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Finding adapted solutions of forward–backward stochastic differential equations: method of continuation*

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Summary. The notion of *bridge* is introduced for systems of coupled forward–backward stochastic differential equations (FBSDEs, for short). This notion helps us to unify the method of continuation in finding adapted solutions to such FBSDEs over *any* finite time durations. It is proved that if two FBSDEs are *linked* by a bridge, then they have the same unique solvability. Consequently, by constructing appropriate bridges, we obtain several classes of uniquely solvable FBSDEs.

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1. Introduction

In this paper, we consider the following system of coupled forward–backward stochastic differential equations (FBSDEs for short):

$$
\begin{aligned}
&dX(t) = b(t,X(t),Y(t),Z(t))\,dt + \sigma(t,X(t),Y(t),Z(t))\,dW(t)\,,\\
&dY(t) = h(t,X(t),Y(t),Z(t))\,dt + Z(t)\,dW(t)\,, \qquad (1.1)\\
&X(0) = x, \qquad Y(T) = g(X(T))\,.
\end{aligned}
$$

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Here, $W(t)$ is a d-dimensional Brownian motion defined on some complete probability space $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\geqq 0}, P)$, satisfying the *usual conditions* (see Sect. 2 or 8 for details). Processes $X(t), Y(t)$ and $Z(t)$ are taking values in $\mathbb{R}^n$, $\mathbb{R}^m$ and $\mathbb{R}^{m\times d}$, respectively. These are processes that we are looking for and they are required to be $\{\mathscr{F}_t\}_{t\geqq 0}$-adapted. Functions b, σ, h and g are given and they are all allowed to depend on $\omega \in \Omega$; for the notational simplicity, we have suppressed ω and we will do so below. The vector $x \in \mathbb{R}^n$ is given as the initial value of $X(\cdot)$.

It is seen that (1.1) is actually a system of stochastic differential equations for the processes $X(t)$ and $Y(t)$. The process $Z(t)$ seems extra. However, we point out that it is the presence of this process that makes it possible for us to find *adapted* processes $X(t)$ and $Y(t)$ to satisfy (1.1). This is an essential feature that the FBSDEs have. See, e.g., 13, 10, 9 for more details.

There are several methods in studying the solvability of above FBSDEs (1.1). Here, by solvability, we mean to find adapted processes $(X(\cdot), Y(\cdot), Z(\cdot))$ satisfying (1.1) (see Definition 2.2 for details). In 1, 15, a Picard type iteration was used. In that approach, since the contraction mapping theorem was applied, the time duration T was assumed to be sufficiently small in order to enforce certain map to be contractive. It was pointed out in 1 that for some cases, if T is large, the system might have no adapted solutions. For the problem in *any* finite time duration, Ma and Yong 10 firstly used the stochastic optimal control theory reducing the solvability of FBSDEs to the existence of nonempty nodal set for the viscosity solutions to certain Hamilton–Jacobi–Bellman equations; and in some cases, such nodal sets were proved to be nonempty. This gives a positive answer to the solvability for some classes of FBSDEs in *any* finite time duration. In 9, inspired by 10, Ma, Protter and Yong introduced a method called the Four-Step-Scheme to attack the problem. In this approach, the solvability problem has been reduced to the solvability of some system of parabolic partial differential equations. As a matter of fact, this is the reverse procedure of the classical Feynman–Kac Formula (which transforms the solvability of PDEs to that of SDEs). It is interesting that by the Four-Step-Scheme, some relations among the processes $X(\cdot)$, $Y(\cdot)$ and $Z(\cdot)$ can be established. In 4, some special case of FBSDEs with $T=\infty$ was discussed by using the similar approach of 9. See 3 for some other related aspects. However, In 10, 9, 4, 3, the co-efficients have to be deterministic and the diffusion coefficient σ in the forward equation (see (1.1)) has to be nondegenerate (in 3, a very special degenerate case was treated) and independent of $Z(t)$. Recently, some further work is undergoing along this direction, in which, the coefficients are now allowed to be random and σ can be degenerate. However, the equations have to be linear and some other conditions are imposed for certain technical reasons 11, 12. On the other hand, Hu and Peng 7 and Peng and Wu 16 discussed the problem by another approach. They introduced certain monotonicity conditions under which, the solvability was established in any time duration.

Unfortunately, the monotonicity conditions that imposed in 7, 16 are not satisfied by many readily solvable *decoupled* FBSDEs (see an example in Sect. 5).

By observing the approaches of 7, 16, we find that the main idea is to use the *method of continuation*. This method has been widely used in proving the existence of solutions to elliptic partial differential equations (see 5, for example). In this paper, we are going to make this approach for FBSDEs more systematic. The key fact is that by using the method of continuation, you may start with a known solvable FBSDE to "reach" another class of FBSDEs, which are not known if it is solvable and now one can prove that it is solvable. The way of "reach" is to apply Itô's formula together with some sort of "monotonicity" or "coercivity" conditions to get certain a priori estimates. More precisely, we will apply Itô's formula to the function

$$\left\langle \Phi(t) \begin{pmatrix} \hat{X}(t) \\ \hat{Y}(t) \end{pmatrix}, \begin{pmatrix} \hat{X}(t) \\ \hat{Y}(t) \end{pmatrix} \right\rangle , \tag{1.2}$$

with some suitable C^1 symmetric matrix valued function $\Phi(\cdot)$, and $\hat{X}(\cdot)$ and $\hat{Y}(\cdot)$ are the differences of the first two components of the two possible adapted solutions to (1.1). Here, instead of just considering the cross term between $\hat{X}(t)$ and $\hat{Y}(t)$ (like in 7, 16), we consider $(\hat{X}(t), \hat{Y}(t))$ as a whole. Further, the t-dependence of the function $\Phi(\cdot)$ will give us some additional advantage. In this paper, we call such a $\Phi(\cdot)$ (with certain properties) a *bridge*. This notion plays a central role in our approach. It turns out that if two FBSDEs are *linked* by a bridge, then, they have the same unique solvability. By constructing suitable bridges, we will obtain several interesting classes of uniquely solvable FBSDEs. Relevant results of 13, 1, 15, 7, 16 are recovered.

We would like to mention that the study of backward SDEs can be traced back to Bismut 2. A systematic treatment concerning this matter was carried out in 13. The readers are referred to the survey paper 6 for more details and references concerning BSDEs. For a different approach for (linear) FBSDEs, see 17.

The rest of this paper is organized as follows. In Section 2, we make some preliminaries and state the main result. A class of nonsolvable FBSDEs are presented. Section 3 is devoted to the proof of the main result. The method of continuation is carried out there. Some properties of the bridge are discussed in Sect. 4. In Sects. 5 and 6, we construct some bridges for certain classes of FBSDEs, which gives the unique solvability for these equations by our main result.

After this paper has been completed, we have received the preprint 14 of Pardoux and Tang, in which, under some structural conditions, they proved the existence and uniqueness of adapted solutions to FBSDEs, among some other things. Essentially, their results say that if the coupling of the forward and backward parts are not very "strong" (one way or another), then the FBSDEs are solvable. We do not have such restrictions. Our approach is different from theirs.

2. Preliminaries and the main result

We let $\mathbb{R}^n$ be the n-dimensional Euclidean space with the usual Euclidean norm $|\cdot|$ and the usual Euclidean inner product $\langle\cdot,\cdot\rangle$. Also, let $\mathbb{R}^{m\times d}$ be the Hilbert space consists of all $(m \times d)$-matrices with the inner product

$$\langle A,B\rangle \triangleq \operatorname{tr}\{AB^T\}, \quad \forall A, B\in \mathbb{R}^{m\times d}. \tag{2.1}$$

Thus, the norm $|A|$ of A induced by inner product (2.1) is given by $|A| = \sqrt{\operatorname{tr}\{AA^T\}}$. We see that if $\|A\|$ denotes the usual norm of the matrix A (regarding it as a linear operator), then $|\cdot|$ and $\|\cdot\|$ are equivalent since $\mathbb{R}^{m\times d}$ is a finite dimensional space. More precisely, we have the estimates:

$$\|A\| \triangleq \sqrt{\max \sigma(AA^T)} \leqq \sqrt{\operatorname{tr}\{AA^T\}} = |A| \leqq \sqrt{m \wedge d}\|A\|, \quad \forall A \in \mathbb{R}^{m\times d}, \tag{2.2}$$

where $\sigma(AA^T)$ is the set of all eigenvalues of AA^T and $m \wedge d = \min\{m,d\}$. We will see that in our discussion, the norm $|\cdot|$ in $\mathbb{R}^{m\times d}$ induced by (2.1) is more convenient. Next, let S^n be the set of all $(n \times n)$ symmetric matrices. In what follows, whenever A is a square matrix, (with λ being a scalar), by $A + \lambda$, we mean $A + \lambda I$. For any $A\in S^n$, by $A \geqq \delta$, we mean that $A - \delta$ is positive semidefinite. The meaning of $A \leqq -\delta$ is similar. For simplicity of notation, we will denote $M = \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{m\times d}$; a generic point in M is denoted by $\theta = (x,y,z)$ with $x\in\mathbb{R}^n$, $y \in \mathbb{R}^m$ and $z\in\mathbb{R}^{m\times d}$. The norm in M is defined by (note (2.2) for the norm $|z|$)

$$|\theta| \triangleq \{|x|^2 + |y|^2 + |z|^2\}^{1/2}, \quad \forall\theta \equiv (x,y,z)\in M. \tag{2.3}$$

Similarly, we will use $\Theta = (X,Y,Z)$, and so on.

Now, we let $T > 0$ be fixed and $(\Omega,\mathscr{F},P)$ be a fixed complete probability space on which is defined a d-dimensional standard Brownian motion $W = \{W(t)\colon\ t \in [0,T]\}$. We further assume that the filtration $\{\mathscr{F}_t\}_{t\geqq 0}$ is generated by W, augmented by all the P-null sets in $\mathscr{F}$ so that $t \mapsto \mathscr{F}_t$ is continuous. For any sub-σ-field $\mathscr{G}$ of $\mathscr{F}$, we denote $L^2_{\mathscr{G}}(\Omega;\mathbb{R}^m)$ to be the set of all $\mathscr{G}$-measurable $\mathbb{R}^m$-valued square-integrable random variables. Let $L^2_{\mathscr{F}}(0,T;\mathbb{R}^n)$ be the set of all $\{\mathscr{F}_t\}_{t\geqq 0}$-progressively measurable processes $X(\cdot)$ valued in $\mathbb{R}^n$ such that

$$\int_0^T E|X(t)|^2\,dt < \infty.$$

Also, we let $L^2_{\mathscr{F}}(\Omega;C([0,T];\mathbb{R}^n))$ be the set of all $\{\mathscr{F}_t\}_{t\geqq 0}$-progressively measurable continuous processes $X(\cdot)$ valued in $\mathbb{R}^n$, such that

$$E \sup_{t\in[0,T]} |X(t)|^2 < \infty.$$

Further, we define

$$\mathscr{M}[0,T] \triangleq L^2_{\mathscr{F}}(\Omega;C([0,T];\mathbb{R}^n)) \times L^2_{\mathscr{F}}(\Omega;C([0,T];\mathbb{R}^m)) \times L^2_{\mathscr{F}}(0,T;\mathbb{R}^{m\times d}). \tag{2.4}$$

The norm of this space is defined by

$$\|(X(\cdot),Y(\cdot),Z(\cdot))\| = \left\{ E \sup_{t\in[0,T]} |X(t)|^2 + E \sup_{t\in[0,T]} |Y(t)|^2 + E\int_0^T |Z(t)|^2\,dt \right\}^{1/2},$$
$$\forall (X(\cdot),Y(\cdot),Z(\cdot)) \in \mathscr{M}[0,T]\,. \tag{2.5}$$

According to the notation of the space M, we will also use $\Theta(t) \equiv (X(t), Y(t), Z(t))$, and so on. It is easy to see that $\mathscr{M}[0,T]$ is a Banach space under the norm (2.5).

Next, we let $L^2_{\mathscr{F}}(0,T;W^{1,\infty}(M;\mathbb{R}^n\times\mathbb{R}^{n\times d}\times\mathbb{R}^m))$ be the set of all functions $f:[0,T]\times M\times\Omega\to\mathbb{R}^n\times\mathbb{R}^{n\times d}\times\mathbb{R}^m$, such that for any fixed $\theta\in M$, $(t,\omega)\mapsto f(t,\theta;\omega)$ is $\{\mathscr{F}_t\}_{t\geqq 0}$-progressively measurable, $f(\cdot,0;\cdot)\in L^2_{\mathscr{F}}(0,T;\mathbb{R}^n\times\mathbb{R}^{n\times d}\times\mathbb{R}^m)$ and there exists a constant $L>0$, such that

$$|f(t,\theta;\omega) - f(t;\bar\theta;\omega)| \leqq L|\theta-\bar\theta|,\ \forall\theta,\ \bar\theta\in M,\ t\in[0,T],\ \text{a.s.}$$

We may similarly define $L^2_{\mathscr{F}_T}(\Omega;W^{1,\infty}(\mathbb{R}^n;\mathbb{R}^m))$. Denote

$$H[0,T] = L^2_{\mathscr{F}}(0,T;W^{1,\infty}(M;\mathbb{R}^n\times\mathbb{R}^{n\times d}\times\mathbb{R}^m))\times L^2_{\mathscr{F}_T}(\Omega;W^{1,\infty}(\mathbb{R}^n;\mathbb{R}^m)). \tag{2.6}$$

Any generic element in $H[0,T]$ is denoted by $\Gamma\equiv(b,\sigma,h,g)$. Thus,

$$\Gamma\equiv(b,\sigma,h,g)\in H[0,T] \Leftrightarrow \begin{cases} b\in L^2_{\mathscr{F}}(0,T;W^{1,\infty}(M;\mathbb{R}^n))\,, \\ \sigma\in L^2_{\mathscr{F}}(0,T;W^{1,\infty}(M;\mathbb{R}^{n\times d}))\,, \\ h\in L^2_{\mathscr{F}}(0,T;W^{1,\infty}(M;\mathbb{R}^m))\,, \\ g\in L^2_{\mathscr{F}_T}(\Omega;W^{1,\infty}(\mathbb{R}^n;\mathbb{R}^m))\,, \end{cases} \tag{2.7}$$

where the space $L^2_{\mathscr{F}}(0,T;W^{1,\infty}(M;\mathbb{R}^n))$, etc. are defined in an obvious way.

Finally, we denote

$$\mathscr{H}[0,T] = L^2_{\mathscr{F}}(0,T;\mathbb{R}^n)\times L^2_{\mathscr{F}}(0,T;\mathbb{R}^{n\times d})\times L^2_{\mathscr{F}}(0,T;\mathbb{R}^m)\times L^2_{\mathscr{F}_T}(\Omega;\mathbb{R}^m)\,. \tag{2.8}$$

An element in $\mathscr{H}[0,T]$ is denoted by $\gamma\equiv(b_0,\sigma_0,h_0,g_0)$ with $b_0\in L^2_{\mathscr{F}}(0,T;\mathbb{R}^n)$, $\sigma_0\in L^2_{\mathscr{F}}(0,T;\mathbb{R}^{n\times d})$, $h_0\in L^2_{\mathscr{F}}(0,T;\mathbb{R}^m)$ and $g_0\in L^2_{\mathscr{F}_T}(\Omega;\mathbb{R}^m)$. We note that the range of the elements in $H[0,T]$ and $\mathscr{H}[0,T]$ are all in $\mathbb{R}^n\times\mathbb{R}^{n\times d}\times\mathbb{R}^m\times\mathbb{R}^m$. Hence, for any $\Gamma\equiv(b,\sigma,h,g)\in H[0,T]$ and $\gamma=(b_0,\sigma_0,h_0,g_0)\in\mathscr{H}[0,T]$, we can define

$$\Gamma+\gamma = (b+b_0,\sigma+\sigma_0,h+h_0,g+g_0)\in H[0,T]\,. \tag{2.9}$$

Now, for any $\Gamma\equiv(b,\sigma,h,g)\in H[0,T]$, $\gamma\equiv(b_0,\sigma_0,h_0,g_0)\in\mathscr{H}[0,T]$ and $x\in\mathbb{R}^n$, we associate them with the following FBSDEs on $[0,T]$:

$$\begin{aligned} &dX(t) = \{b(t,\Theta(t)) + b_0(t)\}\,dt + \{\sigma(t,\Theta(t))+\sigma_0(t)\}\,dW(t)\,, \\ &dY(t) = \{h(t,\Theta(t)) + h_0(t)\}\,dt + Z(t)\,dW(t)\,, \\ &X(0)=x, \qquad Y(T) = g(X(T)) + g_0\,, \end{aligned} \tag{2.10$_{\Gamma,\gamma,x}$}$$

with $\Theta(t) \equiv (X(t), Y(t), Z(t))$. In what follows, sometimes, we will simply identify the FBSDEs $(2.10)_{\Gamma,\gamma,x}$ with (Γ,γ,x) or even with Γ (since γ and x are not essential). Let us now introduce the following definition.

