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Probability Theory and Related Fields

Points of increase of the Brownian sheet

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Summary. It is well-known that Brownian motion has no points of increase. We show that an analogous statement for the Brownian sheet is false. More precisely, for the standard Brownian sheet in the positive quadrant, we prove that there exist monotone curves along which the sheet has a point of increase.

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1 Introduction

A famous result of Dvoretsky, Erdös and Kakutani [3] asserts that with probability one, sample paths of a Brownian motion have no points of increase. More precisely, if $(B(u),\ u \in \mathbb{R}_+)$ is a Brownian motion, then there does not exist a continuous, monotone and injective function $f : [-1, 1] \to \mathbb{R}_+$ such that $B(f(u)) < B(f(0))$ if $u < 0$ and $B(f(u)) > B(f(0))$ if $u > 0$. This paper shows that an analogous statement for the Brownian sheet is false.

The first result in this direction was obtained by Mountford [7]. Notice that the result of [3] can be restated as follows: almost surely, for each $q \in \mathbb{R}$, no component of $\{u \in \mathbb{R}_+\colon B(u) > q\}$ has an endpoint in common with a component of $\{u \in \mathbb{R}_+\colon B(u) < q\}$. In [7], Mountford proved that this statement is false for the Brownian sheet, namely, with positive probability, there exists a component of $\{t \in [1,2]^2\colon W(t) > 1\}$ and a component of $\{t \in [1,2]^2\colon W(t) < 1\}$ with a common boundary point (if $[1,2]^2$ is replaced by the positive quadrant, then this occurs with probability one).

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Since in the plane, a point t in the boundary of a component is not necessarily accessible along a curve with one endpoint at t but otherwise contained in that component, Mountford's result left open the question of whether or not there exist curves along which the Brownian sheet has points of increase. In this paper, we use a variation on Mountford's technique to prove this stronger statement, and the main result of this paper is the following theorem.

Theorem 1. *Let* $(W(t),\ t \in \mathbb{R}_+^2)$ *be a standard Brownian sheet defined on a probability space* $(\Omega, \mathscr{F}, P)$. *For each* $q \in \mathbb{R}$, *there exists a continuous non-decreasing random function* $\gamma : [-1,1] \times \Omega \to [1,2]^2$ *such that with positive probability*, $W(\gamma(u)) < q$ *if* $u < 0$ *and* $W(\gamma(u)) > q$ *if* $u > 0$.

In particular, the theorem asserts that the point $\gamma(0)$ is a point of increase of the sheet along the curve $(\gamma(u),\ -1 \leqq u \leqq 1)$. One can ask whether there exists a continuous monotone curve γ along which $u \mapsto W(\gamma(u))$ is increasing. The answer is no, since γ and $W \circ \gamma$ would be simultaneously differentiable at infinitely many points, and the result of [1] shows that simultaneous differentiability cannot occur even at a single point. The question of whether there exist straight lines along which W has a point of increase remains open.

With little effort, Theorem 1 can be refined as follows: with positive probability, there exists a continuous non-decreasing function $\gamma : [-1,1] \to [1,2]^2$ such that $\gamma(-1) = (1,1)$, $\gamma(1) = (2,2)$, $W(\gamma(-1)) < q$, $W(\gamma(1)) > q$, and $W(\gamma(u)) = q$ for exactly one element $u \in [-1,1]$ (see Remark 14).

A recommended first pass through the paper is as follows. First read Sect. 2. Then browse through Sect. 3 to get some feel for the statements but without checking the proofs. Go on to read the first part of the proof of Lemma 2 in Sect. 4, through the end of Sect. 4.2, referring back to the statements in Sect. 3 as needed. Finally, go through the arguments in Sect. 3 and complete the verification of the proof of Lemma 2 in Sect. 4.3.

2 The basic estimates

The set $\mathbb{R}_+^2$ is endowed with the (partial) order $\leqq$ defined by

$$s = (s_1, s_2) \leqq t = (t_1, t_2) \ \Leftrightarrow\ s_1 \leqq t_1 \text{ and } s_2 \leqq t_2 .$$

A convenient norm on $\mathbb{R}^2$ is $|t| = |t_1| + |t_2|$. An *increasing curve* is a totally ordered and connected subset of $\mathbb{R}_+^2$. A (canonically parameterized) *increasing path* (resp. *decreasing path*) Γ is a continuous function defined on some interval of $\mathbb{R}$ with values in $\mathbb{R}_+^2$ with the property that $\Gamma(u) \leqq \Gamma(v)$ (resp. $\Gamma(u) \geqq \Gamma(v)$) when $u \leqq v$ and $|\Gamma(u) - \Gamma(v)| = |u - v|$. Recall [8, Theorem 2.7] that a set is an increasing curve if and only if it is the image of an increasing path. Moreover, increasing paths are Lipschitz functions, therefore, when equipped with the topology of uniform convergence, the set of increasing paths defined on a compact interval with values in a compact set is compact.

Recall that a standard Brownian sheet is a mean-zero continuous Gaussian process $W = (W(t),\ t \in \mathbb{R}^2_+)$, defined on some probability space $(\Omega, \mathscr{F}, P)$, with the covariance

$$E(W(s)W(t)) = \min(s_1, t_1) \min(s_2, t_2)\ ,$$

for all $s = (s_1, s_2)$ and $t = (t_1, t_2)$ in $\mathbb{R}^2_+$. It is well known [9] that the restriction of W to horizontal or vertical lines yields a Brownian motion. More precisely, $W(t_1, \cdot)$ (resp. $W(\cdot, t_2)$) is a Brownian motion with speed t_1 (resp. t_2). In this paper, we use the term *Brownian motion* to refer to any Brownian motion with speed between $\frac{1}{2}$ and 3. Recall also that *white noise* is the vector-measure W defined on the bounded Borel sets of $\mathbb{R}^2_+$ with values in $L^2(\Omega, \mathscr{F}, P)$ such that $W([0, t_1] \times [0, t_2]) = W(t_1, t_2)$, for all $(t_1, t_2) \in \mathbb{R}^2_+$. A basic property of white noise is that $E(W(A)W(B)) = m(A \cap B)$, where m denotes Lebesgue measure.

Throughout this paper, $q \in \mathbb{R}$ is fixed and $c > 0$, $u_0 > 0$ will be constants whose values shall be determined later (see the beginning of Sect. 4). For $u \geqq 0$, define

$$g(u) = c\, u^{3/4}\ . \tag{1}$$

For each $t = (t_1, t_2) \in [1, 2]^2$ and $n \in \mathbb{N}$, we shall define in Sect. 3 a random increasing path $(u, \omega) \mapsto \Gamma^n_t(u, \omega)$ and a random decreasing path $(u, \omega) \mapsto \hat{\Gamma}^n_t(u, \omega)$ on $[0, u_0] \times \Omega$, both with canonical parameterization and various other properties and, in particular, such that

$$\Gamma^n_t(0, \cdot) = \hat{\Gamma}^n_t(0, \cdot) = t, \qquad \Gamma^n_t(2^{-2n}) = (t_1 + 2^{-2n}, t_2)\ ,$$
$$\hat{\Gamma}^n_t(2^{-2n}) = (t_1 - 2^{-2n}, t_2)\ .$$

We will use these paths to define sets $F_0(t, n)$, $F_1(t, n)$, $\hat{F}_1(t, n)$, and $F(t, n)$ so that

$$F_0(t,n) = \{W(\Gamma^n_t(2^{-2n})) \in]q + 2^{-n}, q + 2^{-n+1}[,\ W(\hat{\Gamma}^n_t(2^{-2n})) \in]q - 2^{-n+1}, q - 2^{-n}[\}\ , \tag{2}$$

$$F_1(t,n) \subset \{W(\Gamma^n_t(u)) - W(\Gamma^n_t(2^{-2n})) \geqq g(u) - 2^{-n},\ \text{for } 2^{-n} \leqq u \leqq u_0\}\ , \tag{3}$$

$$\hat{F}_1(t,n) \subset \{W(\hat{\Gamma}^n_t(u)) - W(\hat{\Gamma}^n_t(2^{-2n})) \leqq -g(u) + 2^{-n},\ \text{for } 2^{-n} \leqq u \leqq u_0\}\ , \tag{4}$$

$$F(t,n) = F_0(t,n) \cap F_1(t,n) \cap \hat{F}_1(t,n)$$

(the definition of the sets $F_1(t,n)$ and $\hat{F}_1(t,n)$ will be given in Sect. 4).

Let D_{2n} be the set of points in $[1, 2]^2$ for which both coordinates are dyadic rationals of order $2n$. For $i, j \in \{0, \ldots, n\}$ with $i \leqq j$, let $E_{i,j}$ be the set of couples (s, t) of elements of D_{2n} such that

$$2^{-2(j+1)} \leqq \inf(|s_1 - t_1|, |s_2 - t_2|) \leqq 2^{-2j} \quad \text{and}$$
$$2^{-2(i+1)} \leqq \sup(|s_1 - t_1|, |s_2 - t_2|) \leqq 2^{-2i}\ .$$

In the case where i or j equals n, we replace $2^{-2(n+1)}$ by 0.

It will be shown that the definitions of $\Gamma_t^n(u,\cdot)$ and $\hat{\Gamma}_t^n(u,\cdot)$ are such that the following lemma, analogous to Lemma 2.4 of [7], holds.

Lemma 2. *Given q and the constants c and u_0 as defined in* (34), *there exist constants $K>0$ and $\alpha\in\,]0,1[$ (depending on q, c, and u_0) such that for all large $n\in\mathbb{N}$,*

(a) $P(F_1(t,n)) \geqq K2^{-\alpha n}$ *and* $P(\hat{F}_1(t,n)) \geqq K2^{-\alpha n}$, *for all* $t\in[1,2]^2$;
(b) $P(F(t,n)) \geqq K2^{-(1+2\alpha)n}$, *for all* $t\in[1,2]^2$;
(c) $P(F(s,n)\cap F(t,n)) \leqq K2^{-(1+2\alpha)n}2^{-(n-i)-2\alpha(n-j)}$, *for all* $(s,t)\in E_{i,j}$, $0 \leqq i \leqq j \leqq n$.

This lemma, along with the construction of the paths Γ_t^n and $\hat{\Gamma}_t^n$, are the heart of the paper, for they easily lead to a proof of Theorem 1.

Proof of Theorem 1. Let $D'_{2n} = D_{2n}\cap[5/4,7/4]^2$, and let $X_n(\omega)$ be the number of elements $t\in D'_{2n}$ such that $\omega\in F(t,n)$. We shall show that

$$E(X_n) \geqq K2^{(3-2\alpha)n} \quad \text{and} \quad E(X_n^2) \leqq 2K2^{2(3-2\alpha)n}\,. \tag{5}$$

Indeed, applying Lemma 2(b), we see that

$$E(X_n) = \sum_{t\in D'_{2n}} P(F(t,n)) \geqq (2^{2n-1})^2 K2^{-(1+2\alpha)n} = K2^{(3-2\alpha)n}\,.$$

Moreover, noticing that the cardinality of $E_{i,j}$ is bounded by $(2^{2n})^2 2^{2(n-i)}2^{2(n-j)}$, we can apply Lemma 2(c) to get

$$\begin{aligned} E(X_n^2) &= \sum_{s,t\in D'_{2n}} P(F(s,n)\cap F(t,n)) \\ &\leqq \sum_{i=0}^{n}\sum_{j=i}^{n}\sum_{(s,t)\in E_{i,j}} K2^{-(1+2\alpha)n}\,2^{-(n-i)-2\alpha(n-j)} \\ &= K2^{-(2+4\alpha)n}\sum_{i=0}^{n}\sum_{j=i}^{n} 2^i\,2^{2\alpha j}(2^{2n})^2\,2^{2(n-i)}\,2^{2(n-j)} \\ &= K2^{(6-4\alpha)n}\sum_{i=0}^{n}2^{-i}\sum_{j=i}^{n}2^{2(\alpha-1)j} \\ &\leqq 2K2^{(6-4\alpha)n}\,. \end{aligned}$$

In the last inequality, we have used the fact that $\alpha<1$. This proves the inequalities in (5).

