((\alpha, z-x) - (n-m-1)\log\rho(\alpha))\ . \end{aligned}$$

Differentiating both sides of (4.4) in α one arrives to (4.3). Set

$$Y_k(m,n) = \Lambda^{-(n-m)}\int Z(x,n;y,m)W_k(n-m, y-x)\,dy\ ,$$

where I omit in this notation the dependence on x, then by (4.3),

$$(4.5) \quad \begin{aligned} &E_Q(Y_k(m,n)|\mathscr{F}^{n-1}_{-\infty}) \\ &\quad = \Lambda^{-(n-m)}\int\int Z(x,m;v,n-1)p(y-v)E_Q(e^{F_{n-1}(v,y)}) \\ &\qquad \times W_k(n-m, y-x)\,dv\,dy \\ &\quad = \Lambda^{(-n-m-1)}\int Z(x,m;v,n-1)\int q_0(y-v)W_k(n-m, y-x)\,dy\,dv \\ &\quad = Y_k(m,n-1)\ , \end{aligned}$$

and so $Y_k(m,n)$ is a martingale in n and, similarly, it is a backward martingale in m.

Next, I shall prove that Q-a.s.,

$$(4.6) \qquad \lim_{n\to\infty} n^{-\|k\|/2}Y_k(m,n) = 0\ ,$$

provided $\|k\| \geqq 1$. By Kronecker's lemma, (4.6) would follow if the series

$$\sum_{n=m+1}^{\infty}(n-m)^{-\|k\|/2}(Y_k(m,n) - Y_k(m,n-1))$$

converges Q-a.s. Since by (4.5) this series contains uncorrelated terms then it converges Q-a.s. if

$$(4.7) \qquad \sum_{n=m+1}^{\infty}(n-m)^{-\|k\|}E_Q(Y_k(m,n) - Y_k(m,n-1))^2 < \infty\ .$$

In order to obtain (4.7) I write $e^{F_{n-1}(v,y)} = \Lambda h_{n-1}(v,y) + \Lambda(y-v)$ and proceed

$$
\begin{aligned}
(4.8)\qquad E_Q(Y_k(m,n))^2 &= \Lambda^{-2(n-m)} E_Q \int\int Z(x,m;y,n) W_k(n-m,y-x) \\
&\qquad \times Z(x,m;y',n) W_k(n-m,y'-x)\,dy\,dy' \\
&= \Lambda^{-2(n-m-1)} E_Q \int\int\int\int Z(x,m;v,n-1) q_0(y-v) \\
&\quad \times W_k(n-m,y-x) Z(x,m;v',n-1) q_0(y'-v') \\
&\quad \times W_k(n-m,y'-x)\,dy\,dy'\,dv\,dv' + \Lambda^{-2(n-m-1)} \\
&\quad \times E_Q \int\int\int\int Z(x,m;v,n-1)\, p(y-v) h_{n-1}(v,y) \\
&\quad \times W_k(n-m,y-x) Z(x,m;v',n-1)\, p(y'-v') h_{n-1}(v',y') \\
&\quad \times W_k(n-m,y'-x)\,dy\,dy'\,dv\,dv' \\
&= E_Q(Y_k(m,n-1))^2 + \Lambda^{-2(n-m-1)} E_Q \int\int\int\int Z(x,m;v,n-1) \\
&\quad \times W_k(n-m,y-x) Z(x,m;v',n-1) W_k(n-m,y'-x) \\
&\quad \times p(y-v)\, p(y'-z') E_Q h_{n-1}(v,y) h_{n-1}(v',y')\,dv\,dv'\,dy\,dy' \,.
\end{aligned}
$$

It is clear that

$$
(4.9)\quad W_k(n-m,y-x) = \sum_{\tilde{k}+\tilde{\tilde{k}}=k} C_{\tilde{k},\tilde{\tilde{k}}} W_{\tilde{k}}(n-m-1,v-x) W_{\tilde{\tilde{k}}}(1,y-v)\,.
$$

By (2.21) and (3.12) if $\tilde{k} + \tilde{\tilde{k}} = k$ and $\tilde{k}' + \tilde{\tilde{k}}' = k$ then

$$
\begin{aligned}
(4.10)\quad & \Lambda^{-2(n-m-1)} E_Q \int\int\int\int Z(x,m;v,n-1) Z(x,m;v',n-1) \\
& \times |W_{\tilde{k}}(n-m-1,v-x)||W_{\tilde{k}'}(n-m-1,v'-x)| \\
& \times |W_{\tilde{\tilde{k}}}(1,y-v)||W_{\tilde{\tilde{k}}'}(1,y'-v')|\, p(y-v)\, p(y'-z') \\
& \times |E_Q h_{n-1}(v,y) h_{n-1}(v',y')|\,dv\,dv'\,dy\,dy' \\
& \leqq C_4 \Lambda^{-2(n-m-1)} \int ((W_{\tilde{k}}(n-m-1,v-x))^2 E_Q(Z(x,m;v,n-1))^2 \\
& \quad + (W_{\tilde{k}'}(n-m-1,v'-x))^2 E_Q(Z(x,m;v',n-1))^2) \\
& \quad \times \min(1, |v-v'|^{-(d+\delta_0)})\,dv\,dv' \\
& \leqq C_5 (n-m)^{-d/2} \int (W_{\tilde{k}}(n-m-1,v-x))^2 \\
& \quad + ((W_{\tilde{k}'}(n-m-1,v-x))^2) \sum_{r=0}^{n-m-1} \kappa^r q_r(n-m-1,v-x)\,dv
\end{aligned}
$$

for some constants C_4 and C_5 depending on k but independent of $n-m$. In view of (2.7) and (2.10) it follows from Theorems 15 and 16 in Sect. 3.4 of

[Pe] that there exists a $\gamma_2 > 0$ such that

$$(4.11) \qquad \int_{\{|v| \geqq R\sqrt{l}\}} q_r(l,v)dv \leqq e^{-\gamma_2 R} \quad \text{for all } R > 0 \text{ and } l \geqq 1$$

and γ_2 can be estimated explicitly in terms of C_1, Λ, and γ_1 from (2.7) and (2.8). This together with the property (i) of coefficients A_k yield that the right hand side of (4.10) can be estimated by $C_6(n-m)^{\|k\|-d/2}$, and so by (4.5) and (4.8),

$$(4.12) \qquad \begin{aligned} &(n-m)^{-\|k\|} E_Q(Y_k(m,n) - Y_k(m,n-1))^2 \\ &= (n-m)^{-\|k\|}(E_Q(Y_k(m,n))^2 - E_Q(Y_k(m,n-1))^2) \\ &\leqq C_7(n-m)^{-d/2} \end{aligned}$$

for some constants C_6 and C_7 depending on k but independent of $n-m$. Since $d \geqq 3$ this implies (4.7), and so (4.6) follows.