Definition 2.1. *A process* $\Theta(\cdot) \equiv (X(\cdot), Y(\cdot), Z(\cdot)) \in \mathscr{M}[0,T]$ *is called an* adapted solution *of* $(2.10)_{\Gamma,\gamma,x}$, *if the following holds for any* $t \in [0,T]$, *almost surely.*

$$
\begin{aligned}
X(t) &= x + \int_0^t \{b(t,\Theta(s)) + b_0(s)\}\,ds + \int_0^t \{\sigma(t,\Theta(s)) + \sigma_0(s)\}\,dW(s)\,,\\
Y(t) &= g(X(T)) + g_0 - \int_t^T \{h(t,\Theta(s)) + h_0(s)\}\,ds - \int_t^T Z(s)\,dW(s)\,.
\end{aligned}
\tag{2.11$_{\Gamma,\gamma,x}$}
$$

When $(2.10)_{\Gamma,\gamma,x}$ admits a unique adapted solution, we say that $(2.10)_{\Gamma,\gamma,x}$ is *uniquely solvable.*

We see that $(2.11)_{\Gamma,\gamma,x}$ is the integral form of $(2.10)_{\Gamma,\gamma,x}$. In what follows, we will not distinguish $(2.10)_{\Gamma,\gamma,x}$ and $(2.11)_{\Gamma,\gamma,x}$.

Definition 2.2. *Let* $T > 0$. *A* $\Gamma \in H[0,T]$ *is said to be* solvable *if for any* $x \in \mathbb{R}^n$ *and* $\gamma \in \mathscr{H}[0,T]$, Eq. $(2.10)_{\Gamma,\gamma,x}$ admits a unique adapted solution $\Theta(\cdot) \in \mathscr{M}[0,T]$. The set of all $\Gamma \in H[0,T]$ that is solvable is denoted by $\mathscr{S}[0,T]$. Any $\Gamma \in H[0,T] \setminus \mathscr{S}[0,T]$ is said to be *nonsolvable.*

We recall that there are several examples of nonsolvable FBSDEs presented in 1 and 16. Also, in 10, it was proved that if in (1.1), the term $Z(t)\,dW(t)$ in the backward part is replaced by $\hat{\sigma}(X(t), Y(t), Z(t))\,dW(t)$, and $\hat{\sigma}$ has a "small" range, then, (1.1) could be nonsolvable. Here, we are going to present another type of result, which gives a big class of nonsolvable FBSDEs with an extremely simple proof.

Proposition 2.3. *Let the following two-point boundary value problem for a system of linear ordinary differential equations admit no solutions*:

$$
\begin{aligned}
&\begin{pmatrix} \dot{X}(t) \\ \dot{Y}(t) \end{pmatrix} = \mathscr{A} \begin{pmatrix} X(t) \\ Y(t) \end{pmatrix},\\
&X(0) = x, \qquad Y(T) = GX(T)\,,
\end{aligned}
\tag{2.12}
$$

where $\mathscr{A}$ *and* G *are certain matrices. Then, for any* $\sigma \in L^2_{\mathscr{F}}(0,T; W^{1,\infty}(M; \mathbb{R}^{n\times d}))$, *the following FBSDEs:*

$$
\begin{aligned}
&d\begin{pmatrix} X(t) \\ Y(t) \end{pmatrix} = \mathscr{A} \begin{pmatrix} X(t) \\ Y(t) \end{pmatrix} dt + \begin{pmatrix} \sigma(t,\Theta(t)) \\ Z(t) \end{pmatrix} dW(t)\,,\\
&X(0) = x, \qquad Y(T) = GX(T)\,,
\end{aligned}
\tag{2.13}
$$

admits no adapted solutions.

Proof. Suppose (2.13) admits an adapted solution $\Theta(\cdot) \equiv (X(\cdot), Y(\cdot), Z(\cdot))$. Then, $(EX(\cdot), EY(\cdot))$ is a solution of (2.12), a contradiction. This proves the assertion. □

There are many examples of systems like (2.12) which does not admit a solution. Here is a very simple one: ($n = m = 1$)

$$\begin{aligned} &\dot{X} = Y\,, \\ &\dot{Y} = -X\,, \\ X(0) = x, &\qquad Y(T) = -X(T)\,. \end{aligned} \tag{2.14}$$

We can easily show that for $T = k\pi + \frac{3}{4}\pi$ (k, nonnegative integer), the above two-point boundary value problem does not admit a solution for any $x \in \mathbb{R}\backslash\{0\}$ and it admits infinitely many solutions for $x = 0$. A consequence of Proposition 2.3 and the above example (2.14) is the following conclusion.

Corollary 2.4. *For any* $T > 0$, $H[0,T] \neq \mathscr{S}[0,T]$; *that is*, *nonsolvable* FBSDEs *exist over any time durations.*

Proof. From (2.14) and time scaling, we can construct a nonsolvable two-point boundary value problem for a system of linear ordinary differential equations of (2.12) type with the unknowns X and Y taking values in $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively. Then, Proposition 2.3 applies. □

Some more interesting comments related to the above example will be given later. Also, in Sect. 6, we will say something more about (2.12) and (2.13).

Now, let us introduce the following notions, which will play the central role in this paper.

Definition 2.5. *Let* $T > 0$ and $\Gamma \equiv (b,\sigma,h,g) \in H[0,T]$. *A* C^1 function

$$\Phi \equiv \begin{pmatrix} A & B^T \\ B & C \end{pmatrix} : [0,T] \to S^{n+m}\,,$$

with $A : [0,T] \to S^n$, $B : [0,T] \to \mathbb{R}^{m\times n}$ and $C : [0,T] \to S^m$, is called a *bridge* extending from Γ (defined on $[0,T]$) if there exist some constants K, $\delta > 0$, *such that*

$$\begin{aligned} &C(T) \leqq 0, \qquad A(t) \geqq 0, \quad \forall t \in [0,T]\,, \\ &\Phi(0) \leqq K \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}\,, \end{aligned} \tag{2.15}$$

and either (2.16) and (2.17) or (2.16)′ and (2.17)′ hold:

$$\left\langle \Phi(T) \begin{pmatrix} x - \bar{x} \\ g(x) - g(\bar{x}) \end{pmatrix}, \begin{pmatrix} x - \bar{x} \\ g(x) - g(\bar{x}) \end{pmatrix} \right\rangle \geqq \delta|x - \bar{x}|^2, \quad \forall x,\ \bar{x} \in \mathbb{R}^n\,. \tag{2.16}$$

$$\begin{aligned} &\left\langle \dot{\Phi}(t) \begin{pmatrix} x - \bar{x} \\ y - \bar{y} \end{pmatrix}, \begin{pmatrix} x - \bar{x} \\ y - \bar{y} \end{pmatrix} \right\rangle + 2\left\langle \Phi(t) \begin{pmatrix} x - \bar{x} \\ y - \bar{y} \end{pmatrix}, \begin{pmatrix} b(t,\theta) - b(t,\bar{\theta}) \\ h(t,\theta) - h(t,\bar{\theta}) \end{pmatrix} \right\rangle \\ &\quad + \left\langle \Phi(t) \begin{pmatrix} \sigma(t,\theta) - \sigma(t,\bar{\theta}) \\ z - \bar{z} \end{pmatrix}, \begin{pmatrix} \sigma(t,\theta) - \sigma(t,\bar{\theta}) \\ z - \bar{z} \end{pmatrix} \right\rangle \\ &\qquad \leqq -\delta|x - \bar{x}|^2, \forall \theta,\ \bar{\theta} \in M, \ \text{a.e. } t \in [0,T], \ \text{a.s.} \end{aligned} \tag{2.17}$$

$$\left\langle \Phi(T)\begin{pmatrix} x-\bar{x} \\ g(x)-g(\bar{x}) \end{pmatrix}, \begin{pmatrix} x-\bar{x} \\ g(x)-g(\bar{x}) \end{pmatrix} \right\rangle \geqq 0, \quad \forall x,\ \bar{x}\in \mathbb{R}^n . \tag{2.16$'$}$$

$$\left\langle \dot{\Phi}(t)\begin{pmatrix} x-\bar{x} \\ y-\bar{y} \end{pmatrix}, \begin{pmatrix} x-\bar{x} \\ y-\bar{y} \end{pmatrix} \right\rangle + 2\left\langle \Phi(t)\begin{pmatrix} x-\bar{x} \\ y-\bar{y} \end{pmatrix}, \begin{pmatrix} b(t,\theta)-b(t,\bar{\theta}) \\ h(t,\theta)-h(t,\bar{\theta}) \end{pmatrix} \right\rangle$$
$$+\left\langle \Phi(t)\begin{pmatrix} \sigma(t,\theta)-\sigma(t,\bar{\theta}) \\ z-\bar{z} \end{pmatrix}, \begin{pmatrix} \sigma(t,\theta)-\sigma(t,\bar{\theta}) \\ z-\bar{z} \end{pmatrix} \right\rangle \leqq -\delta\{|y-\bar{y}|^2+|z-\bar{z}|^2\}\ ,$$
$$\forall \theta,\ \bar{\theta}\in M,\ \text{a.e. } t\in[0,T],\ \text{a.s.} \tag{2.17$'$}$$

If (2.15)–(2.17) (*resp.* (2.15), (2.16)$'$ and (2.17)$'$) hold, we call Φ a *type* (I) (resp. *type* (II)) *bridge* extending from Γ (defined on $[0,T]$). The set of all type (I) and type (II) bridges extending from Γ (defined on $[0,T]$) are denoted by $\mathscr{B}_I(\Gamma;[0,T])$ and $\mathscr{B}_{II}(\Gamma;[0,T])$, respectively. Finally, we let

$$\begin{aligned} &\mathscr{B}(\Gamma;[0,T]) = \mathscr{B}_I(\Gamma;[0,T]) \cup \mathscr{B}_{II}(\Gamma;[0,T])\ , \\ &\mathscr{B}^s(\Gamma;[0,T]) = \mathscr{B}_I(\Gamma;[0,T]) \cap \mathscr{B}_{II}(\Gamma;[0,T])\ . \end{aligned} \tag{2.18}$$

Any element $\Phi\in\mathscr{B}^s(\Gamma;[0,T])$ is called a *strong bridge* extending from Γ (defined on $[0,T]$).

Definition 2.6. *Let* $T>0$ and $\Gamma,\ \bar{\Gamma}\in H[0,T]$. We say that they are *linked by a direct bridge* if

$$\{\mathscr{B}_I(\Gamma;[0,T])\cap\mathscr{B}_I(\bar{\Gamma};[0,T])\}\cup\{\mathscr{B}_{II}(\Gamma;[0,T])\cap\mathscr{B}_{II}(\bar{\Gamma};[0,T])\}\neq\emptyset\ ; \tag{2.19}$$

and we say that they are *linked by a bridge,* if there are $\Gamma_1,\ldots,\Gamma_k\in H[0,T]$, such that with $\Gamma_0=\Gamma$ and $\Gamma_{k+1}=\bar{\Gamma}$, it holds

$$\begin{aligned} &\{\mathscr{B}_I(\Gamma_i;[0,T])\cap\mathscr{B}_I(\Gamma_{i+1};[0,T])\} \\ &\quad\cup\{\mathscr{B}_{II}(\Gamma_i;[0,T])\cap\mathscr{B}_{II}(\Gamma_{i+1};[0,T])\}\neq\emptyset, \quad 0\leqq i\leqq k\ . \end{aligned} \tag{2.20}$$

We may similarly define the notion that Γ and $\bar{\Gamma}$ are linked by a (direct) strong bridge. Our main result can be stated as follows:

Theorem 2.7. *Let* $T>0$ *and* $\Gamma_1,\Gamma_2\in H[0,T]$ *be linked by a bridge. Then,* $\Gamma_1\in\mathscr{S}[0,T]$ *if and only if* $\Gamma_2\in\mathscr{S}[0,T]$.

The above theorem tells us that if the FBSDEs associated with Γ_1 is solvable, so is the one associated with Γ_2, provided Γ_1 and Γ_2 are linked by a bridge. In applications, one of the FBSDEs is known to be solvable and the other is to be solved. Then, the problem is reduced to construct appropriate bridges. We will see this later.

To conclude this section, let us give the following result, which enables us to enlarge the class of solvable FBSDEs in a simple way. This result will be useful in Sect. 6.

Proposition 2.8. *Let* $T > 0$ *and* (1.1) *be solvable on* $[0, T]$. *Then, for any* $\beta \in \mathbb{R}$, *the following* FBSDE *is also solvable on* $[0, T]$:

$$\begin{aligned} dX(t) &= \{\beta X(t) + e^{\beta t} b(t, e^{-\beta t}\Theta(t))\}\, dt + e^{\beta t}\sigma(t, e^{-\beta t}\Theta(t))\, dW(t)\,, \\ dY(t) &= \{\beta Y(t) + e^{\beta t} h(t, e^{-\beta t}\Theta(t))\}\, dt + Z(t)\, dW(t)\,, \\ X(0) &= x, \qquad Y(T) = e^{\beta T} g(e^{-\beta T} X(T))\,. \end{aligned} \tag{2.21}$$

Proof. Let $\Theta(t) \equiv (X(t), Y(t), Z(t)) \in \mathscr{M}[0, T]$ be a solution of (1.1). Then, we can check that $\tilde{\Theta}(t) \triangleq e^{\beta t}\Theta(t)$ is a solution of (2.21). □

3. Method of continuation

In this section, we are going to prove Theorem 2.7.

Let $\Gamma_i \equiv (b_i, \sigma_i, h_i, g_i) \in H[0, T] (i = 1, 2)$. Clearly, by induction, it suffices to prove our theorem for the case that Γ_1 and Γ_2 are linked by a direct bridge. We now assume this. For any $\gamma \equiv (b_0(\cdot), \sigma_0(\cdot), h_0(\cdot), g_0) \in \mathscr{H}[0, T], x \in \mathbb{R}^n$ and $\alpha \in [0, 1]$, we consider the following system of FBSDEs:

$$\begin{aligned} dX(t) &= \{(1-\alpha) b_1(t, \Theta(t)) + \alpha b_2(t, \Theta(t)) + b_0(t)\}\, dt \\ &\quad + \{(1-\alpha)\sigma_1(t, \Theta(t)) + \alpha\sigma_2(t, \Theta(t)) + \sigma_0(t)\}\, dW(t)\,, \\ dY(t) &= \{(1-\alpha) h_1(t, \Theta(t)) + \alpha h_2(t, \Theta(t)) + h_0(t)\}\, dt \\ &\quad + Z(t)\, dW(t)\,, \\ X(0) &= x, \qquad Y(T) = (1-\alpha) g_1(X(T)) + \alpha g_2(X(T)) + g_0\,. \end{aligned} \tag*{$(3.1)^{\alpha}_{\gamma,x}$}$$

We may give the definition of the (adapted) solutions to above system $(3.1)^{\alpha}_{\gamma,x}$ similar to Definition 2.1. It is clear that $(3.1)^{0}_{\gamma,x}$ and $(3.1)^{1}_{\gamma,x}$ coincide with $(2.10)_{\Gamma_1,\gamma,x}$ and $(2.10)_{\Gamma_2,\gamma,x}$, respectively. Let us assume that $(3.1)^{0}_{\gamma,x}$ is uniquely solvable for any $\gamma \in \mathscr{H}[0, T]$ and $x \in \mathbb{R}^n$. We want to prove the unique solvability of $(3.1)^{1}_{\gamma,x}$ for all $\gamma \in \mathscr{H}[0, T]$ and $x \in \mathbb{R}^n$.

Now, let us explain our main idea. We start to solve $(3.1)^{0}_{\gamma,x}$, i.e., $(2.10)_{\Gamma_1,\gamma,x}$, which is possible by our assumption. We show that there exists a fixed step-length $\varepsilon_0 > 0$, such that if for some $\alpha \in [0, 1)$, $(3.1)^{\alpha}_{\gamma,x}$ is uniquely solvable for any $\gamma \in \mathscr{H}[0, T]$ and $x \in \mathbb{R}^n$, then the same conclusion holds for α being replaced by $\alpha + \varepsilon \leqq 1$ with $\varepsilon \in [0, \varepsilon_0]$. Once this has been proved, we can increase the parameter α step by step and finally reach $\alpha = 1$, which gives the unique solvability of system $(2.10)_{\Gamma_2,\gamma,x}$. This idea is adopted from [7, 16]. For solving partial differential equations, such a method, called the *method of continuation*, is standard and has been frequently used (see [5], for example).

We now establish some a priori estimates for the solutions of $(3.1)^{\alpha}_{\gamma,x}$, which will be crucial below.