From the lower bound on $E(X_n)$ in (5), we conclude in particular that $E(X_n^2)>0$ for each n. In addition, (5) implies that for some finite constant $C>0$,

$$E(X_n^2) \leqq CE(X_n)^2 = CE(X_n I_{\{X_n>0\}})^2\,. \tag{6}$$

We can now use a standard argument which can be found for instance in [4]: applying the Cauchy–Schwarz inequality to the right-hand side of (6), we conclude that $E(X_n^2) \leqq CE(X_n^2)P\{X_n > 0\}$, and therefore $P\{X_n > 0\} \geqq 1/C$, for all $n \in \mathbb{N}$. By Fatou's Lemma,

$$P\left(\limsup_{n\to\infty} \{X_n > 0\}\right) \geqq \limsup_{n\to\infty} P\{X_n > 0\} \geqq 1/C > 0 .$$

Let $G = \limsup_{n\to\infty}\{X_n > 0\}$ and fix $\omega \in G$. There is a sequence $n_k \uparrow \infty$ such that $\omega \in \{X_{n_k} > 0\}$ for all k, that is, there exists a sequence $t_k \in [5/4, 7/4]^2$ such that $\omega \in F(t_k, n_k)$ for all k. Consider the sequence of paths $(\Gamma_{t_k}^{n_k}(\omega),\ k \in \mathbb{N})$ and $(\hat{\Gamma}_{t_k}^{n_k}(\omega),\ k \in \mathbb{N})$. By taking a subsequence, we can assume that (t_k) converges to $t \in [5/4, 7/4]^2$, and $(\Gamma_{t_k}^{n_k}(\omega))$ and $(\hat{\Gamma}_{t_k}^{n_k}(\omega))$ converge uniformly to paths $\Gamma(\omega)$ and $\hat{\Gamma}(\omega)$, respectively. For $0 \leqq u \leqq u_0$, let $\gamma(-u,\omega) = \hat{\Gamma}(u,\omega)$ and $\gamma(u,\omega) = \Gamma(u,\omega)$ if $\omega \in G$, and let $\gamma(\cdot,\omega)$ be an arbitrary increasing path if $\omega \in \Omega \backslash G$. Then $|\gamma(\pm u,\omega) - \gamma(0,\omega)| = u$ for $0 \leqq u \leqq u_0$. From (2)–(4) (with t replaced by t_k and n by n_k), we conclude that for $\omega \in G$, $W(\gamma(-u,\omega),\omega) \leqq q - g(u)$ and $W(\gamma(u,\omega),\omega) \geqq q + g(u)$ for $0 < u \leqq u_0$. If the range of γ is not contained in $[1,2]^2$, this can be achieved by truncating its image and reparameterizing. This proves the theorem. □

3 The construction of the paths Γ_t^n and $\hat{\Gamma}_t^n$

Our task is now reduced to constructing the paths Γ_t^n and $\hat{\Gamma}_t^n$ and proving Lemma 2. The construction relies on several preliminaries.

For $t \in \mathbb{R}_+^2$, set $\mathscr{F}_t = \sigma\{W(s),\ s \leqq t\}$. A random variable T with values in $\mathbb{R}_+^2$ is a *stopping point* provided $\{T \leqq t\} \in \mathscr{F}_t$, for all $t \in \mathbb{R}_+^2$. Given a stopping point T, $\mathscr{F}_T$ denotes the sigma-field $\{F \in \mathscr{F} : F \cap \{T \leqq t\} \in \mathscr{F}_t$, for all $t \in \mathbb{R}_+^2\}$.

An observation that appears in Kendall [5] and Dalang and Walsh [2] is that in the neighborhood of an element $t \in \mathbb{R}_+^2$, the Brownian sheet behaves like the sum of two independent diffusions. More precisely, for all $u, v \geqq 0$,

$$W(t_1 + u, t_2 + v) = W(t) + B_1(u) + B_2(v) + \varepsilon(u,v) , \qquad (7)$$

where $(B_1(u))$ and $(B_2(v))$ are Brownian motions (with variance $t_2 u$ and $t_1 v$, respectively), and $(\varepsilon(u,v))$ is a Brownian sheet, and all three processes are independent. When u and v are small, the term $\varepsilon(u,v)$ is of order $(uv)^{1/2}$, which is much smaller than the typical value of $B_1(u) + B_2(v)$, which is of order $u^{1/2} + v^{1/2}$.

We use the following notation for simple curves that connect two points. If $s \leqq t$, we let $\langle s,t\rangle^h$ denote the segment $[s_1,t_1] \times \{t_2\}$ if $s_2 = t_2$, and the union of the two segments $\{s_1\} \times [s_2,t_2]$ and $[s_1,t_1] \times \{t_2\}$ if $s_2 < t_2$. Similarly, $\langle s,t\rangle^v$ denotes the segment $\{s_1\} \times [s_2,t_2]$ if $s_1 = t_1$ and the union of the two segments $[s_1,t_1] \times \{s_2\}$ and $\{t_1\} \times [s_2,t_2]$ if $s_1 < t_1$. $\langle s,t\rangle$ stands for either of these two paths.

3.1 The probability of doubling the distance to q

If $B = (B(u), u \in \mathbb{R}_+)$ is a standard Brownian motion and if $B(v_0) = q + r$ for some $v_0 \in \mathbb{R}_+$ and $r > 0$, then the probability that B reaches level $q + 2r$ before level q (after time v_0) is 2^{-1}. Moreover, if $r = 2^{-n}$, then the probability of reaching level $q + 1$ before level q is 2^{-n}.

Now suppose T is a stopping point and $W(T) = q + r$. What is the probability that there exists an increasing path Γ starting at T along which W reaches level $q + 2r$ before level q? We will show that this probability is $\geqq 2^{-\alpha}$ for some $\alpha \in]0, 1[$, by constructing a particular path which achieves this bound. The main idea is that either level $q + 2r$ is reached as we move horizontally to the right away from T, which occurs with probability 2^{-1}, or this occurs as we move vertically up from T, giving an additional opportunity of reaching level $q + 2r$.

By repeating the construction from level $q + 2r$, we see that if $r = 2^{-n}$, then the probability that there exists an increasing path starting at T along which W reaches level $q + 1$ before level q is $\geqq \theta 2^{-\alpha n}$, for some $\theta > 0$, which is orders of magnitude larger than 2^{-n}. This is the crucial observation that led to the results of [7], and which is the intuitive reason behind the difference in behavior of Brownian motions and Brownian sheets with regard to points of increase: the Brownian sheet has a much higher chance of escaping to a high level (along some increasing path) than does a Brownian motion.

Of course, some care must be taken to ensure that we reach level $q + 2r$ at a stopping point, and we also want to ensure that when this level is reached, W has grown at a guaranteed rate. Therefore the actual construction is somewhat more involved. To estimate the probability of reaching level $q + 2r$ before level q, we introduce the following notation. Given a stopping point $T = (T_1, T_2)$ with values in $[1/2, 3]^2$, let

$$W_1^T(u) = W(T_1 + u, T_2) - W(T) \quad \text{and} \quad W_2^T(v) = W(T_1, T_2 + v) - W(T) .$$

Recall [9] that these processes are conditionally independent, and conditionally independent of $\mathscr{F}_T$, given T. More precisely, given T, W_1^T (resp. W_2^T) is a Brownian motion with speed T_2 (resp. T_1). Since the stopping points we will consider in this paper satisfy $\frac{1}{2} \leqq T_1 \leqq 3$ a.s. and $\frac{1}{2} \leqq T_2 \leqq 3$ a.s., these speeds are between $\frac{1}{2}$ and 3.

If $S \geqq T$ is also a stopping point, then we set

$$\Delta_{]T,S]} W = W(S) - W(S_1, T_2) - W(S_2, T_1) + W(T) .$$

For $r \in]0, 1[$, let

$$S_1^T = \inf\{u \geqq 0 : W_1^T(u) \in \{-r + r^{3/2}, r\}\} ,$$

$$S_2^T = \begin{cases} 0 & \text{if } W_1^T(S_1^T) = r , \\ \inf\{v \geqq 0 : W_2^T(v) \in \{-r + r^{3/2}, 2r - r^{3/2}\}\} & \text{if } W_1^T(S_1^T) = -r + r^{3/2} . \end{cases}$$

Clearly, S_1^T and S_2^T depend on r, even though the notation does not indicate this explicitly. Notice that since W_1^T and W_2^T are (time-changed) Brownian motions, the probability that they hit one level before another is the same as for standard Brownian motion, and so for small r,

$$P\{W_1^T(S_1^T) = r\} = (r - r^{3/2})/(2r - r^{3/2}) \simeq \tfrac{1}{2} ,$$

$$P\{W_2^T(S_2^T) = 2r - r^{3/2} \mid W_1^T(S_1^T) = -r + r^{3/2}\} = (r - r^{3/2})/(3r - 2r^{3/2}) \simeq \tfrac{1}{3} .$$

Also, given $W(T) = q + r$, if $W_1^T(S_1^T) = r$, which occurs with probability $\simeq \frac{1}{2}$, then

$$W(T_1 + S_1^T, T_2) = W(T) + r = q + 2r ,$$

while if $W_1^T(S_1^T) = -r + r^{3/2}$ and $W_2^T(S_2^T) = 2r - r^{3/2}$, then

$$W(T_1 + S_1^T, T_2 + S_2^T) \simeq W(T) - r + r^{3/2} + 2r - r^{3/2} = q + 2r .$$

Therefore, for small $r > 0$, with probability approximately equal to

$$\tfrac{1}{2} + (1 - \tfrac{1}{2})\tfrac{1}{3} = \tfrac{2}{3} ,$$

W reaches approximately level $q + 2r$ before $q + r^{3/2}$ along the path $\langle T, (T_1 + S_1^T, T_2 + S_2^T)\rangle^h$.

Since $W(T_1 + S_1^T, T_2 + S_2^T)$ is not exactly equal to $q + 2r$ when $S_2^T > 0$, one additional step is needed. Set

$$\varphi_1(T) = \inf\{u \geqq S_1^T : W(T_1 + u, T_2 + S_2^T) - W(T) = r\}$$

and

$$\psi^h(T, r) = (T_1 + \varphi_1(T), T_2 + S_2^T) .$$

Observe that $\varphi_1(T) = S_1^T$ when $S_2^T = 0$. Moreover, notice that $\psi^h(T, r)$ is a stopping point and that given $W(T) = q + r$, with probability approximately 2/3, $W(\psi^h(T, r)) = q + 2r$ and along the path $\langle T, \psi^h(T, r)\rangle^h$, W reaches level $q + 2r$ before $q + r^{3/2}$.

The construction of $\psi^h(T, r)$ privileges the horizontal direction. By exchanging the roles of the coordinates and privileging the vertical direction, we define analogously a stopping point $\psi^v(T, r)$ with similar properties.

In order to make the statement "with probability approximately 2/3" precise, we introduce the following notation. Let B be a standard Brownian motion, and let $U_{a,b} = \inf\{u \geqq 0\colon B(u) \in \{a, b\}\}$. Recall [6, Theorem 4.1.1] that $P\{U_{-1,1} \in [u, u + \tau]\} \leqq \tau$ for all $u > 0$ and $\tau > 0$. For $M > 0$ and $a > 0$, define

$$p_1(M, a) = P\{U_{-1+a,1} \in [1/M, M],\ B(U_{-1+a,1}) = 1\} ,$$

$$p_2(M, a) = P\{U_{-1+a,2-a} \in [1/M, M],\ B(U_{-1+a,2-a}) = 2 - a\} ,$$

$$p(M, a) = p_1(M, a) + (1 - p_1(M, a))p_2(M, a) .$$

Notice that $\lim_{M\to\infty, a\downarrow 0} p(M, a) = \frac{2}{3}$.

Throughout this paper, we fix $M > 0$ and $a_0 > 0$ so that

$$p(M, a) > 1/2 \quad \text{for } 0 \leqq a \leqq a_0$$

and we set

$$p_0 = p(M, 0) . \tag{8}$$

Constants whose existence is affirmed generally depend on M and a_0.

