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Probability Theory and Related Fields

A central limit theorem for "critical" first-passage percolation in two dimensions

Harry Kesten[1,*], **Yu Zhang**[2,**]

[1] Department of Mathematics, White Hall, Cornell University, Ithaca, NY 14853, USA
2 (e-mail: kesten@math.cornell.edu)
[2] Department of Mathematics, University of Colorado, Colorado Springs, CO 80933, USA
(e-mail: yzhang@vision.uccs.edu)

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Summary. Consider (independent) first-passage percolation on the edges of $\mathbb{Z}^2$. Denote the passage time of the edge e in $\mathbb{Z}^2$ by $t(e)$, and assume that $P\{t(e) = 0\} = 1/2$, $P\{0 < t(e) < C_0\} = 0$ for some constant $C_0 > 0$ and that $E[t^\delta(e)] < \infty$ for some $\delta > 4$. Denote by $b_{0,n}$ the passage time from $\mathbf{0}$ to the halfplane $\{(x, y)\colon x \geqq n\}$, and by $T(\mathbf{0}, nu)$ the passage time from $\mathbf{0}$ to the nearest lattice point to nu, for u a unit vector. We prove that there exist constants $0 < C_1$, $C_2 < \infty$ and γ_n such that $C_1(\log n)^{1/2} \leqq \gamma_n \leqq C_2(\log n)^{1/2}$ and such that $\gamma_n^{-1}[b_{0,n} - Eb_{0,n}]$ and $(\sqrt{2}\gamma_n)^{-1}[T(\mathbf{0}, nu) - ET(\mathbf{0}, nu)]$ converge in distribution to a standard normal variable (as $n \to \infty$, u fixed).

A similar result holds for the site version of first-passage percolation on $\mathbb{Z}^2$, when the common distribution of the passage times $\{t(v)\}$ of the vertices satisfies $P\{t(v) = 0\} = 1 - P\{t(v) \geqq C_0\} = p_c(\mathbb{Z}^2, \text{site}) :=$ critical probability of site percolation on $\mathbb{Z}^2$, and $E[t^\delta(u)] < \infty$ for some $\delta > 4$.

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1. Introduction

Let $\mathscr{E}$ be the set of edges between nearest neighbors on $\mathbb{Z}^2$, and let $\{t(e)\colon e \in \mathscr{E}\}$ be an i.i.d. family of positive random variables with common distribution function F. For any path π on $\mathbb{Z}^2$, which successively traverses the edges $e_1, \ldots, e_p$ we define the *passage time of* π as

$$T(\pi) = \sum_{i=1}^{p} t(e_i) . \tag{1.1}$$

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The passage time between two vertex sets $A, B \subset \mathbb{Z}^2$ is defined as

$$T(A,B) = \inf\{T(\pi)\colon\ \pi \text{ a path connecting some vertex of } A \text{ with some vertex of } B\}\ . \tag{1.2}$$

Standard first-passage percolation (see Hammersley and Welsh 1965; Smythe and Wierman 1978; Kesten 1986, 1987) studies among other quantities the asymptotic behavior of

$$a_{0,n} := T(\mathbf{0}, n\xi_1) \quad \text{and} \quad b_{0,n} := T(\mathbf{0}, H_n)\ ,$$

where ξ_1 is the first coordinate vector and H_n is the halfplane

$$H_n = \{(x,y)\colon\ x \geqq n\}\ . \tag{1.3}$$

$a_{0,n}$ is called a *point to point passage time* and $b_{0,n}$ a *point to line passage time*. As a generalization of $a_{0,n}$ one also considers the point to point passage time $T(\mathbf{0}, nu)$ which (with some abuse of notation) is the passage time from $\{\mathbf{0}\}$ to the nearest point on $\mathbb{Z}^2$ to nu, for any unit vector u. If several points of $\mathbb{Z}^2$ minimize the distance to nu, then we take $T(\mathbf{0}, nu) = T(\mathbf{0}, A)$ with A equal to the set of vertices of $\mathbb{Z}^2$ with minimal distance to nu.

In this paper we shall restrict ourselves to what we call *critical* first-passage percolation, that is, we assume that

$$F(0) = P\{t(e) = 0\} = p_c(\mathbb{Z}^2, \text{bond}) = \tfrac{1}{2}\ . \tag{1.4}$$

Here $p_c(\mathbb{Z}^2,\text{bond})$ stands for the critical probability for bond percolation, which is known to equal 1/2 (see Kesten (1982, Chap. 3) or Grimmett (1989, Chap. 9)). It is appropriate to call this "critical" first-passage percolation because there is a transition in the behavior of the passage times at $F(0) = p_c$. For $F(0) < p_c$ (and mild additional conditions on F), $a_{0,n}/n$ and $b_{0,n}/n$ converge almost surely to a strictly positive constant, while for $F(0) > p_c$, the families $\{a_{0,n}\}$ and $\{b_{0,n}\}$ are tight (see Kesten (1986, Theorem 6.1) and Zhang and Zhang (1984)). For $F(0) = p_c$, $\{a_{0,n}\}$ and $\{b_{0,n}\}$ have an intermediate behavior (see Remark (i) below). In addition to (1.4) we shall assume that

$$E[t^\delta(e)] < \infty \quad \text{for some } \delta > 4 \tag{1.5}$$

and that there exists a $C_0 > 0$ such that

$$P\{0 < t(e) < C_0\} = 0\ , \tag{1.6}$$

so that the possible strictly positive values of $t(e)$ are bounded away from 0. Our principal result is the following central limit theorem for $b_{0,n}$, $a_{0,n}$, and more generally, $T(\mathbf{0}, nu)$.

Theorem. *If* (1.4)–(1.6) *hold, then there exist constants* $0 < C_1,\ C_2 < \infty$ *and* γ_n *such that*

$$C_1(\log n)^{1/2} \leqq \gamma_n \leqq C_2(\log n)^{1/2}, \quad n \geqq 2, \tag{1.7}$$

and such that

$$\frac{b_{0,n} - Eb_{0,n}}{\gamma_n} \to N(0,1) \quad \textit{in distribution}, \tag{1.8}$$

and for each unit vector u

$$\frac{T(\mathbf{0}, nu) - ET(\mathbf{0}, nu)}{\sqrt{2}\gamma_n} \to N(0,1) \quad \textit{in distribution}, \tag{1.9}$$

where $N(0,1)$ *is a standard normal variable* (*with mean* 0 *and variance* 1).

Remarks. (i) Chayes et al. (1986) proved that in the special case when

$$P\{t(e) = 0\} = P\{t(e) = 1\} = \tfrac{1}{2},$$

there exists constants $0 < C_3,\ C_4 < \infty$ for which

$$C_3 \log n \leqq Eb_{0,n} \leqq C_4 \log n,$$

and for each unit vector u

$$C_3 \log n \leqq ET(\mathbf{0}, nu) \leqq C_4 \log n.$$

It is expected that even (under (1.4)–(1.6) only)

$$\frac{Eb_{0,n}}{\log n} \quad \text{and} \quad \frac{ET(\mathbf{0}, nu)}{\log n} \tag{1.10}$$

converge to finite, strictly positive limits (as $n \to \infty$). One also expects that

$$\frac{\gamma_n^2}{\log n}, \qquad \frac{\mathrm{Var}(b_{0,n})}{\log n} \quad \text{and} \quad \frac{\mathrm{Var}(T(\mathbf{0}, nu))}{\log n} \tag{1.11}$$

converge to finite, strictly positive limits. Possibly the semi-explicit expressions for γ_n (see (2.63) below) will eventually help to prove that the expressions in (1.11) have limits.

(ii) In the course of proving the theorem we also prove a central limit theorem for a related quantity. Let

$$S(n) = [-n, n]^2 \tag{1.12}$$

be the square of size $2n$ centered at $\mathbf{0}$ and $\partial S(n)$ its boundary. Further define $c_n = T(\mathbf{0}, \partial S(n))$. Then we prove that

$$\frac{c_n - Ec_n}{\gamma_n} \to N(0,1) \quad \text{in distribution}. \tag{1.13}$$

The estimates which we develop can also be used to prove a strong law of large numbers for $b_{0,n}$: If (1.4) and (1.6) hold, and

$$E[t^{\delta}(e)] < \infty \quad \text{for some } \delta > 2, \tag{1.14}$$

then

$$\frac{b_{0,n}}{Eb_{0,n}} \to 1 \quad \text{w.p. } 1 . \tag{1.15}$$

We shall not give this proof here, though.
(iii) It is likely that conditions (1.5) and (1.6) can be weakened somewhat. However, (1.4) and (1.5) by themselves are not sufficient to guarantee the central limit theorems (1.8) and (1.9), even when the $t(e)$ are bounded above; one needs some condition which prevents the occurrence of many very small (but strictly positive) $t(e)$. In fact, Zhang (1995) shows that the common distribution F of the $t(e)$ can be chosen such that (1.4) holds and such that F has compact support, but such that there exists with probability 1 an infinite path whose total passage time is finite. It is not hard to see that for such F,

$$b_{0,n} \text{ is bounded with probability } 1 , \tag{1.16}$$

and

$$\{T(\mathbf{0}, nu)\}_{n \geqq 1} \text{ is a tight family} . \tag{1.17}$$

Thus (1.8) and (1.9) fail for such an F.
(iv) A similar central limit theorem holds in the site model for first-passage percolation on $\mathbb{Z}^2$. One now assigns i.i.d. passage times to the *vertices* of $\mathbb{Z}^2$. If $t(v)$ denotes the passage time of the vertex v, and π is a path on $\mathbb{Z}^2$ containing the vertices $v_0, \ldots, v_p$, then one replaces $T(\pi)$ in (1.1) by

$$T(\pi) = \sum_{i=1}^{p} t(v_i) .$$

The other definitions need no change. If now

$$F(0) = P\{t(v) = 0\} = p_c(\mathbb{Z}^2, \text{site})$$

(the critical probability for site percolation on $\mathbb{Z}^2$),

$$E[t^{\delta}(v)] < \infty \quad \text{for some } \delta > 4$$

and

$$P\{0 < t(v) < C_0\} = 0 ,$$

then (1.7)–(1.9) again hold. We shall restrict ourselves here to the bond problem, but no significant changes in the proof are necessary to treat the site problem.