Lemma 3.1. *Let* $\alpha \in [0, 1]$. *Let* $\Theta(\cdot) \triangleq (X(\cdot), Y(\cdot), Z(\cdot))$ *and* $\bar{\Theta}(\cdot) \triangleq (\bar{X}(\cdot), \bar{Y}(\cdot), \bar{Z}(\cdot))$ *be adapted solutions of* $(3.1)^{\alpha}_{\gamma,x}$ *and* $(3.1)^{\alpha}_{\bar{\gamma},\bar{x}}$, *respectively, with* $\gamma = (b_0, \sigma_0, h_0, g_0)$, $\bar{\gamma} = (\bar{b}_0, \bar{\sigma}_0, \bar{h}_0, \bar{g}_0) \in \mathscr{H}[0, T]$ *and* x, $\bar{x} \in \mathbb{R}^n$. *Then, the* *following estimate holds*:

$$
\begin{aligned}
&\|\Theta(\cdot) - \bar{\Theta}(\cdot)\|^2_{\mathscr{M}[0,T]} \\
&\quad \equiv E \sup_{t\in[0,T]} |X(t) - \bar{X}(t)|^2 + E \sup_{t\in[0,T]} |Y(t) - \bar{Y}(t)|^2 + E \int_0^T |Z(t) - \bar{Z}(t)|^2 \, dt \\
&\quad \leqq C\Bigg\{ |x - \bar{x}|^2 + E|g_0 - \bar{g}_0|^2 \\
&\qquad + E \int_0^T \{|b_0(t) - \bar{b}_0(t)|^2 + |\sigma_0(t) - \bar{\sigma}_0(t)|^2 + |h_0(t) - \bar{h}_0(t)|^2\} \, dt \Bigg\}.
\end{aligned}
\tag{3.2}
$$

Proof. We denote

$$
\begin{aligned}
&\hat{X}(t) = X(t) - \bar{X}(t), \quad \hat{Y}(t) = Y(t) - \bar{Y}(t), \\
&\hat{Z}(t) = Z(t) - \bar{Z}(t), \quad \hat{\Theta}(t) = \Theta(t) - \bar{\Theta}(t), \\
&\left.\begin{aligned}
\hat{b}_i(t) &= b_i(t, \Theta(t)) - b_i(t, \bar{\Theta}(t)), \\
\hat{\sigma}_i(t) &= \sigma_i(t, \Theta(t)) - \sigma_i(t, \bar{\Theta}(t)), \\
\hat{h}_i(t) &= h_i(t, \Theta(t)) - h_i(t, \bar{\Theta}(t)), \\
\hat{g}_i(T) &= g_i(X(T)) - g_i(\bar{X}(T)),
\end{aligned}\right\} \quad i = 1, 2, \\
&\hat{b}_0(t) = b_0(t) - \bar{b}_0(t), \quad \hat{\sigma}_0(t) = \sigma_0(t) - \bar{\sigma}_0(t), \\
&\hat{h}_0(t) = h_0(t) - \bar{h}_0(t), \quad \hat{g}_0 = g_0 - \bar{g}_0, \quad \hat{x} = x - \bar{x}.
\end{aligned}
\tag{3.3}
$$

Note that $\Gamma_i \in H[0,T]$ implies that all the functions b_i, σ_i, h_i, g_i are uniformly Lipschitz continuous. Suppose the common Lipschitz constant is $L > 0$. Now, applying Itô's formula to $|\hat{X}(t)|^2$, we obtain that

$$
\begin{aligned}
|\hat{X}(t)|^2 &= |\hat{x}|^2 + 2\int_0^t \langle \hat{X}(s), (1-\alpha)\,\hat{b}_1(s) + \alpha\,\hat{b}_2(s) + \hat{b}_0(s) \rangle \, ds \\
&\quad + \int_0^t |(1-\alpha)\hat{\sigma}_1(s) + \alpha\hat{\sigma}_2(s) + \hat{\sigma}_0(s)|^2 \, ds \\
&\quad + 2\int_0^t \langle \hat{X}(s), [(1-\alpha)\hat{\sigma}_1(s) + \alpha\hat{\sigma}_2(s) + \hat{\sigma}_0(s)] \, dW(s) \rangle \\
&\leqq |\hat{x}|^2 + C\int_0^t |\hat{X}(s)| \{ |\hat{X}(s)| + |\hat{Y}(s)| + |\hat{Z}(s)| + |\hat{b}_0(s)| \} \, ds \\
&\quad + C\int_0^t \{ |\hat{X}(s)| + |\hat{Y}(s)| + |\hat{Z}(s)| + |\hat{\sigma}_0(s)| \}^2 \, ds \\
&\quad + 2\int_0^t \langle \hat{X}(s), [(1-\alpha)\hat{\sigma}_1(s) + \alpha\hat{\sigma}_2(s) + \hat{\sigma}_0(s)] \, dW(s) \rangle,
\end{aligned}
\tag{3.4}
$$

with some constant $C > 0$. Hereafter, C will be some generic constant, which can be different from line to line. By taking the expectation and using Gronwall's inequality, we obtain

$$
E|\hat{X}(t)|^2 \leqq CE\left\{ |\hat{x}|^2 + \int_0^T \{ |\hat{Y}(t)|^2 + |\hat{Z}(t)|^2 + |\hat{b}_0(t)|^2 + |\hat{\sigma}_0(t)|^2 \} \, dt \right\},
\tag{3.5}
$$

with some constant $C = C(L,T)$. Next, applying Burkholder–Davis–Gundy's inequality [8] to (3.4) (note (3.5)), one has that

$$E \sup_{t\in[0,T]} |\hat{X}(t)|^2 \leqq C\left\{|\hat{x}|^2 + E\int_0^T \{|\hat{Y}(t)|^2 + |\hat{Z}(t)|^2 + |\hat{b}_0(t)|^2 + |\hat{\sigma}_0(t)|^2\}\,dt\right\} . \tag{3.6}$$

On the other hand, by applying Itô's formula to $|\hat{Y}(t)|^2$, we have

$$\begin{aligned}
&|\hat{Y}(t)|^2 + \int_t^T |\hat{Z}(s)|^2\,ds \\
&\quad = |\hat{Y}(T)|^2 - 2\int_t^T \langle \hat{Y}(s), (1-\alpha)\,\hat{h}_1(s) + \alpha\,\hat{h}_2(s) + \hat{h}_0(s)\rangle\,ds \\
&\qquad -2\int_t^T \langle \hat{Y}(s), \hat{Z}(s)\,dW(s)\rangle \\
&\quad \leqq C\left\{|\hat{X}(T)|^2 + |\hat{g}_0|^2 + \int_t^T \{|\hat{X}(s)|^2 + |\hat{Y}(s)|^2 + |\hat{h}_0(s)|^2\}^2\,ds\right\} \\
&\qquad -\frac{1}{2}\int_t^T |\hat{Z}(s)|^2\,ds - 2\int_t^T \langle \hat{Y}(s), \hat{Z}(s)\,dW(s)\rangle .
\end{aligned} \tag{3.7}$$

Similar to the procedure of getting (3.6), we obtain

$$\begin{aligned}
&E \sup_{t\in[0,T]} |\hat{Y}(t)|^2 + E\int_0^T |\hat{Z}(t)|^2\,dt \\
&\quad \leqq CE\left\{|\hat{X}(T)|^2 + |\hat{g}_0|^2 + \int_0^T \{|\hat{X}(t)|^2 + |\hat{h}_0(t)|^2\}\,dt\right\}.
\end{aligned} \tag{3.8}$$

We emphasize that the constants C appeared in (3.6) and (3.8) only depend on L and T. Also, in deriving these two estimates, only the condition $\Gamma_i \in H[0,T]$ has been used. Now, we apply Itô's formula to

$$\left\langle \Phi(t)\begin{pmatrix}\hat{X}(t)\\ \hat{Y}(t)\end{pmatrix}, \begin{pmatrix}\hat{X}(t)\\ \hat{Y}(t)\end{pmatrix}\right\rangle .$$

It follows that

$$\begin{aligned}
&E\left\langle \Phi(T)\begin{pmatrix}\hat{X}(T)\\ \hat{Y}(T)\end{pmatrix}, \begin{pmatrix}\hat{X}(T)\\ \hat{Y}(T)\end{pmatrix}\right\rangle - E\left\langle \Phi(0)\begin{pmatrix}\hat{x}\\ \hat{Y}(0)\end{pmatrix}, \begin{pmatrix}\hat{x}\\ \hat{Y}(0)\end{pmatrix}\right\rangle \\
&\quad = E\int_0^T \left\{\left\langle \dot{\Phi}(t)\begin{pmatrix}\hat{X}(t)\\ \hat{Y}(t)\end{pmatrix}, \begin{pmatrix}\hat{X}(t)\\ \hat{Y}(t)\end{pmatrix}\right\rangle\right. \\
&\qquad +2\left\langle \Phi(t)\begin{pmatrix}\hat{X}(t)\\ \hat{Y}(t)\end{pmatrix}, \begin{pmatrix}(1-\alpha)\,\hat{b}_1(t) + \alpha\,\hat{b}_2(t) + \hat{b}_0(t)\\ (1-\alpha)\,\hat{h}_1(t) + \alpha\,\hat{h}_2(t) + \hat{h}_0(t)\end{pmatrix}\right\rangle \\
&\qquad + \left\langle \Phi(t)\begin{pmatrix}(1-\alpha)\hat{\sigma}_1(t) + \alpha\hat{\sigma}_2(t) + \hat{\sigma}_0(t)\\ \hat{Z}(t)\end{pmatrix},\right. \\
&\qquad\qquad \left.\left.\begin{pmatrix}(1-\alpha)\hat{\sigma}_1(t) + \alpha\hat{\sigma}_2(t) + \hat{\sigma}_0(t)\\ \hat{Z}(t)\end{pmatrix}\right\rangle\right\}\,dt .
\end{aligned} \tag{3.9}$$

Let us separate two cases.

Case 1. Suppse $\Phi \in \mathscr{B}_I(\Gamma_i;[0,T])$ $(i=1,2)$. In this case, we have

$$
\begin{aligned}
F(\alpha) &\triangleq \left\langle \Phi(T)\begin{pmatrix} \hat{X}(T) \\ (1-\alpha)\hat{g}_1(T)+\alpha\hat{g}_2(T)\end{pmatrix}, \begin{pmatrix} \hat{X}(T) \\ (1-\alpha)\hat{g}_1(T)+\alpha\hat{g}_2(T)\end{pmatrix}\right\rangle \\
&= \langle A(T)\hat{X}(T),\hat{X}(T)\rangle + 2\langle B(T)\hat{X}(T),(1-\alpha)\hat{g}_1(T)+\alpha\hat{g}_2(T)\rangle \\
&\quad + \langle C(T)\{(1-\alpha)\hat{g}_1(T)+\alpha\hat{g}_2(T)\},(1-\alpha)\hat{g}_1(T)+\alpha\hat{g}_2(T)\rangle \\
&= \alpha^2\langle C(T)\{\hat{g}_2(T)-\hat{g}_1(T)\},\{\hat{g}_2(T)-\hat{g}_1(T)\}\rangle \\
&\quad + \alpha\{\cdots\}+\{\cdots\} \geqq \delta|\hat{X}(T)|^2, \quad \forall\alpha\in[0,1]\,,
\end{aligned}
\tag{3.10}
$$

where $\{\cdots\}$ are terms that do not depend on α. The above holds because $C(T) \leqq 0$ implies that $F(\alpha)$ is concave in α, whereas (2.16) tells us that (recall $\Phi \in \mathscr{B}_I(\Gamma_i;[0,T]), i=1,2$)

$$
F(0), F(1) \geqq \delta|\hat{X}(T)|^2\,. \tag{3.11}
$$

Then, (3.10) follows easily. Similarly, we have

$$
\begin{aligned}
f(\alpha) &\triangleq \left\langle \dot{\Phi}(t)\begin{pmatrix} \hat{X}(t) \\ \hat{Y}(t)\end{pmatrix}, \begin{pmatrix} \hat{X}(t) \\ \hat{Y}(t)\end{pmatrix}\right\rangle \\
&\quad +2\left\langle \Phi(t)\begin{pmatrix} \hat{X}(t) \\ \hat{Y}(t)\end{pmatrix}, \begin{pmatrix} (1-\alpha)\,\hat{b}_1(t)+\alpha\,\hat{b}_2(t) \\ (1-\alpha)\,\hat{h}_1(t)+\alpha\,\hat{h}_2(t)\end{pmatrix}\right\rangle \\
&\quad + \left\langle \Phi(t)\begin{pmatrix} (1-\alpha)\hat{\sigma}_1(t)+\alpha\hat{\sigma}_2(t) \\ \hat{Z}(t)\end{pmatrix}, \begin{pmatrix} (1-\alpha)\hat{\sigma}_1(t)+\alpha\hat{\sigma}_2(t) \\ \hat{Z}(t)\end{pmatrix}\right\rangle \\
&= \alpha^2\langle A(t)\{\hat{\sigma}_2(t)-\hat{\sigma}_1(t)\},\hat{\sigma}_2(t)-\hat{\sigma}_1(t)\rangle \\
&\quad +\alpha\{\cdots\}+\{\cdots\} \leqq -\delta|\hat{X}(t)|^2, \quad \forall\alpha\in[0,1]\,,
\end{aligned}
\tag{3.12}
$$

since now $A(t) \geqq 0$ which implies $f(\alpha)$ is convex in α. Then, we have

$$
\begin{aligned}
&\text{Left-hand side of (3.9)} \\
&= E\left\{\langle A(T)\hat{X}(T),\hat{X}(T)\rangle\right. \\
&\quad +2\langle B(T)\hat{X}(T),(1-\alpha)\hat{g}_1(T)+\alpha\hat{g}_2(T)+\hat{g}_0\rangle \\
&\quad \left. +\langle C(T)\{(1-\alpha)\hat{g}_1(T)+\alpha\hat{g}_2(T)+\hat{g}_0\},(1-\alpha)\hat{g}_1(T)+\alpha\hat{g}_2(T)+\hat{g}_0\rangle\right\} \\
&\quad -E\left\langle \Phi(0)\begin{pmatrix} \hat{x} \\ \hat{Y}(0)\end{pmatrix}, \begin{pmatrix} \hat{x} \\ \hat{Y}(0)\end{pmatrix}\right\rangle \\
&\geqq \delta E|\hat{X}(T)|^2 - 2|B(T)|E(|\hat{X}(T)||\hat{g}_0|) - 2L|C(T)|E(|\hat{X}(T)||\hat{g}_0|) \\
&\quad -|C(T)|E|\hat{g}_0|^2 - K|\hat{x}|^2 \\
&\geqq \tfrac{1}{2}\delta E|\hat{X}(T)|^2 - C\{|\hat{x}|^2+E|\hat{g}_0|^2\}\,.
\end{aligned}
\tag{3.13}
$$

Here, the constant $C > 0$ only depends on $K, L, \delta, |B(T)|$ and $|C(T)|$. Similarly, we have the following estimate for the right-hand side of (3.9).

$$
\begin{aligned}
&\text{Right-hand side of (3.9)}\\
&\quad \leqq E\int_0^T \left\{ -\delta|\hat{X}(t)|^2\,dt + 2\left\langle \Phi(t)\begin{pmatrix}\hat{X}(t)\\ \hat{Y}(t)\end{pmatrix}, \begin{pmatrix}\hat{b}_0(t)\\ \hat{h}_0(t)\end{pmatrix}\right\rangle \right.\\
&\qquad +2\left\langle \Phi(t)\begin{pmatrix}(1-\alpha)\hat{\sigma}_1(t)+\alpha\hat{\sigma}_2(t)\\ \hat{Z}(t)\end{pmatrix}, \begin{pmatrix}\hat{\sigma}_0(t)\\ 0\end{pmatrix}\right\rangle\\
&\qquad \left. +\left\langle \Phi(t)\begin{pmatrix}\hat{\sigma}_0(t)\\ 0\end{pmatrix}, \begin{pmatrix}\hat{\sigma}_0(t)\\ 0\end{pmatrix}\right\rangle\right\} dt\\
&\quad \leqq -\frac{\delta}{2}E\int_0^T |\hat{X}(t)|^2\,dt + C_\varepsilon E\int_0^T \{|\hat{b}_0(t)|^2 + |\hat{\sigma}_0(t)|^2 + |\hat{h}_0(t)|^2\}\,dt\\
&\qquad +\varepsilon E\int_0^T \{|\hat{Y}(t)|^2 + |\hat{Z}(t)|^2\}\,dt\,,
\end{aligned}
\tag{3.14}
$$

with the constant $C_\varepsilon > 0$ only depending on the bounds of $|\Phi(t)|$, as well as δ, L and the undetermined small positive number $\varepsilon > 0$. Combining (3.13) and (3.14) and noting (3.8), we have

$$
\begin{aligned}
&E|\hat{X}(T)|^2 + E\int_0^T |\hat{X}(t)|^2\,dt\\
&\quad \leqq C_\varepsilon\left\{|\hat{x}|^2 + E|\hat{g}_0|^2 + E\int_0^T \{|\hat{b}_0(t)|^2 + |\hat{\sigma}_0(t)|^2 + |\hat{h}_0(t)|^2\}\,dt\right\}\\
&\qquad + \frac{2\varepsilon}{\delta}E\int_0^T \{|\hat{Y}(t)|^2 + |\hat{Z}(t)|^2\}\,dt\\
&\quad \leqq C_\varepsilon\left\{|\hat{x}|^2 + E|\hat{g}_0|^2 + E\int_0^T \{|\hat{b}_0(t)|^2 + |\hat{\sigma}_0(t)|^2 + |\hat{h}_0(t)|^2\}\,dt\right\}\\
&\qquad + \varepsilon\bar{C}E\left\{|\hat{X}(T)|^2 + |\hat{g}_0|^2 + \int_0^T \{|\hat{X}(t)|^2 + |\hat{h}_0(t)|^2\}\,dt\right\},
\end{aligned}
\tag{3.15}
$$

with the constant $\bar{C}$ independent of $\varepsilon > 0$, and C_ε might be different from that appeared in (3.14). Thus, we may choose suitable $\varepsilon > 0$, such that

$$
\begin{aligned}
&E|\hat{X}(T)|^2 + E\int_0^T |\hat{X}(t)|^2\,dt\\
&\quad \leqq CE\left\{|\hat{x}|^2 + |\hat{g}_0|^2 + \int_0^T \{|\hat{b}_0(t)|^2 + |\hat{\sigma}_0(t)|^2 + |\hat{h}_0(t)|^2\}\,dt\right\}.
\end{aligned}
\tag{3.16}
$$

Then, return to (3.8), we obtain

$$E \sup_{t\in[0,T]} |\hat{Y}(t)|^2 + E\int_0^T |\hat{Z}(t)|^2\,dt \leqq CE\left\{|\hat{x}|^2 + |\hat{g}_0|^2 + \int_0^T \{|\hat{b}_0(t)|^2 + |\hat{\sigma}_0(t)|^2 + |\hat{h}_0(t)|^2\}\,dt\right\}. \quad (3.17)$$

Finally, by (3.6), we have

$$E \sup_{t\in[0,T]} |\hat{X}(t)|^2 \leqq CE\left\{|\hat{x}|^2 + |\hat{g}_0|^2 + \int_0^T \{|\hat{b}_0(t)|^2 + |\hat{\sigma}_0(t)|^2 + |\hat{h}_0(t)|^2\}\,dt\right\}. \quad (3.18)$$

Hence, (3.2) follows from (3.17) and (3.18).