Lemma 3. *Let T be a stopping point with values in $[\frac{1}{2}, 3]^2$. For $r > 0$, set*

$$G(T,r) = \left\{ \frac{r^2}{MT_2} \leqq S_1^T \leqq \frac{Mr^2}{T_2} \right\}$$
$$\cap \left(\{S_2^T = 0\} \cup \left\{ \frac{r^2}{MT_1} \leqq S_2^T \leqq \frac{Mr^2}{T_1}, \ \frac{r^2}{MT_2} \leqq \varphi_1(T) \leqq \frac{2Mr^2}{T_2} \right\} \right)$$
$$\cap \{ W(\cdot) - W(T) > -r + r^{3/2} \ \text{on} \ \langle T, \psi^h(T,r) \rangle^h \} .$$

Then $G(T,r)$ is conditionally independent of $\mathscr{F}_T$ given T and there exists $C > 0$ and $r_0 > 0$ such that for all $r \in]0, r_0[$,

$$|P(G(T,r) \mid \mathscr{F}_T) - p_0| \leqq Cr^{1/4} .$$

Remark 4. The event $G(T,r)$ describes the following situation. The process W_1^T first hits $-r + r^{3/2}$ or r during the time interval $[r^2/(MT_2), Mr^2/T_2]$. If it hits r first, then $S_2^T = 0$ and the third event in the definition of $G(T,r)$ necessarily occurs. However, if W_1^T hits $-r + r^{3/2}$ first, then W_2^T must hit $2r - r^{3/2}$ or $-r + r^{3/2}$ during the time interval $[r^2/(MT_1), Mr^2/T_1]$ and there are similar constraints on $\varphi_1(T)$. In order that the third event in the definition of $G(T,r)$ occur, it must be the case that W_2^T first hits $2r - r^{3/2}$, along the segment $\langle (T_1, T_2 + S_2^T), (T_1 + S_1^T, T_2 + S_2^T) \rangle^h$, $W(\cdot) - W(T) > -r + r^{3/2}$, and along $\langle (T_1 + S_1^T, T_2 + S_2^T), \psi^h(T,r) \rangle^h$, $W(\cdot) - W(T)$ hits r before $-r + r^{3/2}$. In particular, on $G(T,r)$, $W(\psi^h(T,r)) = W(T) + r$.

Proof of Lemma 3. Let

$$H(T,r) = \left\{ \frac{r^2}{MT_2} \leqq S_1^T \leqq \frac{Mr^2}{T_2} \right\}$$
$$\cap \left(\{S_2^T = 0\} \cup \left\{ \frac{r^2}{MT_1} \leqq S_2^T \leqq \frac{Mr^2}{T_1} \right\} \right)$$
$$\cap \{ W_1^T(S_1^T) + W_2^T(S_2^T) = r \} .$$

Further, define

$$\tilde{S}_1^T = \inf\{ u \geqq 0 : W_1^T(u) \in \{-r, r\} \} ,$$
$$\tilde{S}_2^T = \begin{cases} 0 & \text{if } W_1^T(\tilde{S}_1^T) = r , \\ \inf\{ v \geqq 0 : W_2^T(v) \in \{-r, 2r\} \} & \text{if } W_1^T(\tilde{S}_1^T) = -r , \end{cases}$$

and let $\tilde{H}(T,r)$ be defined in the same way as $H(T,r)$ but with S_1^T and S_2^T replaced by $\tilde{S}_1^T$ and $\tilde{S}_2^T$, respectively.

By Brownian scaling, observe that $P(\tilde{H}(T,r) \mid T) \equiv p_0$. Also note that the stopping times $\tilde{S}_i^T$ are greater than or equal to S_i^T and that if $W_1^T(S_1^T)$ takes value r, then so must $W_1^T(\tilde{S}_1^T)$. In addition, if $W_2^T(S_2^T) = 2r - r^{3/2}$, then it is highly probable that $W_2^T(\tilde{S}_2^T) = 2r$.

To prove the lemma, we shall show that there is a constant C such that

$$|P(H(T,r)\mid T)-P(\tilde{H}(T,r)\mid T)|\leqq Cr^{1/4} \tag{9}$$

and

$$|P(G(T,r)\mid T)-P(H(T,r)\mid T)|\leqq Cr^{1/2}\,. \tag{10}$$

For this, we first check that $\tilde{H}(T,r)\backslash H(T,r)$ is contained in

$$\left\{S_1^T<\frac{r^2}{MT_2}\leqq \tilde{S}_1^T,\ W_1^T(S_1^T)=-r+r^{3/2},\ W_1^T(\tilde{S}_1^T)=-r\right\} \tag{11}$$

$$\cup\,\{W_1^T(S_1^T)=-r+r^{3/2},\ W_1^T(\tilde{S}_1^T)=r\} \tag{12}$$

$$\cup\left\{S_2^T<\frac{r^2}{MT_1}\leqq \tilde{S}_2^T,\ W_2^T(S_2^T)=2r-r^{3/2},\ W_2^T(\tilde{S}_2^T)=2r\right\} \tag{13}$$

$$\cup\,\{W_2^T(S_2^T)=-r+r^{3/2},\ W_2^T(\tilde{S}_2^T)=2r\}\,. \tag{14}$$

Indeed, with obvious notations, we can write

$$H(T,r)=A_1\cap A_2\cap A_3,\qquad \tilde{H}(T,r)=\tilde{A}_1\cap\tilde{A}_2\cap\tilde{A}_3\,,$$

and so $\tilde{H}(T,r)\backslash H(T,r)$ is the union of the three events

$$(\tilde{A}_1\backslash A_1)\cap\tilde{A}_2\cap\tilde{A}_3,\qquad \tilde{A}_1\cap(\tilde{A}_2\backslash A_2)\cap\tilde{A}_3,\qquad \tilde{A}_1\cap\tilde{A}_2\cap(\tilde{A}_3\backslash A_3)\,. \tag{15}$$

On the first event in (15), $S_1^T<r^2/(MT_2)\leqq\tilde{S}_1^T$, so $W_1^T(S_1^T)=-r+r^{3/2}$, and therefore (11) or (12) occurs. On the second event in (15), $S_2^T>0$, so $W_1^T(S_1^T)=-r+r^{3/2}$. Moreover, either $\tilde{S}_2^T=0$, in which case $W_1^T(\tilde{S}_1^T)=r$ and (12) occurs, or $S_2^T<r^2/(MT_1)\leqq\tilde{S}_2^T$, in which case $W_1^T(\tilde{S}_1^T)=-r$. Because $\tilde{A}_3$ occurs, $W_2^T(\tilde{S}_2^T)$ must equal $2r$, and so (13) or (14) occurs. On the last event in (15), $W_1^T(\tilde{S}_1^T)=r$ or $W_1^T(\tilde{S}_1^T)=-r$ and $W_2^T(\tilde{S}_2^T)=2r$, and both $W_1^T(S_1^T)=-r+r^{3/2}$ and $W_2^T(S_2^T)=-r+r^{3/2}$. So either (12) or (14) occurs.

We now use similar arguments to show that $H(T,r)\backslash\tilde{H}(T,r)$ is contained in

$$\left\{S_1^T\leqq\frac{Mr^2}{T_2}<\tilde{S}_1^T,\ W_1^T(S_1^T)=-r+r^{3/2},\ W_1^T(\tilde{S}_1^T)=-r\right\} \tag{16}$$

$$\cup\,\{W_1^T(S_1^T)=-r+r^{3/2},\ W_1^T(\tilde{S}_1^T)=r\} \tag{17}$$

$$\cup\left\{S_2^T\leqq\frac{Mr^2}{T_1}<\tilde{S}_2^T,\ W_2^T(S_2^T)=2r-r^{3/2},\ W_2^T(\tilde{S}_2^T)=2r\right\} \tag{18}$$

$$\cup\,\{W_2^T(S_2^T)=2r-r^{3/2},\ W_2^T(\tilde{S}_2^T)=-r\}\,. \tag{19}$$

Indeed, with the notations above, $H(T,r)\backslash\tilde{H}(T,r)$ is the union of the three events

$$(A_1\backslash\tilde{A}_1)\cap A_2\cap A_3,\qquad A_1\cap(A_2\backslash\tilde{A}_2)\cap A_3,\qquad A_1\cap A_2\cap(A_3\backslash\tilde{A}_3)\,. \tag{20}$$

On the first of these events, $S_1^T \leqq Mr^2/T_2 < \tilde{S}_1^T$, so $W_1^T(S_1^T) = -r + r^{3/2}$, and therefore either (16) or (17) occurs. On the second event in (20), $\tilde{S}_2^T > 0$, so $W_1^T(\tilde{S}_1^T) = -r$ and thus $W_1^T(S_1^T) = -r + r^{3/2}$. Because A_3 occurs, $W_2^T(S_2^T)$ must equal $2r - r^{3/2}$. Since $\tilde{S}_2^T$ does not belong to $[r^2/(MT_1), Mr^2/T_1]$ on this event, $S_2^T \leqq Mr^2/T_1 < \tilde{S}_2^T$ and therefore either (18) or (19) occurs. On the last event in (20), both $W_1^T(\tilde{S}_1^T) = -r$ and $W_2^T(\tilde{S}_2^T) = -r$. Therefore $W_1^T(S_1^T) = -r + r^{3/2}$ and so $W_2^T(S_2^T) = 2r - r^{3/2}$. Therefore the last event is contained in (19).

To prove (9), note that the events in (12) and (14) are independent of T, and by Brownian scaling and the strong Markov property, their probabilities are equal to the probability that a standard Brownian motion B hits $2r - r^{3/2}$ (resp. $3r - r^{3/2}$) before $-r^{3/2}$, and therefore are bounded by $Cr^{\frac{1}{2}}$. Likewise for (17) and (19).

We bound the conditional probability given T of the set in (11) as follows. It is less than

$$P\left\{\frac{r^2}{MT_2} \leqq \tilde{S}_1^T \leqq \frac{r^2}{MT_2} + \frac{r^{5/2}}{T_2}\right\} + P\left\{\tilde{S}_1^T - S_1^T \geqq \frac{r^{5/2}}{T_2}\right\}.$$

The first term is bounded by $Cr^{1/2}$ while the second term is bounded by

$$P\left\{\inf_{0 \leqq u \leqq r^{5/2}} B(u) \geqq -r^{3/2}\right\} \leqq Cr^{1/4}\,. \tag{21}$$

The probabilities of the events in (13), (16), and (18) can be bounded in a similar way. This proves (9).

We now turn to (10). Let $\varepsilon_T(u,v) = \Delta_{]T,T+(u,v)]}W$ and $T' = (T_1 + S_1^T, T_2 + S_2^T)$. Observe from the definitions that $G(T,r) \subset H(T,r)$, and that $H(T,r)\backslash G(T,r)$ can only occur because of an unfortunate behavior of W along the horizontal half-line with left endpoint at $(T_1, T_2 + S_2^T)$. In particular, $H(T,r)\backslash G(T,r)$ is contained in the union of two events:

(a) $\inf_{0 \leqq u \leqq Mr^2/T_2} \varepsilon_T(u, S_2^T) \leqq -2r + r^{3/2}$ and $S_2^T \leqq Mr^2/T_1$;

(b) $S_1^T \leqq Mr^2/T_2$, $S_2^T \leqq Mr^2/T_1$ and starting from level $W(T') - W(T) = r + \varepsilon_T(S_1^T, S_2^T)$, $W(T_1' + \cdot, T_2') - W(T)$ hits level $-r + r^{3/2}$ before level r, or takes at least time Mr^2/T_2 to hit $\{-r + r^{3/2},\ r\}$.

Since $\frac{1}{2} \leqq T_i \leqq 3$ for $i = 1, 2$, given $S_2^T \leqq Mr^2/T_1$, $\varepsilon_T(\cdot, S_2^T)$ is a Brownian motion with speed at most $Mr^2/T_1 \leqq 2Mr^2$, so the probability in (a) is bounded by

$$P\left\{2Mr^2 \inf_{0 \leqq u \leqq 1} B(u) \leqq -2r + r^{3/2}\right\} \leqq \exp\left(-\frac{1}{2M^2r^2}\right) \leqq Cr^{1/2}$$

for small r. As for the probability in (b), it is bounded by the probability that a Brownian motion B started at level $r + 2Mr^2Z$, where Z is a standard Normal independent of B, hits level $-r + r^{3/2}$ before level r, or hits level r first but $\varphi_1(T) \geqq 2Mr^2$. This probability is bounded by the sum of two terms.

For small r, the first term is less than

$$\int_0^\infty \frac{2Mr^2 z}{2r - r^{3/2}} P\{Z \in \mathrm{d}z\} \leqq Cr \ .$$

The second term is bounded by standard Brownian motion inequalities similar to (21). This proves (10) and completes the proof of Lemma 3. □

3.2 Defining the path Γ_t^n

For $t \in [1,2]^2$ and $r \in]0,1[$, set $T_t^0 = t + (2^{-2n}, 0)$, and for $i \geqq 1$, let

$$T_t^i = \begin{cases} \psi^h(T_t^{i-1}, 2^{i-1}r) & \text{if } i \text{ is odd} \ , \\ \psi^v(T_t^{i-1}, 2^{i-1}r) & \text{if } i \text{ is even} \ . \end{cases}$$

The alternation between even and odd is merely a simple way of ensuring that both coordinates of T_t^i grow at the same rate.