Idea of the Proof. We give here an outline of our proof for $b_{0,n}$. Following percolation terminology we call an edge *e open* if $t(e) = 0$, and *closed* if $t(e) \geqq C_0$. A *circuit surrounding* $\mathbf{0}$ is a simple closed curve C consisting of edges of $\mathbb{Z}^2$, and separating $\mathbf{0}$ from ∞ (that is, such that each continuous path from $\mathbf{0}$ to ∞ must intersect C). It has been known since Harris (1960), that (1.4) implies that with probability 1 there exist infinitely many open circuits surrounding $\mathbf{0}$. Denote by $\mathscr{C}_p$ the "innermost" open circuit surrounding $\mathbf{0}$ which lies outside $S(2^p)$. (See the next section for a detailed definition of "innermost" and of the actual $\mathscr{C}_p$ used in the proof; the above definition will be slightly modified.) Then any two vertices v' and v'' on $\mathscr{C}_p$ are connected by a path which is part of $\mathscr{C}_p$ and has zero passage time (because $\mathscr{C}_p$ is open). Therefore $T(\mathbf{0}, v)$ has the same value for every v on $\mathscr{C}_p$. For similar reasons

$$T(\mathbf{0}, \mathscr{C}_q) = \sum_{p=0}^{q} T(\mathscr{C}_{p-1}, \mathscr{C}_p) \tag{1.18}$$

(where we took $\mathscr{C}_{-1} = \{\mathbf{0}\}$). If

$$2^{q-1} < n \leqq 2^q \, , \tag{1.19}$$

then $b_{0,n}$ will be shown to be well approximated by $T(0, \mathscr{C}_q)$ (see (2.69)). Now it is also well known that for a given circuit C surrounding $\mathbf{0}$ and lying outside $S(2^p)$, the event $\{\mathscr{C}_p = C\}$ depends only on the $t(e)$ with $e \in \overline{C} \setminus S(2^p)$, where

$$\overline{C} = C \cup \text{ interior of } C \, . \tag{1.20}$$

In fact, once $\mathscr{C}_{p-1}$ is given, one does not need to know the $t(e)$ for any e in the interior of $\mathscr{C}_{p-1}$ to determine $\mathscr{C}_p$. From this it is not hard to obtain that the random variables $\{(T(\mathscr{C}_{p-1}, \mathscr{C}_p), \mathscr{C}_p)\}_{p \geqq 0}$ form a Markov chain. Moreover, as we shall essentially prove, the family $\{T(\mathscr{C}_{p-1}, \mathscr{C}_p)\}_{p \geqq 0}$ is tight (see the lines following (2.54)). This Markov property is the intuitive reason why $b_{0,n} - Eb_{0,n}$ has a normal limit distribution. Unfortunately, the above Markov chain is not recurrent (since $\mathscr{C}_p$ lies outside $S(2^p)$), so the central limit theorems for Markov chains which we know do not apply. We therefore base our proof on another representation for $b_{0,n} - Eb_{0,n}$ as a sum of martingale differences. To this end we define

$$\mathscr{F}_p = \sigma\text{-field generated by } \mathscr{C}_p \text{ and } \{t(e) \colon e \in \overline{\mathscr{C}_p}\} \, . \tag{1.21}$$

(but as stated, the $\mathscr{C}_p$ will be redefined slightly). $\mathscr{F}_p$ contains unions of sets of the form $\{\mathscr{C}_p = C, (t(e_1), \ldots, t(e_k)) \in B\}$ for C a circuit surrounding $\mathbf{0}$ outside $S(2^p)$, and $e_1, \ldots, e_k \in \overline{C}$, B a k-dimensional Borel set. Then

$$b_{0,n} - Eb_{0,n} = \sum_{p=0}^{q} [E\{b_{0,n} \mid \mathscr{F}_p\} - E\{b_{0,n} \mid \mathscr{F}_{p-1}\}] \, . \tag{1.22}$$

The variables

$$\zeta_p := E\{b_{0,n} \mid \mathscr{F}_p\} - E\{b_{0,n} \mid \mathscr{F}_{p-1}\}$$

are clearly martingale differences, and will be seen to be closely related to the $T(\mathscr{C}_{p-1}, \mathscr{C}_p)$ (see Lemma 2), and we will prove (in Lemma 1) that truncated versions of ζ_{p_1} and ζ_{p_2} are independent for $|p_1 - p_2|$ large. This will allow us to apply an existing central limit theorem for martingales (see McLeish 1974) to obtain (1.8).

2. Proof of Theorem

For the time being we shall assume that n is a power of 2, say

$$n = 2^q \, . \tag{2.1}$$

Later we show how to deal with general n. It also simplifies some estimates to restrict the $\mathscr{C}_p$ to certain annuli. This is done as follows. With $S(n)$ as in (1.12) define

$$A(p) = S(2^{p+1}) \setminus S(2^p), \quad p \geqq 0 , \tag{2.2}$$

the annulus between $S(2^{p+1})$ and $S(2^p)$. Next define for $p \geqq 0$

$$m(p) = \inf\{t \in \{p, p+1, \dots\} : A(t) \text{ contains an open circuit surrounding } \mathbf{0}\} . \tag{2.3}$$

Note that $m(p) \geqq p$, but that we can have $m(p) = m(p') \geqq p'$ for some $p' > p$. This happens precisely when

$$\text{there is no open circuit surrounding } \mathbf{0} \text{ in any of the annuli } A(p), A(p+1), \dots, A(p'-1) . \tag{2.4}$$

Now, when C is a simple closed curve in $\mathbb{R}^2$, $\mathrm{int}(C)$, the *interior of* C, is the bounded component of $\mathbb{R}^2 \setminus C$. *The exterior of* C, $\mathrm{ext}(C)$, will be the unbounded component of $\mathbb{R}^2 \setminus C$. In particular for $\mathscr{C}$ a circuit on $\mathbb{Z}^2$ which surrounds $\mathbf{0}$, $\mathrm{int}(\mathscr{C})$ is defined. If $\{C_\gamma\}_{\gamma\in\Gamma}$ is some collection of circuits on $\mathbb{Z}^2$ which surround $\mathbf{0}$, then the *innermost circuit* of this family is that circuit C_{γ_0} in the family for which $\mathrm{int}(C_\gamma)$ is minimal, that is for which

$$\mathrm{int}(C_{\gamma_0}) \subset \mathrm{int}(C_\gamma) \quad \text{for all } \gamma \in \Gamma . \tag{2.5}$$

Of course not every family $\{C_\gamma\}$ has an innermost circuit. However, if for some fixed m, $\{C_\gamma\}_{\gamma\in\Gamma}$ is the collection of all open circuits in $A(m)$ which surround $\mathbf{0}$, and if this collection is nonempty, then it has an innermost circuit. This can be seen by the arguments in Harris (1960) or the Appendix of Kesten (1980) or Proposition 2.3 in Kesten (1982). Thus we can define

$$\mathscr{C}_p = \text{innermost open circuit surrounding } \mathbf{0} \text{ in } A(m(p)), \quad p \geqq 0 , \tag{2.6}$$

and $\mathscr{C}_{-1} = \{\mathbf{0}\}$. This is the definition we shall use, rather than the temporary one at the end of the Introduction. The $\mathscr{C}_p$ of (2.6) are fairly well constrained. They all have to belong to one of the annuli $A(m)$, or in particular (by virtue of (2.1)) each $\mathscr{C}_p$ either lies entirely in $S(n) = S(2^q)$ or entirely outside $S(n)$. Moreover, by definition $p_1 \leqq p_2$ implies $m(p_1) \leqq m(p_2)$ and therefore either $m(p_1) = m(p_2)$ and $\mathscr{C}_{p_1} = \mathscr{C}_{p_2}$, or

$$m(p_1) < m(p_2) \quad \text{and} \quad \mathscr{C}_{p_1} \subset A(m(p_1)) \subset S(m(p_2)) \subset \mathrm{int}(\mathscr{C}_{p_2}) . \tag{2.7}$$

For the $\mathscr{C}_p$ of (2.6) we define

$$\overline{\mathscr{C}}_p = \mathscr{C}_p \cup \mathrm{int}(\mathscr{C}_p)$$

and then the σ-fields $\mathscr{F}_p$ as in (1.21). Note that $\mathscr{F}_{-1}$ is the trivial σ-field, because $\mathscr{C}_{-1} = \{\mathbf{0}\}$ and $\mathrm{int}(\mathscr{C}_{-1}) = \emptyset$. Furthermore, we shall use v for $m(q)$, with q as in (2.1). Thus v will depend on n, and so will $\mathscr{C}_v$ and Δ_p below. Rather than prove the central limit theorem for $b_{0,n}$ directly, we first prove that $T(\mathbf{0}, \mathscr{C}_v) - ET(\mathbf{0}, \mathscr{C}_v)$ (suitably normalized) has a normal limit law. To this end we observe that $\mathscr{C}_v$ and $T(\mathbf{0}, \mathscr{C}_v)$ are $\mathscr{F}_q$-measurable. Moreover, by (2.7),

$\mathscr{C}_{p-1} \subset \overline{\mathscr{C}}_p$, and therefore $\mathscr{F}_{p-1} \subset \mathscr{F}_p$. Thus

$$T(\mathbf{0}, \mathscr{C}_v) - ET(\mathbf{0}, \mathscr{C}_v) = \sum_{p=0}^{q} [E\{T(\mathbf{0}, \mathscr{C}_v) \mid \mathscr{F}_p\} - E\{T(\mathbf{0}, \mathscr{C}_v) \mid \mathscr{F}_{p-1}\}]$$

$$= \sum_{p=0}^{q} \Delta_p , \tag{2.8}$$

with

$$\Delta_p = \Delta_{p,q} = E\{T(\mathbf{0}, \mathscr{C}_v) \mid \mathscr{F}_p\} - E\{T(\mathbf{0}, \mathscr{C}_v) \mid \mathscr{F}_{p-1}\} . \tag{2.9}$$

Clearly, $E\{\Delta_p \mid \mathscr{F}_{p-1}\} = 0$, so that the right hand side of (2.8) is a sum of martingale differences, and we are going to apply a central limit theorem for martingales to this. We shall use the version of McLeish (1974).