Case 2. Let $\Phi \in \mathscr{B}_{II}(\Gamma_i;[0,T])$ $(i = 1,2)$ now. In this case, we still have (3.6), (3.8) and (3.9). Further, we have inequalities similar to (3.10) and (3.12) with $|\hat{X}(T)|^2$ and $|\hat{X}(t)|^2$ replaced by 0 and $|\hat{Y}(t)|^2 + |\hat{Z}(t)|^2$, respectively. Thus, it follows that

$$\text{Left-hand side of (3.9)} \geqq -\varepsilon E|\hat{X}(T)|^2 - C_\varepsilon\{|\hat{x}|^2 + E|\hat{g}_0|^2\}\,, \quad (3.19)$$

with the constant $C_\varepsilon > 0$ depending on $K, L, \delta, |B(T)|, |C(T)|$, and the undetermined constant $\varepsilon > 0$. Whereas,

$$\begin{aligned}&\text{Right-hand side of (3.9)}\\ &\quad\leqq -\frac{\delta}{2}E\int_0^T \{|\hat{Y}(t)|^2 + |\hat{Z}(t)|^2\}\,dt\\ &\qquad + \varepsilon E\int_0^T |\hat{X}(t)|^2\,dt + C_\varepsilon E\int_0^T \{|\hat{b}_0(t)|^2 + |\hat{\sigma}_0(t)|^2 + |\hat{h}_0(t)|^2\}\,dt\,. \quad (3.20)\end{aligned}$$

Now, combining (3.19) and (3.20) and using (3.6), we obtain (for suitable choice of $\varepsilon > 0$)

$$E\int_0^T \{|\hat{Y}(t)|^2 + |\hat{Z}(t)|^2\}\,dt \leqq CE\left\{|\hat{x}|^2 + |\hat{g}_0|^2 + \int_0^T \{|\hat{b}_0(t)|^2 + |\hat{\sigma}_0(t)|^2 + |\hat{h}_0(t)|^2\}\,dt\right\}. \quad (3.21)$$

Finally, by (3.6) and (3.8) again, we obtain the estimate (3.2). □

An easy and interesting consequence of Lemma 3.1 is the following.

Corollary 3.2. *Let* $\Gamma \in H[0,T]$ *with* $\mathscr{B}(\Gamma;[0,T]) \neq \phi$. *Then, for any* $\gamma \in \mathscr{H}[0,T]$ *and* $x \in \mathbb{R}^n$, $(2.10)_{\Gamma,\gamma,x}$ *admits at most one adapted solution.*

Proof. We take $\Gamma_1 = \Gamma_2 = \Gamma$ in Lemma 3.1. Then, (3.2) gives the uniqueness. □

From Corollary 3.2, we see that for the Γ associated with (2.14), $\mathscr{B}(\Gamma;[0,T]) = \phi$ for $T = k\pi + \frac{3}{4}\pi,\ k \geqq 0$.

Now, we prove the following continuation lemma.

Lemma 3.3. *Let $\Gamma_1, \Gamma_2 \in H[0,T]$ be linked by a direct bridge. Then, there exists an absolute constant $\varepsilon_0 > 0$, such that if for some $\alpha \in [0,1]$, $(3.1)^{\alpha}_{\gamma,x}$ is uniquely solvable for any $\gamma \in \mathscr{H}[0,T]$ and $x \in \mathbb{R}^n$, then the same is true for $(3.1)^{\alpha+\varepsilon}_{\gamma,x}$ with $\varepsilon \in [0,\varepsilon_0]$, $\alpha + \varepsilon \leqq 1$.*

Proof. Let $\varepsilon_0 > 0$ be undetermined. Let $\varepsilon \in [0,\varepsilon_0]$. For $k \geqq 0$, we successively solve the following systems for $\Theta^k(t) \stackrel{\Delta}{=} (X^k(t), Y^k(t), Z^k(t))$: (compare $(3.1)^{\alpha+\varepsilon}_{\gamma,x}$)

$$\begin{aligned}
&\Theta^0(t) \stackrel{\Delta}{=} (X^0(t), Y^0(t), Z^0(t)) \equiv 0\,,\\
&dX^{k+1}(t) = \{(1-\alpha)b_1(t,\Theta^{k+1}(t)) + \alpha b_2(t,\Theta^{k+1}(t))\\
&\qquad\qquad - \varepsilon b_1(t,\Theta^k(t)) + \varepsilon b_2(t,\Theta^k(t)) + b_0(t)\}\,dt\\
&\qquad\qquad + \{(1-\alpha)\sigma_1(t,\Theta^{k+1}(t)) + \alpha\sigma_2(t,\Theta^{k+1}(t))\\
&\qquad\qquad - \varepsilon\sigma_1(t,\Theta^k(t)) + \varepsilon\sigma_2(t,\Theta^k(t)) + \sigma_0(t)\}\,dW(t)\,,\\
&dY^{k+1}(t) = \{(1-\alpha)h_1(t,\Theta^{k+1}(t)) + \alpha h_2(t,\Theta^{k+1}(t))\\
&\qquad\qquad - \varepsilon h_1(t,\Theta^k(t)) + \varepsilon h_2(t,\Theta^k(t)) + h_0(t)\}\,dt + Z^{k+1}(t)\,dW(t)\,,\\
&X^{k+1}(0) = x,\\
&Y^{k+1}(T) = (1-\alpha)g_1(X^{k+1}(T)) + \alpha g_2(X^{k+1}(T))\\
&\qquad\qquad - \varepsilon g_1(X^k(T)) + \varepsilon g_2(X^k(T)) + g_0\,.
\end{aligned} \tag{3.22$^{\alpha+\varepsilon}_{\gamma,x}$}$$

By our assumption, the above systems are uniquely solvable. We now apply Lemma 3.1 to $\Theta^{k+1}(\cdot)$ and $\Theta^k(\cdot)$. It follows that

$$\begin{aligned}
&E\sup_{t\in[0,T]}|X^{k+1}(t) - X^k(t)|^2 + E\sup_{t\in[0,T]}|Y^{k+1}(t) - Y^k(t)|^2\\
&\quad + E\int_0^T |Z^{k+1}(t) - Z^k(t)|^2\,dt\\
&\leqq C\left\{\varepsilon^2 E|X^k(T) - X^{k-1}(T)|^2 + \varepsilon^2 E\int_0^T |\Theta^k(t) - \Theta^{k-1}(t)|^2\,dt\right\}\\
&\leqq \varepsilon^2 C_0\left\{E\sup_{t\in[0,T]}|X^k(t) - X^{k-1}(t)|^2 + E\sup_{t\in[0,T]}|Y^k(t) - Y^{k-1}(t)|^2\right.\\
&\quad\left. + E\int_0^T |Z^k(t) - Z^{k-1}(t)|^2\,dt\right\}.
\end{aligned} \tag{3.23}$$

We note that the constant $C_0 > 0$ appearing in (3.23) is independent of α and ε. Hence, if we choose $\varepsilon_0 > 0$ so that $\varepsilon_0^2 C_0 < \frac{1}{4}$, then for any $\varepsilon \in [0,\varepsilon_0]$, we have the following estimate:

$$\|\Theta^{k+1}(\cdot) - \Theta^k(\cdot)\|_{\mathscr{M}[0,T]} \leqq \tfrac{1}{2}\|\Theta^k(\cdot) - \Theta^{k-1}(\cdot)\|_{\mathscr{M}[0,T]}, \quad \forall k \geqq 1\,. \tag{3.24}$$

This implies that the sequence $\{\Theta^k(\cdot)\}$ is Cauchy in the Banach space $\mathscr{M}[0,T]$. Hence, it admits a limit. Clearly, this limit is an adapted solution to $(3.1)^{\alpha+\varepsilon}_{\gamma,x}$. Uniqueness follows from Corollary 3.2 immediately. □

Now, we are ready to give a proof of our main result.

Proof of Theorem 2.7. We know that it suffices to consider the case that Γ_1 and Γ_2 are linked by a direct bridge. Let us assume that $(2.10)_{\Gamma_1,\gamma,x}$ is uniquely solvable for any $\gamma \in \mathscr{H}[0,T]$ and $x \in \mathbb{R}^n$. This means that $(3.1)^0_{\gamma,x}$ is uniquely solvable. By Lemma 3.3, we can then solve $(3.1)^{\alpha}_{\gamma,x}$ uniquely for any $\alpha \in [0,1]$. In particular, $(3.1)^1_{\gamma,x}$, which is $(2.10)_{\Gamma_2,\gamma,x}$, is uniquely solvable. This proves Theorem 2.7. □

Actually, we have proved something more than Theorem 2.7. From Lemma 3.1, we see that if $\Gamma \in \mathscr{S}[0,T]$ and $\mathscr{B}(\Gamma;[0,T]) \neq \phi$, then the corresponding stability estimate (3.2) holds for the FBSDEs associated with Γ.

4. Properties of the bridges

In this section, we are going to explore some interesting properties of the bridges.

Proposition 4.1. *Let $T > 0$.*

(i) *For any $\Gamma \in H[0,T]$, the set $\mathscr{B}_I(\Gamma;[0,T])$ is convex whenever it is nonempty. Moreover,*

$$\mathscr{B}_I(\Gamma;[0,T]) = \mathscr{B}_I(\Gamma+\gamma;[0,T]), \quad \forall \gamma \in \mathscr{H}[0,T]\,. \tag{4.1}$$

(ii) *For any $\Gamma_1,\Gamma_2 \in H[0,T]$, it holds*

$$\mathscr{B}_I(\Gamma_1;[0,T]) \cap \mathscr{B}_I(\Gamma_2;[0,T]) \subseteq \bigcap_{\alpha\in[0,1]} \mathscr{B}_I(\alpha\Gamma_1 + (1-\alpha)\Gamma_2;[0,T])\,. \tag{4.2}$$

Proof. (i) The convexity of $\mathscr{B}_I(\Gamma;[0,T])$ is clear from (2.15)–(2.17). Conclusion (4.1) also follows easily from the definition of the bridge.

(ii) The proof follows from (3.10), (3.12) and the fact that $\mathscr{B}_I(\Gamma;[0,T])$ is convex. □

It is clear that the same conclusions as Proposition 4.1 hold for $\mathscr{B}_{II}(\Gamma;[0,T])$ and $\mathscr{B}^s(\Gamma;[0,T])$.

As a consequence of (4.2), we see that if $\Gamma_1,\Gamma_2 \in H[0,T]$, then

$$\begin{aligned}&\mathscr{B}_I(\alpha\Gamma_1 + \beta\Gamma_2;[0,T]) = \phi, \quad \text{for some } \alpha,\beta > 0,\\ &\quad \Rightarrow \mathscr{B}_I(\Gamma_1;[0,T]) \cap \mathscr{B}_I(\Gamma_2;[0,T]) = \phi\,.\end{aligned} \tag{4.3}$$

This means that for such a case, Γ_1 and Γ_2 are *not* linked by a direct bridge (of type (I)). Let us look at a concrete example. Let $\Gamma_i = (b_i, \sigma_i, h_i, g_i) \in H[0,T]$, $i = 1,2,3$, with

$$\begin{pmatrix} b_1 \\ h_1 \end{pmatrix} = \begin{pmatrix} -\lambda & 0 \\ -1 & -v \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}, \qquad \begin{pmatrix} b_2 \\ h_2 \end{pmatrix} = \begin{pmatrix} \lambda & 1 \\ 0 & v \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix},$$
$$\begin{pmatrix} b_3 \\ h_3 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}, \qquad \sigma_1 = \sigma_2 = \sigma_3 = 0, \qquad g_1 = g_2 = g_3 = -x\,, \tag{4.4}$$

with $\lambda, v \in \mathbb{R}$. Clearly, it holds

$$\Gamma_3 = \Gamma_1 + \Gamma_2\,. \tag{4.5}$$

By the remark right after Corollary 3.2, we know that $\mathscr{B}(\Gamma_3; [0,T]) = \phi$. Thus, it follows from (4.5) and (4.3) that Γ_1 and Γ_2 are not linked by a direct bridge. However, we see that the FBSDEs associated with Γ_1 is decoupled and thus it is uniquely solvable (see Sect. 5, or [13]). In Sect. 6, we will show that for suitable choice of λ and v, $\Gamma_2 \in \mathscr{S}[0,T]$. Hence, we find two elements in $\mathscr{S}[0,T]$ that are *not* linked by a direct bridge. This means Γ_1 and Γ_2 are *not* very "close".

Next, for any $b_1, b_2 \in L^2_{\mathscr{F}}(0,T; W^{1,\infty}(M, \mathbb{R}^n))$, we define

$$\|b_1 - b_2\|_0(t) = \operatorname*{esssup}_{\omega\in\Omega} \sup_{\theta,\bar\theta\in M} \frac{|b_1(t,\theta;\omega) - b_1(t,\bar\theta;\omega) - b_2(t,\theta;\omega) + b_2(t,\bar\theta;\omega)|}{|\theta - \bar\theta|}\,. \tag{4.6}$$

We define $\|h_1 - h_2\|_0(t)$ and $\|\sigma_1 - \sigma_2\|_0(t)$ similarly. For $g_1, g_2 \in L^2_{\mathscr{F}_T}(\Omega; W^{1,\infty}(\mathbb{R}^n; \mathbb{R}^m))$, we define

$$\|g_1 - g_2\|_0 = \operatorname*{esssup}_{\omega\in\Omega} \sup_{x,\,\bar x\in\mathbb{R}^n} \frac{|g_1(x;\omega) - g_1(\bar x;\omega) - g_2(x;\omega) + g_2(\bar x;\omega)|}{|x - \bar x|}\,. \tag{4.7}$$

Then, for any $\Gamma_i = (b_i, \sigma_i, h_i, g_i) \in H[0,T]$ $(i = 1,2)$, set

$$\|\Gamma_1 - \Gamma_2\|_0(t) = \|b_1 - b_2\|_0(t) + \|\sigma_1 - \sigma_2\|_0(t) + \|h_1 - h_2\|_0(t) + \|g_1 - g_2\|_0\,. \tag{4.8}$$

Note that $\|\cdot\|_0(t)$ is just a family of semi-norms (parameterized by $t \in [0,T]$). As a matter of fact, $\|\Gamma_1 - \Gamma_2\|_0(t) = 0$ for all $t \in [0,T]$ if and only if

$$\Gamma_2 = \Gamma_1 + \gamma\,, \tag{4.9}$$

for some $\gamma \in \mathscr{H}[0,T]$.

Theorem 4.2. *Let $T > 0$ and $\Gamma \in H[0,T]$. Let $\Phi \in \mathscr{B}^s(\Gamma; [0,T])$. Then, there exists an $\varepsilon > 0$, such that for any $\Gamma' \in H[0,T]$ with*

$$\|\Gamma - \Gamma'\|_0(t) < \varepsilon, \quad \forall t \in [0,T]\,, \tag{4.10}$$

we have $\Phi \in \mathscr{B}^s(\Gamma'; [0,T])$.