Let $\Gamma(t,r)$ be the union of the paths $\langle T_t^{i-1}, T_t^i\rangle^h$ if i is odd, $\langle T_t^{i-1}, T_t^i\rangle^v$ if i is even, $i \geqq 1$. We define Γ_t^n as follows.

Definition. Γ_t^n *is the (canonically parameterized) increasing path whose image is the union of* $\langle t, t + (2^{-2n}, 0)\rangle^h$ *and* $\Gamma(t, 2^{-n})$.

In order to establish properties of this path, for $t \in [1,2]^2$ and $k \in \mathbb{N}$, consider the set

$$J(t,r,k) = \bigcap_{i=0}^{k} G(T_t^i, 2^i r) \ . \tag{22}$$

Lemma 5. *There exists an integer* k_0 *such that for all* $t \in [1,2]^2$, $r > 0$, *and* $j \in \{1,2\}$, *if* $k \geqq 1$, *then*

$$(T_t^k - t)_j \leqq 2^{2(k+k_0)} r^2 \quad \textit{on } J(t,r,k-1) \ ,$$

and if $k \geqq 2$, *then*

$$(T_t^k - t)_j \geqq 2^{2(k-k_0)} r^2 \quad \textit{on } J(t,r,k-1), \ \textit{provided } 2^{2(k+k_0)} r^2 \leqq 1 \ . \tag{23}$$

Proof. Because of the constraints of the form $\varphi_1(T) \leqq 2Mr^2/T_1$ and similar constraints on S_2^T, the maximum value on $J(t,r,k-1)$ of $(T_t^k - t)_j$ is

$$\sum_{i=0}^{k} 2M(2^i r)^2 = \frac{2}{3} M r^2 (4^{k+1} - 1) \ ,$$

and this quantity is $\leqq 2^{2(k+k_0)} r^2$ provided (essentially) k_0 is such that $4^{k_0} \geqq 8M/3$.

Suppose that $2^{2(k+k_0)} r^2 \leqq 1$. When $k \geqq 2$, given that either step k or step $k-1$ privileges direction j, $j = 1,2$, $(T_t^k - t)_j$ is at least equal to $(2^{k-1}r)^2/(M(T_t^k)_j)$. As we have just seen that $(T_t^k)_j \leqq t_j + 2^{2(k+k_0)} r^2$, and the right-hand side is $\leqq 3$ by hypothesis, we conclude that $(T_t^k - t)_j \geqq (2^{k-1}r)^2/(3M)$, and this quantity is $\geqq 2^{2(k-k_0)} r^2$ provided $4^{k_0} \geqq 12M$. □

Remark 6. Let k_0 be as in Lemma 5, fix $k_2 \geqq k_0$ and suppose $J(t, 2^{-n}, n - k_2)$ occurs. Then on $\langle T_t^0, T_t^1 \rangle^h$, $W(\cdot) - W(T_t^0) > -2^{-n} + 2^{-3n/2}$ and $W(T_t^1) - W(T_t^0) = 2^{-n}$. Similarly, on $\langle T_t^i, T_t^{i+1} \rangle$, $W(\cdot) - W(T_t^i) > -2^{-n+i} + 2^{3(-n+i)/2}$ and $W(T_t^{i+1}) - W(T_t^i) = 2^{-n+i}$. Therefore,

$$W(T_t^i) - W(T_t^0) = 2^{-n+i-1} + \cdots + 2^{-n} = 2^{-n+i} - 2^{-n} ,$$

and on $\langle T_t^i, T_t^{i+1} \rangle$,

$$\begin{aligned} W(\cdot) - W(T_t^0) &= W(\cdot) - W(T_t^i) + W(T_t^i) - W(T_t^0) \\ &\geqq -2^{-n+i} + 2^{3(-n+i)/2} + 2^{-n+i} - 2^{-n} \\ &= 2^{3(-n+i)/2} - 2^{-n} . \end{aligned}$$

This implies that on $J(t, 2^{-n}, n - k_2)$, the process $(W(\Gamma_t^n(u)) - W(\Gamma_t^n(2^{-2n}))$, $2^{-2n} \leqq u \leqq |T_t^{n-k_2}|)$ has "risen" from level 0 to level $2^{-k_2} - 2^{-n}$ without going below level -2^{-n}, and in fact, has grown at a guaranteed rate. Indeed, for small $u > 0$ (but large enough relative to 2^{-2n}, for instance $u > 2^{-n}$), $T_t^k \leqq \Gamma_t^n(u)$ occurs if $2^{2(k+k_0)}(2^{-n})^2 \leqq u$, that is, if $k \leqq n - k_0 + \log_2 u^{1/2}$. Therefore, if k is the integer part of $n - k_0 + \log_2 u^{1/2}$, then

$$\begin{aligned} W(\Gamma_t^n(u)) - W(\Gamma_t^n(2^{-2n})) &= W(\Gamma_t^n(u)) - W(T_t^k) + W(T_t^k) - W(\Gamma_t^n(2^{-2n})) \\ &\geqq -2^{-n+k} + 2^{3(-n+k)/2} + 2^{-n+k} - 2^{-n} \\ &\geqq 2^{-3(k_0+1)/2} u^{3/4} - 2^{-n} . \end{aligned}$$

The condition in (23) is satisfied when $k = n - k_2$ and $r = 2^{-n}$, so for large n, $|T_t^{n-k_2} - t| \geqq 2^{2(n-2k_2)} 2^{-2n} = 2^{-4k_2}$ by Lemma 5. In particular, the portion of the path Γ_t^n with extremities t and $T_t^{n-k_2}$ is guaranteed to have length at least 2^{-4k_2}, and $J(t, 2^{-n}, n - k_2)$ is contained in the event on the right-hand side of (3) provided $u_0 \leqq 2^{-4k_2}$ and the constant c that appears in (1) is $\leqq 2^{-3(k_0+1)/2}$.

Proposition 7. *Let p_0 be as in (8) and let $\alpha \in]0, 1[$ be defined by the relation $2^{-\alpha} = p_0$. There exist positive contants θ and Θ and an integer k_1 such that for all $t \in [1, 2]^2$, for all large n and all $k \in \{0, \dots, n - k_1\}$,*

$$\theta\, 2^{-\alpha k} \leqq P(J(t, 2^{-n}, k)) \leqq \Theta\, 2^{-\alpha k} .$$

Proof. Let r_0 be as in Lemma 3 and k_0 be as in Lemma 5. By Lemma 5, if $2^{2(k+k_0-n)} \leqq 1$, which is the case if $k_1 \geqq k_0$ and $k \leqq n - k_1$, then $T_t^k \in [1, 3]^2$. Using Lemma 3 and repeated conditioning on $\mathscr{F}_{T_t^i}$, $i = k, \dots, 0$, as well as the fact that $J(t, 2^{-n}, 0)$ is independent of $\mathscr{F}_{T_t^0}$, we see that if $2^{k-n} < r_0$, which is the case if $k_1 > -\log_2 r_0$ and $k \in \{0, \dots, n - k_1\}$, then

$$\prod_{i=0}^{k} (p_0 - C2^{(i-n)/4}) \leqq P(J(t, 2^{-n}, k)) \leqq \prod_{i=0}^{k} (p_0 + C2^{(i-n)/4}) .$$

The right-hand side is equal to

$$p_0^{k+1} \prod_{i=0}^{k} \left(1 + \frac{C}{p_0} 2^{(i-n)/4}\right) \leqq p_0^{k+1} \exp\left(\frac{C}{p_0} \sum_{i=0}^{k} 2^{(i-n)/4}\right) ,$$

and the sum in the exponential is $\leqq 30$ for $k \leqq n$. The left-hand side is equal to

$$p_0^{k+1} \prod_{i=0}^{k} \left(1 - \frac{C}{p_0} 2^{(i-n)/4}\right) \geqq p_0^{k+1} \exp\left(\frac{C}{2p_0} \sum_{i=0}^{k} 2^{(i-n)/4}\right) .$$

We have used the elementary inequality $1 - x \geqq e^{-x/2}$ for small positive x, say $0 \leqq x \leqq x_0$. The last inequality is therefore justified provided (C/p_0). $2^{(i-n)/2} \leqq x_0$, or provided $i \leqq n - k_1$, where k_1 is any integer greater than $-2 \log_2 (x_0 p_0/C)$. □

3.3 *Constructing the path* $\hat{\Gamma}_t^n$

The construction of $\hat{\Gamma}_t^n$ is similar to that of Γ_t^n. However, since $\hat{\Gamma}_t^n$ is decreasing, there is less independence to be used than in the construction of Γ_t^n, and therefore this construction requires some additional effort.

In order to construct the path $\hat{\Gamma}_t^n$, given a random point T and $r > 0$, set

$$\hat{W}_1^T(u) = W(T) - W(T_1 - u, T_2), \qquad \hat{W}_2^T(v) = W(T) - W(T_1, T_2 - v) ,$$

and let

$$\hat{S}_1^T = \inf\{u \geqq 0 : \hat{W}_1^T(u) \in \{-r + r^{3/2}, r\}\} ,$$
$$\hat{S}_2^T = \begin{cases} 0 & \text{if } \hat{W}_1^T(S_1^T) = r , \\ \inf\{v \geqq 0 : \hat{W}_2^T(v) \in \{-r + r^{3/2}, 2r - r^{3/2}\}\} & \text{if } \hat{W}_1^T(S_1^T) = -r + r^{3/2} , \end{cases}$$

and

$$\hat{\varphi}_1(T) = \inf\{u \geqq \hat{S}_1^T : W(T) - W(T_1 - u, T_2 - \hat{S}_2^T) = r\}$$

and

$$\hat{\psi}^h(T, r) = (T_1 - \hat{\varphi}_1(T), T_2 - \hat{S}_2^T) .$$

Observe that $\hat{\varphi}_1(T) = \hat{S}_1^T$ when $\hat{S}_2^T = 0$.

As in the definition of ψ^h, the construction of $\hat{\psi}^h(T, r)$ privileges the horizontal direction. By exchanging the roles of the coordinates and privileging the vertical direction, we define analogously a random point $\hat{\psi}^v(T, r)$ with similar properties.

For $t \in [1, 2]^2$, set $\hat{T}_t^0 = t - (2^{-2n}, 0)$, and for $i \geqq 1$, let

$$\hat{T}_t^i = \begin{cases} \hat{\psi}^h(\hat{T}_t^{i-1}, 2^{i-1} r) & \text{if } i \text{ is odd} , \\ \hat{\psi}^v(\hat{T}_t^{i-1}, 2^{i-1} r) & \text{if } i \text{ is even} . \end{cases}$$

Let $\hat{\Gamma}(t,r)$ be the union of the paths $\langle \hat{T}_t^i, \hat{T}_t^{i-1}\rangle^v$ if i is odd, $\langle \hat{T}_t^i, \hat{T}_t^{i-1}\rangle^h$ if i is even, $i \geqq 1$. We define the path $\hat{\Gamma}_t^n$ as follows.

Definition. *$\hat{\Gamma}_t^n$ is the (canonically parameterized) decreasing path whose image is $\hat{\Gamma}(t, 2^{-n})$.*

In order to establish estimates concerning $\hat{\Gamma}_t^n$, a decomposition of $W(t_1 - u, t_2 - v)$ analogous to (7) is needed. We could use the decomposition given in [2, Sect. 2], but for our purposes it is more convenient to proceed as follows. Observe that

$$W(t_1 - u, t_2 - v) = W(t) - \hat{W}_1^t(u) - \hat{W}_2^t(v) + \hat{\varepsilon}_t(u,v)\,,$$

where $\hat{\varepsilon}_t(u,v) = \Delta_{R(u,v)}W$ and $R(u,v) =]t_1 - u, t_1] \times]t_2 - v, t_2]$. However, the two processes $\hat{W}_1^t$ and $\hat{W}_2^t$ are *not* independent, which was one key feature of the decomposition (7). Fix $r > 0$. A decomposition which does yield independent terms is

$$W(t_1 - u, t_2 - v) = W(t) - \tilde{W}_1^t(u) - X_1^t(u) - \tilde{W}_2^t(v) - X_2^t(v) + \hat{\varepsilon}_t(u,v)\,,$$

where

$$\tilde{W}_1^t(u) = W(t_1, t_2 - 4Mr^2) - W(t_1 - u, t_2 - 4Mr^2)\,,$$
$$\tilde{W}_2^t(v) = W(t_1 - 4Mr^2, t_2) - W(t_1 - 4Mr^2, t_2 - v)\,,$$

and

$$X_1^t(u) = \Delta_{]t_1 - u, t_1] \times]t_2 - 4Mr^2, t_2]}W, \qquad X_2^t(v) = \Delta_{]t_1 - 4Mr^2, t_1] \times]t_2 - v, t_2]}W\,.$$

The processes $(\tilde{W}_1^t(u),\ 0 \leqq u \leqq 4Mr^2)$ and $(\tilde{W}_2^t(v),\ 0 \leqq v \leqq 4Mr^2)$ are independent, and the other processes are all comparatively small. More precisely, by the scaling properties of Brownian motion and the Brownian sheet,

$$P\left\{\sup_{0 \leqq u,v \leqq 4Mr^2} (|X_1^t(u)|, |X_2^t(v)|, |\hat{\varepsilon}_t(u,v)|) \geqq r^{5/3}\right\} \leqq K \exp\left(-\frac{r^{-2/3}}{32M^2}\right).$$

Of course, we need this type of decomposition at random times as well as at fixed times. For $(0,0) \leqq s \leqq t$, define the sigma-field

$$\mathscr{G}_s^t = \sigma\{\Delta_R W,\ R \subset ([s_1, t_1] \times [0, t_2]) \cup ([0, t_1] \times [s_2, t_2])\}\,. \tag{24}$$

Assume now that T is a random point such that $T \leqq t$ a.s. and $\{T \geqq s\} \in \mathscr{G}_s^t$ for all $0 \leqq s \leqq t$. Notice that in this case, $\hat{\psi}^h(T,r)$ is a random point with the property $\{\hat{\psi}^h(T,r) \geqq s\} \in \mathscr{G}_s^t$, and that given $W(T) = q - r$, with probability approximately 2/3, $W(\hat{\psi}^h(T,r)) = q - 2r$ and along $\langle \psi^h(T,r), T\rangle^v$, W reaches level $q - 2r$ before $q - r^{3/2}$. To make this statement precise, we introduce the following notation.