To apply the theorem of McLeish we are going to show that Δ_p depends only on $\mathscr{C}_{p-1}$, $\mathscr{C}_p$ and the $t(e)$ with $e \subset \overline{\mathscr{C}}_p \setminus \mathrm{int}(\mathscr{C}_{p-1})$; this will give us a certain amount of independence between the Δ_p. In addition we have to estimate the tail of the distribution of Δ_p. To do this, we have to become a bit more specific about our probability space. Without loss of generality we take this to be

$$\Omega = \prod_{e \in \mathscr{E}} [0, \infty)$$

with the σ-field $\mathscr{B}$ generated by the cylinder sets. A typical point of Ω is denoted by $\omega = \{\omega_e\}_{e \in \mathscr{E}}$ and $t(e) = t(e, \omega)$ is the e-th coordinate function, that is $t(e, \omega) = \omega_e$. The probability measure P on Ω is

$$\prod_{e \in \mathscr{E}} \mu_{F,e} ,$$

where each $\mu_{F,e}$ is the measure on $[0, \infty)$ with distribution function F. E denotes the expectation operator with respect to P. It will also be necessary to introduce a copy $(\Omega', \mathscr{B}', P')$ of $(\Omega, \mathscr{B}, P)$. The generic element of Ω' is denoted by ω' and expectation with respect to P' is denoted by E'. When necessary, we give an argument ω or ω' to our random variables, to make clear that we regard them as a function on Ω or Ω'. For instance, $\mathscr{C}_p(\omega)$ is the circuit $\mathscr{C}_p$ in the configuration $\{t(e, \omega)\}_{e \in \mathscr{E}}$. Unfortunately this leads to some cumbersome expressions, but this seems unavoidable. For instance to determine

$$T(\mathscr{C}_{p-1}(\omega), \mathscr{C}_{v(\omega')}(\omega'))(\omega') \tag{2.10}$$

in (2.11) below one first determines $\mathscr{C}_{p-1}(\omega)$ in the configuration ω, and then $v(\omega')$ and $\mathscr{C}_{v(\omega')}(\omega')$ in the configuration ω'. With the two circuits $\mathscr{C}_{p-1}(\omega)$ and $\mathscr{C}_{v(\omega')}(\omega')$ fixed, (2.10) is the passage time between them in the configuration ω'.

Lemma 1

$$\Delta_p(\omega) = T(\mathscr{C}_{p-1}(\omega), \mathscr{C}_p(\omega))(\omega) + E'T(\mathscr{C}_p(\omega), \mathscr{C}_{v(\omega')}(\omega'))(\omega')$$

$$- E'T(\mathscr{C}_{p-1}(\omega), \mathscr{C}_{v(\omega')}(\omega'))(\omega') . \tag{2.11}$$

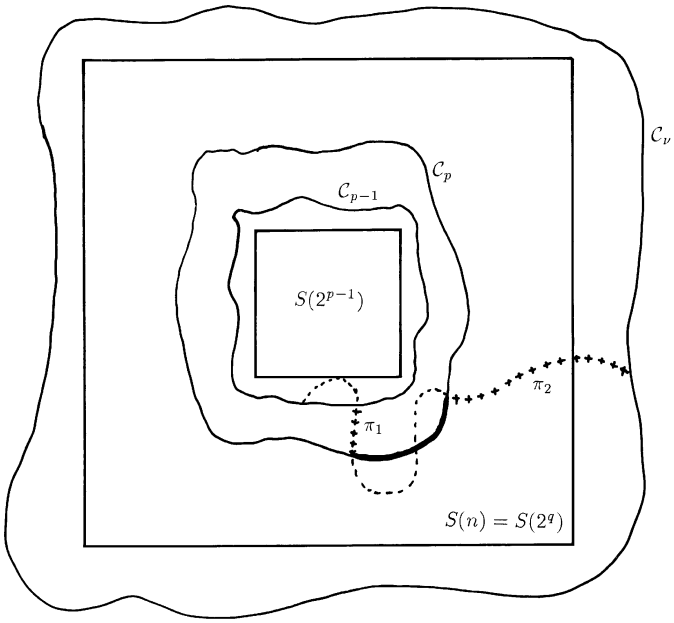

Fig. 1. Illustration of $\mathscr{C}_{p-1}$, $\mathscr{C}_p$, $\mathscr{C}_\nu$ and a path π from $\mathscr{C}_{p-1}$ to $\mathscr{C}_\nu$. π is the path indicated as $---$ or $+++$. The pieces indicated by $+++$ are π_1 and π_2. The boldly drawn arc of $\mathscr{C}_p$ connects π_1 to π_2 at no cost in passage time.

Moreover, $\Delta_p(\omega)$ *depends only on* $\mathscr{C}_{p-1}(\omega)$, $\mathscr{C}_p(\omega)$ *and* $\{t(e,\omega)\colon e \in \overline{\mathscr{C}}_p(\omega) \setminus \mathrm{int}(\mathscr{C}_{p-1}(\omega))\}$. *Finally, if* $g(\cdot)$ *is a deterministic function such that* $g(p) \geqq p$, *then the random variables*

$$\Delta_{p_i} I[m(p_i) \leqq g(p_i)], \quad 1 \leqq i \leqq \ell\,, \tag{2.12}$$

on Ω *are independent, provided*

$$0 \leqq p_1 < p_2 < \cdots < p_\ell \leqq q \quad \text{and} \quad p_{i+1} \geqq g(p_i) + 2\,. \tag{2.13}$$

Proof. Fix a configuration ω. Any path π from $\mathscr{C}_{p-1}$ to $\mathscr{C}_\nu$ must intersect $\mathscr{C}_p$, by virtue of (2.7). (We include here the case $\mathscr{C}_{p-1} = \mathscr{C}_p$, in which $T(\mathscr{C}_{p-1}, \mathscr{C}_p) = 0$.) If π is such a path on $\mathbb{Z}^2$, let π_1 be the piece of π from its last intersection with $\mathscr{C}_{p-1}$ to its first intersection with $\mathscr{C}_p$ (see Fig. 1). Also, let π_2 be the piece of π from its last intersection with $\mathscr{C}_p$ to its first intersection with $\mathscr{C}_\nu$. Then

$$T(\pi) \geqq T(\pi_1) + T(\pi_2) \geqq T(\mathscr{C}_{p-1}, \mathscr{C}_p) + T(\mathscr{C}_p, \mathscr{C}_\nu)\,.$$

Taking the inf over π, we obtain

$$T(\mathscr{C}_{p-1}, \mathscr{C}_\nu) \geqq T(\mathscr{C}_{p-1}, \mathscr{C}_p) + T(\mathscr{C}_p, \mathscr{C}_\nu)\,. \tag{2.14}$$

Conversely, if π_1 is a path from $\mathscr{C}_{p-1}$ to $\mathscr{C}_p$ with

$$T(\pi_1) = T(\mathscr{C}_{p-1}, \mathscr{C}_p)$$

and π_2 is a path from $\mathscr{C}_p$ to $\mathscr{C}_v$ with

$$T(\pi_2) = T(\mathscr{C}_p, \mathscr{C}_v) ,$$

then let π be the path from $\mathscr{C}_{p-1}$ to $\mathscr{C}_v$ which is the concatenation of π_1, an arc of $\mathscr{C}_p$ connecting π_1 to π_2, and π_2. Then

$$T(\pi) = T(\pi_1) + T(\pi_2) = T(\mathscr{C}_{p-1}, \mathscr{C}_p) + T(\mathscr{C}_p, \mathscr{C}_v) ,$$

because the open arc connecting π_1 to π_2 has zero passage time. Thus

$$T(\mathscr{C}_{p-1}, \mathscr{C}_v) \leqq T(\mathscr{C}_{p-1}, \mathscr{C}_p) + T(\mathscr{C}_p, \mathscr{C}_v) .$$

Together with (2.14) this gives

$$T(\mathscr{C}_{p-1}, \mathscr{C}_v) = T(\mathscr{C}_{p-1}, \mathscr{C}_p) + T(\mathscr{C}_p, \mathscr{C}_v) . \tag{2.15}$$

We also see that any path π from $\mathscr{C}_{p-1}$ to $\mathscr{C}_p$ has passage time $T(\pi)$ at least as large as the piece of π from its last intersection with $\mathscr{C}_{p-1}$ to its first intersection with $\mathscr{C}_p$. Therefore

$$T(\mathscr{C}_{p-1}, \mathscr{C}_p) = \inf\{T(\pi_1) \colon \pi_1 \text{ a path from } \mathscr{C}_{p-1} \text{ to } \mathscr{C}_p \text{ in } \overline{\mathscr{C}}_p \setminus (\mathscr{C}_{p-1})\} . \tag{2.16}$$

Thus $T(\mathscr{C}_{p-1}, \mathscr{C}_p)(\omega)$ depends only on $\mathscr{C}_{p-1}(\omega)$, $\mathscr{C}_p(\omega)$ and the $t(e, \omega)$ with $e \subset \overline{\mathscr{C}}_p \setminus \text{int}(\mathscr{C}_{p-1})$, but not on the $t(e, \omega)$ with $e \subset \text{int}(\mathscr{C}_{p-1})$ or e which lie (except possibly for an endpoint) in

$$\text{ext}(\mathscr{C}_p) := \text{complement of } \overline{\mathscr{C}}_p$$

(once $\mathscr{C}_{p-1}$ and $\mathscr{C}_p$ are fixed). In particular $T(\mathscr{C}_{p-1}, \mathscr{C}_p)$ is $\mathscr{F}_p$-measurable. By similar arguments we obtain

$$T(\mathbf{0}, \mathscr{C}_v) = T(\mathbf{0}, \mathscr{C}_{p-1}) + T(\mathscr{C}_{p-1}, \mathscr{C}_p) + T(\mathscr{C}_p, \mathscr{C}_v) \tag{2.17}$$

and the fact that $T(\mathbf{0}, \mathscr{C}_{p-1})$ is $\mathscr{F}_{p-1}$-measurable. Finally, once $\mathscr{C}_p$ is fixed, $T(\mathscr{C}_p, \mathscr{C}_v)$ depends only on $t(e, \omega)$ for edges e which lie (with the possible exception of an endpoint) in $\text{ext}(\mathscr{C}_p)$. Given $\mathscr{C}_p$ and $\{t(e, \omega) \colon e \subset \overline{\mathscr{C}}_p\}$, the latter edges in $\text{ext}(\mathscr{C}_p)$ (except for endpoints) are conditionally independent of $\{t(e, \omega) \colon e \in \overline{\mathscr{C}}_p\}$. Therefore

$$E\{T(\mathscr{C}_p, \mathscr{C}_v) \,|\, \mathscr{F}_p\}(\omega)$$

is simply the integral of $T(\mathscr{C}_p, \mathscr{C}_v)$ with respect to the distribution of the $t(e)$ with $e \subset \text{ext}(\mathscr{C}_p(\omega))$ (except for endpoints). In other words

$$E\{T(\mathscr{C}_p, \mathscr{C}_v) \,|\, \mathscr{F}_p\}(\omega) = E' T(\mathscr{C}_p(\omega), \mathscr{C}_{v(\omega')}(\omega'))(\omega') . \tag{2.18}$$