Proof. Let $\Gamma = (b, \sigma, h, g)$ and $\Gamma' = (b', \sigma', h', g')$. Suppose $\Phi \in \mathscr{B}^s(\Gamma; [0, T])$. Then, for some $K, \delta > 0$, (2.15)–(2.17) and (2.16)′–(2.17)′ hold. Now, we denote (for any $\theta, \bar{\theta} \in M$)

$$\begin{aligned}
&\hat{x} = x - \bar{x}, \qquad \hat{\theta} = \theta - \bar{\theta}\,.\\
&\hat{b} = b(t,\theta) - b(t,\bar{\theta}), \qquad \hat{\sigma} = \sigma(t,\theta) - \sigma(t,\bar{\theta})\,,\\
&\hat{h} = h(t,\theta) - h(t,\bar{\theta}), \qquad \hat{g} = g(x) - g(\bar{x})\,,\\
&\hat{b}' = b'(t,\theta) - b'(t,\bar{\theta}), \qquad \hat{\sigma}' = \sigma'(t,\theta) - \sigma'(t,\bar{\theta})\,,\\
&\hat{h}' = h'(t,\theta) - h'(t,\bar{\theta}), \qquad \hat{g}' = g'(x) - g'(\bar{x})\,.
\end{aligned} \tag{4.11}$$

Then, one has

$$|\hat{g}' - \hat{g}| = |g'(x) - g'(\bar{x}) - g(x) + g(\bar{x})| \leqq \|g' - g\|_0 |\hat{x}|\,. \tag{4.12}$$

Similarly, we have

$$\begin{aligned}
|\hat{b}' - \hat{b}| &\leqq \|b' - b\|_0(t)|\hat{\theta}|\,,\\
|\hat{\sigma}' - \hat{\sigma}| &\leqq \|\sigma' - \sigma\|_0(t)|\hat{\theta}|\,,\\
|\hat{h}' - \hat{h}| &\leqq \|h' - h\|_0(t)|\hat{\theta}|\,.
\end{aligned} \tag{4.13}$$

Hence, it follows that

$$\begin{aligned}
&\left\langle \Phi(T)\begin{pmatrix}\hat{x}\\ \hat{g}'\end{pmatrix}, \begin{pmatrix}\hat{x}\\ \hat{g}'\end{pmatrix}\right\rangle\\
&\quad = \left\langle \Phi(T)\begin{pmatrix}\hat{x}\\ \hat{g}\end{pmatrix}, \begin{pmatrix}\hat{x}\\ \hat{g}\end{pmatrix}\right\rangle\\
&\qquad + 2\left\langle \Phi(T)\begin{pmatrix}\hat{x}\\ \hat{g}\end{pmatrix}, \begin{pmatrix}0\\ \hat{g}' - \hat{g}\end{pmatrix}\right\rangle + \left\langle \Phi(T)\begin{pmatrix}0\\ \hat{g}' - \hat{g}\end{pmatrix}, \begin{pmatrix}0\\ \hat{g}' - \hat{g}\end{pmatrix}\right\rangle\\
&\quad \geqq \delta|\hat{x}|^2 + 2\langle B(T)\hat{x}, \hat{g}' - \hat{g}\rangle + \langle C(T)(\hat{g}' + \hat{g}), \hat{g}' - \hat{g}\rangle\\
&\quad \geqq \{\delta - 2|B(T)|\,\|g' - g\|_0 - |C(T)|\,\|g' + g\|_0\|g' - g\|_0\}|\hat{x}|^2 \geqq \tfrac{1}{2}\delta|\hat{x}|^2\,,
\end{aligned} \tag{4.14}$$

provided $\|g' - g\|_0$ is small enough. Similarly, we have the following:

$$\begin{aligned}
&\left\langle \dot{\Phi}(t)\begin{pmatrix}\hat{x}\\ \hat{y}\end{pmatrix}, \begin{pmatrix}\hat{x}\\ \hat{y}\end{pmatrix}\right\rangle + 2\left\langle \Phi(t)\begin{pmatrix}\hat{x}\\ \hat{y}\end{pmatrix}, \begin{pmatrix}\hat{b}'\\ \hat{h}'\end{pmatrix}\right\rangle + \left\langle \Phi(t)\begin{pmatrix}\hat{\sigma}'\\ \hat{z}\end{pmatrix}, \begin{pmatrix}\hat{\sigma}'\\ \hat{z}\end{pmatrix}\right\rangle\\
&\quad \leqq -\delta|\hat{\theta}|^2 + 2\left\langle \Phi(t)\begin{pmatrix}\hat{x}\\ \hat{y}\end{pmatrix}, \begin{pmatrix}\hat{b}' - \hat{b}\\ \hat{h}' - \hat{h}\end{pmatrix}\right\rangle + 2\left\langle \Phi(t)\begin{pmatrix}\hat{\sigma}\\ \hat{z}\end{pmatrix}, \begin{pmatrix}\hat{\sigma}' - \hat{\sigma}\\ 0\end{pmatrix}\right\rangle\\
&\qquad + \left\langle \Phi(t)\begin{pmatrix}\hat{\sigma}' - \hat{\sigma}\\ 0\end{pmatrix}, \begin{pmatrix}\hat{\sigma}' - \hat{\sigma}\\ 0\end{pmatrix}\right\rangle
\end{aligned}$$

$$\leqq -\delta|\hat{\theta}|^2 + 2\langle A(t)\hat{x} + B(t)^T\hat{y}, \hat{b}' - \hat{b}\rangle + 2\langle B(t)\hat{x} + C(t)\hat{y}, \hat{h}' - \hat{h}\rangle$$

$$+ 2\langle B(t)^T\hat{z}, \hat{\sigma}' - \hat{\sigma}\rangle + \langle A(t)(\hat{\sigma}' + \hat{\sigma}), \hat{\sigma}' - \hat{\sigma}\rangle$$

$$\leqq \{-\delta + 2(|A(t)| + |B(t)|)\|b' - b\|_0(t) + 2(|B(t)| + |C(t)|)\|h' - h\|_0(t)$$

$$+ 2|B(t)|\|\sigma' - \sigma\|_0(t) + |A(t)|\|\sigma' + \sigma\|_0(t)\|\sigma' - \sigma\|_0(t)\}|\hat{\theta}|^2. \quad (4.15)$$

Then, our assertion follows. □

The above result tells us that if the equation associated with Γ is solvable and Γ admits a strong bridge, then all the equations "nearby" are solvable. This is a kind of stability result.

Remark 4.3. We see from (4.14) and (4.15) that the condition (4.10) can be replaced by

$$\begin{aligned} &2(|B(T)| + |C(T)|\|g' + g\|_0)\|g' - g\|_0 < \delta, \\ &\sup_{t\in[0,T]} \{2(|A(t)| + |B(t)|)\|b' - b\|_0(t) + 2(|B(t)| + |C(t)|)\|h' - h\|_0(t) \\ &\qquad + [2|B(t)| + |A(t)|\|\sigma' + \sigma\|_0(t)]\|\sigma' - \sigma\|_0(t)\} < \delta, \end{aligned} \quad (4.16)$$

where $\delta > 0$ is the one appeared in the definition of the bridge (see Definition 2.5). Actually, (4.16) can further be replaced by the following even weaker conditions:

$$\begin{aligned} &2\langle B(T)\hat{x}, \hat{g}' - \hat{g}\rangle + \langle C(T)(\hat{g}' + \hat{g}), \hat{g}' - \hat{g}\rangle > -\delta|\hat{x}|^2, \quad \forall x, \bar{x} \in \mathbb{R}^n, \\ &\sup_{t\in[0,T]} \{2\langle A(t)\hat{x} + B(t)^T\hat{y}, \hat{b}' - \hat{b}\rangle + 2\langle B(t)\hat{x} + C(t)\hat{y}, \hat{h}' - \hat{h}\rangle \\ &\qquad + 2\langle B(t)^T\hat{z}, \hat{\sigma}' - \hat{\sigma}\rangle + \langle A(t)(\hat{\sigma}' + \hat{\sigma}), \hat{\sigma}' - \hat{\sigma}\rangle\} \\ &\qquad < \delta|\hat{\theta}|^2, \quad \forall \theta, \bar{\theta} \in M. \end{aligned} \quad (4.17)$$

The above means that if the perturbation is made not necessarily small but in the right direction, the solvability will be kept. This observation will be useful later.

To conclude this section, we present the following simple proposition.

Proposition 4.4. *Let* $T > 0$, $\Gamma \equiv (b, \sigma, h, g) \in H[0, T]$ *and* $\Phi \in \mathscr{B}_I(\Gamma; [0, T])$. *Let* $\beta \in \mathbb{R}$ *and*

$$\begin{aligned} &\tilde{\Phi}(t) = e^{2\beta t}\Phi(t), \quad t \in [0, T], \\ &\tilde{\Gamma} = (b - \beta x, \sigma, h - \beta y, g) \in H[0, T]. \end{aligned} \quad (4.18)$$

Then, $\tilde{\Phi} \in \mathscr{B}_I(\tilde{\Gamma}; [0, T])$.

The proof is immediate. Clearly, the similar conclusion holds if we replace $\mathscr{B}_I(\Gamma; [0, T])$ by $\mathscr{B}_{II}(\Gamma; [0, T]), \mathscr{B}(\Gamma; [0, T])$ or $\mathscr{B}^s(\Gamma; [0, T])$.

5. Solvability of FBSDEs

In this section, we are going to prove the unique solvability of some FBSDEs by constructing appropriate bridges. In what follows, we denote $\Gamma_0 = (0,0,0,0) \in H[0,T]$. Clearly, the FBSDEs associated with Γ_0 is trivially solvable. Thus, hereafter, we will refer to the FBSDEs associated with Γ_0 as the *trivial* FBSDEs. Now, let us present the following result.

Proposition 5.1. *Let* $T > 0$ *and* $\Gamma_0 = (0,0,0,0) \in H[0,T]$. *Then,*

$$\Phi \triangleq \begin{pmatrix} A & B^T \\ B & C \end{pmatrix} \in \mathscr{B}^s(\Gamma_0;[0,T])$$

if and only if

$$\begin{array}{ll} C(0) < 0, & A(T) > 0\,, \\ \dot{\Phi}(t) < 0, & \forall t \in [0,T]\,. \end{array} \tag{5.1}$$

Proof. By Definition 2.5, we know that $\Phi \in \mathscr{B}^s(\Gamma_0;[0,T])$ if and only if (2.15)–(2.17) and (2.16)$'$ and (2.17)$'$ hold. These are equivalent to the following:

$$\begin{cases} C(0) \leqq -\delta, & A(T) \geqq \delta\,, \\ \dot{\Phi}(t) \leqq -\delta, & \forall t \in [0,T]\,, \end{cases} \tag{5.2}$$

for some $\delta > 0$. We note that under condition $C(0) < 0$, the second inequality in (2.15) is always true for sufficiently large $K > 0$. Then, we see easily that $\Phi \in \mathscr{B}^s(\Gamma_0;[0,T])$ is characterized by (5.1) since $\delta > 0$ can be arbitrarily small. □

From the above, we also have the following characterization:

$$\begin{aligned} \mathscr{B}^s(\Gamma_0;[0,T]) = \Bigg\{ Q - \int_0^{\cdot} \Psi(s)\,ds \,\Bigg|\, & 0 < \Psi(\cdot) = \begin{pmatrix} \Psi_1(\cdot) & \Psi_2(\cdot)^T \\ \Psi_2(\cdot) & \Psi_3(\cdot) \end{pmatrix} \in C([0,T];S^{n+m}), \\ & Q = \begin{pmatrix} Q_1 & Q_2^T \\ Q_2 & Q_3 \end{pmatrix} \in S^{n+m},\ Q_3 < 0,\ Q_1 - \int_0^T \Psi_1(s)\,ds > 0 \Bigg\}\,. \end{aligned} \tag{5.3}$$

A useful consequence of Proposition 5.1 is the following.

Corollary 5.2. *Let* $\Gamma = (b,\sigma,h,g) \in H[0,T]$ *admits a bridge* $\Phi \in \mathscr{B}(\Gamma;[0,T])$ *satisfying* (5.1)*. Then,* $\Gamma \in \mathscr{S}[0,T]$*.*

Proof. In this case, we see that $\Phi \in \mathscr{B}(\Gamma_0;[0,T]) \cap \mathscr{B}(\Gamma;[0,T])$. Since $\Gamma_0 \in \mathscr{S}[0,T]$, Theorem 2.7 applies. □

Next, we would like to discuss several concrete cases.

1. Decoupled case. Let $\Gamma \equiv (b,\sigma,h,g) \in H[0,T]$ such that

$$\begin{cases} b(t,x,y,z) \equiv b(t,x), \\ \sigma(t,x,y,z) \equiv \sigma(t,x), \end{cases} \quad \forall (t,x,y,z) \in [0,T] \times M\,. \tag{5.4}$$

We see that the associated FBSDE is decoupled, which is known to be solvable by [13]. The following result recovers this conclusion.

Proposition 5.3. *Let* $T > 0$, $\Gamma_0 \equiv (0,0,0,0) \in H[0,T]$ *and* $\Gamma \equiv (b,\sigma,h,g) \in H[0,T]$ *satisfying* (5.4)*. Then,*

$$\mathscr{B}^s(\Gamma_0;[0,T]) \cap \mathscr{B}^s(\Gamma;[0,T]) \neq \phi \,. \tag{5.5}$$

Consequently, $\Gamma \in \mathscr{S}[0,T]$.

Proof. We take

$$\begin{aligned} &\Phi(t) = \begin{pmatrix} a(t)I & 0 \\ 0 & c(t)I \end{pmatrix} , \\ &a(t) = A_0 e^{A_0(T-t)}, \quad c(t) = -C_0 e^{C_0 t}, \qquad t \in [0,T] \,, \end{aligned} \tag{5.6}$$

where A_0, $C_0 > 0$ are undetermined constants. We first check that this $\Phi \in \mathscr{B}^s(\Gamma_0;[0,T])$. In fact,

$$\begin{aligned} &c(0) = -C_0 < 0, \qquad a(T) = A_0 > 0 \,, \\ &\dot{a}(t) = -A_0^2 e^{A_0(T-t)} < 0, \quad t \in [0,T] \,, \\ &\dot{c}(t) = -C_0^2 e^{C_0 t} < 0, \quad t \in [0,T] \,. \end{aligned} \tag{5.7}$$

Thus, by Proposition 5.1, we see that $\Phi \in \mathscr{B}^s(\Gamma_0;[0,T])$. Next, we show that $\Phi \in \mathscr{B}^s(\Gamma;[0,T])$ for suitable choice of A_0 and C_0. To this end, we let L be the common Lipschitz constant for b,σ,h and g. We note that (5.7) implies (2.15). Thus, it is enough to have

$$a(T) + L^2 c(T) \geqq \delta \,, \tag{5.8}$$

and

$$\begin{aligned} &\dot{a}(t)|x-\bar{x}|^2 + \dot{c}(t)|y-\bar{y}|^2 + c(t)|z-\bar{z}|^2 + 2a(t)\langle x-\bar{x}, b(t,x) - b(t,\bar{x})\rangle \\ &\quad + a(t)|\sigma(t,x) - \sigma(t,\bar{x})|^2 + 2c(t)\langle y-\bar{y}, h(t,x,y,z) - h(t,\bar{x},\bar{y},\bar{z})\rangle \\ &\qquad \leqq -\delta\{|x-\bar{x}|^2 + |y-\bar{y}|^2 + |z-\bar{z}|^2\} \,, \\ &\qquad \forall t \in [0,T],\ x,\bar{x} \in \mathbb{R}^n,\ y,\ \bar{y} \in \mathbb{R}^m,\ z,\bar{z} \in \mathbb{R}^{m\times d},\ \text{a.s.} \end{aligned} \tag{5.9}$$

Let us first look at (5.9). We note that

$$\begin{aligned} &\text{Left-hand side of (5.9)} \\ &\quad \leqq \dot{a}(t)|x-\bar{x}|^2 + \dot{c}(t)|y-\bar{y}|^2 + c(t)|z-\bar{z}|^2 \\ &\qquad + 2a(t)L|x-\bar{x}|^2 + a(t)L^2|x-\bar{x}|^2 \\ &\qquad + 2|c(t)|L|y-\bar{y}|\{|x-\bar{x}| + |y-\bar{y}| + |z-\bar{z}|\} \\ &\quad \leqq \{\dot{a}(t) + 2a(t)L + a(t)L^2 + |c(t)|L\}|x-\bar{x}|^2 \\ &\qquad + \{\dot{c}(t) + 3|c(t)|L + 2L^2|c(t)|\}|y-\bar{y}|^2 + \tfrac{1}{2}c(t)|z-\bar{z}|^2 \,. \end{aligned} \tag{5.10}$$

Hence, to have (5.9), it suffices to have the following:

$$\begin{aligned}&\dot a(t)+(2L+L^2)a(t)+L|c(t)|\leqq -\delta\,,\\&\dot c(t)+(3L+2L^2)|c(t)|\leqq -\delta, \qquad\qquad \forall t\in[0,T]\,.\\&c(t)\leqq -2\delta\,,\end{aligned}\tag{5.11}$$

Now, we take $a(t)$ and $c(t)$ as in (5.6) and we require

$$\begin{aligned}\dot c(t)+(3L+2L^2)|c(t)| &= -C_0(C_0-3L-2L^2)e^{C_0t}\,,\\&\leqq -C_0(C_0-3L-2L^2)\leqq -\delta, \quad \forall t\in[0,T]\,,\end{aligned}\tag{5.12}$$

and

$$c(t)=-C_0e^{C_0t}\leqq -C_0\leqq -2\delta, \quad \forall t\in[0,T]\,.\tag{5.13}$$

These two are possible if $C_0>0$ is large enough. Next, for this fixed $C_0>0$, we choose $A_0>0$ as follows. We want

$$a(T)+c(T)L^2=A_0e^{A_0(T-t)}-C_0L^2e^{C_0t}\geqq A_0-C_0L^2e^{C_0T}\geqq\delta\,,\tag{5.14}$$

and

$$\begin{aligned}\dot a(t)+(2L+L^2)a(t)+L|c(t)| &= -A_0(A_0-2L-L^2)e^{A_0(T-t)}+LC_0e^{C_0t}\\&\leqq -A_0(A_0-2L-L^2)+LC_0e^{C_0T}\leqq -\delta\,.\end{aligned}\tag{5.15}$$

These are also possible by choosing $A_0>0$ large enough. Hence, (5.8) and (5.11) hold and $\Phi\in\mathscr{B}^s(\Gamma;[0,T])$. □

From the above, we obtain that any decoupled FBSDEs are solvable. In particular, any BSDEs are solvable. This recovers the result of [13]. From Lemma 3.1, we see that the adapted solutions to such equations have the continuous dependence on the data.