Consider the sigma-field

$$\hat{\mathscr{G}}_T^t = \{F \in \mathscr{F} : F \cap \{T \geqq s\} \in \mathscr{G}_s^t\}\,.$$

Given $r > 0$, we set

$$\tilde{W}_1^T(u) = W(T_1, T_2 - 4Mr^2) - W(T_1 - u, T_2 - 4Mr^2)\,,$$
$$\tilde{W}_2^T(v) = W(T_1 - 4Mr^2, T_2) - W(T_1 - 4Mr^2, T_2 - v)\,.$$

Then the processes $(\tilde{W}_1^T(u),\ 0 \leqq u \leqq 4Mr^2)$ and $(\tilde{W}_2^T(v),\ 0 \leqq v \leqq 4Mr^2)$ are conditionally independent, and conditionally independent of $\hat{\mathscr{G}}_T^t$, given T. More precisely, given T, $\tilde{W}_1^T$ (resp. $\tilde{W}_2^T$) is a Brownian motion with speed $T_2 - 4Mr^2$ (resp. $T_1 - 4Mr^2$). In addition,

$$W(T_1 - u, T_2 - v) = W(T) + \tilde{W}_1^T(u) + X_1^T(u) + \tilde{W}_2^T(v) + X_2^T(v) + \hat{\varepsilon}_T(u,v)\,,$$

where X_1^T, X_2^T and $\hat{\varepsilon}_T$ are such that

$$P\left\{\sup_{0 \leqq u,v \leqq 4Mr^2}(|X_1^T(u)|, |X_2^T(v)|, |\hat{\varepsilon}_T(u,v)|) \geqq r^{5/3} \mid \hat{\mathscr{G}}_T^t\right\} \leqq K \exp\left(-\frac{r^{-2/3}}{32M^2}\right). \tag{25}$$

Note that the random points $\hat{T}_t^i$ all have the property that $\{\hat{T}_t^i \geqq s\} \in \mathscr{G}_s^t$ for $s \leqq t$.

Lemma 8. *Let p_0 be as in* (8) *and let T be a stopping point with values in $[1/2, 3]^2$. For $r > 0$, set*

$$A(T,r) = \left\{\sup_{0 \leqq u,v \leqq 4Mr^2}(|X_1^T(u)|, |X_2^T(v)|, |\hat{\varepsilon}_T(u,v)|) \leqq r^{5/3}\right\},$$
$$\kappa_j^T = T_j - 4Mr^2,\ j = 1,2\,,$$

and

$$\hat{G}(T,r) = \left\{\frac{r^2}{M\kappa_2^T} \leqq \hat{S}_1^T \leqq \frac{Mr^2}{\kappa_2^T}\right\}$$
$$\cap \left(\{\hat{S}_2^T = 0\} \cup \left\{\frac{r^2}{M\kappa_1^T} \leqq \hat{S}_2^T \leqq \frac{Mr^2}{\kappa_1^T},\ \frac{r^2}{M\kappa_1^T} \leqq \hat{\varphi}_1(T) \leqq \frac{2Mr^2}{\kappa_1^T}\right\}\right)$$
$$\cap \{W(T) - W(\cdot) > -r + r^{3/2} \text{ on } \langle\hat{\psi}^h(T,r), T\rangle^v\}\,.$$

Then $\hat{G}(T,r)$ is conditionally independent of $\hat{\mathscr{G}}_T^t$ given T and there exists $C > 0$ and $r_0 > 0$ such that for all $r \in]0, r_0[$,

$$|P(\hat{G}(T,r) \mid \hat{\mathscr{G}}_T^t \vee A(T,r)) - p_0| \leqq Cr^{1/4} \quad \text{on } A(T,r)\,. \tag{26}$$

The proof of this lemma uses the following property of Brownian motion, which provides a bound on the difference of hitting probabilities for a Brownian motion B and a small perturbation of B.

Lemma 9. *Let B be a Brownian motion of speed at least $1/4$. For $r > 0$, set $Z(u) = B(u) + Y(u)$, where $|Y(u)| \leqq r^{5/3}$. For X equal to Z or B and r' equal to r or $2r - r^{3/2}$, set*

$$T^X = \inf\{u \geqq 0 : X(u) \in \{-r + r^{3/2},\ r'\}\},$$

and for $x \in \{-r + r^{3/2},\ r'\}$, let

$$\Lambda(X,x) = \{X(T^X) = x,\ T^X > M\}, \qquad \Lambda'(X,x) = \{X(T^X) = x,\ T^X < M\}.$$

Then there exists a constant K such that for small r and $x \in \{-r + r^{3/2},\ r'\}$,

$$P((\Lambda(B,x) \triangle \Lambda(Z,x)) \cup (\Lambda'(B,x) \triangle \Lambda'(Z,x))) \leqq Kr^{2/3}. \tag{27}$$

Proof. Each symmetric difference is the union of two terms, and therefore the probability on the left-hand side of (27) is bounded by the sum of four terms. We only bound one of them, namely, we show that

$$P((\Lambda'(B,x)\backslash\Lambda'(Z,x)) \leqq Kr^{2/3},$$

since the three other terms can be handled in a similar way. We also only consider the case where $x = r$.

Observe that the event $\Lambda'(B,r)\backslash\Lambda'(Z,r)$ is contained in the union of the three events

$$\Lambda_1 = \{T^B \in [M - r^{2/3}, M]\}$$
$$\Lambda_2 = \{T^B < M - r^{2/3},\ B(T^B) = r,\ Z(T^Z) = -r + r^{3/2}\},$$
$$\Lambda_3 = \{T^B < M - r^{2/3},\ B(T^B) = r,\ Z(T^Z) = r,\ T^Z > M\}.$$

Now $P(\Lambda_1) \leqq Cr^{2/3}$ by [6, Theorem 4.1.1]. Also, writing Z in terms of B and using the bound $|Y(u)| \leqq r^{5/3}$, we see that $P(\Lambda_2)$ is bounded by the probability that a Brownian motion started at level r hits level $-r(1 - r^{\frac{1}{2}} - r^{2/3})$ before level $r(1 + r^{2/3})$, which is $\leqq Cr^{2/3}$. Finally, $P(\Lambda_3)$ is bounded by

$$P\left\{\max_{0 \leqq u \leqq r^{2/3}} B(u) \leqq r^{5/3}\right\} = P\left\{\max_{0 \leqq u \leqq 1} B(u) \leqq r^{4/3}\right\} \leqq Cr^{2/3}. \qquad \square$$

Proof of Lemma 8. The constant r_0 will be chosen so that $16Mr_0^2 \leqq 1$. For $0 < r < r_0$, let

$$\hat{H}(T,r) = \left\{\frac{r^2}{\kappa_2^T M} \leqq \hat{S}_1^T \leqq \frac{Mr^2}{\kappa_2^T}\right\}$$
$$\cap \left(\{\hat{S}_2^T = 0\} \cup \left\{\frac{r^2}{\kappa_1^T M} \leqq \hat{S}_2^T \leqq \frac{Mr^2}{\kappa_1^T}\right\}\right)$$
$$\cap \{\hat{W}_1^T(\hat{S}_1^T) + \hat{W}_2^T(\hat{S}_2^T) = r\}.$$

Further, define $\tilde{S}_1^T$ and $\tilde{S}_2^T$ in the same way as $\hat{S}_1^T$ and $\hat{S}_2^T$, but with $\hat{W}$ replaced by $\tilde{W}$, and let $\tilde{H}(T,r)$ be defined in the same way as $\hat{H}(T,r)$, but with $\hat{S}$ replaced by $\tilde{S}$ and $\hat{W}$ by $\tilde{W}$. The speeds κ_j^T of the Brownian motions $\tilde{W}_j$ are at least 1/4 by the choice of r_0. Moreover, on $A(T,r)$, the difference Y_j between $\hat{W}_j^T$ and $\tilde{W}_j^T$ satisfies the bound on Y in Lemma 9. From this lemma, we conclude that for small r,

$$|P(\hat{H}(T,r) \mid \mathscr{G}_T^t \vee A(T,r)) - P(\tilde{H}(T,r) \mid \mathscr{G}_T^t \vee A(T,r))| \leqq Cr^{2/3} \quad \text{on } A(T,r)\,. \tag{28}$$

Let

$$\bar{S}_1^T = \inf\{u \geqq 0 : \tilde{W}_1^T(u) \in \{-r,r\}\}\,,$$

$$\bar{S}_2^T = \begin{cases} 0 & \text{if } \tilde{W}_1^T(\bar{S}_1^T) = r\,, \\ \inf\{v \geqq 0 : \tilde{W}_2^T(v) \in \{-r,2r\}\} & \text{if } \tilde{W}_1^T(\bar{S}_1^T) = -r\,. \end{cases}$$

Let $\bar{H}(T,r)$ be defined in the same way as $\hat{H}(T,r)$, but with $\hat{S}$ replaced by $\bar{S}$ and $\hat{W}$ by $\tilde{W}$. As in (9) and (10), for small r, we have

$$|P(\tilde{H}(T,r) \mid \hat{\mathscr{G}}_T^t \vee A(T,r)) - P(\bar{H}(T,r) \mid \hat{\mathscr{G}}_T^t \vee A(T,r))| \leqq Cr^{1/4}\,, \tag{29}$$

$$|P(\hat{G}(T,r) \mid \hat{\mathscr{G}}_T^t \vee A(T,r)) - P(\hat{H}(T,r) \mid \hat{\mathscr{G}}_T^t \vee A(T,r))| \leqq Cr^{1/2}\,. \tag{30}$$

To help the reader with the notation, we point out that (30) accounts for the difference between the requested behavior of the sheet along the path $\langle\hat{\psi}^h(T,r),T\rangle^v$ and its behavior on the horizontal and vertical lines through T, (28) replaces the actual increments along these lines by increments of independent processes, and (29) replaces the hitting value $-r + r^{3/2}$ by the value $-r$, which makes it possible to use Brownian scaling.

Indeed, by Brownian scaling, $P(\bar{H}(T,r) \mid \hat{\mathscr{G}}_T^t) \equiv p_0$ and $\bar{H}(T,r)$ is independent of $A(T,r)$, so the conclusion follows by applying the triangle inequality to (26) and using (28)–(30). □

For $t \in [1,2]^2$, $r > 0$ and $k \in \mathbb{N}$, let

$$\hat{J}(t,r,k) = \bigcap_{i=0}^{k} \hat{G}(\hat{T}_t^i, 2^i r)\,. \tag{31}$$

Statements analogous to those of Lemma 5 and Remark 6 are again valid, as we now show.