Combined with (2.17) and the preceding remarks this gives

$$\begin{aligned} E\{T(\mathbf{0}, \mathscr{C}_v) \,|\, \mathscr{F}_p\}(\omega) = {} & T(\mathbf{0}, \mathscr{C}_{p-1}(\omega))(\omega) + T(\mathscr{C}_{p-1}(\omega), \mathscr{C}_p(\omega))(\omega) \\ & + E' T(\mathscr{C}_p(\omega), \mathscr{C}_{v(\omega')}(\omega'))(\omega') . \end{aligned} \tag{2.19}$$

Essentially the same proof shows that

$$E\{T(\mathbf{0},\mathscr{C}_v(\omega))\,|\,\mathscr{F}_{p-1}\} = T(\mathbf{0},\mathscr{C}_{p-1}(\omega))(\omega) + E'T(\mathscr{C}_{p-1}(\omega),\mathscr{C}_{v(\omega')}(\omega'))(\omega')\,. \tag{2.20}$$

(2.11) now follows by subtracting (2.20) from (2.19).

Our preceding observations also show that the right hand side of (2.11) is determined once we know $\mathscr{C}_{p-1}(\omega)$, $\mathscr{C}_p(\omega)$ and $\{t(e,\omega)\colon e \subset \overline{\mathscr{C}_p}(\omega)\setminus \text{int}(\mathscr{C}_{p-1}(\omega))\}$.

Finally, by the definition of $m(p)$, $I[m(p) \leqq g(p)]$ depends only on $t(e,\omega)$ with

$$e \subset S(2^{g(p)+1})\setminus S(2^p)\,.$$

Moreover, if $I[m(p) \leqq g(p)] = 1$, then

$$\mathscr{C}_p \subset A(m(p)) \subset S(2^{g(p)+1})\setminus S(2^p)\,.$$

Also, if $p \geqq 1$, by definition, $m(p-1) \geqq p-1$, and $\mathscr{C}_{p-1} \subset$ complement of $S(2^{p-1})$. Therefore $\mathscr{C}_{p-1}$ and $\mathscr{C}_p$ both lie in

$$S(2^{g(p)+1})\setminus S(2^{p-1})\,. \tag{2.21}$$

Consequently, if $p \geqq 1$, $\Delta_p I[m(p) \leqq g(p)]$ is determined by $\{t(e,\omega)\colon e$ in the set (2.21)$\}$. For $p=0$, (2.21) should be interpreted as $S(2^{g(0)+1})$. If (2.13) holds, then the regions

$$S(2^{g(p_i)+1})\setminus S(2^{p_i-1}),\quad 1 \leqq i \leqq \ell\,,$$

are disjoint, and the independence of the random variables in (2.12) follows from this. □

To estimate the distribution of Δ_p, we must refine the representation (2.11). It is tempting to write

$$T(\mathscr{C}_{p-1}(\omega),\mathscr{C}_{v(\omega')}(\omega'))(\omega') = T(\mathscr{C}_{p-1}(\omega),\mathscr{C}_p(\omega))(\omega') + T(\mathscr{C}_p(\omega),\mathscr{C}_{v(\omega')}(\omega'))(\omega')\,. \tag{2.22}$$

This, however, is *not* true, because $\mathscr{C}_p(\omega)$ is not necessarily an open circuit in the configuration ω'. The following provides a replacement for (2.22).

Lemma 2 *Define*

$$\ell(p,\omega,\omega') = m(m(p,\omega)+1,\omega')\,. \tag{2.23}$$

Then

$$\Delta_p(\omega) = T(\mathscr{C}_{p-1}(\omega),\mathscr{C}_p(\omega))(\omega) + E'T(\mathscr{C}_p(\omega),\mathscr{C}_{\ell(p,\omega,\omega')}(\omega'))(\omega') - E'T(\mathscr{C}_{p-1}(\omega),\mathscr{C}_{\ell(p,\omega,\omega')}(\omega'))(\omega')\,. \tag{2.24}$$

Proof. The main difficulty here is to understand the notation. $\ell(p,\omega,\omega')$ is found as follows. First one determines $m(p,\omega)$. Then one finds the smallest $t \geqq m(p,\omega)+1$ for which there is an open circuit surrounding $\mathbf{0}$ in $A(t)$ in the configuration ω'. This value of t is $\ell(p,\omega,\omega')$ and the innermost open circuit surrounding $\mathbf{0}$ in the annulus $A(\ell(p,\omega,\omega'))$ in ω' is $\mathscr{C}_{\ell(p,\omega,\omega')}(\omega')$. In particular, this circuit is open in ω'. Moreover $\ell(p,\omega,\omega') \geqq m(p,\omega)+1$, so that

$$\mathscr{C}_p(\omega) = \mathscr{C}_{m(p,\omega)}(\omega) \subset A(m(p,\omega)) \subset \mathrm{int}(\mathscr{C}_{\ell(p,\omega,\omega')}(\omega')) \,. \tag{2.25}$$

We now obtain essentially by the same proof as for (2.15), that

$$T(\mathscr{C}_p(\omega),\mathscr{C}_{v(\omega')}(\omega'))(\omega') = T(\mathscr{C}_p(\omega),\mathscr{C}_{\ell(p,\omega,\omega')}(\omega'))(\omega') + T(\mathscr{C}_{\ell(p,\omega,\omega')}(\omega'),\mathscr{C}_{v(\omega')}(\omega'))(\omega') \,. \tag{2.26}$$

For the same reasons

$$T(\mathscr{C}_{p-1}(\omega),\mathscr{C}_{v(\omega')}(\omega'))(\omega') = T(\mathscr{C}_{p-1}(\omega),\mathscr{C}_{\ell(p,\omega,\omega')}(\omega'))(\omega') + T(\mathscr{C}_{\ell(p,\omega,\omega')}(\omega'),\mathscr{C}_{v(\omega')}(\omega'))(\omega') \,. \tag{2.27}$$

Substitution of (2.26) and (2.27) into the right hand side of (2.11) yields (2.24). □

Lemma 3 *There exist constants $C_i \in (0,\infty)$ such that for $q \geqq 1$ (with δ as in* (1.5))

$$P\{m(p) - p \geqq t\} \leqq e^{-C_5 t}, \quad t,\ p \geqq 0\,, \tag{2.28}$$

$$P\{|\Delta_p| \geqq x\} \leqq C_6 x^{-\delta/2}, \quad x \geqq 0, \quad 0 \leqq p \leqq q\,, \tag{2.29}$$

$$P\left\{\max_{0\leqq p\leqq q} |\Delta_p| \geqq \varepsilon q^{1/2}\right\} \leqq 2C_6\varepsilon^{-\delta/2}q^{1-\delta/4}\,, \tag{2.30}$$

$$E\left\{\max_{0\leqq p\leqq q} \Delta_p^2\right\} \leqq C_7 q\,, \tag{2.31}$$

$$C_8 q \leqq \sum_{p=0}^{q} E\Delta_p^2 \leqq C_9 q\,. \tag{2.32}$$

Proof. The event $m(p) - p \geqq t$ occurs if and only if (2.4) occurs with $p' = p + t$. But it is well known (see Smythe and Wierman (1978, Sect. 3.4) or Kesten (1982, Corollary 6.1) or Grimmett (1989, Theorem 9.70)) that there exists a constant $C_5 > 0$ for which

$$P\{\text{there is no open circuit surrounding } \mathbf{0} \text{ in } A(j)\} \leqq e^{-C_5}, \quad j \geqq 0\,. \tag{2.33}$$

Since the annuli $A(j)$ are disjoint, the events in the left hand side of (2.33) for distinct j are independent. Thus (2.28) holds.

As a consequence of (2.28) we also have for each fixed ω

$$P'\{\ell(p,\omega,\omega') \geqq m(p,\omega) + 1 + t\} \leqq e^{-C_5 t}\,. \tag{2.34}$$

To obtain (2.29) we first observe that

$$\mathscr{C}_{p-1}(\omega) \subset \overline{\mathscr{C}}_p(\omega) ,$$

and consequently (compare (2.14))

$$T(\mathscr{C}_p(\omega), \mathscr{C}_{\ell(p,\omega,\omega')}(\omega'))(\omega') \leqq T(\mathscr{C}_{p-1}(\omega), \mathscr{C}_{\ell(p,\omega,\omega')}(\omega'))(\omega') .$$

Thus, by (2.24)

$$|\Delta_p(\omega)| \leqq T(\mathscr{C}_{p-1}(\omega), \mathscr{C}_p(\omega))(\omega) + E'T(\mathscr{C}_{p-1}(\omega), \mathscr{C}_{\ell(p,\omega,\omega')}(\omega'))(\omega') . \tag{2.35}$$

We shall now estimate

$$P'\{T(\mathscr{C}_{p-1}(\omega), \mathscr{C}_{\ell(p,\omega,\omega')}(\omega'))(\omega') \geqq y\} \tag{2.36}$$

for fixed ω. To avoid special treatment of the case $p = 0$ we shall interpret $S(2^{-1})$ as $\{\mathbf{0}\}$ and $A(-1)$ as $S(1)$ in this proof. For a square S, ∂S will be its topological boundary and $\text{int}(S)$ the interior of S ($\partial S(2^{-1}) = \{\mathbf{0}\}$, $\text{int}(S(2^{-1})) = \emptyset$).