Next, I proceed similarly to [Bo]. From the properties (i) and (iii) of the coefficients A_k I derive by induction in the degree of polynomials in spatial variables x_i that for any $k = (k_1, \ldots, k_d)$,

$$(4.13) \qquad \prod_{i=1}^{d} x_i^{k_i} = W_k(t,x) - \sum_{j \geqq 1, \|l\|+2j \leqq \|k\|} B_{l,j} W_l(t,x) t^j$$

for some constants $B_{l,j}$. It follows from (4.6) and (4.13) that Q-a.s. the limit

$$(4.14) \qquad \lim_{n\to\infty} \Lambda^{-(n-m)} \int \prod_{j=1}^{d} \left(\frac{y_j - x_j}{\sqrt{n-m}} \right)^{k_j} Z(x,m;y,n)\, dy \stackrel{\text{def}}{=} L_k(x,m)$$

exist and is finite. Furthermore, if $U_k(t,x)$ is obtained from $W_k(t,x)$ by deleting all summands $A_k(i_1, \ldots, i_d, j) x_1^{i_1} \cdots x_d^{i_d} t^j$ with $i_1 + \cdots + i_d + 2j < \|k\|$ then

$$W_k(t,x) - U_k(t,x) = \sum_{\|l\|+2j < \|k\|} B_{l,j} W_l(t,x) t^j \,,$$

and so by (4.6),

$$\lim_{n\to\infty} \Lambda^{-(n-m)} (n-m)^{-\|k\|/2} \int U_k(n-m, y-x) Z(x,m;y,n)\, dy = 0 \quad Q\text{-a.s.}$$

This together with (2.17) yields that

$$(4.15) \qquad \begin{aligned} &\lim_{n\to\infty} (Z(x,m;n))^{-1} \int \sum_{j_1,\ldots,j_d} A_k \left(j_1, \ldots, j_d, \frac{\|k\| - j_1 - \cdots - j_d}{2} \right) \\ &\times \left(\frac{y_1 - x_1}{\sqrt{n-m}} \right)^{j_1} \cdots \left(\frac{y_d - x_d}{\sqrt{n-m}} \right)^{j_d} Z(x,m;y,n)\, dy = 0 \,, \end{aligned}$$

where the sum is taken over all $j_1, \dots, j_d$ such that $\|k\| - j_1 - \cdots - j_d$ is nonnegative and even. Now let $V = (V_1, \dots, V_d)$ be a Gaussian random vector with zero mean and the covariance matrix A given by (2.24). Set

$$\tilde{\rho}(\alpha) = Ee^{(\alpha,V)} = e^{\frac{1}{2}(A\alpha,\alpha)} .$$

Since $E\exp((\alpha, V) - \log \tilde{\rho}(\alpha)) = 1$ then

$$(4.16) \qquad 0 = \frac{\partial^{\|k\|}}{\partial\alpha_1^{k_1} \cdots \alpha_d^{k_d}} E\exp((\alpha, V) - \log \tilde{\rho}(\alpha)) .$$

It is easy to see that the matrices of second derivatives at 0 of both $\rho(\alpha)$ and $\tilde{\rho}(\alpha)$ coincide with A which together with (4.16) and the property (ii) of the coefficients A_k yield

$$(4.17) \qquad E \sum_{j_1,\dots,j_d} A_k \left(j_1, \dots, j_d, \frac{\|k\| - j_1 - \cdots - i_d}{2} \right) V_1^{j_1} \cdots V_d^{j_d} = 0$$

where the sum is taken over the same indices as in (4.15) and E denotes the corresponding expectation. Using the representation (4.13) I conclude from (2.17), (4.15) and (4.17) by induction in the degree of polynomials in spatial variables that the limit in (4.14) is given by

$$L_k(x,m) = \varphi(x,m) \int z_1^{k_1} \cdots z_d^{k_d} r(1,z)\,dz, \quad z = (z_1, \dots, z_d) ,$$

proving (2.27) for f being a polynomial.

Since a Gaussian distribution is determined uniquely by its mixed moments then (see, for instance, [Bi, Theorem 30.2]) (2.27) being true for any polynomial f yields (2.27) for any bounded continuous function. Now let f be a continuous function satisfying (2.26). Then (2.27) holds true for any $f_C = \max(-C, \min(f, C))$, $C > 0$. Set $R_C = \sup\{r : f(z) = f_C(z)\ \forall z$ with $|z| \leqq r\}$. Then for C large enough,

$$(4.18) \quad (Z(x,m;n))^{-1} \int Z(x,m;y,n) \left| f\left(\frac{y-x}{\sqrt{n-m}}\right) - f_C\left(\frac{y-x}{\sqrt{n-m}}\right) \right| dy$$

$$\leqq 2(Z(x,m;n))^{-1} \int_{\mathbb{R}^d \setminus B_{R_C}} Z(x,m;y,n) \left| \frac{y-x}{\sqrt{n-m}} \right|^N dy$$

$$\leqq 2(R_C)^{-N} (Z(x,m;n))^{-1} \int Z(x,m;y,n) \left| \frac{y-x}{\sqrt{n-m}} \right|^{2N} dy ,$$

where $B_R = \{y : |y-x| \leqq R\sqrt{n-m}\}$. Since $R_C \to \infty$ as $C \to \infty$ then applying (2.27) to the polynomial $|z|^{2N}$ I conclude that the right hand side of (4.18) tends to 0 uniformly in n as $C \to \infty$. In addition, clearly,

$$(4.19) \quad \lim_{C\to\infty} \int |f(z) - f_C(z)| r(1,z)\,dz \leqq \lim_{C\to\infty} 2 \int_{|z|>R_C} |f(z)| r(1,z)\,dz = 0$$

which together with the above yield (2.27) for any continuous function satisfying (2.26).

In order to derive (2.29) and (2.30) observe that (4.7) implies also that the limit (4.6) holds true in the $L^2(\Omega, Q)$-sense, as well. Then arguments similar to above yield (2.29) and (2.30) for any f being a polynomial. If f is bounded, say, $|f| \leqq C$ then

$$(4.20)\qquad I(x,m;n) \overset{\text{def}}{=} \Lambda^{-(n-m)} \int Z(x,m;y,n) \left| f\left(\frac{y-x}{\sqrt{n-m}}\right)\right| dy$$
$$\leqq C\Lambda^{-(n-m)} Z(x,m;n)\,.$$

It follows from the above that $I(x,m;n)$ converges Q-a.s. to the right hand side of (2.29) and since by (2.17) the right hand side of (4.20) converges in $L^2(\Omega,Q)$ to $\varphi(x,m)$ it follows that $I(x,m;n)$ converges in the $L^2(\Omega,Q)$-sense, as well, yielding (2.29) for any bounded and continuous function f. Next, let f be a continuous function satisfying (2.28) and introduce f_C and R_C, as above. Then (2.29) holds true for f_C and I estimate by the Cauchy–Schwartz inequality,

$$(4.21)\quad \Lambda^{-2(n-m)} E_Q\left(\int Z(x,m;y,n)\left| f\left(\frac{y-x}{\sqrt{n-m}}\right) - f_C\left(\frac{y-x}{\sqrt{n-m}}\right)\right| dy\right)^2$$
$$\leqq 4 \int\limits_{\mathbb{R}^d\setminus B_{R_C}} e^{-\frac{\delta_0|y-x|}{\sqrt{n-m}}} \left| f\left(\frac{y-x}{\sqrt{n-m}}\right)\right| dy$$
$$\times \int\limits_{\mathbb{R}^d\setminus B_{R_C}} \left| f\left(\frac{y-x}{\sqrt{n-m}}\right)\right| e^{\frac{\delta_0|y-x|}{\sqrt{n-m}}} \Lambda^{-2(n-m)} E_Q(Z(x,m;y,n))^2\, dy\,.$$

From (2.21) and (4.11) it follows that for $\delta_0 < \frac{1}{2}\gamma_2$ the right hand side of (4.21) tends to zero uniformly in n as $C \to \infty$ (and so, $R_C \to \infty$). This together with (4.19), which holds true in view of (2.25) provided δ_0 is sufficiently small, yield (2.29), while (2.30) follows by similar arguments. □