2. Monotone case. Let $\Gamma=(b,\sigma,h,g)\in H[0,T]$ satisfying one of the following *monotonicity* conditions.

(M) Let $m\geqq n$. There exists a matrix $B\in\mathbb{R}^{m\times n}$ such that for some $\beta>0$, it holds that

$$\langle B(x-\bar x),g(x)-g(\bar x)\rangle\geqq\beta|x-\bar x|^2, \quad \forall x,\bar x\in\mathbb{R}^n,\ \text{a.s}\,.\tag{5.16}$$

$$\begin{aligned}&\langle B^T[h(t,\theta)-h(t,\bar\theta)],x-\bar x\rangle+\langle B[b(t,\theta)-b(t,\bar\theta)],y-\bar y\rangle\\&\quad+\langle B[\sigma(t,\theta)-\sigma(t,\bar\theta)],z-\bar z\rangle\leqq -\beta|x-\bar x|^2\,,\\&\qquad\forall t\in[0,T],\ \theta,\bar\theta\in M,\ \text{a.s}\,.\end{aligned}\tag{5.17}$$

(M)$'$ Let $m \leqq n$. There exists a matrix $B \in \mathbb{R}^{m\times n}$ such that for some $\beta > 0$, it holds that

$$\langle B(x-\bar{x}), g(x)-g(\bar{x})\rangle \geqq 0, \quad \forall x, \bar{x} \in \mathbb{R}^n, \text{ a.s}. \tag{5.16$'$}$$

$$\begin{aligned}&\langle B^T[h(t,\theta)-h(t,\bar{\theta})], x-\bar{x}\rangle + \langle B[b(t,\theta)-b(t,\bar{\theta})], y-\bar{y}\rangle \\ &\quad + \langle B[\sigma(t,\theta)-\sigma(t,\bar{\theta})], z-\bar{z}\rangle \leqq -\beta(|y-\bar{y}|^2+|z-\bar{z}|^2), \\ &\quad \forall t\in[0,T],\ \theta,\bar{\theta}\in M, \text{ a.s}. \end{aligned} \tag{5.17$'$}$$

We know that (5.16) means that the function $B^Tg(x)$ is uniformly monotone on $\mathbb{R}^n$, and (5.17) implies that the function $-(B^Th(t,\theta), Bb(t,\theta), B\sigma(t,\theta))$ is monotone on the space M. We may similarly explain the meaning of (5.16)$'$ and (5.17)$'$. Here, we should point out that (5.16) (or (5.17)) implies $m \geqq n$ and (5.17)$'$ implies $m \leqq n$. Hence, these two are different situations.

We now prove the following.

Proposition 5.4. *Let $T > 0$ and $\Gamma \equiv (b,\sigma,h,g) \in H[0,T]$ satisfy (M) (resp. $(M)'$). Then, (5.5) holds. Consequently, $\Gamma \in \mathscr{S}[0,T]$.*

Proof. First, we assume (M) holds. Take

$$\left.\begin{aligned}&\Phi(t) = \begin{pmatrix} A(t) & B(t)^T \\ B(t) & C(t)\end{pmatrix} \\ &A(t) = a(t)I \equiv \delta e^{T-t}I, \\ &B(t) \equiv B, \\ &C(t) = c(t)I \equiv -2\delta C_0 e^{C_0 t}I,\end{aligned}\right\} \quad t\in[0,T], \tag{5.18}$$

with δ, $C_0 > 0$ being undetermined. Since

$$\begin{aligned}&C(0) = -2\delta C_0 I < 0, \\ &A(T) = \delta I > 0, \\ &\dot{\Phi}(t) = \begin{pmatrix} -\delta e^{T-t}I & 0 \\ 0 & -2\delta C_0^2 e^{C_0 t}\end{pmatrix} < 0,\end{aligned} \tag{5.19}$$

by Proposition 5.1, we see that $\Phi \in \mathscr{B}^s(\Gamma_0;[0,T])$. Next, we prove $\Phi \in \mathscr{B}^s(\Gamma;[0,T])$ for suitable choice of δ and C_0. Again, we let L be the common Lipschitz constant for b, σ, h and g. We will choose δ and C_0 so that

$$a(T) + 2\beta + c(T)L^2 \geqq \delta, \tag{5.20}$$

and

$$\begin{aligned}&\dot{a}(t)|x|^2 + \dot{c}(t)|y|^2 + c(t)|z|^2 + 2La(t)|x|(|x|+|y|+|z|) \\ &\quad + 2L|c(t)||y|(|x|+|y|+|z|) + L^2a(t)(|x|+|y|+|z|)^2 \\ &\quad \leqq (2\beta-\delta)|x|^2 - \delta(|y|^2+|z|^2), \quad \forall (t,\theta)\in[0,T]\times M. \end{aligned} \tag{5.21}$$

It is not hard to see that under (5.16) and (5.17), (5.20) implies (2.16), and (5.21) implies (2.15) and (2.17)$'$ (Note (2.16) implies (2.16)$'$). We see that the left-hand side of (5.21) can be controlled by the following:

$$\{\dot{a}(t) + Ka(t) + K|c(t)|\}|x|^2 + \{\dot{c}(t) + K|c(t)| + Ka(t)\}|y|^2 + \left\{\tfrac{1}{2}c(t) + Ka(t)\right\}|z|^2 , \tag{5.22}$$

for some constant $K>0$. Then, for this fixed $K>0$, we now choose δ and C_0. First of all, we require

$$\tfrac{1}{2}c(t) + Ka(t) = -\delta C_0 e^{C_0 t} + K\delta e^{T-t} \leqq -\delta C_0 + K\delta e^{T} \leqq -\delta , \tag{5.23}$$

and

$$\begin{aligned}\dot{c}(t) + K|c(t)| + Ka(t) &= -2\delta C_0^2 e^{C_0 t} + 2KC_0\delta e^{C_0 t} + K\delta e^{T-t} \\ &\leqq -2\delta C_0(C_0 - K) + K\delta e^{T} < -\delta .\end{aligned} \tag{5.24}$$

These two can be achieved by choosing $C_0 > 0$ large enough (independent of $\delta > 0$). Next, we require

$$\begin{aligned}\dot{a}(t) + Ka(t) + K|c(t)| &= -\delta e^{T-t} + K\delta e^{T-t} + 2\delta KC_0 e^{C_0 t} \\ &\leqq -\delta + K\delta e^{T} + 2\delta KC_0 e^{C_0 T} \leqq 2\beta - \delta ,\end{aligned} \tag{5.25}$$

and

$$a(T) + 2\beta + c(T)L^2 = \delta + 2\beta - 2\delta C_0 e^{C_0 T} L^2 \geqq \delta . \tag{5.26}$$

Since $\beta > 0$, (5.25) and (5.26) can be achieved by letting $\delta > 0$ be small enough (note again that the choice of C_0 is independent of $\delta > 0$). Hence, we have (5.20) and (5.21), which proves $\Phi \in \mathscr{B}^s(\Gamma; [0, T])$.

Now, we assume (M)$'$ holds. Take (compare (5.18))

$$\left.\begin{aligned}&\Phi(t) = \begin{pmatrix} A(t) & B(t)^T \\ B(t) & C(t) \end{pmatrix} , \\ &A(t) = a(t)I \equiv \delta A_0 e^{A_0(T-t)} I , \\ &B(t) \equiv B , \\ &C(t) = c(t)I \equiv -\delta e^{t} I ,\end{aligned}\right\} \quad \forall t \in [0, T] , \tag{5.27}$$

with δ, $A_0 > 0$ being undetermined. Note that

$$\begin{aligned}&C(0) = -\delta I < 0 , \\ &A(T) = A_0 I > 0 , \\ &\dot{\Phi}(t) = \begin{pmatrix} -\delta A_0^2 e^{A_0(T-t)} I & 0 \\ 0 & -\delta e^{t} I \end{pmatrix} < 0 .\end{aligned} \tag{5.28}$$

Thus, by Proposition 5.1, we have $\Phi \in \mathscr{B}^s(\Gamma_0; [0, T])$. We now choose the constants δ and A_0. In the present case, we will still require (5.20) and the following instead of (5.21):

$$\begin{aligned}&\dot{a}(t)|x|^2+\dot{c}(t)|y|^2+c(t)|z|^2+2La(t)|x|(|x|+|y|+|z|)\\&\quad+2L|c(t)||y|(|x|+|y|+|z|)+L^2a(t)(|x|+|y|+|z|)^2\\&\qquad\leqq -\delta|x|^2+(2\beta-\delta)\{|y|^2+|z|^2\},\quad \forall(t,\theta)\in[0,T]\times M\ .\end{aligned} \tag{5.29}$$

These two will imply the conclusion $\Phi\in\mathscr{B}^s(\Gamma;[0,T])$. Again the left-hand side of (5.29) can be controlled by (5.22) for some constant $K>0$. Now, we require

$$\begin{aligned}\dot{a}(t)+Ka(t)+K|c(t)| &= -\delta A_0^2e^{A_0(T-t)}+\delta KA_0e^{A_0(T-t)}+K\delta e^t\\&\leqq -\delta A_0(A_0-K)+\delta Ke^T \leqq -\delta,\end{aligned} \tag{5.30}$$

and

$$\begin{aligned}a(T)+c(T)L^2 &= \delta A_0e^{A_0(T-t)}-\delta L^2e^t\\&\geqq \delta(A_0-L^2e^T)>\delta\ .\end{aligned} \tag{5.31}$$

We can choose $A_0>0$ large enough (independent of $\delta>0$) to achieve the above two. Next, we require

$$\tfrac{1}{2}c(t)+Ka(t)\leqq Ka(t)\leqq \delta KA_0e^{A_0T}\leqq 2\beta-\delta\ , \tag{5.32}$$

and

$$\begin{aligned}\dot{c}(t)+K|c(t)|+Ka(t) &= -\delta e^t+K\delta e^t+KA_0\delta e^{A_0(T-t)}\\&\leqq \delta(Ke^T+KA_0e^{A_0T})\leqq 2\beta-\delta\ .\end{aligned} \tag{5.33}$$

These two can be achieved by choosing $\delta>0$ small enough. Hence, we obtain (5.20) and (5.29), which gives $\Phi\in\mathscr{B}^s(\Gamma;[0,T])$. □

We note that Proposition 5.4 recovers the results of 7, 16. It should be pointed out that the above monotone cases do not cover the decoupled case. Here is a simple example.

Let $n=m=1$. We consider the following decoupled FBSDEs:

$$\begin{aligned}&dX(t)=X(t)\,dt+dW(t)\ ,\\&dY(t)=X(t)\,dt+Z(t)\,dW(t)\ ,\\&X(0)=x,\qquad Y(T)=X(T)\ .\end{aligned} \tag{5.34}$$

We can easily check that neither (M) nor (M)$'$ holds. But, (5.34) is uniquely solvable over any finite time duration $[0,T]$.

Remark 5.5. From the above, we see that decoupled and monotone FBSDEs are two different classes of solvable FBSDEs. None of them includes the other. Under our framework, however, these two classes are proved to be linked by direct bridges to the trivial FBSDEs (the one associated with $\Gamma_0=(0,0,0,0)$). Thus, in some sense, these classes of FBSDEs are still very "near" to the trivial FBSDEs. At the present stage, we don't know any classes of solvable FBSDEs that are "far away" from the trivial one.

6. Solvability of FBSDEs (continued)

6.1. A general consideration

In the previous section, we have recovered the unique solvability of two (known) classes of FBSDEs by constructing proper bridges. In this section, we are going to present some results for new classes of FBSDEs.

Let us start with the following linear ordinary differential equation: (compare (2.12))

$$\begin{aligned} &\begin{pmatrix} \dot X(t) \\ \dot Y(t) \end{pmatrix} = \mathscr{A} \begin{pmatrix} X(t) \\ Y(t) \end{pmatrix} + \begin{pmatrix} b_0(t) \\ h_0(t) \end{pmatrix}, \quad t \in [0,T]\,, \\ &X(0) = x, \qquad Y(T) = GX(T) + g_0\,. \end{aligned} \tag{6.1}$$

Here, $\mathscr{A} \in \mathbb{R}^{(n+m)\times(n+m)}, G \in \mathbb{R}^{m\times n}, \begin{pmatrix} b_0(t) \\ h_0(t) \end{pmatrix} \in L^2(0,T;\mathbb{R}^{n+m}),\ x \in \mathbb{R}^n,\ g_0 \in \mathbb{R}^m$. We have the following result.

Lemma 6.1. *Let $T_0 > 0$. Then, the two-point boundary value problem* (6.1) *is uniquely solvable for all $T \in (0,T_0]$ and all (b_0,h_0,g_0,x) if and only if*

$$\det \Psi(t) > 0, \quad \forall t \in [0,T_0]\,, \tag{6.2}$$

where

$$\Psi(T) \triangleq (-G,I)e^{\mathscr{A}T} \begin{pmatrix} 0 \\ I \end{pmatrix}\,. \tag{6.3}$$

This result should not be new. Since we are not able to find a proper reference, for reader's convenience, we present a proof here.

Proof. We know that (6.1) is equivalent to the following:

$$\begin{pmatrix} X(t) \\ Y(t) \end{pmatrix} = e^{\mathscr{A}t} \begin{pmatrix} x \\ Y(0) \end{pmatrix} + \int_0^t e^{\mathscr{A}(t-s)} \begin{pmatrix} b_0(s) \\ h_0(s) \end{pmatrix} ds, \quad t \in [0,T]\,, \tag{6.4}$$

together with the following condition (which will be used to determine the unknown $Y(0)$)

$$\begin{aligned} g_0 &= (-G,I) \begin{pmatrix} X(T) \\ Y(T) \end{pmatrix} \\ &= (-G,I)e^{\mathscr{A}T} \begin{pmatrix} 0 \\ I \end{pmatrix} Y(0) \\ &\quad +(-G,I) \left\{ e^{\mathscr{A}T} \begin{pmatrix} x \\ 0 \end{pmatrix} + \int_0^T e^{\mathscr{A}(T-s)} \begin{pmatrix} b_0(s) \\ h_0(s) \end{pmatrix} ds \right\}. \end{aligned} \tag{6.5}$$

Thus, by (6.3), we see that

$$\Psi(T)Y(0) = g_0 - \left\{(-G,I)\left[e^{\mathscr{A}T}\begin{pmatrix} x \\ 0 \end{pmatrix} + \int_0^T e^{\mathscr{A}(T-s)}\begin{pmatrix} b_0(s) \\ h_0(s) \end{pmatrix} ds\right]\right\}. \tag{6.6}$$

This implies that for a given $T > 0$, (6.1) is solvable for all (b_0, h_0, g_0, x) if and only if $\det \Psi(T) \neq 0$. Hence, noting $\det \Psi(0) = 1$, we obtain that (6.2) is a necessary and sufficient condition for (6.1) to be uniquely solvable for all $T \in (0, T_0]$ and all (b_0, h_0, g_0, x). □

Now, we consider the following FBSDEs:

$$\begin{aligned} d\begin{pmatrix} X(t) \\ Y(t) \end{pmatrix} &= \left\{\mathscr{A}\begin{pmatrix} X(t) \\ Y(t) \end{pmatrix} + \begin{pmatrix} b_0(t) \\ h_0(t) \end{pmatrix}\right\} dt \\ &\quad + \begin{pmatrix} \sigma_0(t) \\ Z(t) \end{pmatrix} dW(t), \quad t \in [0,T]\,, \\ x(0) = x, &\qquad Y(T) = GX(T) + g_0\,. \end{aligned} \tag{6.7}$$

where, $\mathscr{A}$, G and x are the same as those in (6.1) and now $\gamma \equiv (b_0, \sigma_0, h_0, g_0) \in \mathscr{H}[0,T]$ (see (2.8)). This is a class of linear FBSDEs. Clearly, it is not necessarily decoupled nor monotone. Thus, it is a new class of FBSDEs (although it is not very complicated).