When ω is fixed, then so is $m = m(p,\omega)$ and $\mathscr{C}_p(\omega) \subset A(m)$. Also for any $r \geqq \ell(p,\omega,\omega') + 1$,

$$\mathscr{C}_{\ell(p,\omega,\omega')} \subset A(\ell(p,\omega,\omega')) \subset S(2^r) .$$

Furthermore

$$S(2^{p-1}) \subset \text{int}\,\mathscr{C}_{p-1}(\omega) \subset \overline{\mathscr{C}}_{\ell(p,\omega,\omega')} .$$

This shows that any path on $\mathbb{Z}^2$ from $\partial S(2^{p-1})$ to $\partial S(2^r)$ must intersect $\mathscr{C}_{p-1}(\omega)$ and $\mathscr{C}_{\ell(p,\omega,\omega')}$. Therefore, we have just as in (2.14) that

$$T(\mathscr{C}_{p-1}(\omega), \mathscr{C}_{\ell(p,\omega,\omega')}(\omega'))(\omega') \leqq T(\partial S(2^{p-1}), \partial S(2^r))(\omega') . \tag{2.37}$$

It follows that for $t = 0, 1, 2, \dots$

$$\begin{aligned} &P'\{T(\mathscr{C}_{p-1}(\omega), \mathscr{C}_{\ell(p,\omega,\omega')}(\omega'))(\omega') \geqq y\} \\ &\quad \leqq P'\{\ell(p,\omega,\omega') \geqq m(p,\omega) + 1 + t\} \\ &\qquad + P'\{T(\partial S(2^{p-1}), \partial S(2^{m(p,\omega)+1+t}))(\omega') \geqq y\} \\ &\quad \leqq e^{-C_5 t} + P'\{T(\partial S(2^{p-1}), \partial S(2^{m(p,\omega)+1+t})(\omega') \geqq y\} . \end{aligned} \tag{2.38}$$

In order to estimate the last probability, we shall choose the $t(e,\omega')$ with $e \in S(2^{m+1+t}) \setminus \text{int}(S(2^{p-1}))$ in two stages. First we decide for each edge e whether it is open or closed in the configuration ω' (i.e., whether $t(e,\omega') = 0$ or $t(e,\omega') > 0$). Then we will pick the values of $t(e,\omega')$ for the closed edges e with the conditional distribution of $t(e,\omega')$, given $t(e,\omega') > 0$. Clearly the contributions to the passage time $T(\partial S(2^{p-1}), \partial S(2^{m+1-t}))$ come only from the closed edges. We are therefore first going to look for some general $k > j \geqq -1$ for a path from $\partial S(2^j)$ to $\partial S(2^k)$ which contains few closed edges. To this end we introduce the dual graph of $\mathbb{Z}^2$. This dual graph can be identified with $\mathbb{Z}^2 + (\frac{1}{2}, \frac{1}{2})$, with edges between nearest neighbors again. Each edge $\widetilde{e}$ of this dual graph bisects a unique edge e of $\mathbb{Z}^2$. For any configuration ω, we call $\widetilde{e}$ open or closed, when the corresponding edge e is open or closed, respectively. By a theorem of Whitney (see Smythe and Wierman (1978, Sect. 2.1) or Kesten (1982, Proposition 2.1, Corollary 2.2)) the minimal edge sets of $\mathbb{Z}^2$ which separate $\partial S(2^j)$ from $\partial S(2^k)$ are precisely the edge sets on $\mathbb{Z}^2$ whose bisecting edges on $\mathbb{Z}^2 + (\frac{1}{2}, \frac{1}{2})$ form a closed dual circuit in $S(2^k) \setminus S(2^j)$ which surrounds $S(2^j)$. Now define

$$\kappa(j,k,\omega') = \text{minimal number of closed edges in any path from } \partial S(2^j) \text{ to } \partial S(2^k) \text{ (in configuration } \omega')$$

$$\rho(j,k,\omega') = \text{maximal number of edge-disjoint closed dual circuits which surround } S(2^j) \text{ in } S(2^k) \setminus S(2^j) \text{ (in configuration } \omega')$$

(recall our convention that $S(2^{-1}) = \{\mathbf{0}\}$). It seems to be well known that

$$\kappa(j,k,\omega') = \rho(j,k,\omega') \,. \tag{2.39}$$

Since we could not find a reference with proof for this fact, we indicate a proof in the appendix.

Next we must estimate the tail of the distribution of ρ. Chayes et al. (1986) in their proof of Theorem 3.3 show that for some constant $C_{10} < \infty$

$$E'\{\rho(j,k,\omega')\} \leqq C_{10}(k-j) \,. \tag{2.40}$$

Markov's inequality therefore shows that

$$P'\{\rho(j,k,\omega') \geqq 2C_{10}(k-j)\} \leqq \tfrac{1}{2} \,. \tag{2.41}$$

To improve this estimate we introduce the following events in Ω':

$$\begin{aligned} G(y) = G(y,j,k) &= \{\rho(j,k,\omega') \geqq \lfloor y \rfloor\} \\ &= \{\text{there exists at least } \lfloor y \rfloor \text{ disjoint closed dual circuits surrounding } S(2^j) \text{ in } S(2^k) \setminus S(2^j)\} \,. \end{aligned} \tag{2.42}$$

Now for any events $G_1, \ldots, G_r \subset \Omega'$, which depend only on finitely many of the variables

$$J(e) := I[t(e,\omega') \text{ is open}] \,, \tag{2.43}$$

it holds that

$$P'\{G_1 \,\square\, G_2 \cdots \square\, G_r\} \leqq \prod_{i=1}^{r} P'\{G_i\} \,, \tag{2.44}$$

where $G_1 \,\square\, G_2 \cdots \square\, G_r$ is the event that $G_1, \ldots, G_r$ occur disjointly, as defined in van den Berg and Kesten (1985); see also Grimmett (1989, p. 31). The general inequality (2.44) was only proved recently in Reimer (1994), but we shall apply this with

$$G_i = G(2C_{10}(k-j)+1)$$

which is a decreasing event (i.e., its characteristic function is a decreasing function of the $J(e)$). For this special situation (2.44) already follows from van den Berg and Kesten (1985); see also Grimmett (1989, Eq. (2.14)). It is easily seen that from the definition (2.42) that

$$G(r[2C_{10}(k-j)+1]) \subset G(2C_{10}(k-j)+1)\square \cdots \square G(2C_{10}(k-j)+1) \tag{2.45}$$

(with r events G on the right hand side). We shall take

$$r = \lfloor C_{11}y[2C_{10}(k-j)+1]^{-1}\rfloor \geqq C_{12}\frac{y}{k-j} - 1 \tag{2.46}$$

with

$$C_{11} = \left[4 \int_{(0,\infty)} x\, dF(x)\right]^{-1} . \tag{2.47}$$

We then obtain

$$\begin{aligned}
&P'\{\text{in the configuration } \omega' \text{ there does not exist a path from } \partial S(2^j)\\
&\quad \text{to } \partial S(2^k) \text{ with fewer than } r[2C_{10}(k-j)+1] \text{ closed edges}\}\\
&\quad \leqq P'\{\rho \geqq r[2C_{10}(k-j)+1]\} \quad (\text{by } (2.39))\\
&\quad \leqq P'\{G(r[2C_{10}(k-j)+1])\}\\
&\quad \leqq [P'\{G(2C_{10}(k-j)+1)\}]^r \quad (\text{by } (2.45) \text{ and } (2.44))\\
&\quad \leqq 2^{-r} \quad (\text{by } (2.41)) \leqq 2 \cdot 2^{-C_{12}y/(k-j)} .
\end{aligned} \tag{2.48}$$

Finally, if the event in the left hand side of (2.48) fails, then there exists in ω' a path π from $\partial S(2^j)$ to $\partial S(2^k)$ with no more than

$$s := \lfloor r[2C_{10}(k-j)+1]\rfloor$$

closed edges. We now choose $t(e,\omega')$ for the closed edges. Then, given π, the conditional probability that $T(\pi) \geqq y$ is at most $P\{S_s > y\}$, where

$$S_s = \sum_1^s X_i ,$$

and the X_i are i.i.d. random variables with the conditional distribution of $t(e,\omega')$, given that $t(e,\omega') > 0$. Thus

$$EX_i = [1-F(0)]^{-1} \int_{(0,\infty)} x\, dF(x) = 2 \int_{(0,\infty)} x\, dF(x) ,$$

and, by virtue of (2.47),

$$ES_s = 2s \int_{(0,\infty)} x\, dF(x) \leqq \frac{1}{2} y .$$

Markov's inequality and moment estimates for martingales (cf. Gut 1988, Theorem I.5.1) now show that for some constants C_{13}, C_{14}

$$\begin{aligned}
P\{S_s \geqq y\} &\leqq P\left\{S_s - ES_s \geqq \tfrac{1}{2}y\right\} \leqq 2^\delta y^{-\delta} E|S_s - ES_s|^\delta\\
&\leqq C_{13}\frac{s^{\delta/2}}{y^\delta} \leqq C_{14}\frac{1}{y^{\delta/2}} .
\end{aligned} \tag{2.49}$$

Combining those estimates we obtain for $y \geqq 0$

$$
\begin{aligned}
&P'\{T(\partial S(2^j), \partial S(2^k) \geqq y\} \\
&\quad \leqq \text{left hand side of (2.48)} + P\{S_s \geqq y\} \\
&\quad \leqq 2 \cdot 2^{-C_{12} y / (k-j)} + C_{14} y^{-\delta/2}, \quad -1 \leqq j < k . \qquad (2.50)
\end{aligned}
$$