Lemma 10. *There exists an integer k_0 such that for all $t \in [1,2]^2$, all large n and all $j \in \{1,2\}$, if $k \in \{0,\ldots,n-k_0\}$, then*

$$(t - \hat{T}_t^k)_j \leqq 2^{2(k+k_0-n)}\,,$$

and if $k \in \{2,\ldots,n-k_0\}$, then

$$(t - \hat{T}_t^k)_j \geqq 2^{2(k-k_0-n)} \quad \text{on } \hat{J}(t,2^{-n},k-1)\,.$$

Proof. Pick k_0 large enough so that $2^{2k_0} \geqq 2^5 M$. Let $\kappa_j^k = (\hat{T}_t^k)_j - 4M(2^{k-n})^2$. Note that for $n > k_0$, $\kappa_j^0 \geqq 1 - 4M2^{-2n} \geqq \frac{1}{2}$ by the choice of k_0. Therefore, from the bounds on $\hat{S}_j^T$ and $\hat{\varphi}_j(T)$ in Lemma 8, $(t - \hat{T}_j^1) \leqq 2M2^{-2n}/(1/2)$, and this quantity is $\leqq 2^{2(k_0-n)}$ because $2^{2k_0} \geqq 4M$.

Therefore, for $n > k_0$, when $k = 1$, the inequalities

$$(t - \hat{T}_t^k)_j \leqq 2^{2(k+k_0-n-1)} \quad \text{and} \quad \kappa_j^{k-1} \geqq \tfrac{1}{2} \tag{32}$$

are satisfied on $\hat{J}(t, 2^{-n}, k-1)$. We proceed by induction on k to show that (32) is valid on $\hat{J}(t, 2^{-n}, k-1)$ for all $k \leqq n - k_0$.

Suppose that (32) holds for k and show that it holds for $k+1$ (assuming $k + 1 \leqq n - k_0$). Using (32), we see that on $\hat{J}(t, 2^{-n}, k-1)$,

$$\begin{aligned} \kappa_j^k &= (\hat{T}_t^k)_j - 4M2^{2(k-n)} \geqq t_j - 2^{2(k+k_0-n-1)} - 4M2^{-2k_0} \\ &\geqq 1 - 2^{-2} - 2^{-3} \geqq \tfrac{1}{2}\,. \end{aligned}$$

From (32) and the bounds in Lemma 8, we now conclude that on $\hat{J}(t, 2^{-n}, k)$,

$$\begin{aligned} (t - \hat{T}_t^{k+1})_j &= t_j - (\hat{T}_t^k)_j + (\hat{T}_t^k)_j - (\hat{T}_t^{k+1})_j \\ &\leqq 2^{2(k+k_0-n-1)} + 2M(2^{k+1-n})^2/(1/2) \\ &\leqq 2^{2(k+k_0-n-1)} + 2^{2k_0-3}\, 2^{2(k+1-n)} \\ &\leqq 2^{2(k+1+k_0-n-1)}\,. \end{aligned}$$

This proves (32), which gives the first conclusion of the lemma.

To prove the second conclusion, observe that when $k \geqq 2$, because either step k or step $k-1$ privileges direction j, $j \in \{1,2\}$, $(t - \hat{T}_t^k)_j$ is at least equal on $\hat{J}(t, 2^{-n}, k-1)$ to $(2^{k-1}2^{-n})^2/(M\kappa)$, where $\kappa \leqq 2$, so

$$(t - \hat{T}_t^k)_j \geqq 2^{2(k-n-1)}/(2M) \geqq 2^{2(k-k_0-n)}$$

because $2^{2k_0} \geqq 8M$. □

The analogue of Proposition 7 also remains valid, as we now show.

Proposition 11. *Let* p_0 *be as in* (8) *and let* $\alpha \in]0,1[$ *be such that* $2^{-\alpha} = p_0$. *There exist positive constants* θ *and* Θ *and an integer* k_1 *such that for all* $t \in [1,2]^2$, *for all large* n *and all* $k \in \{0, \ldots, n - k_1\}$,

$$\theta\, 2^{-\alpha k} \leqq P(\hat{J}(t, 2^{-n}, k)) \leqq \Theta\, 2^{-\alpha k}\,.$$

Proof. The proof is analogous to that of Proposition 7, with an added complication due to the fact that the inequality in (26) is only valid on $A(T, r)$. Let r_0 be as in Lemma 8. If $2^{-k_1} < r_0$, then for $i \leqq n - k_1$, the conclusion of Lemma 8 applies to $r = 2^{i-n}$. In this case, for $k \leqq n - k_1$,

$$P(\hat{J}(t, 2^{-n}, k)) \geqq P\left(\bigcap_{i=1}^{k} (\hat{G}(\hat{T}_t^i, 2^{i-n}) \cap A(\hat{T}_t^i, 2^{i-n}))\right).$$

By repeated conditioning and use of independence, Lemma 8 and (25) imply that the right-hand side is

$$\geqq \prod_{i=0}^{k} (p_0 - C2^{(i-n)/4}) \left(1 - \exp\left(-\frac{2^{-2(i-n)/3}}{32M^2}\right)\right) \geqq \theta\, 2^{-\alpha k}$$

(the last inequality uses the bounds $1 - x \geqq e^{-x/2}$ and $\exp(-r^{-2/3}/(32M^2)) \leqq r$ for small positive x and r).

In order to prove the upper bound, set $\tau(\omega) = \inf\{i \geqq 0\colon \omega \in \Omega \backslash A(\hat{T}_t^i, 2^{i-n})\}$, and observe that since $\{\tau > k\} = \bigcap_{i=1}^{k} A(\hat{T}_t^i, 2^{i-n})$, Lemma 8 implies that

$$P(\hat{J}(t, 2^{-n}, k) \cap \{\tau > k\}) \leqq \Theta\, 2^{-\alpha k}\,,$$

while for $i = 0, \ldots, k$,

$$\begin{aligned} P(\hat{J}(t, 2^{-n}, k) \cap \{\tau = i\}) &\leqq P(\hat{J}(t, 2^{-n}, i) \cap \{\tau = i\}) \\ &\leqq K \exp\left(-\frac{2^{-2(i-n)/3}}{32M^2}\right) \prod_{l=0}^{i} (p_0 + C2^{(l-n)/4}) \\ &\leqq K' \exp\left(-\frac{2^{2(n-i)/3}}{32M^2}\right) 2^{-\alpha i}\,. \end{aligned}$$

Fix $\beta > \alpha$. For sufficiently large $x \in \mathbb{R}$, say $x \geqq x_0$, $\exp(-2^{2x/3}/(32M^2)) < 2^{-\beta x}$. Assume that $2^{-k_1} < r_0$ and that $2^{2k_1/3} \geqq x_0$. Then for $i \in \{0, \ldots, k\}$ and $k \leqq n - k_1$,

$$\exp(-2^{2(n-i)/3}/(32M^2))\, 2^{-\alpha i} \leqq 2^{-\beta n + (\beta - \alpha) i}\,.$$

Summing over $i = 0, \ldots, k$, we see that if $k \leqq n - k_1$, then

$$P(\hat{J}(t, 2^{-n}, k) \cap \{\tau \leqq k\}) \leqq K'\, 2^{-\beta n}\, 2^{(\beta-\alpha)k} \leqq K'\, 2^{-\alpha k}\, 2^{-\beta k_1} \leqq \Theta\, 2^{-\alpha k}. \qquad \square$$

Lemma 12. *Let k_0 be the largest of the integers so denoted in Lemmas* 5 *and* 10. *For all $t \in [1,2]^2$, all large n and all $k \in \{1, \ldots, n - k_0\}$,*

$$J(t, 2^{-n}, k-1) \in \mathscr{G}_{t+(2^{-2n},0)}^{t+2^{2(k+k_0-n)}(1,1)}\,, \tag{33}$$

and

$$\hat{J}(t, 2^{-n}, k-1) \in \mathscr{G}_{t-2^{2(k+k_0-n)}(1,1)}^{t-(2^{-2n},0)}\,.$$

Proof. The event $J(t, 2^{-n}, k-1)$ is determined by increments of W in $[t_1 + 2^{-2n}, T_1^k] \times [0, T_2^k] \cup [0, T_1^k] \times [t_2, T_2^k]$, therefore, by Lemma 5, (33) holds. The proof of the statement concerning $\hat{J}(t, 2^{-n}, k-1)$ uses Lemma 10 and is analogous. $\square$

4 Proof of Lemma 2

This section is devoted to the proof of Lemma 2. The first two statements in this lemma are simpler than the third.

Let M be as indicated just before Lemma 3 and define α as in Propositions 7 and 11. Let k_2 be such that $k_2 - 2$ is the maximum of the integers denoted k_0 in Lemmas 5 and 10 and k_1 in Propositions 7 and 11. Let

$$c = 2^{-3(k_2+1)/2} \quad \text{and} \quad u_0 = 2^{-4k_2}\,. \tag{34}$$

Recall that the constant c appears in the definition of the function g in (1) and u_0 appears in the sets on the right-hand sides of (3) and (4). Let θ (resp. Θ) be positive constants smaller (resp. larger) than those denoted by the same symbols in Propositions 7 and 11. Define $J(t,r,k)$ and $\hat{J}(t,r,k)$ as in (22) and (31). Finally, as promised in the introduction, we define the two events $F_1(t,n)$ and $\hat{F}_1(t,n)$ by

$$F_1(t,n) = J(t, 2^{-n}, n-k_2) \quad \text{and} \quad \hat{F}_1(t,n) = \hat{J}(t, 2^{-n}, n-k_2)\,.$$

The constants k_2, θ and Θ have been chosen so that the conclusions of Propositions 7, 11 and Lemmas 5, 10 and 12 hold for all large n and with $k = n - k_2$. Because $n - k_2 + k_0 - n \leqq -2$, Lemmas 5 and 10, imply that for $t \in [1,2]^2$ and $k \leqq n - k_2$,

$$T_t^k \in [1,3]^2 \text{ on } F_1(t,n) \quad \text{and} \quad \hat{T}_t^k \in \left[\frac{3}{4}, 2\right]^2 \text{ on } \hat{F}_1(t,n)\,.$$

Secondly, by the last paragraph of Remark 6, on $F_1(t,n)$ (resp. $\hat{F}_1(t,n)$), the portion of the path Γ_t^n (resp. $\hat{\Gamma}_t^n$) defined in Sect. 3.2 (resp. 3.3.) with extremities t and $T_t^{n-k_2}$ (resp. $\hat{T}_t^{n-k_2}$) has length at least u_0, and the inclusions in (3) and (4) are satisfied.

4.1 Proof of (a) and (b) of Lemma 2

By Proposition 7, for all $t \in [1,2]^2$ and for all large n,

$$P(F_1(t,n)) = P(J(t, 2^{-n}, n-k_2)) \geqq \theta\, 2^{-\alpha(n-k_2)} = \theta\, 2^{\alpha k_2} 2^{-\alpha n}\,.$$

A similar inequality is valid for $P(\hat{F}_1(t,n))$ by Proposition 11. This proves (a).