We now study the unique solvability of this class of FBSDEs. First of all, similar to Proposition 2.3, we see that if (6.1) does not admit a solution, then, (6.7) does not admit an adapted solution. We now give some further results.

Theorem 6.2. *Let* $T_0 > 0$. *Then*, FBSDEs (6.7) *is uniquely solvable for all* $T \in (0, T_0]$ *and all* $x \in \mathbb{R}^n$ *and* $\gamma \equiv (b_0, \sigma_0, h_0, g_0) \in \mathscr{H}[0,T]$ *if and only if* (6.2) *holds.*

Proof. By Lemma 6.1, we have the necessity. Let us now prove the sufficiency. To this end, let us first make the following observation. For any $\gamma = (b_0, \sigma_0, h_0, g_0) \in \mathscr{H}[0,T]$ and $x \in \mathbb{R}^n$, suppose $(X, Y, Z) \in \mathscr{M}[0,T]$ is an adapted solution of (6.7). Then, we have

$$\begin{aligned} \begin{pmatrix} X(t) \\ Y(t) \end{pmatrix} &= e^{\mathscr{A}t}\begin{pmatrix} x \\ Y(0) \end{pmatrix} + \int_0^t e^{\mathscr{A}(t-s)}\begin{pmatrix} b_0(s) \\ h_0(s) \end{pmatrix} ds \\ &\quad + \int_0^t e^{\mathscr{A}(t-s)}\begin{pmatrix} \sigma_0(s) \\ Z(s) \end{pmatrix} dW(s), \quad t \in [0,T]\,. \end{aligned} \tag{6.8}$$

Here, $Y(0)$ is undetermined. We want the following to be satisfied:

$$\begin{aligned} g_0 = (-G,I)\begin{pmatrix} X(T) \\ Y(T) \end{pmatrix} &= \Psi(T)Y(0) + (-G,I)\begin{pmatrix} x \\ 0 \end{pmatrix} \\ + (-G,I)&\left\{\int_0^T e^{\mathscr{A}(T-s)}\begin{pmatrix} b_0(s) \\ \sigma_0(s) \end{pmatrix} ds + \int_0^T e^{\mathscr{A}(T-s)}\begin{pmatrix} \sigma_0(s) \\ Z(s) \end{pmatrix} dW(s)\right\}. \end{aligned} \tag{6.9}$$

Taking expectation in both sides of the above, we have

$$Eg_0 = \Psi(T)Y(0) + (-G,I)\left\{e^{\mathscr{A}T}\begin{pmatrix} x \\ 0 \end{pmatrix} + \int_0^T e^{\mathscr{A}(T-s)}\begin{pmatrix} Eb_0(s) \\ Eh_0(s) \end{pmatrix} ds\right\}. \quad (6.10)$$

By (6.2), we must have

$$Y(0) = \Psi(T)^{-1}\left\{Eg_0 - (-G,I)\left[e^{\mathscr{A}T}\begin{pmatrix} x \\ 0 \end{pmatrix} + \int_0^T e^{\mathscr{A}(T-s)}\begin{pmatrix} Eb_0(s) \\ Eh_0(s) \end{pmatrix} ds\right]\right\}. \quad (6.11)$$

Based on the above observation, we can now prove the solvability of (6.7) (under condition (6.2)). We define

$$\begin{aligned} \xi \triangleq g_0 - Eg_0 - (-G,I)&\left\{\int_0^T e^{\mathscr{A}(T-s)}\begin{pmatrix} b_0(s) - Eb_0(s) \\ h_0(s) - Eh_0(s) \end{pmatrix} ds\right. \\ &\left. + \int_0^T e^{\mathscr{A}(T-s)}\begin{pmatrix} \sigma_0(s) \\ 0 \end{pmatrix} dW(s)\right\}. \end{aligned} \quad (6.12)$$

It is clear that ξ is $\mathscr{F}_T$-measurable, square-integrable and $E\xi = 0$. Thus, by the Martingale Representation Theorem, there exists a $\tilde{Z}(\cdot) \in L^2_{\mathscr{F}}(0,T;\mathbb{R}^{m\times d})$, such that

$$\xi = \int_0^T \tilde{Z}(s)\,dW(s)\,. \quad (6.13)$$

By (6.2), we may set

$$Z(t) = \Psi(T-t)^{-1}\tilde{Z}(t), \quad t \in [0,T]\,. \quad (6.14)$$

Obviously, $Z(\cdot) \in L^2_{\mathscr{F}}(0,T;\mathbb{R}^{m\times d})$. At the same time, we define $Y(0)$ by (6.11). Then, we see that (6.10) holds and combining (6.10), (6.12) and (6.13), we see that (6.9) holds. Thus, by defining $(X(\cdot),Y(\cdot))$ through (6.8), we obtain that $(X,Y,Z) \in \mathscr{M}[0,T]$ is an adapted solution of (6.7). The uniqueness follows easily from condition (6.2). □

The above theorem tells us that the solvability of (6.1) and (6.7) are equivalent in a proper sense. The condition to check is (6.2).

From Theorem 2.7, we know that if Γ_1 and Γ_2 are linked by a bridge, then Γ_1 and Γ_2 have the same solvability. On the other hand, for a given Γ at hand, Corollary 3.2 tells us that if Γ admits a bridge, then, the FBSDEs associated with Γ admits at most one adapted solution. The existence, however, is not known. The following result tells us something concerning the existence. This result will be useful below.

Proposition 6.3. *Let* $T_0 > 0$ *and* $\Gamma = (b,0,h,g)$ *with*

$$\begin{pmatrix} b(t,\theta) \\ h(t,\theta) \end{pmatrix} = \mathscr{A}\begin{pmatrix} x \\ y \end{pmatrix}, \qquad g(x) = Gx, \qquad \forall (t,\theta) \in [0,T_0]\times M\,. \quad (6.15)$$

Then, $\Gamma \in \mathscr{S}[0,T]$ *for all* $T \in (0,T_0]$ *if* $\mathscr{B}(\Gamma;[0,T]) \neq \phi$, *for all* $T \in (0,T_0]$.

Proof. Since $\mathscr{B}(\Gamma;[0,T]) \neq \phi$, for all $T \in (0,T_0]$, by Corollary 3.2, (6.7) admits at most one solution. By taking $\gamma \equiv (b_0, \sigma_0, h_0, g_0) = 0$ and $x = 0$, we see that the resulting homogeneous equation only admits the zero solution. This is equivalent to that (6.1) with the nonhomogeneous terms being zero only admits the zero solution. Hence, it is necessary that (6.2) holds. Then, by Theorem 6.2, we have $\Gamma \in \mathscr{S}[0,T]$, for all $T \in (0,T_0]$. □

Let us now look at some new class of nonlinear FBSDEs. Recall the semi-norms $\|\cdot\|_0(t)$ defined by (4.5).

Theorem 6.4. *Let $T_0 > 0$ and $\mathscr{A} \in \mathbb{R}^{(n+m)\times(n+m)}$ be given such that* (6.2) *holds. Let $T \in (0,T_0]$. Let $\Gamma \equiv (b,0,h,g)$ be defined by* (6.15). *Further, suppose that $\mathscr{B}^s(\Gamma;[0,T]) \neq \phi$. Then, there exists an $\varepsilon > 0$, such that for all $\beta \in \mathbb{R}$ and $\bar{\Gamma} \equiv (\bar{b}, \bar{\sigma}, \bar{h}, \bar{g}) \in H[0,T]$ with*

$$\|\bar{\Gamma}\|_0(t) < \varepsilon, \quad t \in [0,T], \tag{6.16}$$

the following FBSDEs:

$$\begin{aligned} d\begin{pmatrix} X(t) \\ Y(t) \end{pmatrix} &= \left\{ (\mathscr{A} + \beta I)\begin{pmatrix} X(t) \\ Y(t) \end{pmatrix} + \begin{pmatrix} \bar{b}(t,\Theta(t)) \\ \bar{h}(t,\Theta(t)) \end{pmatrix} \right\} dt \\ &\quad + \begin{pmatrix} \bar{\sigma}(t,\Theta(t)) \\ Z(t) \end{pmatrix} dW(t), \quad t \in [0,T], \\ X(0) = x, &\qquad Y(T) = GX(T) + \bar{g}(X(T)), \end{aligned} \tag{6.17}$$

admits a unique adapted solution $\Theta \equiv (X,Y,Z) \in \mathscr{M}[0,T]$.

Proof. We note that if

$$\tilde{b}(t,\theta) = e^{\beta t}\,\bar{b}(t, e^{-\beta t}\theta), \quad \forall (t,\theta) \in [0,T] \times M, \tag{6.18}$$

then,

$$\|\tilde{b}\|_0(t) = \|\bar{b}\|_0(t), \quad \forall t \in [0,T]. \tag{6.19}$$

Thus, by applying Proposition 2.8 and Theorems 4.2, 2.7 and 6.2, we obtain our conclusion immediately. □

We note that FBSDEs (6.17) is nonlinear and the Lipschitz constants of the coefficients could be large. Also, (6.17) is not necessarily decoupled nor monotone. Thus, Theorem 6.4 gives a new class of nonlinear FBSDEs which are uniquely solvable. On the other hand, by Remark 4.3, we see that condition (6.16) can be replaced by something like (4.16), or even (4.17). This further enlarges the class of FBSDEs covered by (6.17). However, the major problem left is whether we can construct the (strong) bridge for Γ defined by (6.15). In the rest of this section, we will concentrate on this issue.

We now consider the construction of the bridges for (6.7). Let $\Gamma = (b,0,h,g)$ be given by (6.15). Then, $\Phi \in \mathscr{B}^s(\Gamma;[0,T])$ if it is the solution of the following differential equation for some $K, \bar{K}, \delta, \varepsilon > 0$,

$$
\begin{aligned}
&\dot{\Phi}(t) + \mathscr{A}^T \Phi(t) + \Phi(t)\mathscr{A} = -\delta I, \quad t \in [0, T] , \\
&\Phi(0) = \begin{pmatrix} K & 0 \\ 0 & -\bar{K} \end{pmatrix},
\end{aligned}
\tag{6.20}
$$

and the following additional conditions are satisfied:

$$
\begin{aligned}
&(I,0)\Phi(t)\begin{pmatrix} I \\ 0 \end{pmatrix} \geqq 0, \quad (0,I)\Phi(t)\begin{pmatrix} 0 \\ I \end{pmatrix} \leqq -\varepsilon I, \qquad \forall t \in [0,T] , \\
&(I, G^T)\Phi(T)\begin{pmatrix} I \\ G \end{pmatrix} \geqq \varepsilon I \ .
\end{aligned}
\tag{6.21}
$$

This can be proved directly.

We can check that the solution to (6.20) is given by

$$
\Phi(t) = e^{-\mathscr{A}^T t}\begin{pmatrix} K & 0 \\ 0 & -\bar{K} \end{pmatrix} e^{-\mathscr{A} t} - \delta \int_0^t e^{-\mathscr{A}^T s} e^{-\mathscr{A} s} ds, \quad t \in [0,T] \ . \tag{6.22}
$$

Thus, in principle, if we can find constants $K, \bar{K}, \delta, \varepsilon > 0$, such that (6.21) holds, then, we obtain a strong bridge $\Phi(\cdot)$ and Theorem 6.4 applies.

6.2. A case of $n = m = d = 1$

We now, present a concrete case to illustrate the procedure of finding a strong bridge $\Phi(\cdot)$ and the corresponding class of solvable FBSDEs.

We consider the case $m = n = d = 1$ and $\Gamma = (b, 0, h, g)$ is given by

$$
\begin{pmatrix} b(t,x,y,z) \\ h(t,x,y,z) \end{pmatrix} = \mathscr{A}\begin{pmatrix} x \\ y \end{pmatrix} \equiv \begin{pmatrix} -\lambda & \mu \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}, \tag{6.23}
$$

for all $(t,x,y,z) \in [0,\infty) \times \mathbb{R}^3$, with $\lambda, \mu, g \in \mathbb{R}$ being constants satisfying the following:

$$
\lambda, \mu, g > 0, \qquad \frac{1}{2} + \frac{3g\mu}{2\lambda} - g^2 \geqq 0 \ . \tag{6.24}
$$

We point out that conditions (6.24) for the constants λ, μ, g are not necessarily the best. We prefer not to get into the most generality to avoid some complicated computation. Let us now carry out some calculations. First of all

$$
e^{\mathscr{A} t} = \begin{pmatrix} e^{-\lambda t} & \frac{\mu}{\lambda}(1 - e^{-\lambda t}) \\ 0 & 1 \end{pmatrix}. \tag{6.25}
$$

Thus, we easily see that (6.2) holds for all $T_0 > 0$. Hence, Theorem 6.4 applies. We now compute

$$
\begin{aligned}
&e^{-\mathscr{A}^T t}\begin{pmatrix} K & 0 \\ 0 & -\bar{K} \end{pmatrix} e^{-\mathscr{A} t} \\
&\quad = \begin{pmatrix} e^{\lambda t} & 0 \\ \frac{\mu}{\lambda}(1 - e^{-\lambda t}) & 1 \end{pmatrix}\begin{pmatrix} K & 0 \\ 0 & -\bar{K} \end{pmatrix}\begin{pmatrix} e^{\lambda t} & \frac{\mu}{\lambda}(1 - e^{-\lambda t}) \\ 0 & 1 \end{pmatrix} \\
&\quad = \begin{pmatrix} Ke^{2\lambda t} & \frac{K\mu}{\lambda}(e^{\lambda t} - e^{2\lambda t}) \\ \frac{K\mu}{\lambda}(e^{\lambda t} - e^{2\lambda t}) & -\bar{K} + \frac{K\mu^2}{\lambda^2}(1 - e^{\lambda t})^2 \end{pmatrix},
\end{aligned}
\tag{6.26}
$$

and

$$\int_0^t e^{-\mathscr{A}^T s} e^{-\mathscr{A} s} ds = \int_0^t \begin{pmatrix} e^{2\lambda s} & \frac{\mu}{\lambda}(e^{\lambda s} - e^{2\lambda s}) \\ \frac{\mu}{\lambda}(e^{\lambda s} - e^{2\lambda s}) & 1 + \frac{\mu^2}{\lambda^2}(1 - e^{\lambda s})^2 \end{pmatrix} ds$$

$$= \begin{pmatrix} \frac{1}{2\lambda}(e^{2\lambda t} - 1) & -\frac{\mu}{2\lambda^2}(e^{\lambda t} - 1)^2 \\ -\frac{\mu}{2\lambda^2}(e^{\lambda t} - 1)^2 & \frac{\mu^2}{2\lambda^3}(e^{\lambda t} - 2)^2 + \frac{\lambda^2 + \mu^2}{\lambda^2} t - \frac{\mu^2}{2\lambda^3} \end{pmatrix} . \tag{6.27}$$

We let $\bar{K} > 0$ be undetermined and choose

$$K = \frac{3}{4\lambda}, \qquad \delta = 1 . \tag{6.28}$$

Then, we define (note (6.22))

$$\Phi(t) = \begin{pmatrix} A(t) & B(t) \\ B(t) & C(t) \end{pmatrix}$$

with

$$\begin{aligned} A(t) &= Ke^{2\lambda t} - \frac{\delta}{2\lambda}(e^{2\lambda t} - 1) = \left(K - \frac{\delta}{2\lambda}\right) e^{2\lambda t} + \frac{\delta}{2\lambda} \\ &= \frac{1}{4\lambda}(e^{2\lambda t} + 2) , \\ B(t) &= \frac{K\mu}{\lambda}(e^{\lambda t} - e^{2\lambda t}) + \frac{\delta\mu}{2\lambda^2}(e^{\lambda t} - 1)^2 \\ &= -\frac{\mu}{\lambda}\left(K - \frac{\delta}{2\lambda}\right) e^{\lambda t} + \frac{\mu}{\lambda}\left(K - \frac{\delta}{\lambda}\right) e^{\lambda t} + \frac{\delta\mu}{2\lambda^2} \\ &= -\frac{\mu}{4\lambda^2}(e^{2\lambda t} + e^{\lambda t} - 2) . \end{aligned} \tag{6.29}$$

$$\begin{aligned} C(t) &= -\bar{K} + \frac{K\mu^2}{\lambda^2}(e^{\lambda t} - 1)^2 - \frac{\delta\mu^2}{2\lambda^3}(e^{\lambda t} - 2)^2 - \frac{\delta(\lambda^2 + \mu^2)}{\lambda^2} t + \frac{\delta\mu^2}{2\lambda^3} \\ &= \frac{\mu^2}{\lambda^2}\left(K - \frac{\delta}{2\lambda}\right) e^{2\lambda t} - \frac{2\mu^2}{\lambda^2}\left(K - \frac{\delta}{\lambda}\right) e^{\lambda t} - \frac{\mu^2}{\lambda^2}\left(K - \frac{3\delta}{2\lambda}\right) \\ &\quad -\bar{K} - \frac{\delta(\lambda^2 + \mu^2)}{\lambda^2} t \\ &= \frac{\mu^2}{4\lambda^3}(e^{2\lambda t} + 2e^{\lambda t} + 3) - \bar{K} - \frac{\lambda^2 + \mu^2}{\lambda^2} t . \end{aligned}$$

From (6.21), we need the following: ($\varepsilon > 0$ is undetermined)

$$\begin{aligned} & A(t) \geqq 0, \quad C(t) \leqq -\varepsilon, \qquad \forall t \in [0, T] , \\ & A(T) - 2gB(T) + g^2 C(T) \geqq \varepsilon . \end{aligned} \tag{6.30}$$

Let us now look at these requirements separately.