By taking $j = p - 1$, $k = m(p, \omega) + 1 + t$ we are finally led to the desired estimate for (2.36). Indeed, by virtue of (2.38), we have for $t = 0, 1, \ldots$

$$
\begin{aligned}
&P'\{T(\mathscr{C}_{p-1}(\omega), \mathscr{C}_{\ell(p,\omega,\omega')}(\omega'))(\omega') \geqq y\} \\
&\quad \leqq e^{-C_5 t} + 2 \cdot 2^{-C_{12} y / (m-p+t+2)} + C_{14} y^{-\delta/2} . \qquad (2.51)
\end{aligned}
$$

The choice $t = \lfloor \sqrt{y} \rfloor$ now gives that for suitable constants $C_{15}, C_{16} \in (0, \infty)$ and all $\omega \in \Omega$, $y \geqq 0$,

$$
\begin{aligned}
&P'\{T(\mathscr{C}_{p-1}(\omega), \mathscr{C}_{\ell(p,\omega,\omega')}(\omega'))(\omega') \geqq y\} \\
&\quad \leqq C_{15} \exp\left(-C_{16} \frac{y}{m - p + \sqrt{y}}\right) + C_{14} y^{-\delta/2} . \qquad (2.52)
\end{aligned}
$$

Integration of (2.52) over y from 0 to ∞ now shows that the second term in the right hand side of (2.35) is at most

$$
C_{17}[m(p, \omega) - p + 1] . \qquad (2.53)
$$

In view of (2.35) we therefore have for $t = 0, 1, \ldots, \lfloor x/2C_{17} \rfloor$

$$
\begin{aligned}
P\{|\Delta_p(\omega)| \geqq x\} &\leqq P\{m(p, \omega) - p \geqq t\} \\
&\quad + P\{T(\mathscr{C}_{p-1}(\omega), \mathscr{C}_p(\omega))(\omega) \geqq x/2,\ m(p, \omega) - p < t\} . \qquad (2.54)
\end{aligned}
$$

Analogously to (2.37) and (2.38) we see that the second term in the right hand side here is at most

$$
P\{T(\partial S(2^{p-1}), \partial S(2^{p+t}))(\omega) \geqq x/2\} \leqq 2 \cdot 2^{-C_{12} x / 2(t+1)} + C_{14} 2^{\delta/2} x^{-\delta/2} \quad \text{(see (2.50))} .
$$

Therefore, (2.54) and (2.28) give for $t \leqq x/2C_{17}$

$$
P\{|\Delta_p(\omega)| \geqq x\} \leqq e^{-C_5 t} + 2 \cdot 2^{-C_{12} x / 2(t+1)} + C_{14} 2^{\delta/2} x^{-\delta/2} . \qquad (2.55)
$$

(2.29) follows by taking $t = \lfloor \sqrt{x} \rfloor$.

With (2.29) proven, (2.30) and (2.31) are immediate. Also the second inequality in (2.32) is now obvious. Lastly we prove the first inequality in (2.32). By (2.24) and the estimate (2.53) for the last term in the right hand side, we have

$$
\Delta_p(\omega) \geqq T(\mathscr{C}_{p-1}(\omega), \mathscr{C}_p(\omega))(\omega) - C_{17}[m(p, \omega) - p + 1] .
$$

Now we observe that a path which crosses k closed dual circuits has a passsage time of at least $C_0 k$ (by (1.6)). Therefore, for $p+2 \leqq q$

$$
\begin{aligned}
E\Delta_p^2 &\geqq P\{\Delta_p \geqq 1\} \geqq P\{m(p,\omega) = p+1 \text{ and } \\
&\qquad T(\mathscr{C}_{p-1}(\omega), \mathscr{C}_p(\omega))(\omega) \geqq 2C_{17}+1\} \\
&\geqq P\{\mathscr{C}_{p-1}(\omega) \subset A(p-1), \text{ there is no open circuit in } A(p), \text{ but there} \\
&\qquad \text{exist at least } C_0^{-1}(2C_{17}+1) \text{ edge-disjoint closed dual circuits} \\
&\qquad \text{surrounding } S(2^{p-1}) \text{ in } A(p), \text{ and there is an open circuit in} \\
&\qquad A(p+1)\}\,.
\end{aligned} \tag{2.56}
$$

By the independence of the edges in $A(p-1)$, $A(p)$, $A(p+1)$ and the Harris–FKG inequality (see Kesten (1982, Proposition 4.1) or Grimmett (1989, Theorem 2.4))

$$
\begin{aligned}
E\Delta_p^2 &\geqq P\{\text{there exist open circuits surrounding } \mathbf{0} \text{ in } A(p-1) \text{ and } A(p+1)\} \\
&\quad \times P\{\text{there does not exist an open circuit surrounding } \mathbf{0} \text{ in } A(p)\} \\
&\quad \times P\{\text{there exist at least } C_0^{-1}(2C_{17}+1) \text{ edge-disjoint closed dual} \\
&\qquad \text{circuits surrounding } S(2^{p-1}) \text{ in } A(p)\}\,.
\end{aligned} \tag{2.57}
$$

The Russo–Seymour–Welsh Theorem implies that the right hand side of (2.57) is bounded away from zero (see Smythe and Wierman (1978, Lemma 3.5), Kesten (1982, Theorem 6.1) or Grimmett (1989, Sect. 9.7)). Thus (2.32) follows. □

It is now easy to prove a preliminary central limit theorem for $T(\mathbf{0}, \mathscr{C}_{m(q)})$ as $q \to \infty$.

Lemma 4 *As* $q \to \infty$,

$$
\frac{T(\mathbf{0}, \mathscr{C}_{m(q)}) - ET(\mathbf{0}, \mathscr{C}_{m(q)})}{\left[\sum_{p=0}^{q} E\Delta_{p,q}^2\right]^{1/2}} \to N(0,1) \quad \textit{in distribution}\,. \tag{2.58}
$$

Proof. Recall that we have written v for $m(q)$ in the preceding lemmas. Therefore, if we set

$$
X_{p,q} = \frac{\Delta_{p,q}}{\left[\sum_{p=0}^{q} E\Delta_{p,q}^2\right]^{1/2}}\,,
$$

then the left hand side of (2.58) is

$$
\frac{T(\mathbf{0}, \mathscr{C}_v) - ET(\mathbf{0}, \mathscr{C}_v)}{\left[\sum_{p=0}^{q} E\Delta_{p,q}^2\right]^{1/2}} = \sum_{p=0}^{q} X_{p,q}\,.
$$

We now apply Theorem 2.3 in McLeish (1974). Since

$$
|X_{p,q}| \leqq |\Delta_{p,q}|/[C_8 q]^{1/2}\,,
$$

by virtue of (2.32), the conditions (2.3a) and (2.3b) of McLeish are implied by (2.30) and (2.31) (recall that $\delta > 4$). It remains to verify his condition (2.3c), that is

$$\sum_{p=0}^{q} X_{p,q}^2 \to 1 \quad \text{in probability} . \tag{2.59}$$

By (2.32) this is equivalent to

$$\frac{1}{q}\sum_{p=0}^{q} [\Delta_{p,q}^2 - E\Delta_{p,q}^2] \to 0 \quad \text{in probability} . \tag{2.60}$$

This is a kind of weak law of large numbers which we prove by standard arguments. First we replace $\Delta_{p,q}$ by

$$\widetilde{\Delta}_{p,q} := \Delta_{p,q} I\left[m(p) \leqq p + \frac{3}{C_5}\log q \text{ and } |\Delta_{p,q}| \leqq q^{2/\delta}\log q\right] .$$

Then

$$\begin{aligned} &P\{\Delta_{p,q} \neq \widetilde{\Delta}_{p,q} \text{ for some } p \leqq q\} \\ &\quad \leqq \sum_{p=0}^{q} P\left\{m(p) - p \geqq \frac{3}{C_5}\log q\right\} + \sum_{p=0}^{q} P\{|\Delta_{p,q}| > q^{2/\delta}\log q\} \\ &\quad \leqq (q+1)e^{-3\log q} + 2C_6(\log q)^{-\delta/2} \quad \text{(by (2.28) and (2.29))} \\ &\quad \to 0 . \end{aligned}$$

Moreover, $E|\Delta_{p,q}|^{1+\delta/4}$ is bounded, by virtue of (2.29) and $\delta > 4$, so that

$$\begin{aligned} \sum_{p=0}^{q} [E\Delta_{p,q}^2 - E\widetilde{\Delta}_{p,q}^2] &= \sum_{p=0}^{q} E\left\{\Delta_{p,q}^2 I\left[m(p) - p > \frac{3}{C_5}\log q \text{ or} \right.\right. \\ &\qquad\qquad \left.\left. |\Delta_{p,q}| > q^{2/\delta}\log q\right]\right\} \\ &\leqq \sum_{p=0}^{q} [E|\Delta_{p,q}|^{1+\delta/4}]^{8/(4+\delta)}\left[P\left\{m(p) > p + \frac{3}{C_5}\log q\right\}\right. \\ &\qquad \left. + P\{|\Delta_{p,q}| > q^{2/\delta}\log q\}\right]^{(\delta-4)/(4+\delta)} \\ &\leqq C_{18}(q+1)[q^{-3} + C_6 q^{-1}(\log q)^{-\delta/2}]^{(\delta-4)/(4+\delta)} \\ &= o(q) . \end{aligned}$$

It therefore suffices for (2.60) to show that

$$\frac{1}{q}\sum_{p=0}^{q} \left[\widetilde{\Delta}_{p,q}^2 - E\widetilde{\Delta}_{p,q}^2\right] \to 0 \quad \text{in probability} . \tag{2.61}$$

(2.61) follows easily from Chebyshev's inequality. Indeed,

$$\begin{aligned} \widetilde{\Delta}_{p,q} &= \Delta_{p,q} I\left[m(p) \leqq p + \frac{3}{C_5}\log q\right] \\ &\quad \times I\left[\left|\Delta_{p,q} I\left[m(p) \leqq p + \frac{3}{C_5}\log q\right]\right| \leqq q^{2/\delta}\log q\right] , \end{aligned}$$

so that, by Lemma 1, $\widetilde{\Delta}_{p,q}$ and $\widetilde{\Delta}_{r,q}$ are independent, whenever $|p-r| \geqq (3/C_5)\log q + 2$. Moreover, uniformly in $0 \leqq p \leqq q$

$$\begin{aligned}\mathrm{Var}(\widetilde{\Delta}^2_{p,q}) &\leqq E\{\widetilde{\Delta}^4_{p,q}\} \leqq 4 \int_{0\leqq x\leqq q^{2/\delta}\log q} x^3 P\{|\Delta_{p,q}| \geqq x\}dx \\ &\leqq 4 \int_{0\leqq x\leqq q^{2/\delta}\log q} x^3(C_6 x^{-\delta/2} \wedge 1)dx \quad \text{(by (2.29))} \\ &\leqq C_{19}[1 + q^{8/\delta-1}(\log q)^{5-\delta/2}] \,.\end{aligned}$$