5 Continuous time case

In this section I shall derive Theorems 2.4–2.6 and a part of Theorem 2.7. For each $\delta > 0$ set

$$(5.1)\quad \varphi_\delta(x,s) = 1 + \sum_{r\geqq 1}\ \sum_{0\leqq k_1<k_2<\cdots<k_r} \int q_0(k_1\delta, z-x) U^s_\delta(\mathbf{k}^{(r)}; z, v)\, dz\, dv\,,$$

and

$$(5.2)\qquad \psi_\delta(y,t) = 1 + \sum_{r\geqq 1}\ \sum_{k_1<k_2<\cdots<k_r<0} \int\int U^t_\delta(\mathbf{k}^{(r)}; z, v)$$
$$\times q_0(-(k_r+1)\delta, y-v)\, dz\, dv$$

where for $\mathbf{k}^{(r)}=(k_1,\ldots,k_r)$ with $k_1<\cdots<k_r$,

$$(5.3)\quad U_\delta^s(\mathbf{k}^{(r)},z,v)=\int\cdots\int p(\delta,\tilde z_1-z)h(z,s+k_1\delta,\tilde z_1,s+(k_1+1)\delta)$$
$$\times q_0((k_2-k_1-1)\delta,z_2-\tilde z_1)p(\delta,\tilde z_2-z_2)$$
$$\times h(z_2,s+k_2\delta,\tilde z_2,s+(k_2+1)\delta)$$
$$\times\cdots\times q_0((k_r-k_{r-1}-1)\delta,z_r-\tilde z_{r-1})p(\delta,v-z_r)$$
$$\times h(z_r,s+k_r\delta,v,s+(k_r+1)\delta)$$
$$\times d\tilde z_1\,dz_2\,d\tilde z_2\cdots dz_{r-1}\,d\tilde z_{r-1}\,dz_r\ .$$

By (2.31) and (2.33) for any $n\in\mathbb{Z}_+$,

$$(5.4)\quad p(\delta,v-z)h(z,s+k\delta,v,s+(k+1)\delta)$$
$$=\sum_{r=1}^{n}\sum_{0\leqq l_1<l_2<\cdots<l_r<n}\int\cdots\int q_0(l_1\delta/n,v_1-z)\,p(\delta/n,\,\tilde v_1-v_1)$$
$$\times h(v_1,s+(kn+l_1)\delta/n,\tilde v_1,s+(kn+l_1+1)\delta/n)$$
$$\times q_0((l_2-l_1-1)\delta/n,v_2-\,\tilde v_1)\,p(\delta/n,\,\tilde v_2-v_2)$$
$$\times h(v_2,(kn+l_2)\delta/n,\tilde v_2,(kn+l_2+1)\delta/n)\cdots p(\delta/n,\,\tilde v_r-v_r)$$
$$\times h(v_r,(kn+l_r)\delta/n,\,\tilde v_r,(kn+l_r+1)\delta/n)q_0((n-l_r)\delta/n,v-\,\tilde v_r)$$
$$\times dv_1\,d\tilde v_1\cdots dv_r\,d\tilde v_r.$$

It follows that for any rational δ,

$$(5.5)\qquad \varphi_\delta(x,s)=\varphi_1(x,s)\overset{\text{def}}{=}\varphi(x,s)\quad\text{and}\quad \psi_\delta(y,t)=\psi_1(y,t)\overset{\text{def}}{=}\psi(y,t)\ .$$

In the same way as in (2.17) and (2.18), just from the $L^2(\Omega,Q)$-convergence of the series for $\varphi_\delta(x,s)$ and $\psi_\delta(x,s)$ it follows from (5.5) that for any t and a rational δ,

$$(5.6)\qquad \underset{n\to\infty}{\text{L.i.m.}}\,\Lambda^{-n\delta}Z(x,t;t+n\delta)=\varphi(x,t)\ ,$$

and similarly for $\psi(y,t)$. Observe also that for any $1\leqq u<2$ the inequalities (2.36) and (2.37) yield

$$(5.7)\qquad E_Q(Z(x,s;y,s+u))^2\ \leqq\ C_1\Lambda^{2u}(\varepsilon+C_1)e^{-2\gamma_1|y-x|}$$

and

$$(5.8)\qquad E_Q(Z(x,s;s+u))^2\ \leqq\ C_1\Lambda^{2u}(\varepsilon+C_1)c^{-2}\ ,$$

where c is the same as in (2.10). Now (5.7) together with (5.6) give in the same way as in (3.18) that for any t and a rational $r>0$,

$$(5.9)\qquad \varphi(x,t-r)=\Lambda^{-r}\int Z(x,t-r;y,t)\varphi(y,t)\,dy\ .$$

Since $Z(x,s;y,t)$ is $L^2(\Omega,Q)$-continuous in x,s,y,t and $p(u,w)$ is continuous in u,w then it follows easily from the definitions (5.1)–(5.3) that both $\varphi(x,s)$ and $\psi(y,t)$ are $L^2(\Omega,Q)$-continuous in x,s,y,t. Thus I can pass to the limit in (5.9) as $r \to t-s$ and obtain (2.40). The proof of (2.41) is the same.

Observe that by stationarity $E_Q(\Lambda^{-n}Z(x,t-n;t)-\varphi(x,t-n))^2$ depends only on n but not on t. Thus setting $n_{s,t}=-1+$ integral part of $(t-s)$ and using (2.17), (2.40), (5.7), and (5.8) I obtain similarly to (3.21) that

$$(5.10)\quad \underset{t-\infty}{\text{L.i.m.}}\,\Lambda^{-(t-s)}Z(x,s;t)=\underset{t\to\infty}{\text{L.i.m.}}\int\Lambda^{-(t-s-n_{s,t})}Z(x,s;v,t-n_{s,t})$$

$$\times\Lambda^{-n_{s,t}}Z(v,t-n_{s,t};t)\,dv$$

$$=\underset{t\to\infty}{\text{L.i.m.}}\int\Lambda^{-(t-s-n_{s,t})}\varphi(v,t-n_{s,t})\,dv=\varphi(x,s)\,.$$

The convergence in (5.10) is also Q-a.s. since $\Lambda^{-(t-s)}Z(x,s;t)$ is a martingale in t. The proof of (2.39) is the same.

In order to obtain (2.42) one can do direct estimates as in Sect. 3 for rational $t-s$ and pass to an appropriate limit. I shall proceed in a simpler way. For $t-s<2$, (2.42) follows from (5.7). For $t-s\geqq 2$ I can write by (2.21) and the Cauchy–Schwartz inequality that

$$(5.11)\quad E_Q(Z(x,s;y,t))^2$$

$$=E_Q\left(\int\frac{Z(x,s;v,t-n_{s,t})}{Z(x,s;t-n_{s,t})}(Z(x,s;t-n_{s,t})Z(v,t-n_{s,t};y,t))\,dv\right)^2$$

$$\leqq E_Q(Z(x,s;t-n_{s,t})\int Z(x,s;v,t-n_{s,t})(Z(v,t-n_{s,t};y,t))^2\,dv)$$

$$=\int E_Q(Z(x,s;t-n_{s,t})Z(x,s;v,t-n_{s,t}))E_Q(Z(v,t-n_{s,t};y,t))^2\,dv$$

$$\leqq C\Lambda^{2n_{s,t}}n_{s,t}^{-d/2}\sum_{r=0}^{n_{s,t}}\kappa^r(E_Q(Z(x,s;t-n_{s,t}))^2)^{1/2}\,dv$$

$$\times\int q_r(n_{s,t},y-v)(E_Q(Z(x,s;v,t-n_{s,t}))^2)^{1/2}\,dv\,.$$

Now (5.7), (5.8), and (5.11) yield (2.42) and (2.43). □

Next, I shall derive Theorem 2.5 employing the arguments of Sect. 4 for the continuous time. Let W_k and ρ be defined by (4.1) and (4.2) where $q_0(x)=q_0(1,x)$ with q_0 defined before (2.35). Observe that if $\rho_t(\alpha)=\int\exp(\alpha,z)\,q_0(t,z)dz$ then $\rho_{t+s}(\alpha)=\rho_t(\alpha)\rho_s(\alpha)$, and so by continuity, $\rho_t(\alpha)=(\rho(\alpha))^t$. Hence, for $s<u<t$ one has,

$$\int q_0(t-u,y-v)\exp((\alpha,y-x)-(t-s)\log\rho(\alpha))\,dy$$

$$=\exp((\alpha,v-x)-(u-s)\log\rho(\alpha))\,.$$

Differentiating both parts of this equality in α I obtain

$$\int q_0(t-u,y-v)W_k(t-s,y-x)\,dy=W_k(u-s,v-x)\,.$$

Set

$$Y_k(s,t) = \Lambda^{-(t-s)} \int Z(x,s;y,t) W_k(t-s, y-x)\, dy ,$$

where, again, I omit in this notation the dependence on x. Then for $s<u<t$ similarly to (4.5),

$$\begin{aligned} E_Q(Y_k(s,t)|\mathscr{F}^u_{-\infty}) &= \Lambda^{-(t-s)} \int\int Z(x,s;v,u) E_Q Z(v,u;y,t) W_k(t-s,y-x)\, dv\, dy \\ &= \Lambda^{-(u-s)} \int\int Z(x,s;v,u) q_0(t-u, y-v) W_k(t-s, y-x)\, dy\, dv \\ &= Y_k(s,u) , \end{aligned}$$

where $\mathscr{F}^t_s$ is the σ-algebra generated by all $Z(x,u;y,\tilde{u})$ with $s \leqq u < \tilde{u} \leqq t$, and so $Y_k(s,t)$ is a martingale in t and a backward martingale in s.