By (24) and Lemma 12, $F_1(t,n)$ is independent of $F_0(t,n)$ and $\hat{F}_1(t,n)$, so by Proposition 7,

$$P(F(t,n)) \geqq P(\hat{F}_1(t,n) \cap F_0(t,n))\, \theta\, 2^{-\alpha(n-k_2)}\,.$$

Let $t_0 = (\frac{1}{2}, \frac{1}{2})$, $t_n' = (t_1 - 2^{-2n}, t_2)$, $Z_n = W(t_n') - W(t_0)$ and $Z_n' = W(\Gamma_t^n(2^{-2n})) - W(t_n')$. Then Z_n is measurable with respect to $\mathscr{G}_{t_0}^{t_n'}$, $\hat{F}_1(t,n) \in \mathscr{G}_{t_0}^{t_n'}$ (by Lemma 12) and $W(t_0)$ and Z_n' are independent of this sigma-field. The probability that $W(t_0)$ is in a particular interval of length 2^{-n} contained in $[q-2, q+2]$

is $\geqq \kappa 2^{-n}$, where $\kappa > 0$, and by Brownian scaling, the probability that Z'_n is in such an interval is $\geqq \kappa' > 0$. Therefore $P(\hat{F}_1(t,n) \cap F_0(t,n))$ is

$$\geqq P(\hat{F}_1(t,n) \cap \{|Z_n| \leqq 1\} \cap \{Z_n + W(t_0) \in]q - 2^{-n+1}, q - 2^{-n}[\}$$
$$\cap \{Z_n + W(t_0) + Z'_n \in]q + 2^{-n}, q + 2^{-n+1}[\}$$
$$\geqq \kappa\kappa' \, 2^{-n} \, P(\hat{F}_1(t,n) \cap \{|Z_n| \leqq 1\}) .$$

Now on $\hat{F}_1(t,n)$, $\{|Z_n| \leqq 1\} = \{|2^{-k_2} + W(\hat{T}_t^{n-k_2}) - W(t_0)| \leqq 1\}$, and this last event is independent of $\hat{F}_1(t,n)$ and has probability bounded below by a positive constant (because the interval $[-2^{-k_2} - 1, -2^{-k_2} + 1]$ contains 0 and the variance of $W(\hat{T}_t^{n-k_2}) - W(t_0)$ is $\geqq (1/4)^2$). It follows that

$$P(F(t,n)) \geqq K' \, 2^{-n} \, (2^{-\alpha(n-k_2)})^2 = K 2^{-(1+2\alpha)n} .$$

This proves (b). □

4.2 *Proof of (c) of Lemma 2 when $s \leqq t$*

Fix $s \leqq t$. Then the event $F(s,n) \cap F(t,n)$ is contained in the intersection of six events, to which we will apply the estimates of Sect. 3. More precisely, for $n > k_2 + 2$, assuming that $0 \leqq i \leqq j \leqq n - k_2 - 2$ (the case where $n - k_2 - 2 < j \leqq n$ will be treated below), $(s,t) \in E_{i,j}$ and $t_2 - s_2 \leqq t_1 - s_1$, it is contained in

$$\hat{J}(s, 2^{-n}, n - k_2) \cap F_0(s,n) \cap J(s, 2^{-n}, n - j - k_2 - 2)$$
$$\cap \, \hat{J}(t, 2^{-n}, n - j - k_2 - 2) \cap F_0(t,n) \cap J(t, 2^{-n}, n - k_2) . \qquad (35)$$

By Lemma 12, the last event is independent of the others and has probability $\leqq \Theta 2^{-\alpha(n-k_2)}$ by Proposition 7. Notice that $n - j - k_2 - 1 + k_0 - n \leqq -j - 2$, so by Lemma 12,

$$J(s, 2^{-n}, n - j - k_2 - 2) \in \mathscr{G}_{s+(2^{-2n},0)}^{s+2^{-2j-4}(1,1)}$$

and

$$\hat{J}(t, 2^{-n}, n - j - k_2 - 2) \in \mathscr{G}_{t-2^{-2j-4}(1,1)}^{t-(2^{-2n},0)} .$$

Let $t'_n = (t_1 - 2^{-2n}, t_2)$. From Fig. 4.2, it is easy to see that the variable $W(t'_n)$ is equal to the sum of a random variable that is correlated with the first four events in (35) and an independent Gaussian random variable (namely $W([s_1 + 2^{-2j-4}, t_1 - 2^{-2j-4}] \times [0,1])$) with mean 0 and variance at least $2^{-2(i+1)} - 2^{-2j-3} \geqq 2^{-2i-3}$, and therefore the conditional probability of $F_0(t,n)$ given the remaining four events is $\leqq 2^{-n}/2^{-i-2}$. Also, $s_2 + 2^{-2j-4} < t_2 - 2^{-2j-4}$ because $(s,t) \in E_{i,j}$, so by Lemma 12, $\hat{J}(t, 2^{-n}, n - j - k_2 - 2)$ is independent of the first three events in (35) and has probability $\leqq \Theta 2^{-\alpha(n-j-k_2-2)}$ by Proposition 11.

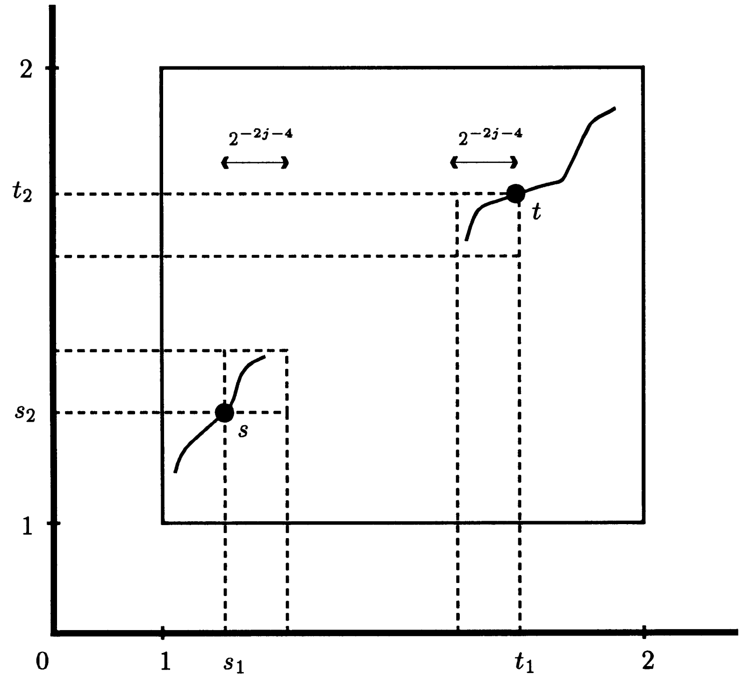

Fig. 1. Disposition of s and t in Sect. 4.2

By writing $W(s_1 - 2^{-2n}, s_2) = W(\frac{1}{2}, \frac{1}{2}) + Z$ and observing that $W(\frac{1}{2}, \frac{1}{2})$ is independent of $\hat{J}(s, 2^{-n}, n - k_2)$, we bound the probability of the remaining intersection of three events in a similar way, and we conclude that

$$P(F(s,n) \cap F(t,n)) \leqq K' 2^{-(n-i)} (2^{-\alpha(n-k_2)} 2^{-\alpha(n-j-k_2-2)})^2 2^{-n} \\ = K 2^{-(1+2\alpha)n} 2^{-(n-i)-2\alpha(n-j)} . \tag{36}$$

This proves (c) for $s \leqq t$ and $0 \leqq i \leqq j \leqq n - k_2 - 2$.

If $n - k_2 - 2 < j \leqq n$ and $i \leqq n - 2$, we omit the third and fourth events in (35). The independent random variable used in the decomposition of $W(t_n')$ is now $W([s_1 + 2^{-2n}, t_1 - 2^{-2n}] \times [0,1])$, which has mean 0 and variance $\geqq 2^{-2(i+1)} - 2\,2^{-2n} \geqq 2^{-2i-3}$ by the assumption on i. Inequality (36) becomes

$$P(F(s,n) \cap F(t,n)) \leqq K 2^{-(1+2\alpha)n} 2^{-(n-i)} . \tag{37}$$

However, $n - j < k_2 + 2$, so $2^{-2\alpha(n-j)} 2^{2\alpha(k_2+2)} \geqq 1$, and therefore we can increase the constant K in (37) by a factor of $2^{2\alpha(k_2+2)}$ to get the desired inequality.

If $n - 1 \leqq i \leqq j \leqq n$, we omit the third, fourth and fifth events in (35). Inequality (36) becomes

$$P(F(s,n) \cap F(t,n)) \leqq K 2^{-(1+2\alpha)n} .$$

However, arguing as in the previous case, we can increase the constant K by a factor of $2^{2\alpha(k_2+2)} \cdot 2$ to get the desired inequality. □

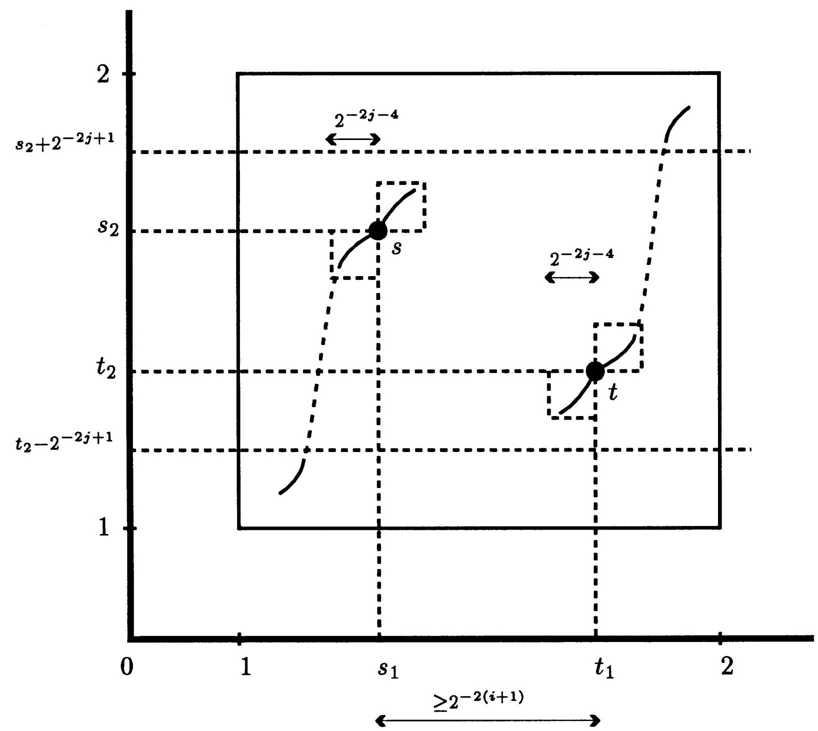

Fig. 2. Disposition of s and t in Sect. 4.3 (note that $s_2 - t_2 \geqslant 2^{-2(j+1)}$)

4.3 End of the proof of part (c) of Lemma 2

It remains to consider the case where neither $s \leqq t$ nor $t \leqq s$. Consider $n > k_2 + 2$. We assume without loss of generality that $s_1 < t_1$, $s_2 > t_2$ and that $2^{-2(j+1)} < s_2 - t_2 \leqq 2^{-2j}$ and $2^{-2(i+1)} < t_1 - s_1 \leqq 2^{-2i}$, where $0 \leqq i \leqq j \leqq n - k_2 - 2$ (the case where $n - k_2 - 2 < j \leqq n$ is easily handled as in Sect. 4.2). By definition,

$$\begin{aligned} P(F(s,n) \cap F(t,n)) \leqq\ & P(\hat{J}(s,2^{-n},n-k_2) \cap J(s,2^{-n},n-j-k_2-2) \\ & \cap \hat{J}(t,2^{-n},n-j-k_2-2) \cap J(t,2^{-n},n-k_2) \\ & \cap F_0(s,n) \cap F_0(t,n))\,. \end{aligned} \tag{38}$$

As in Sect. 4.2, from Fig. 2, we see that the variable $W(t)$ is equal to the sum of a random variable that is correlated with the first five events in (38) and an independent Gaussian random variable with mean 0 and variance at least 2^{-2i-3}, and we conclude that the conditional probability of $F_0(t,n)$ given the remaining five events is $\leqq 2^{-n}/2^{-i-2}$. Similarly, the conditional probability of $F_0(s,n)$ given the remaining four events is $\leqq 2^{-n}$. Therefore, $P(F(s,n) \cap F(t,n))$ is

$$\begin{aligned} \leqq\ & 4\,2^{-n}2^{-(n-i)}P(\hat{J}(s,2^{-n},n-k_2) \cap J(s,2^{-n},n-j-k_2-2) \\ & \cap \hat{J}(t,2^{-n},n-j-k_2-2) \cap J(t,2^{-n},n-k_2))\,. \end{aligned} \tag{39}$$

In order to reduce the dependence between the remaining events in the intersection above, we remove some of the events which define the J's and $\hat{J}$'s. By Lemma 5, on $J(t, 2^{-n}, n-k_2)$, $(T_t^{n-j+k_0+1})_2 \geqq t_2 + 2^{-2j+2} \geqq s_2 + 2^{-2j+1}$, and similarly, on $\hat{J}(s, 2^{-n}, n-k_2)$, $(\hat{T}_s^{n-j+k_0+1})_2 \leqq t_2 - 2^{-2j+1}$. Therefore the last probability above is

$$\begin{aligned}
&\leqq P(\hat{J}(s, 2^{-n}, n-j-k_2-2) \cap J(s, 2^{-n}, n-j-k_2-2) \\
&\quad \cap \hat{J}(t, 2^{-n}, n-j-k_2-2) \cap J(t, 2^{-n}, n-j-k_2-2) \\
&\quad \cap \{(\hat{T}_s^{n-j+k_0+1})_2 \leqq t_2 - 2^{-2j+1}\} \cap \{(T_t^{n-j+k_0+1})_2 \geqq s_2 + 2^{-2j+1}\} \\
&\quad \cap \hat{J}(\hat{T}_s^{n-j+k_0+1}, 2^{-(j-k_0-1)}, j-k_0-k_2-1) \\
&\quad \cap J(T_s^{n-j+k_0+1}, 2^{-(j-k_0-1)}, j-k_0-k_2-1)) .
\end{aligned}$$

Given the two events on the third line of the right-hand side, the last two events are conditionally independent of the previous ones, and therefore, by a slight extension of Propositions 7 and 11 to appropriate random times, the right-hand side above is

$$\begin{aligned}
&\leqq \Theta^2\, 2^{-2\alpha(j-k_0-k_2-1)} P(\hat{J}(s, 2^{-n}, n-j-k_2-2) \cap J(s, 2^{-n}, n-j-k_2-2) \\
&\quad \cap \hat{J}(t, 2^{-n}, n-j-k_2-2) \cap J(t, 2^{-n}, n-j-k_2-2)) . \qquad (40)
\end{aligned}$$

The four remaining events are still not quite independent. For instance, for $k \leqq n-j-k_2-2$, the event $G(T_s^k, 2^{k-n})$ enters into the definition of $J(s, 2^{-n}, n-j-k_2-2)$, and $G(T_t^k, 2^{k-n})$ enters into the definition of $J(t, 2^{-n}, n-j-k_2-2)$. These events involve increments of W over non-disjoint regions (see Fig. 3), the area of their intersection being bounded by $C\, 2^{4(k-n)+2}$. The key observation is that the contributions of the increments of W over this intersection is small, typically of order $2^{2(k-n)}$, while $W(T_t^k)$ and $W(T_s^k)$ are much larger, of order 2^{k-n}.