Therefore

$$\begin{aligned}&\mathrm{Var}\left(\sum_{p=0}^{q} [\widetilde{\Delta}^2_{p,q} - E\widetilde{\Delta}^2_{p,q}]\right) \\ &\leqq 2\sum_{p=0}^{q} \sum_{p\leqq r\leqq p+(3/C_5)\log q+2} \left[\mathrm{Var}\left(\widetilde{\Delta}^2_{p,q}\right)\mathrm{Var}\left(\widetilde{\Delta}^2_{r,q}\right)\right]^{1/2} \\ &\leqq C_{20}(q+1)\log q \; q^{8/\delta-1}(\log q)^{5-\delta/2} \\ &\leqq 2C_{20}q^{8/\delta}(\log q)^{6-\delta/2} = o(q^2)\,.\end{aligned}$$

Thus (2.61) holds and the lemma follows from Theorem 2.3 in McLeish (1974). □

We are now ready to prove our main results. To do this we drop the requirement that n is a power of 2. For

$$2^{q-1} < n \leqq 2^q \tag{2.62}$$

we define

$$\gamma_n = \left\{\sum_{p=0}^{q} E\Delta^2_{p,q}\right\}^{1/2} . \tag{2.63}$$

Proof of (1.7) *and* (1.8). It is clear from (2.32) that the γ_n of (2.63) satisfies (1.7).

For (1.8) we note first that $\mathscr{C}_{m(q)}$ surrounds $\mathbf{0}$, but lies outside $S(2^q)$, and hence outside $S(n)$ when (2.62) holds. Thus $\mathscr{C}_{m(q)}$ must contain points in the half plane $H_n = \{(x,y)\colon x \geqq n\}$ and therefore

$$0 \leqq b_{0,n} \leqq T(\mathbf{0}, \mathscr{C}_{m(q)}) \tag{2.64}$$

(recall that the passage time from $\mathbf{0}$ to v is the same for all $v \in \mathscr{C}_{m(q)}$). In the opposite direction, if for some k

$$\mathscr{C}_{m(q-k)} \subset S(2^{q-1}) \subset S(n)\,, \tag{2.65}$$

then any path from $\mathbf{0}$ to H_n must intersect $\mathscr{C}_{m(q-k)}$ and *a fortiori*

$$b_{0,n} \geqq T(\mathbf{0}, \mathscr{C}_{m(q-k)}) = T(\mathbf{0}, \mathscr{C}_{m(q)}) - T(\mathscr{C}_{m(q-k)}, \mathscr{C}_{m(q)}) \tag{2.66}$$

(compare (2.15)). Moreover, when $m(q) \leqq q + t$, as in (2.37),

$$T(\mathscr{C}_{m(q-k)}, \mathscr{C}_{m(q)}) \leqq T(\partial S(2^{q-k}), \partial S(2^{q+t+1}))\,. \tag{2.67}$$

(2.64)–(2.67) and (2.50) show that when (2.62) holds, then for all $x \geqq 0$, $k \leqq q$, $t \geqq 0$

$$\begin{aligned} &P\{|b_{0,n} - T(\mathbf{0}, \mathscr{C}_{m(q)})| \geqq x\} \\ &\quad \leqq P\{m(q-k) \geqq q-1\} + P\{m(q) \geqq q+t\} \\ &\qquad + P\{T(\partial S(2^{q-k}), \partial S(2^{q+t+1}) \geqq x\} \\ &\quad \leqq e^{-C_5(k-1)} + e^{-C_5 t} + 2 \cdot 2^{-C_{12}x\,/\,(k+t+1)} + C_{14}x^{-\delta/2} . \end{aligned} \tag{2.68}$$

We can take $k = t = \lfloor \sqrt{x} \rfloor$, provided this is $\leqq q$. Thus, for some constant $C_{21} < \infty$ and $x \leqq q^2$ we have

$$P\{|b_{0,n} - T(\mathbf{0}, \mathscr{C}_{m(q)})| \geqq x\} \leqq C_{21}x^{-\delta/2} . \tag{2.69}$$

Actually this estimate remains valid even for $x \geqq q^2$. In this case we see from (2.64) that

$$\begin{aligned} P\{|b_{0,n} - T(\mathbf{0}, \mathscr{C}_{m(q)})| \geqq x\} &\leqq P\{T(\mathbf{0}, \mathscr{C}_{m(q)}) \geqq x\} \\ &\leqq P\{m(q) \geqq q+t\} + P\{T(\mathbf{0}, \partial S(2^{q+t}) \geqq x\} \\ &\leqq e^{-C_5 t} + 2 \cdot 2^{-C_{12}x/(q+t+1)} + C_{14}x^{-\delta/2} \end{aligned}$$

(by (2.28) and (2.50)). When $t = \lfloor \sqrt{x} \rfloor \geqq q-1$, this again gives (2.69).

It follows from (2.69) that, as $n \to \infty$ and q chosen to satisfy (2.62),

$$\frac{b_{0,n} - T(\mathbf{0}, \mathscr{C}_{m(q)})}{\gamma_n} \to 0 \quad \text{in probability} . \tag{2.70}$$

Since

$$\gamma_n = \gamma_{2^q} = \left\{ \sum_{p=0}^{q} E\Delta_{p,q}^2 \right\}^{1/2}$$

under (2.62), we conclude from Lemma 4 that

$$\frac{b_{0,n} - ET(\mathbf{0}, \mathscr{C}_{m(q)})}{\gamma_n} \to N(0,1) \quad \text{in distribution} .$$

In order to obtain (1.8) we merely have to observe that also

$$Eb_{0,n} - ET(\mathbf{0}, \mathscr{C}_{m(q)}) \text{ is bounded} ,$$

by virtue of (2.69). □

Remark. (v) Define

$$c_n = T(\mathbf{0}, \partial S(n)) . \tag{2.71}$$

Because any path from $\mathbf{0}$ to H_n must intersect $\partial S(n)$, we have

$$c_n \leqq b_{0,n} . \tag{2.72}$$

But also

$$T(\mathbf{0}, \partial S(2^{q-k})) \leqq c_n \tag{2.73}$$

whenever (2.65) occurs. Thus, as in the last proof

$$P\{|c_n - T(\mathbf{0}, \mathscr{C}_{m(q)})| \geqq x\} \leqq C_{21}x^{-\delta/2} \tag{2.74}$$

and also

$$\frac{c_n - Ec_n}{\gamma_n} \to N(0,1) \quad \text{in distribution}\,. \tag{2.75}$$

Moreover, we see from (2.74) and (2.69) that

$$Eb_{0,n} - Ec_n = O(1)\,. \tag{2.76}$$

Similarly, one can use (2.83) below to show that

$$ET(\mathbf{0}, nu) - 2Ec_n = O(1) \text{ for every fixed unit vector } u\,. \tag{2.77}$$

Proof of (1.9). Let $u = (u_1, u_2)$ be a fixed unit vector. Without loss of generality let $0 \leqq u_2 \leqq u_1 \leqq 1$ (and hence $u_1 \geqq 2^{-1/2}$). Define r by

$$2^{r-1} < \tfrac{1}{2}nu_1 \leqq 2^r\,. \tag{2.78}$$

Then

$$q - 3 \leqq r \leqq q\,, \tag{2.79}$$

because of (2.62) and $u_1 \geqq 2^{-1/2}$. Now consider the two squares

$$S' = S(2^{r-1}) \quad \text{and} \quad S'' = nu + S(2^{r-1})\,.$$

These two squares are disjoint and $\mathbf{0} \in S'$, $nu \in S''$. Therefore any path from $\mathbf{0}$ to nu contains the piece from $\mathbf{0}$ to its first intersection with $\partial S'$ and the piece from its last intersection with $\partial S''$ to nu. Thus

$$T(\mathbf{0}, nu) \geqq T(\mathbf{0}, \partial S') + T(nu, \partial S'')\,. \tag{2.80}$$

To obtain an estimate in the other direction, consider the annuli $A(p), A(p+1), \ldots$ with $p \geqq q+2$. Since $n \leqq 2^q$, $|u| = 1$,

$$S' \cup S'' \subset S(2^p) \tag{2.81}$$

for each such p. Recall that

$$m(q+2) = \inf\{p \geqq q+2\colon \text{there exists an open circuit surrounding } \mathbf{0} \text{ in } A(p)\}\,,$$

and let $\mathscr{C} = \mathscr{C}_{m(q+2)}$. Then by (2.81), $\mathscr{C}$ surrounds both S' and S'', hence $\mathbf{0}$ and nu. We can now connect $\mathbf{0}$ to nu by connecting $\mathbf{0}$ and nu to $\mathscr{C}$, and by adding an arc of $\mathscr{C}$ (see Fig. 2). In addition,

$$\partial S(2^{m(q+2)+1}) \subset \operatorname{int}(nu + S(2^{m(q+1)+2}))\,.$$

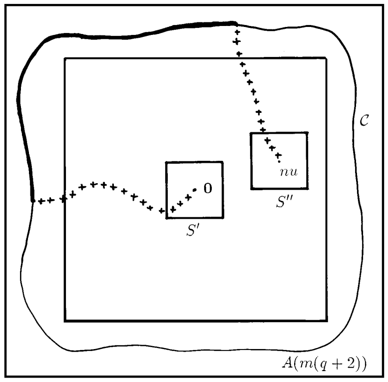

Fig. 2. The two squares S' and S'' and the circuit $\mathscr{C}$ (not drawn to scale). The two paths indicated by $+++$ have passage times $T(\mathbf{0},\mathscr{C})$ and $T(nu,\mathscr{C})$, respectively. Together with the boldly drawn arc of $\mathscr{C}$ they connect $\mathbf{0}$ to nu.