I shall show next that Q-a.s.,

$$\lim_{t\to\infty} (t-s)^{-\|k\|/2} Y_k(s,t) = 0 , \tag{5.12}$$

provided $\|k\| \geqq 1$. First, similarly to the discrete time case Q-a.s.,

$$\lim_{t\to\infty} (t-s)^{-\|k\|/2} Y_k(s, s+n_{s,t}) = 0 . \tag{5.13}$$

Now put $M_{k,s,n}(u) = Y_k(s,s+n+u) - Y_k(s,s+n)$ which is, clearly, a martingale in u. Thus if $M_{k,s,n} = \sup_{0\leqq u<1} |M_{k,s,n}(u)|$ then by the L^2 Doob inequality (see, for instance, [RY, p. 52]),

$$E_Q(M_{k,s,n})^2 \leqq 4E_Q(M_{k,s,n}(1))^2 \leqq 8E_Q(Y_k(0,1))^2 + 8E_Q(Y_k(0,0))^2 . \tag{5.14}$$

By the Cauchy–Schwartz inequality

$$\begin{aligned} E_Q(Y_k(s,t))^2 &\leqq \Lambda^{-2(t-s)} E_Q Z(x,s;t) \int Z(x,s;y,t) |W_k(t-s,y-x)|^2\, dy \\ &\leqq \Lambda^{-2(t-s)} (E_Q(Z(x,s;t))^2)^{1/2} \int (E_Q(Z(x,s;y,t))^2)^{1/2} \\ &\quad \times |W_k(t-s,y-x)|^2\, dy . \end{aligned}$$

Since W_k is a polynomial then this together with (5.7) and (5.8) yield that the right hand side of (5.14) is finite. Thus for any $\varepsilon>0$,

$$\sum_{n=1}^{\infty} Q\{(M_{k,s,n})^2 \geqq \varepsilon n\} = \sum_{n=1}^{\infty} Q\{(M_{k,0,0})^2 \geqq \varepsilon n\} \leqq 1 + \frac{1}{\varepsilon} E_Q(M_{k,0,0})^2 < \infty .$$

By the Borel–Cantelli lemma this yields that Q-a.s.,

$$\lim_{n\to\infty} \frac{1}{\sqrt{n}} |M_{k,s,n}| = 0$$

which together with (5.13) gives (5.12). Since both the last limit and (5.13) hold true also in the $L^2(\Omega,Q)$-sense I conclude that the limit in (5.12) remains true in the $L^2(\Omega,Q)$-sense, as well. The rest of the proof of Theorem 2.5 is the same as in Sect. 4 for the discrete time case. □

In order to establish Theorem 2.6 I merely have to check the conditions of Theorems 2.4 and 2.5 when $Z(x,s;y,t)$ is defined by (2.49). Since the distribution of the time scaled Wiener process W_u is invariant under the space the time shifts and W is independent of Ψ which is spatially and temporarily homogeneous then the distribution of $Z(x,s;y,t)$ depends only on $y-x$ and $t-s$. In addition, the invertibility of W_u implies that $Z(x,s;y,t)$ and $Z(y,s;x,t)$ have the same distributions. The independence of time increments of $\Psi(x,t)$ together with the definition of stochastic integrals with spatial parameters in [Ku2] yield that the random fields $Z(\cdot,s;\cdot,t)$ and $Z(\cdot,\tilde{s};\cdot,\tilde{t})$ are independent if (s,t) and $(\tilde{s},\tilde{t})$ are disjoint intervals. By the Markov property of W_u the "random Chapman–Kolmogorov" formula (2.31) holds true, as well.

Let $\zeta=\zeta(u)$ be a continuous curve in $\mathbb{R}^d$ and for each constant $C>0$ set

$$X^s_{C,\zeta}(t)=\exp\int_s^t C\Psi(\zeta(u),du)\ .$$

By the generalized Ito formula (see [Ku2, Sect. 3.3]),

$$dX^s_{C,\zeta}(t)=CX^s_{C,\zeta}(t)\Psi(\zeta(t),dt)+\tfrac{1}{2}C^2X^s_{C,\zeta}(t)A(0)\,dt\ , \tag{5.15}$$

where $A(x,y)=A(y-x)$ is the spatial covariance function of $\Psi(x,t)$. Thus

$$E_QX^s_{C,\zeta}(t)=\frac{1}{2}C^2A(0)\int_s^t E_QX^s_{C,\zeta}(u)\,du+1\ . \tag{5.16}$$

Hence,

$$E_QX^s_{C,\zeta}(t)=\exp(\tfrac{1}{2}C^2A(0)(t-s))\ , \tag{5.17}$$

and so by (2.49),

$$\Lambda(u,w)=(p(u,w))^{-1}E_QZ(0,0;w,u)=\Lambda(u)=1\ , \tag{5.18}$$

which gives (2.36). Similarly, if $\eta=\eta(u)$ is another continuous curve in $\mathbb{R}^d$ then using (5.15) and rules for stochastic integrals with spatial parameters from Sects. 3.2 and 3.3 of [Ku2] I derive

$$\begin{aligned} E_QX^s_{C,\zeta}(t)X^s_{C,\eta}(t)=C^2\int_s^t E_QX^s_{C,\zeta}(u)X^s_{C,\eta}(u)A(\zeta_u-\eta_u)\,du\\ +C^2A(0)\int_s^t E_QX^s_{C,\zeta}(u)X^s_{C,\eta}(u)\,du+1\ ,\end{aligned} \tag{5.19}$$

and so by (5.17),

$$\begin{aligned}&\mathrm{Cov}_Q(X^s_{C,\zeta}(t),X^s_{C,\eta}(t))\\ &\quad=E_Q(X^s_{C,\zeta}(t)-E_QX^s_{C,\zeta}(t))(X^s_{C,\eta}(t)-E_QX^s_{C,\eta}(t))\\ &\quad=\exp(C^2A(0)(t-s))\left(\exp\left(C^2\int_s^t A(\zeta_u-\eta_u)\,du\right)-1\right)\end{aligned} \tag{5.20}$$

where Cov_Q denotes the covariance of random variables on (Ω,Q).