To make this observation precise, we must introduce several sigma-fields. If T is a random point with the property that $\{T \leqq u\} \in \mathscr{G}_t^u$ for all $u \geqq t$, then we set

$$\mathscr{G}_t^T = \{F \in \mathscr{F} : F \cap \{T \leqq u\} \in \mathscr{G}_t^u\} .$$

Other sigma-fields of interest are

$$\mathscr{G}_k = \mathscr{G}_s^{T_s^k} \vee \hat{\mathscr{G}}_{\hat{T}_s^k}^s \vee \mathscr{G}_t^{T_t^k} \vee \hat{\mathscr{G}}_{\hat{T}_t^k}^t ,$$

$$\mathscr{H}_k = \mathscr{G}_s^{T_s^{k+1}} \vee \hat{\mathscr{G}}_{\hat{T}_s^{k+1}}^s \vee \mathscr{G}_t^{T_t^k} \vee \hat{\mathscr{G}}_{\hat{T}_t^k}^t .$$

Define the rectangles

$$R^k(u) = [(T_s^k)_1, (T_s^k)_1 + u] \times [(\hat{T}_t^k)_2, (T_t^k)_2] ,$$

$$\hat{R}^k(u) = [(\hat{T}_s^k)_1 - u, (\hat{T}_s^k)_1] \times [(\hat{T}_t^k)_2, (T_t^k)_2] ,$$

$$Q^k(v) = [(\hat{T}_s^{k+1})_1, (T_s^{k+1})_1] \times [(T_t^k)_2, (T_t^k)_2 + v] ,$$

$$\hat{Q}^k(v) = [(\hat{T}_s^{k+1})_1, (T_s^{k+1})_1] \times [(\hat{T}_t^k)_2 - v, (\hat{T}_t^k)_2] ,$$

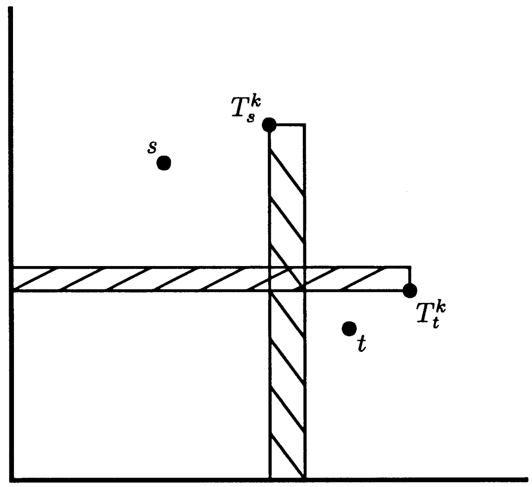

Fig. 3. The overlapping increments

and the processes

$$Y_1(u) = \Delta_{R^k(u)} W, \qquad \hat{Y}_1(u) = \Delta_{\hat{R}^k(u)} W,$$
$$Y_2(v) = \Delta_{Q^k(v)} W, \qquad \hat{Y}_2(v) = \Delta_{\hat{Q}^k(v)} W.$$

Consider the events

$$A^1(k) = \{\exists u,\ u' \in [0, 4M2^{2(k-n)}] : \sup(Y^1(u') - Y^1(u)\,,$$
$$\hat{Y}^1(u') - \hat{Y}^1(u)) \geqq (2^{k-n})^{5/3}\}\,,$$
$$A^2(k) = \{\exists v,\ v' \in [0, 4M2^{2(k-n)}] : \sup(Y^2(v') - Y^2(v)\,,$$
$$\hat{Y}^2(v') - \hat{Y}^2(v)) \geqq (2^{k-n})^{5/3}\}\,.$$

Since the maximal variance of an increment appearing in the definition of $A^1(k)$ and $A^2(k)$ is $(4M2^{2(k-n)})^2$, and the typical increment is of order $2^{2(k-n)}$, a standard calculation (increments are bounded by $\max Y^i - \min Y^i$) shows that

$$\begin{aligned} P(A^2(k)\,|\,\mathscr{G}_k) &\leqq \exp(-C(2^{n-k})^{2/3}) \quad \text{on } A^2(k-1)^c\,, \\ P(A^1(k)\,|\,\mathscr{H}_{k-1}) &\leqq \exp(-C(2^{n-k})^{2/3}) \quad \text{on } A^1(k-1)^c\,. \end{aligned} \tag{41}$$

Lemma 13. *There exists a constant C such that*

$$|P(G(T_s^k, 2^{k-n})\,|\,\mathscr{G}_k) - p_0| \leqq C(2^{k-n})^{1/4} \quad \text{on } A^1(k)^c$$

and

$$|P(G(T_t^k, 2^{k-n})\,|\,\mathscr{H}_k) - p_0| \leqq C(2^{k-n})^{1/4} \quad \text{on } A^2(k)^c\,.$$

The same inequalities hold if G *is replaced by* $\hat{G}$, T_s^k *by* $\hat{T}_s^k$, *and* T_t^k *by* $\hat{T}_t^k$.

Proof. We only prove the first inequality since the others are similar. Also, we assume that k is even, so that $T_s^{k+1} = \psi^h(T_s^k, 2^{k-n})$; indeed, the other case is analogous. Set $\tilde{B}(u) = W_1^{T_s^k} - Y^1(u)$ and observe that $\tilde{B}(\cdot)$ is independent of $\mathscr{G}_k$ and its distribution is that of a Brownian motion. Moreover, $|Y^1(u)| \leqq (2^{k-n})^{5/3}$ on $A^1(k)$, so we can apply Lemma 9 to see that conditional hitting probabilities for $W_1^{T_s^k}(\cdot)$ given $\mathscr{G}_k$ differ from the same hitting probabilities for $\tilde{B}$ by no more than $C(2^{k-n})^{2/3}$. The same occurs with the remaining hitting probabilities. Together with Lemma 3, we get the desired estimate. □

We now continue the calculation started in (38) and (40). According to these and the definition of the events $J(\cdot,\cdot,\cdot)$, the probability on the right-hand side of (40) is bounded by

$$\begin{aligned} &P(\hat{G}(\hat{T}_s^0, 2^{-n}) \cap \hat{G}(\hat{T}_s^1, 2^{-n+1}) \cap \cdots \cap \hat{G}(\hat{T}_s^{n-j-k_2-2}, 2^{-j-k_2-2}) \\ &\quad \cap G(T_s^0, 2^{-n}) \cap G(T_s^1, 2^{-n+1}) \cap \cdots \cap G(T_s^{n-j-k_2-2}, 2^{-j-k_2-2}) \\ &\quad \cap \hat{G}(\hat{T}_t^0, 2^{-n}) \cap \hat{G}(\hat{T}_t^1, 2^{-n+1}) \cap \cdots \cap \hat{G}(\hat{T}_t^{n-j-k_2-2}, 2^{-j-k_2-2}) \\ &\quad \cap G(T_t^0, 2^{-n}) \cap G(T_t^1, 2^{-n+1}) \cap \cdots \cap G(T_t^{n-j-k_2-2}, 2^{-j-k_2-2}))\,. \end{aligned} \tag{42}$$

We have to distinguish whether or not one of the events $A^i(k)$ occurs. Set $\tau(\omega) = \inf\{k \geqq 0 : \omega \in A^1(k) \cup A^2(k)\}$. The set $\{\tau > n - j - k_2 - 2\}$ is precisely the set $\bigcup_{k=0}^{n-j-k_0-2} (A^1(k) \cup A^2(k))^c$, and on this set, the probability of the intersection in (42) can be bounded by iterated conditioning using Lemma 13, yielding the bound $K2^{-4\alpha(n-j)}$. Following the estimate used in the proof of Proposition 11, consider $\beta > \alpha$. For $l = 0, \ldots, n - j - k_2 - 2$, on $\{\tau = l\}$, we remove in (42) all events after column l and we use Lemma 13 and (41) to get the bound

$$2^{-4\alpha l} \exp(-C2^{2(n-l)/3}) \leqq 2^{-4\beta n + 4(\beta - \alpha) l}\,.$$

Summing over $0 \leqq l \leqq n - j - k_2 - 2$ yields the bound $K' 2^{-4\alpha(n-j)}$. This bound, together with (39) and (40), yields that

$$\begin{aligned} P(F(s,n) \cap F(t,n)) &\leqq K'' 2^{-n} 2^{-(n-i)} 2^{-2\alpha(j-k_0-k_2-1)} 2^{-4\alpha(n-j)} \\ &= K 2^{-(1+2\alpha)n} 2^{-(n-i)} 2^{-2\alpha(n-j)}\,, \end{aligned}$$

which completes the proof of Lemma 2. □

Remark 14. A variation on Theorem 1 would be the following statement: with positive probability, there exists a continuous non-decreasing random function $\gamma : [-1,1] \times \Omega \to [1,2]^2$ such that $\gamma(-1) = (1,1)$, $\gamma(1) = (2,2)$, $W(\gamma(-1)) < q$, $W(\gamma(1)) > q$, and $W(\gamma(u)) = q$ for exactly one element $u \in [-1,1]$. To see why this statement is true, extend the path Γ_t^n from $\Gamma_t^n(u_0)$ to $(2,2) = \Gamma_t^n(v_0)$ by one vertical segment followed by one horizontal segment, and similarly, extend $\hat{\Gamma}_t^n$ from $\hat{\Gamma}_t^n(u_0)$ to $(1,1) = \hat{\Gamma}_t^n(\hat{v}_0)$ (note that $v_0 = |(2,2) - t|$ and $\hat{v}_0 =$

$|t-(1,1)|$). There are subsets $G_1(t,n)$ and $\hat{G}_1(t,n)$ of $F_1(t,n)$ and $\hat{F}_1(t,n)$, respectively, such that

$$G_1(t,n) \subset \{W(\Gamma_t^n(u)) - W(\Gamma_t^n(2^{-2n})) \geqq g(u) - 2^{-n},\ \text{for } 2^{-n} \leqq u \leqq v_0\}\ ,$$
$$\hat{G}_1(t,n) \subset \{W(\hat{\Gamma}_t^n(u)) - W(\Gamma_t^n(2^{-2n})) \leqq -g(u) + 2^{-n},\ \text{for } 2^{-n} \leqq u \leqq \hat{v}_0\}$$

(the right-hand sides of the inclusions are similar to (3) and (4), except that u is in the interval $[2^{-n}, v_0]$ instead of $[2^{-n}, u_0]$). Let $G(t,n) = F_0(t,n) \cap G_1(t,n) \cap \hat{G}_1(t,n)$. Since on $F_1(t,n)$, $W(\Gamma_t^n(u_0)) \geqq g(u_0) > 0$, given that $F_1(t,n)$ occurs, the probability that $G_1(t,n)$ occurs is just the probability that the sheet restricted to Γ_t^n does not hit zero during $[u_0, v_0]$, which is greater than the probability that a Brownian motion started at $g(u_0)$ does not hit 0 before time $v_0 - u_0$, and is therefore bounded below by a positive constant that does not depend on t or n. In particular, Lemma 2(a) and (b) remain valid with $F_1(t,n)$ replaced by $G_1(t,n)$ and $F(t,n)$ by $G(t,n)$. Lemma 2(c) also clearly remains valid, since $G(s,n) \cap G(t,n) \subset F(s,n) \cap F(t,n)$. Therefore the proof of Theorem 1 carries over to prove the claimed statement.

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