This shows that

$$\begin{aligned} T(\mathbf{0},nu) &\leqq T(\mathbf{0},\mathscr{C}) + T(nu,\mathscr{C}) \\ &\leqq T(\mathbf{0},\partial S(2^{m(q+2)+1})) + T(nu, nu + \partial S(2^{m(q+2)+2})) \,. \end{aligned}$$

Together with (2.80) this gives in the by now familiar way

$$\begin{aligned} &P\{|T(\mathbf{0},nu) - T(\mathbf{0},\partial S') - T(nu,\partial S'')| \geq 2x\} \\ &\quad \leq P\{|T(\mathbf{0},\partial S') - T(\mathbf{0},\partial S(2^{m(q+2)+1}))| \geq x\} \\ &\qquad + P\{|T(nu,\partial S'') - T(nu, nu+\partial S(2^{m(q+2)+2}))| \geq x\} \\ &\quad = P\{|T(\mathbf{0},\partial S(2^{r-1})) - T(\mathbf{0},\partial S(2^{m(q+2)+1}))| \geq x\} \\ &\qquad + P\{T(\mathbf{0},\partial S(2^{r-1})) - T(\mathbf{0},\partial S(2^{m(q+2)+2}))| \geq x\} \\ &\quad \leq 2P\{m(q+2) \geq q+2+t\} + 2P\{m(q-k) \geq q-4\} \\ &\qquad + P\{T(\partial S(2^{q-k}),\partial S(2^{q+3+t})) \geq x\} + P\{T(\partial S(2^{q-k}),\partial S(2^{q+4+t})) \geq x\} \\ &\quad \leq 2e^{-C_5 t} + 2e^{-C_5(k-4)} + 4 \cdot 2^{-C_{12}x\,/\,(k+t+4)} + 2C_{14}x^{-\delta/2} \qquad (2.82) \end{aligned}$$

(by (2.28), (2.50) and (2.79) and translation invariance). Again taking $t = k = \lfloor \sqrt{x} \rfloor$ we find for some constant $C_{22} < \infty$

$$P\{|T(\mathbf{0},nu) - T(\mathbf{0},\partial S') - T(nu,\partial S'')| \geqq 2x\} \leqq C_{22}x^{-\delta/2} \,. \qquad (2.83)$$

Because $T(\mathbf{0},\partial S')$ and $T(nu,\partial S'')$ are independent (recall that S' and S'' are disjoint), and both have the distribution of $c_{2^{r-1}} = T(\mathbf{0},\partial S')$ it follows that

$$\frac{1}{\sqrt{q}}[T(\mathbf{0},nu) - T(\mathbf{0},\partial S') - T(nu,\partial S'')] \to 0 \quad \text{in probability} \qquad (2.84)$$

and (see (2.75))

$$\frac{T(\mathbf{0},nu) - 2Ec_{2^{r-1}}}{\sqrt{2}\gamma_{2^{r-1}}} \to N(0,1) \quad \text{in distribution} \,. \qquad (2.85)$$

It remains to show that

$$\frac{\gamma_{2^{r-1}}}{\gamma_n} \to 1 \tag{2.86}$$

and

$$ET(\mathbf{0}, nu) - 2Ec_{2^{r-1}} = O(1)\ . \tag{2.87}$$

(2.86) follows from the fact that if, for some fixed k,

$$2^{q-k} < \tilde{n} \leqq n \leqq 2^q\ ,$$

then

$$c_{2^{q-k}} \leqq c_{\tilde{n}} \leqq c_n \leqq c_{2^q}\ ,$$

and

$$\frac{1}{\sqrt{q}}[c_{2^q} - c_{2^{q-k}}] \to 0 \text{ in probability as } q \to \infty$$

for the same reasons as (2.68) and (2.69). Together with (2.75) for $n = 2^q$ and for $n = 2^{q-k}$ this forces

$$\frac{1}{\sqrt{q}}[Ec_n - Ec_{\tilde{n}}] \to 0 \quad \text{and} \quad \frac{\gamma_n}{\gamma_{\tilde{n}}} \to 1\ .$$

(2.86) is a special case of this because of (2.79). (2.87) follows immediately form (2.83). □

3. Appendix

Some graph theory.

We indicate here how to prove (2.39). Fix a configuration ω'. Let $\mathscr{D}$ be a dual circuit (i.e., a circuit on $(\frac{1}{2}, \frac{1}{2}) + \mathbb{Z}^2$) which surrounds $S(2^j)$ in $S(2^k) \setminus S(2^j)$. Let $B = B(\mathscr{D})$ be the collection of all vertices of $\mathbb{Z}^2$ which are in the exterior of $\mathscr{D}$ and which are incident to an edge of $\mathbb{Z}^2$ which intersects $\mathscr{D}$. Let $K(B)$ be the open cluster of B outside $\mathscr{D}$, that is the collection of all vertices of $\mathbb{Z}^2$ which are connected to B by an open path of $\mathbb{Z}^2$ in the exterior of $\mathscr{D}$. If K contains vertices in $\partial S(2^k)$ or outside $S(2^k)$, then we do not need the next step. In the other case, when $K \subset$ interior of $S(2^k)$, let D be the "outer boundary" of K, that is the collection of edges $\{u, v\}$ of $\mathbb{Z}^2$ with one endpoint u in K and the other endpoint u outside K and such that v is connected to ∞ by a path on $\mathbb{Z}^2$ which does not intersect K. Then one can see that D separates $\mathscr{D}$ from $\partial S(2^k)$, and by Whitney's theorem (see Smythe and Wierman (1978, Sect. 2.1) or Kesten (1982, Proposition 2.1, Corollary 2.2)) the edges dual to the edges in D form a circuit $\mathscr{D}'$ in $\mathscr{D} \cup \mathrm{ext}(\mathscr{D})$ which surrounds $\mathscr{D}$ and $S(2^j)$. Moreover, by construction, all edges of D and $\mathscr{D}'$ are closed. In other words $\mathscr{D}'$ is a closed dual circuit surrounding $S(2^j)$; see Fig. 3.

We now start with $B_0 = \partial S(2^j)$ and form $K(B_0)$ and its outer boundary, $\mathscr{D}_1$ say. Given $\mathscr{D}_i \subset S(2^k) \setminus S(2^j)$ we form $B(D_i)$, $K(B(D_i))$ and the outer

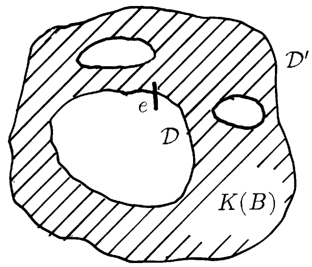

Fig. 3. The inner circuit is $\mathscr{D}$, with an edge e of B indicated. The outer circuit is $\mathscr{D}'$. $K(B)$ is the hatched region; it may have "holes", as indicated.

boundary of the latter, which we call $\mathscr{D}_{i+1}$. We stop this process at the first i for which $K(B_i)$ contains points on $S(2^k)$ or outside $S(2^k)$. Say this happens for $i = \tau$. Thus $\mathscr{D}_1, \ldots, \mathscr{D}_\tau$ are edge-disjoint closed dual circuits in $S(2^k) \setminus S(2^j)$ and we shall now argue that $\tau \geqq \kappa(j,k)$. This will prove that

$$\rho(j,k) \geqq \tau \geqq \kappa(j,k)\,, \tag{3.1}$$

which is the difficult half of (2.39).

To prove our claim, let π be a path on $\mathbb{Z}^2$ from $\partial S(2^j)$ t2o $\partial S(2^k)$ which contains the minimal number of closed edges, $\kappa(j,k)$. Without loss of generality π lies in $S(2^k) \setminus S(2^j)$ except for one of its endpoints on $\partial S(2^j)$. Let the closed edges of π be $e_1, \ldots, e_\kappa$ in the order in which they occur as one traverses π from $\partial S(2^j)$ to $\partial S(2^k)$. If $\kappa = 0$ there is nothing to prove. If $\kappa \geqq 1$, then $K(B_0)$ cannot intersect $\partial S(2^k)$ since all points of $K(B_0)$ have a zero passage time to $B_0 = \partial S(2^j)$. Thus $\mathscr{D}_1$ will exist and be in $S(2^k) \setminus S(2^j)$. $\mathscr{D}_1$ separates $S(2^j)$ from $S(2^k)$ and must therefore intersect an edge of π which is necessarily closed (since $\mathscr{D}_1$ is closed). Let $\mathscr{D}_1$ intersect π in the edge $\{u,v\}$, which by the preceding is one of $e_1, \ldots, e_\kappa$. We claim that it must be e_1. Indeed, if $\{u,v\} = e_p$ with $p \geqq 2$, and $u \in K(B_0)$, then by definition of $K(B_0)$, u can be connected to $\partial S(2^j)$ by an open path π'. Then π', followed by the piece of π consisting of e_p and the edges after it, forms a path from $\partial S(2^j)$ to $\partial S(2^k)$ which only contains the closed edges $e_p, e_{p+1}, \ldots, e_n$. This contradicts the minimality of π. Thus indeed e_1 must lie in the circuit $\mathscr{D}_1$. Furthermore, the same argument shows that the minimality of π guarantees that π *only* has the edge e_1 in common with $\mathscr{D}_1$. Thus, from v on, π lies in the exterior of $\mathscr{D}_1$.

One can now repeat this argument and prove successively that e_p must lie in $\mathscr{D}_p$ for $1 \leqq p \leqq \kappa$. Thus there must exist κ circuits and $\tau \geqq \kappa$. This yields (3.1).

The converse inequality, $\kappa(j,k) \geqq \rho(j,k)$, is trivial. Indeed, if there are ρ edge-disjoint closed dual circuits surrounding $S(2^j)$ in $S(2^k) \setminus S(2^j)$, then any path on $\mathbb{Z}^2$ from $\partial S(2^j)$ to $\partial S(2^k)$ must intersect those ρ circuits, and hence must contain at least ρ closed edges.

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