Set $L = \exp((\lambda^2/8v^2)A(0))$. I represent next the Brownian bridge $\beta^{x\to y}$ from x to y on $[0, t-s]$ in the form

$$\beta_u^{x\to y} = x + \frac{u}{t-s}(y-x) + \left(W_u - \frac{u}{t-s}W_{t-s}\right) \tag{5.21}$$

and for any $R>0$ and $x, y \in \mathbb{R}^d, s, t \in \mathbb{R},\ s < t$ I put

$$f_R(x,s;y,t) = L^{-(t-s)}E_0\left(\mathbb{I}_{|W_u|\leqq R, \forall u\in[0,t-s]} \exp\int_s^t \frac{\lambda}{2v}\Psi(\beta_{t-u}^{x\to y},\, du)\right)$$

and

$$h_R(x,s;y,t) = (f_R(x,s;y,t) - E_Q f_R(x,s;y,t))$$

where $\mathbb{I}_A = 1$ if the event A occurs and $= 0$, otherwise. Then $h_\infty(x,s;y,t) = h(x,s;y,t)$. Choosing two independent copies $\tilde{W}$ and $\hat{W}$ of the process W and the corresponding independent Brownian bridges $\tilde{\beta}^{x\to y}$ and $\hat{\beta}^{v\to w}$ on $[0,t-s]$, I derive from (5.20) that

$$\begin{aligned} (5.22)\quad & E_Q h_{R_1}(x,s;y,t)h_{R_2}(v,s;w,t) \\ &= L^{-2(t-s)}E_{0,0}\Bigg(\mathbb{I}_{|\tilde{W}_u|\leqq R_1,\forall u\in[0,t-s]}\ \mathbb{I}_{|\hat{W}_u|\leqq R_2,\forall u\in[0,t-s]} \\ &\quad \times \mathrm{Cov}_Q\left(\exp\left(\int_s^t \frac{\lambda}{2v}\Psi(\tilde{\beta}_{t-u}^{x\to y}, du)\right)\exp\left(\int_s^t \frac{\lambda}{2v}\Psi(\hat{\beta}_{t-u}^{v\to w},\, du)\right)\right)\Bigg) \\ &= E_{0,0}\Bigg(\mathbb{I}_{|\tilde{W}_u|\leqq R_1,\forall u\in[0,t-s]}\mathbb{I}_{|\hat{W}_u|\leqq R_2,\forall u\in[0,t-s]} \\ &\quad \times \left(\exp\left(\frac{\lambda^2}{4v^2}\int_s^t A(\tilde{\beta}_{t-u}^{x\to y} - \hat{\beta}_{t-u}^{v\to w})\,du\right) - 1\right)\Bigg) \end{aligned}$$

where $E_{0,0}$ is the expectation for the two-component process $(\tilde{W}_u, \hat{W}_u)$ starting at $(0,0)$. Observe that if $\sup_{0\leqq u\leqq t-s}|\tilde{W}_u| \leqq \frac{1}{8}|v|$ and $\sup_{0\leqq u\leqq t-s}|\hat{W}_u| \leqq \frac{1}{8}|v|$ then for the corresponding Brownian bridges I have

$$\inf_{0\leqq u\leqq t-s}|\tilde{\beta}_u^{0\to z} - \hat{\beta}_u^{v\to w}| \geqq \frac{1}{2}|v| - (|z| + |w-v|)\,. \tag{5.23}$$

If $a(r) = \sup_{|x|\geqq r} A(x)$ then (5.23) together with (5.22) yield that for any $u \in [1,2)$, (5.24)

$$\begin{aligned} (5.24)\quad & E_Q h_{\frac{1}{8}|v|}(0,s;z,s+u)h_{\frac{1}{8}|v|}(v,s;w,s+u) \\ &\leqq \left(\exp\left(\frac{\lambda^2}{2v^2}a\left(\frac{1}{2}|v| - |z| - |w-v|\right)\right) - 1\right)\,. \end{aligned}$$

It is clear that there exists $C>0$ and $\gamma>0$ such that for any $v \in \mathbb{R}^d$ and $0 \leqq u \leqq 2$,

$$P_0\left\{\sup_{0\leqq r\leqq u}|W_r| > \frac{1}{8}|v|\right\} \leqq Ce^{-\gamma|v|}\,. \tag{5.25}$$

Let B be any event defined in terms of the process $W_r,\ 0 \leqq r \leqq u$. Set $\eta = \exp\int_s^t \frac{\lambda}{2v}\Psi(\beta_{t-u}^{x\to y}, du)$. Then by (5.17) and the Cauchy–Schwartz inequality

$$E_Q(E_0 \mathbb{I}_B \eta)^2 = (P_0(B))^2 E_Q\left(\frac{1}{P_0(B)}E_x \mathbb{I}_B \eta\right)^2 \leqq (P_0(B))^2 E_Q \frac{1}{P_0(B)} E_0 \mathbb{I}_B \eta^2 = (P_0(B))^2 \exp\left(\frac{\lambda^2}{2v^2}A(0)u\right). \tag{5.26}$$

Again by (5.17),

$$(E_Q E_0 \mathbb{I}_B \eta)^2 = (P_0(B))^2 \exp\left(\frac{\lambda^2}{4v^2}A(0)u\right). \tag{5.27}$$

Setting either $x=0,\ y=z$ or $x=v,\ y=w$ I obtain from (5.25)–(5.27) that

$$E_Q(h(x,s;y,s+u) - h_{\frac{1}{8}|v|}(x,s;y,s+u))^2 \leqq C^2 e^{-2\gamma|v|}\left(\exp\left(\frac{\lambda^2}{2v^2}A(0)u\right) - \exp\left(\frac{\lambda^2}{4v^2}A(0)u\right)\right). \tag{5.28}$$

This together with (5.24) yield (2.37) provided (2.51) holds true with sufficiently small ε, which completes the proof of Theorem 2.6. □

In order to derive Theorem 2.7 I first observe that the representation (2.48) of solutions w of (2.47) (which is a correct way of writing (1.3)), as well as the formulas (2.52) and (2.53) for solutions of (1.2) and (1.1), respectively, follow from Sect. 6.2 in [Ku2]. I just have to check that $\psi(y,t)$ is slowly increasing as $|y|\to\infty$ in the sense of [Ku2, p. 300], and so in view of Theorem 6.2.5 in [Ku2] it can be used as an initial condition for the stochastic heat equation (2.47). Namely, I shall show that Q-a.s.,

$$\lim_{|y|\to\infty}\frac{\psi(y,t)}{1+|y|^d} = \lim_{|x|\to\infty}\frac{\varphi(x,s)}{1+|x|^d} = 0\,. \tag{5.29}$$

I shall deal only with ψ since the proof for φ is similar. Set $K = \{y=(y_1,\dots,y_d)\in\mathbb{R}^d : 0 \leqq y_i < 1\}$, $K_m = m + K$ for $m\in\mathbb{Z}^d$, and $\psi_m(t) = \sup_{x\in K_m}\psi(x,t)$. It suffices to show that

$$E_Q\psi_m^2(t) = E_Q\psi_0^2(0) < \infty\,. \tag{5.30}$$

Indeed, set $\|m\| = \max_i |m_i|$ then by stationarity and the Chebyshev inequality it follows from (5.30) that

$$\sum_{m\in\mathbb{Z}^d} P_Q\{|\psi_m(t)| > \varepsilon\|m\|^d\} = \sum_{k=1}^{\infty}(2k)^d P\{|\psi_0(0)|^2 > \varepsilon^2 k^{2d}\} \leqq 2^d\varepsilon^{-2}E_Q|\psi_0(0)|^2\sum_{k=1}^{\infty}k^{-d} < \infty\,. \tag{5.31}$$

Applying the Borel–Cantelli lemma I conclude that with probability one $\|m\|^{-d}|\psi_m(s)| \leqq \varepsilon$ if $\|m\|$ is large enough and since ε is arbitrary this yields (5.29).