First of all, it is clearly true that $A(t) \geqq 0$ for all $t \in [0, T]$. Next, $C(t) \leqq -\varepsilon$ for all $t \in [0, T]$, if and only if

$$\bar{K} \geqq \varepsilon + \frac{\mu^2}{4\lambda^3}(e^{2\lambda t} + 2e^{\lambda t} + 3) - \frac{\lambda^2 + \mu^2}{\lambda^2} t \triangleq f(t), \quad t \in [0, T] . \tag{6.31}$$

Since $f''(t) \geqq 0$ for all $t \in [0, \infty)$, the function $f(t)$ is convex. Thus, (6.31) holds if and only if

$$\bar{K} \geqq f(0) \vee f(T)\,. \tag{6.32}$$

Finally, we need

$$\begin{aligned}
\varepsilon &\leqq A(T) - 2gB(T) + g^2C(T)\\
&= \frac{1}{4\lambda}(e^{2\lambda T} + 2) + \frac{g\mu}{2\lambda^2}(e^{2\lambda T} + e^{\lambda T} - 2) + \frac{g^2\mu^2}{4\lambda^3}(e^{2\lambda T} + 2e^{\lambda T} + 3)\\
&\quad - g^2\bar{K} - \frac{g^2(\lambda^2+\mu^2)}{\lambda^2}T\\
&= \frac{1}{4\lambda}\left(1 + \frac{g\mu}{\lambda}\right)^2 e^{2\lambda T} + \frac{g\mu}{2\lambda^2}\left(1 + \frac{g\mu}{\lambda}\right)e^{\lambda T} - \frac{g^2(\lambda^2+\mu^2)}{\lambda^2}T\\
&\quad + \frac{1}{2\lambda} - \frac{g\mu}{\lambda^2} + \frac{3g^2\mu^2}{4\lambda^3} - g^2\bar{K}\,.
\end{aligned} \tag{6.33}$$

Thus, we need (note (6.32))

$$\begin{aligned}
F(T) &\overset{\Delta}{=} \frac{1}{4\lambda}\left(1 + \frac{g\mu}{\lambda}\right)^2 e^{2\lambda T} + \frac{g\mu}{2\lambda^2}\left(1 + \frac{g\mu}{\lambda}\right)e^{\lambda T} - \frac{g^2(\lambda^2+\mu^2)}{\lambda^2}T\\
&\quad + \frac{1}{2\lambda} - \frac{g\mu}{\lambda^2} + \frac{3g^2\mu^2}{4\lambda^3} - \varepsilon\\
&\geqq g^2\bar{K} \geqq g^2(f(0) \vee f(T))\,.
\end{aligned} \tag{6.34}$$

We now separate two cases (with $f(T)$ and $f(0)$, respectively). First of all, for $f(T)$, we want

$$\begin{aligned}
0 &\leqq F(T) - g^2 f(T)\\
&= \frac{1}{4\lambda}\left(1 + \frac{2g\mu}{\lambda}\right)e^{2\lambda T} + \frac{g\mu}{2\lambda^2}e^{\lambda T} + \frac{1}{2\lambda} - \frac{g\mu}{\lambda^2} - \varepsilon(1+g^2) \overset{\Delta}{=} \hat{F}(T)\,.
\end{aligned} \tag{6.35}$$

We see that $T \mapsto \hat{F}(T)$ is monotone increasing. Thus, to have the above, it suffices to have

$$0 \leqq \hat{F}(0) = \frac{3}{4\lambda} - \varepsilon(1+g^2)\,. \tag{6.36}$$

Hence, in what follows, we take

$$\varepsilon = \frac{3}{4\lambda(1+g^2)}\,. \tag{6.37}$$

Then, (6.35) holds. Next, we claim that under (6.24) and (6.37), the following holds.

$$F(T) - g^2 f(0) \geqq 0\,. \tag{6.38}$$

In fact, by the choice of ε and by (6.36),

$$F(0) - g^2 f(0) = \hat{F}(0) = 0\,. \tag{6.39}$$

On the other hand,

$$F'(T) = \frac{1}{2}\left(1+\frac{g\mu}{\lambda}\right)^2 e^{2\lambda T} + \frac{g\mu}{2\lambda}\left(1+\frac{g\mu}{\lambda}\right)e^{\lambda T} - \frac{g^2(\lambda^2+\mu^2)}{\lambda^2}\,. \qquad (6.40)$$

Thus, by (6.24), it follows that

$$F'(0) = \left(\frac{1}{2} + \frac{3g\mu}{2\lambda} - g^2\right) \geqq 0\,. \qquad (6.41)$$

Then, by $F''(T) \geqq 0$, together with (6.39) and (6.41), we must have (6.38). Hence, we obtain (6.34). This shows that a strong bridge $\Phi(t)$ has been constructed with K, δ and ε being given by (6.28) and (6.37), respectively, and we may take

$$\bar{K} = f(0) \vee f(T)\,. \qquad (6.42)$$

It is interesting that the $\Phi(\cdot)$ constructed in the above is not in $\mathscr{B}(\Gamma_0;[0,T])$ for any $T>0$ since $\dot{A}(t)>0$. On the other hand, we note that both $A(t)$ and $B(t)$ are independent of T. However, due to the fact that $\bar{K}$ depending on T, $C(t)$ depends on T. But, we claim that there exists a constant $c_0>0$, only depending on λ, μ, g (independent of T), such that

$$\begin{aligned} &-c_0 - f(T) \leqq C(t) \leqq -\frac{3}{4\lambda(1+g^2)}, \quad t\in[0,T]\,,\\ &-c_0 \leqq C(T) \leqq -\frac{3}{4\lambda(1+g^2)}\,, \end{aligned} \qquad (6.43)$$

where $f(t)$ is defined by (6.31). In fact, by (6.29), (6.31), (6.37) and (6.42), we have

$$C(t) = f(t) - f(0)\vee f(T) - \frac{3}{4\lambda(1+g^2)}\,. \qquad (6.44)$$

Clearly, $C(t)$ is convex. Thus,

$$C(t) \leqq C(0)\vee C(T) = -\frac{3}{4\lambda(1+g^2)}, \quad \forall t\in[0,T]\,. \qquad (6.45)$$

On the other hand, by the fact that $f(t)$ is strictly convex and $\lim_{t\to\infty} f(t) = \infty$, we see that there exists a unique $T_0>0$, only depending on λ and μ, such that

$$C(t) \geqq f(T_0) - f(0)\vee f(T) - \frac{3}{4\lambda(1+g^2)}, \quad t\in[0,T]\,. \qquad (6.46)$$

This proves the first relation in (6.43). Next, we see easily that there exists a unique $T_1 > T_0$, such that $f(T_1) = f(0)$, and

$$\begin{aligned} &f(t) \leqq f(0), \quad \forall t\in[0,T_1]\,,\\ &f(t) > f(0), \quad \forall t\in(T_1,\infty)\,. \end{aligned} \qquad (6.47)$$

Hence, we obtain

$$C(T) \geqq f(T_0) - f(0) - \frac{3}{4\lambda(1+g^2)}\,. \qquad (6.48)$$

This proves the second relation in (6.43).

Now, from Remark 4.3 and Theorem 6.4, we know that the following FBSDEs is solvable on $[0,T]$.

$$\begin{aligned} dX(t) &= \{(\beta-\lambda)X(t)+\mu Y(t)+\bar{b}(t,X(t),Y(t),Z(t))\}\,dt \\ &\qquad + \bar{\sigma}(t,X(t),Y(t),Z(t))\,dW(t)\,, \\ dY(t) &= \{\beta Y(t)+\bar{h}(t,X(t),Y(t),Z(t))\}\,dt + Z(t)\,dW(t)\,, \\ X(0) &= x, \qquad Y(T) = -gX(T)+\bar{g}(X(T))\,, \end{aligned} \tag{6.49}$$

where $\lambda,\mu,g>0$ satisfying (6.24), $\beta\in\mathbb{R}$, and $\bar{\Gamma}\equiv(\bar{b},\bar{\sigma},\bar{h},\bar{g})\in H[0,T]$ satisfying

$$\begin{aligned} &2|B(T)|\|\bar{g}\|_0+|C(T)|\|\bar{g}\|_0^2<\varepsilon\wedge 1\,, \\ &\sup_{t\in[0,T]}\{2(|A(t)|+|B(t)|)\|\bar{b}\|_0(t)+2(|B(t)|+|C(t)|)\|\bar{h}\|_0(t) \\ &\qquad +2|B(t)|\|\bar{\sigma}\|_0(t)+|A(t)|\|\bar{\sigma}\|_0(t)^2\}<\varepsilon\wedge 1\,, \end{aligned} \tag{6.50}$$

with $A(\cdot),B(\cdot)$ and $C(\cdot)$ being given by (6.29) and $\varepsilon>0$ being given by (6.37). If we use (4.17), then, (6.50) can be relaxed to the following:

$$\begin{aligned} &2B(T)\hat{x}\,\hat{\bar{g}}+C(T)(\hat{\bar{g}}-2g\hat{x})\hat{\bar{g}}>-(\varepsilon\wedge 1)|\hat{x}|^2, \quad \forall x,\bar{x}\in\mathbb{R}\,, \\ &\sup_{t\in[0,T]}\{2(A(t)\hat{x}+B(t)^T\hat{y})\hat{\bar{b}}+2(B(t)\hat{x}+C(t)\hat{y})\hat{\bar{h}} \\ &\qquad +2B(t)\hat{z}\,\hat{\bar{\sigma}}+A(t)\hat{\bar{\sigma}}^2\}<(\varepsilon\wedge 1)|\hat{\theta}|^2, \quad \forall\theta,\bar{\theta}\in M\,. \end{aligned} \tag{6.51}$$

If $\bar{b},\bar{\sigma},\bar{h}$ and $\bar{g}$ are differentiable, then, we see that (6.51) is equivalent to the following:

$$\begin{aligned} &2B(T)\bar{g}_x(x)+C(T)(\bar{g}_x(x)-2g)\bar{g}_x(x)>-(\varepsilon\wedge 1), \quad \forall x\in\mathbb{R}\,, \\ &\left\{\begin{pmatrix} A(t) & B(t) & 0 \\ B(t) & C(t) & 0 \\ 0 & 0 & B(t)\end{pmatrix}(\nabla\bar{b}(t,\theta),\nabla\bar{h}(t,\theta),\nabla\bar{\sigma}(t,\theta))\right. \\ &\quad + \left\{\begin{pmatrix} A(t) & B(t) & 0 \\ B(t) & C(t) & 0 \\ 0 & 0 & B(t)\end{pmatrix}(\nabla\bar{b}(t,\theta),\nabla\bar{h}(t,\theta),\nabla\bar{\sigma}(t,\theta))\right\}^T \\ &\quad \left. + A(t)\nabla\bar{\sigma}(t,\theta)\{\nabla\bar{\sigma}(t,\theta)\}^T\right\}<\varepsilon\wedge 1, \quad \forall(t,\theta)\in[0,T]\times M\,, \end{aligned} \tag{6.52}$$

where $\nabla\bar{b}(t,\theta)=(\bar{b}_x(t,\theta),\bar{b}_y(t,\theta),\bar{b}_z(t,\theta))^T$, and so on. Some direct computation shows that the first relation in (6.52) is equivalent to the following:

$$\begin{aligned} -r(T) &\stackrel{\Delta}{=} -\sqrt{\frac{\varepsilon\wedge 1}{|C(T)|}+\left(\frac{B(T)}{C(T)}-g\right)^2}-\frac{B(T)}{C(T)}+g \leqq \bar{g}_x(x) \\ &< \sqrt{\frac{\varepsilon\wedge 1}{|C(T)|}+\left(\frac{B(T)}{C(T)}-g\right)^2}-\frac{B(T)}{C(T)}+g, \quad \forall x\in\mathbb{R}\,. \end{aligned} \tag{6.53}$$

By (6.43), we know that $C(T)$ is bounded uniformly in T, while, $B(T) \to -\infty$ as $T \to \infty$ (see (6.29)). Thus, by some calculation, we see that

$$-\sqrt{\frac{\varepsilon \wedge 1}{|C(T)|}} \geqq -r(T) \downarrow -\infty, \quad \text{as } T \to \infty\,, \tag{6.54}$$

and $\bar{g}$ need only to satisfy the following:

$$-r(T) \leqq \bar{g}_x(x) \leqq 0, \quad \forall T \in \mathbb{R}\,. \tag{6.55}$$

Clearly, the larger the T, the weaker the restriction of (6.55). The second condition in (6.52) is also checkable (although it is a little more complicated than the first one). It is not hard to see that the choice of functions $\bar{b}$ and $\bar{\sigma}$ are independent of T as $A(t)$ and $B(t)$ do not depend on T. However, since $C(t)$ depends on T, by some direct calculation, we see that in order FBSDEs (6.49) is solvable for all $T > 0$, we have to restrict ourselves to the case that $\bar{h}(t,\theta) = \bar{h}(t,y)$. Clearly, even with such a restriction, (6.49) is still a very big class of FBSDEs, which are not necessarily decoupled, nor monotone. Also, $\bar{\sigma}$ is allowed to be degenerate. We omit the exact statement of the explicit conditions on $\bar{b}, \bar{\sigma}$ and $\bar{h}$ under which (6.49) is solvable to avoid some lengthy computation. Instead, to conclude our discussion, let us finally look at the following FBSDEs:

$$\begin{aligned} dX(t) &= \{(\beta-\lambda)X(t) + \mu Y(t) + \bar{b}(t,X(t),Y(t),Z(t))\}\,dt \\ &\qquad + \bar{\sigma}(t,X(t),Y(t),Z(t))\,dW(t)\,, \\ dY(t) &= \{\beta Y(t) + h_0(t)\}\,dt + Z(t)\,dW(t)\,, \\ X(0) &= x, \qquad Y(T) = -gX(T) + g_0\,, \end{aligned} \tag{6.56}$$

with $\lambda, \mu, g > 0$ satisfying (6.24) and

$$\sup_{t\in[0,\infty)} \{2(|A(t)| + |B(t)|)\|\bar{b}\|_0(t) + 2|B(t)|\|\bar{\sigma}\|_0(t) + |A(t)|\|\bar{\sigma}\|_0(t)^2\} < \varepsilon \wedge 1\,. \tag{6.57}$$

This is a special case of (6.49) in which $\bar{h} \equiv h_0$ and $\bar{g} \equiv g_0$. Then, by the above analysis, we know that (6.56) is uniquely solvable over any finite time duration $[0,T]$. Condition (6.57) can be carried out explicitly as follows:

$$\begin{aligned} &\{2(e^{2\lambda t}+2) + \frac{2\mu^2}{\lambda}(e^{2\lambda t} + e^{\lambda t} - 2)\}\|\bar{b}\|_0(t) + \frac{2\mu^2}{\lambda}(2^{2\lambda t} + e^{\lambda t} - 2)\|\bar{\sigma}\|_0(t) \\ &\quad + (e^{2\lambda t}+2)\|\bar{\sigma}\|_0(t)^2 < \min\left\{4\lambda, \frac{3}{1+g^2}\right\}, \quad t \in [0,\infty)\,. \end{aligned} \tag{6.58}$$

It is clear that although (6.56) is a special case of (6.49), it is still very general and in particular, it is not necessarily decoupled nor monotone. Also, if we regard (6.56) as a nonlinear perturbation of (6.1) (with $m = n = d = 1$ and (6.23) holds), then the perturbation is not necessarily small (for t not large).

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