By (2.41) and the Cauchy–Schwartz inequality,

$$E_Q\psi_m^2(0) = E_Q\psi_m^2(1) \leqq E_Q \sup_{y\in K_m} \int (\psi(x,0))^2 \hat{Z}(0;y,1)Z(x,0;y,1)\,dx \leqq E_Q(\psi(0,0))^2 E_Q\left(\int \sup_{y\in K_m} Z(x,0;y,1)\,dx\right)^2 \tag{5.32}$$

since $\psi(x,0)$ and $Z(x,0;y,1)$ are independent. In order to estimate the right hand side of (5.32) I use the Brownian bridge representation of $Z(x,s;y,t)$,

$$Z(x,s;y,t) = L^{-(t-s)}p(t-s,y-x)E_0 Y_{s,W}^{x,y}(t) \tag{5.33}$$

where for any continuous curve $\zeta = \zeta_u$, $0 \leqq u \leqq t-s$ satisfying $\zeta_0 = 0$,

$$Y_{s,\zeta}^{x,y}(t) = \exp\int_s^t \frac{\lambda}{2\nu}\Psi(\alpha_{s,\zeta}^{x,y}(u),du) \tag{5.34}$$

with

$$\alpha_{s,\zeta}^{x,y}(u) = x\left(1-\frac{t-u}{t-s}\right) + y\frac{t-u}{t-s} + \zeta_{t-u} - \frac{t-u}{t-s}\zeta_{t-s}\,.$$

Set

$$\tilde{Y}_{s,\zeta}^{x;y,z}(t) = Y_{s,\zeta}^{x,y}(t) - Y_{s,\zeta}^{x,z}(t)\,. \tag{5.35}$$

Employing the generalized Ito formula from Sect. 3.3 of [Ku2] (see also Ch. 4 in [DZ]) in the same way as in (5.15) together with formulas for second moments of stochastic integrals I obtain

$$E_Q(\tilde{Y}_{s,\zeta}^{x;y,z}(t))^2 \leqq I_1 + I_2 + I_3\,, \tag{5.36}$$

where

$$I_1 \leqq \frac{\lambda^4 A^2(0)}{32\nu^4}E_Q\left(\int_s^t \tilde{Y}_{s,\zeta}^{x;y,z}(u)\,du\right)^2 \leqq \frac{\lambda^4 A^2(0)}{32\nu^4}(t-s)\int_s^t E_Q(\tilde{Y}_{s,\zeta}^{x;y,z}(u))^2\,du\,, \tag{5.37}$$

$$I_2 \leqq \frac{\lambda^2}{\nu^2}E_Q\left(\int_s^t \tilde{Y}_{s,\zeta}^{x;y,z}(u)\Psi(\alpha_{s,\zeta}^{x,y}(u),\,du)\right)^2 = \frac{\lambda^2 A(0)}{\nu^2}\int_s^t E_Q(\tilde{Y}_{s,\zeta}^{x;y,z}(u))^2\,du\,, \tag{5.38}$$

and by (5.17),

$$(5.39)\qquad I_3 \leqq \frac{\lambda^2}{\nu^2} E_Q \left(\int_s^t Y^{x;z}_{s,\zeta.}(u)(\Psi(\alpha^{x,y}_{s,\zeta.}(u), du) - \Psi(\alpha^{x,z}_{s,\zeta.}(u), du)) \right)^2$$

$$= E_Q \int_s^t (Y^{x;z}_{s,\zeta.}(u))^2 A\left((y-z)\frac{t-u}{t-s}\right) du$$

$$\leqq \sup_{|v|\leqq 2} |A'(v)|\,|y-z| \exp\left(\frac{\lambda^2}{2\nu^2} A(0)(t-s)\right),$$

provided $|y-z| \leqq 2$. Now (5.36)–(5.39) together with the Gronwall inequality yield that there exists a constant $C > 0$ such that

$$(5.40)\qquad E_Q(\tilde{Y}^{x;y,z}_{s,\zeta.}(t))^2 \leqq C|y-z|e^{C(t-s)},$$

provided, say, $|y-z| \leqq 2$. By Theorem 1.4.1 in [Ku2] I conclude from (5.35) and (5.40) that there exists a constant $\tilde{C} > 0$ such that for any $m \in \mathbb{Z}^d$, $x \in \mathbb{R}^d$, and a continuous curve ζ as above,

$$(5.41)\qquad E_Q\left(\sup_{y\in K_m} Y^{x,y}_{0,\zeta}(1)\right)^2 \leqq \tilde{C}$$

which together with (5.32) and (5.33) yield (5.30). The assertion (2.54) of Theorem 2.7 will be derived in the next section.

6 A factorization theorem and applications

I shall exhibit here a generalization of Sinai's approach from [Si2] which will enable me to complete the proof of Theorem 2.7 though applied to limit theorems it yields weaker results than the method of Sect. 4. The following factorization theorem is the main result here.

Theorem 6.1. *Both in the discrete and in the continuous time cases, if* (2.7) *and* (2.8) *or* (2.36) *and* (2.37), *correspondingly, are satisfied with* ε *small enough then*

$$(6.1)\qquad Z(x,s;y,t) = \Lambda^{t-s} q_0(t-s, y-x)(\varphi(x,s)\psi(y,t) + \delta(x,s;y,t))$$

where for each fixed $R > 0$, $\delta(x,s;y,t) \to 0$ *in* $L^1(\Omega, Q)$ *as* $t-s \to \infty$ *uniformly in* $x, y \in \mathbb{R}^d$ *satisfying* $|y-x| \leqq R\sqrt{t-s}$. *In the discrete time case one can write* $\delta(x,s;y,t) = \varphi(x,s)\delta_1(x,s;y,t) + \psi(y,t)\delta_2(x,s;y,t) + \delta_1(x,s;y,t)\delta_2(x,s;y,t) + \delta_3(x,s;y,t)$ *where for each fixed* $R > 0$, $\delta_i(x,m;y,n) \to 0$, $i = 1,2,3$ *in* $L^2(\Omega, Q)$ *as* $t-s \to \infty$ *uniformly in* x, y *satisfying the above conditions,* $\delta_1(x,s;y,t)$ *is* $\mathscr{F}^t_{-\infty}$*-measurable,* $\delta_2(x,s;y,t)$ *is* $\mathscr{F}^{\infty}_s$*-measurable, and* $\delta_3(x,s;y,t)$ *is* $\mathscr{F}^t_s$*-measurable (recall, that* $\mathscr{F}^t_s$, $-\infty \leqq s \leqq t \leqq \infty$ *is the* σ*-algebra generated by all* $Z(x,u;y,v)$ *with* $s \leqq u \leqq v \leqq t$).

Proof. I start with the discrete time case: $s = m$, $t = n$. By (3.1) I can write

$$\Lambda^{-(n-m)} Z(x, m; y, n) = q_0(n-m, y-x) + Z_1(x, m; y, n) + Z_2(x, m; y, n) \tag{6.2}$$

where $Z_1(x, m; y, n)$ is the partial sum of (3.2) when r runs from 1 to $K \log n$, $Z_2(x, m; y, n) = \tilde{Z}(x, m; y, n) - Z_1(x, m; y, n)$, and I choose $K = (\log(1-\kappa) - 2d)/\log \kappa$. It is clear that $E_Q(Z_2(x, m; y, n))^2 \leqq (n-m)^{-2d}$, provided $n - m \geqq 3$, and so

$$\underset{n-m\to\infty}{\text{L.i.m.}} \frac{Z_2(x, m; y, n)}{q_0(n-m, y-x)} = 0\,. \tag{6.3}$$

Fix $\beta \in (\frac{1}{2}, 1)$ and set

$$\mathscr{T}_{m,n}^{(l)} = \{\mathbf{k}^{(r)} = (k_1, \ldots, k_r) \colon m = k_0 \leqq k_1 < \cdots < k_r < k_{r+1} = n, \text{ there exist } 0 \leqq j_1 < j_2 < \cdots < j_l \leqq r \text{ with } |k_{j_i+1} - k_{j_i}| > (n-m)^\beta \text{ and } |k_{v+1} - k_v| \leqq (n-m)^\beta \text{ if } v \neq j_1, \ldots, j_l\}\,.$$

Clearly,

$$Z_1(x, m; y, n) = \sum_{1 \leqq r \leqq K \log(n-m)} \sum_{l \geqq 1} \sum_{\mathbf{k}^{(r)} \in \mathscr{T}_{m,n}^{(l)}} V(\mathbf{k}^{(r)}; x, m; y, n) \tag{6.4}$$

with $V(\mathbf{k}^{(r)}; x, m; y, n)$ given by (3.3). Set

$$Z^{(1)}(x, m; y, n) = \sum_{1 \leqq r \leqq K \log(n-m)} \sum_{\mathbf{k}^{(r)} \in \mathscr{T}_{m,n}^{(1)}} V(\mathbf{k}^{(r)}; x, m; y, n) \tag{6.5}$$

and $Z^{(2)}(x, m; y, n) = Z_1(x, m; y, n) - Z^{(1)}(x, m; y, n)$. The same arguments as at the end of Sect. 3 which lead to (3.27)–(3.29) yield

$$\begin{aligned} E_Q(Z^{(2)}(x, m; y, n))^2 &= \sum_{r=1}^{n-m} \sum_{l \geqq 2} \sum_{\mathbf{k}^{(r)} \in \mathscr{T}_{m,n}^{(l)}} W(\mathbf{k}^{(r)}; x, m; y, n) \\ &\leqq \sum_{r=1}^{n-m} \kappa_1^r q_r(n-m, y-x) S_{m,n}^{(r)} \\ &\leqq (n-m)^{-d/2} \sum_{r=1}^{n-m} \kappa_1^r S_{m,n}^{(r)}\,, \end{aligned} \tag{6.6}$$

where

$$S_{m,n}^{(r)} = \sum_{l \geqq 2} \sum_{\mathbf{k}^{(r)} \in \mathscr{T}_{m,n}^{(l)}} \alpha(\mathbf{k}^{(r)}; m, n) \tag{6.7}$$

and $k_0 = m$, $k_{r+1} = n$. If $\mathbf{k}^{(r)} \in \mathscr{T}_{m,n}^{(l)}$ with $l \geqq 2$ then there exists j such that

$k_j - m > n^\beta$ and $n - k_j > n^\beta$. By (3.29) and (6.7),

$$S_{m,n}^{(r)} \leqq \sum_{(n-m)^\beta < l < (n-m)-(n-m)^\beta} \left(\sum_{m \leqq k_1 < \cdots < k_{j-1} < l} \alpha(\mathbf{k}^{(j-1)}; m, l) \right) \times \sum_{l < k_1 < \cdots < k_{r-j-1}} \alpha(\mathbf{k}^{(r-j-1)}; l+1, n) \tag{6.8}$$

$$\leqq C_3^{r-2} \sum_{(n-m)^\beta < l < (n-m)-(n-m)^\beta} (l-m)^{-d/2}(n-l-1)^{-d/2}$$

$$= C_3^{r-2}(n-m)^{-d/2}(1-(n-m)^{-1})^{-d/2} \times \sum_{(n-m)^\beta < l < (n-m)-(n-m)^\beta} (l^{-1} + (n-l-1)^{-1})^{d/2}$$

$$\leqq C_3^{r-2}(n-m)^{-d/2} 2^{3d/2} \sum_{l \geqq (n-m)^\beta} l^{-d/2}$$

$$\leqq C_3^{r-2}(n-m)^{-d/2} 2^{3d/2}((n-m)^{-d/2} + 2(d-2)^{-1}(n-m)^{-\beta(d/2-1)}) .$$

If ε in (2.8) is so small that $\kappa = C_3\kappa_1 < 1$ then (2.27), (3.4), (6.7), and (6.8) imply that for each $R > 0$ uniformly in $x, y \in \mathbb{R}^d$ satisfying $|y - x| \leqq R\sqrt{n-m}$,

$$\lim_{n-m\to\infty} (q_0(n-m, y-x))^{-2} E_Q(Z^{(2)}(x, m; y, n))^2 = 0 . \tag{6.9}$$

Thus, it remains to deal with $Z^{(1)}(x, m; y, n)$ given by (6.5). Let $\mathbf{k}^{(r)} \in \mathscr{T}_{m,n}^{(1)}$ and $r \leqq K \log(n-m)$ then

$$|k_{j+1} - k_j| \geqq (n-m) - (n-m)^\beta K \log(n-m) \quad \text{and } |k_j - m| + |n - k_{j+1}| \leqq (n-m)^\beta K \log(n-m) \tag{6.10}$$

for some $j = 0, 1, \ldots, r$ where, again $k_0 = m$ and $k_{r+1} = n$. Let $\frac{1}{2} > \lambda > \beta/2$. Set

$$V_1(\mathbf{k}^{(r)}; x, m; y, n) = \int_{B_x((n-m)^\lambda)} dz\, q_0(k_1 - m, z - x) \times \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} U(\mathbf{k}^{(j)}; z, u) q_0(k_{j+1} - k_j - 1, v - u) U(\mathbf{k}^{(r-j)}; v, w)\, du\, dv \times \int_{B_y((n-m)^\lambda)} q_0(n - k_r - 1, y - w)\, dw \tag{6.11}$$

and $V_2(\mathbf{k}^{(r)};x,m;y,n)=V(\mathbf{k}^{(r)};x,m;y,n)-V_1(\mathbf{k}^{(r)};x,m;y,n)$ where in the case $j=0$ or $j=r$ I denote by $U(k^{(0)};v,w)$ the δ-function at v (i.e. the unit mass sitting at v) and $B_x(r)$ is, again, the r-ball centered at x. By (3.6) for any $z\in B_x((n-m)^\lambda)$ and $u\in B_x((n-m)^\lambda)$,

$$
\begin{aligned}
(6.12)\qquad & q_0(k_{j+1}-k_j-1,u-z)\\
&\quad = q_0(n-m,y-x)(1+\zeta(k_j,z;k_{j+1},u;m,x;n,y))
\end{aligned}
$$

with

$$
(6.13)\qquad \zeta(k_j,z;k_{j+1},u;m,x;n,y) \leqq L_R(n-m)^{-\frac{1}{2}\min(\lambda-\beta/2,1/2-\lambda)}\,,
$$

provided $|y-x| \leqq R\sqrt{n-m}$, where $L_R>0$ depends only on R.

Employing estimates similar to the end of Sect. 3 I derive from (6.13) and (4.11) that for some $C>0$,

$$
\begin{aligned}
(6.14)\qquad & E_Q(V_2(\mathbf{k}^{(r)};x,m;y,n))^2\\
&\leqq C\kappa^r\alpha(\mathbf{k}^{(r)};m,n)\Bigg(\int_{\mathbb{R}^d\setminus B_x((n-m)^\lambda)} q_0(k_j-m,x,v)\,dv\\
&\quad + \int_{\mathbb{R}^d\setminus B_y((n-m)^\lambda)} q_0(n-k_{j+1},w,y)\,dw\Bigg)\\
&\leqq C\kappa^r\alpha(\mathbf{k}^{(r)},m,n)\exp(-\gamma_2(n-m)^{\lambda-\beta/2})
\end{aligned}
$$

and

$$
\begin{aligned}
(6.15)\qquad & (q_0(n-m,y-x))^2E_Q\Bigg(\int_{B_y((n-m)^\lambda)}\int_{B_x((n-m)^\lambda)} q_0(k_1-m,z-x)\\
&\quad\times U(\mathbf{k}^{(j)};z,u)\zeta(k_j,u,k_{j+1},v,m,x;n,y)U(\mathbf{k}^{(r-j)};v,w)\\
&\quad\times q_0(n-k_r-1,y-w)\,dz\,du\,dv\,dw\Bigg)^2\\
&\quad\leqq CL_Rq_0(n-m,y-x)\kappa^r(n-m)^{-\min(\lambda-\beta/2,1/2-\lambda)}\alpha(\mathbf{k}^{(r)},m,n)
\end{aligned}
$$

with $\alpha(\mathbf{k}^{(r)},m,n)$ given by (3.28). Observe that

$$
\begin{aligned}
(6.16)\qquad & \operatorname*{L.i.m.}_{n-m\to\infty}\Bigg|\varphi(x,m)-\sum_{r\geqq 1}\sum_{m\leqq k_1<\cdots<k_r,\max_{0\leqq i\leqq r-1}|k_{i+1}-k_i|\leqq K\log(n-m)}\\
&\quad\int\int q_0(k_1-m,z-x)U(\mathbf{k}^{(r)};z,v)\,dz\,dv\Bigg| = 0
\end{aligned}
$$

and

$$(6.17)\qquad \underset{n-m\to\infty}{\text{L.i.m.}} \Bigg| \psi(y,n) - \sum_{r\geqq 1} \sum_{m\leqq k_1<\cdots<k_r,\max_{0\leqq i\leqq r-1}|k_{i+1}-k_i|\leqq K\log(n-m)} \int\!\!\int U(\mathbf{k}^{(r)};z,v)q_0(n-k_r-1,y-v)\,dz\,dv \Bigg| = 0\ .$$

Now Theorem 6.1 follows for the discrete time case from (3.3)–(3.3) and (6.11)–(6.17).

In order to derive (6.1) for the continuous time case observe that if $t-s$ is an integer then (6.1) follows from the above. Thus by (2.31) and (6.1),

$$(6.18)\quad \Lambda^{-(t-s)}Z(x,s;y,t) = \psi(y,t)\Lambda^{-(t-s-n_{s,t})}\int Z(x,s;v,t-n_{s,t})\varphi(v,t-n_{s,t})q_0(n_{s,t},y-v)\,dv + \Lambda^{-(t-s-n_{s,t})}\int Z(x,s;v,t-n_{s,t})q_0(n_{s,t},y-v)\delta(v,t-n_{s,t};y,t)\,dv\ .$$

Substituting here

$$\delta(v,t-n_{s,t};y,t) = \varphi(v,t-n_{s,t})\delta_1(v,t-n_{s,t};y,t) + \psi(y,t)\delta_2(v,t-n_{s,t};y,t) + \delta_1(v,t-n_{s,t};y,t)\delta_2(v,t-n_{s,t};y,t) + \delta_3(v,t-n_{s,t};y,t)$$

and taking into account that the random field $Z(x,s;v,t-n_{s,t})$ is independent of each of the random fields $\varphi(v,t-n_{s,t})$, $\delta_2(v,t-n_{s,t};y,t)$, and $\delta_3(v,t-n_{s,t};y,t)$ I derive via the Cauchy–Schwartz inequality that the second integral in the right hand side of (6.18) tends to zero in $L^1(\Omega,Q)$-sense as $t-s\to\infty$. The term $Z(x,s;v,t-n_{s,t})$ is small in the $L^2(\Omega,Q)$-sense if $|v-x|$ is large and since by the local limit theorem $q_0(n_{s,t},y-v)$ is close for large $t-s$ (see (3.4)) to $r(n_{s,t},y-v)$ which does not vary much when v varies much less than $\sqrt{n_{s,t}}$ then taking into account that $Z(x,s;v,t-n_{s,t})$ and $\varphi(v,t-n_{s,t})$ are independent I conclude that for large $t-s$ the first integral in (6.18) is close in the $L^2(\Omega,Q)$-sense to

$$q_0(t-s,y-x)\int Z(x,s;v,t-n_{s,t})\varphi(v,t-n_{s,t})\,dv = q_0(t-s,y-x)\varphi(x,s)\ ,$$

which together with the above yield (A.1) and complete the proof of Theorem 6.1. □

Assuming ergodicity of the spatial shift θ_v, $v\in\mathbb{R}^d$ one can derive from Theorem 6.1 a weaker version of Theorems 2.2 and 2.5. Indeed, by (6.1) for any continuous function f satisfying (2.28) I can write up to a small in the appropriate sense error (which can be estimated using (2.21) and (2.42)),

$$(6.19)\qquad \Lambda^{-(t-s)}\int Z(x,s;y,t)f\left(\frac{y-x}{\sqrt{t-s}}\right)dy = \varphi(x,s)\int q_0(t-s,y-x)\psi(y,t)f\left(\frac{y-x}{\sqrt{t-s}}\right)dy + \int q_0(t-s,y-x)\delta(x,s;y,t)f\left(\frac{y-x}{\sqrt{t-s}}\right)dy\ .$$

It is easy to see that as $t-s\to\infty$ the second integral in (6.19) tends to zero in $L^1(\Omega,Q)$-sense. On the other hand, using the ergodic theorem (see, for instance, [Te]) and taking into account that both $q_0(t-s,y-x)$ and $f((y-x)/\sqrt{t-s})$ vary little when y varies much less than $\sqrt{t-s}$, and so $\psi(y,t)$ will average to almost 1 on each large but small relative to $\sqrt{t-s}$ cube, I conclude employing the local limit theorem that as $t-s\to\infty$ the first integral in (6.19) tends in the $L^2(\Omega,Q)$-sense to $\int r(1,z)f(z)\,dz$. This gives (2.29) and (2.45) but with only $L^1(\Omega,Q)$-convergence. In order to derive (2.27) and (2.44) one has to divide (6.19) by a Q-a.s. converging term, which yield only the convergence in probability in (2.27) and (2.44).

Finally, I shall derive (2.54) and (2.55). I shall deal only with w_f since v_g in Remark 2.8 can be treated in the same way. By (6.18) (recall, that in the case of Theorem 2.7 $\Lambda=1$),

$$w_f(y,s;t)=\Lambda^{-(t-s)}\int f(x)Z(x,s;y,t)\,dx=\psi(y,t)I_1^{s,t}+I_2^{s,t}+I_3^{s,t}+I_4^{s,t}+I_5^{s,t}\ ,$$

where

$$I_1^{s,t}=\Lambda^{-(t-s-n_{s,t})}\iint f(x)Z(x,s;v,t-n_{s,t})\varphi(v,t-n_{s,t})q_0(n_{s,t},y-v)\,dx\,dv\ ,$$

and $I_2^{s,t},I_3^{s,t},I_4^{s,t},I_5^{s,t}$ are obtained by integrating the second integral in (6.18) together with $f(x)$ in x and representing $\delta(v,t-n_{s,t};y,t)$ as the sum of 4 terms given in Theorem 6.1. It is not difficult to see by the ergodic theorem that $I_1^{s,t}$ converges in $L^1(\Omega,Q)$ as $t\to\infty$ to $E_Qf(0)$. Indeed, $q_0(n_{s,t},y-v)$ changes very little when v is in a $n_{s,t}^{1/4}$-neighborhood of x and in view of (2.42) the contribution of the integral outside of this neighborhood is very small. This together with (2.40) yield that $I_1^{s,t}$ has the same $L^1(\Omega,Q)$ limit when $t\to\infty$ as the integral $J^{s,t}=\int f(x)\varphi(x,s)q_0(t-s,y-x)\,dx$. Since $f(x)\varphi(x,s)\in L^1(\Omega,Q)$ and $q_0(t-s,y-x)$ changes very little when x varies in a cube of size $(t-s)^{1/4}$, and so $f(x)\varphi(x,s)$ averages in such cubes, it follows by the ergodic theorem that $J^{s,t}$ converges in $L^1(\Omega,Q)$ as $t\to\infty$ to $E_Qf(x)\varphi(x,s)=E_Qf(x)E_Q\varphi(x,s)=E_Qf(0)$ (where I use the independence of $f(x)$ and $\varphi(x,s)$). Next, taking into account that each triple $\{f(x),Z(x,s;v,t-n_{s,t}),\ \varphi(v,t-n_{s,t})\}$, $\{f(x),Z(x,s;v,t-n_{s,t}),\delta_2(v,t-n_{s,t};y,t)\}$, and $\{f(x),Z(x,s;v,t-n_{s,t}),\delta_3(v,t-n_{s,t};y,t)\}$ consists of independent random fields I derive via the Cauchy–Schwartz inequality that $I_2^{s,t},I_3^{s,t},I_4^{s,t},I_5^{s,t}$ converge to zero in $L^1(\Omega,Q)$ as $t\to\infty$. Thus $w_f(y,s;t)-\psi(y,t)E_Qf(0)\to 0$ in probability as $t\to\infty$ (since $\psi(y,t)$ has the same distribution for all t), completing the proof of (2.54) and (2.55). □

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