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# Ergodicity in infinite Hamiltonian systems with conservative noise

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**Summary.** We study the stationary measures of an infinite Hamiltonian system of interacting particles in $\mathbb{R}^3$ subject to a stochastic local perturbation conserving energy and momentum. We prove that the translation invariant measures that are stationary for the deterministic Hamiltonian dynamics, reversible for the stochastic dynamics, and with finite entropy density, are convex combination of "Gibbs" states. This result implies hydrodynamic behavior for the systems under consideration.

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## Introduction

The ergodic problem in Hamiltonian dynamical systems is at the base of equilibrium statistical mechanics. While, beginning with the celebrated Sinai's paper [Si], some results are known for finite systems (see [LW] for a general approach to ergodicity in Hamiltonian systems), very little is known concerning infinite systems (some results are known for special systems with an arbitrary, but finite, number of particles [BLPS]). By ergodicity of an infinite system we mean that convex combinations of Gibbs measures are the only stationary and translation invariant measures, within a reasonably "regular" class.

Furthermore, recent developments in non-equilibrium hydrodynamics (cf. [OVY]) show that the ergodicity of an infinite systems is a main ingredient in the rigorous derivation of Euler equations as a macroscopic description of the conservation laws for the density, the momentum and the energy (at least in the smooth regime of these equations).

Since no results in this direction are present for deterministic systems, it is natural to ask if a stochastic perturbation may help in proving ergodicity.

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The stochastic perturbation should conserve the energy, the momentum and the number of particles of the system, while destroying *locally* the other possible invariant of the motion.

A stochastic perturbation of this type is introduced in [OVY]: any two particles exchange randomly momentum in such a way as to preserve only the total momentum and energy of the two particles. The rate of exchange is assumed to decrease when the distance between the two particles increases, but the range of this random interaction is infinite. Accordingly, any particle is interacting stochastically with any other and the corresponding diffusion on the momenta space of any finite number of particles is elliptic. This permits to characterize the distribution of the momenta of any finite number of particles conditioned to the positions: it must be a uniform measure on the corresponding invariant manifold in the momenta space. The equivalence of ensembles implies that the distribution of the momenta conditioned on the position is a convex combination of "Maxwellians". In addition, one can localize the invariance, under the Hamiltonian dynamics, of the distribution, and prove that the distribution of the positions satisfies the DLR equations with respect to the corresponding interaction.

The purpose of the present paper is to extend the foregoing argument to *finite* range stochastic interactions. Two difficulties arise immediately: one of local and the other of global type.

Locally, restricting oneself to a finite "chain" (or cluster) of particles interacting stochastically, the diffusion on the space of momenta is no longer elliptic; it becomes then necessary to prove that it is, at least, hypoelliptic. This is done quite easily with an inductive argument and in grand generality: only the convexity of the kinetic energy is needed.

The global obstacle is of a more serious nature. The diffusion on the momenta is hypoelliptic only when restricted to chains of interacting particles. But, several clusters of particles, too far apart to interact stochastically, may be present; hence, they could be at "different temperatures". We need the help of the deterministic Hamiltonian dynamics to "connect" distant clusters.

Taking commutators between the vector fields generating the stochastic dynamics and the Hamiltonian generator one obtains a Lie algebra of vector fields large enough to generate all the tangent space to the energy-momentum manifold on the phase space (i.e., position and momentum) of the cluster of the interacting particles. This means that our system is invariant for the dynamics generated by these vector fields (that turn out to be local) which enable, after some work, to produce "cluster deformations" that connect any cluster with the others. Proceeding in such a way we can obtain, in each sufficiently large finite box, a "unique cluster" and consequently prove that the momenta are uniformly distributed.

A further difficulty arises if the kinetic energy is quadratic (i.e., the usual "Gaussian case"). In fact, in this case all the above mentioned dynamics preserve also the center of mass of any finite cluster of particles. To complete the argument in this case it would be necessary to perform cluster deformations that conserve the center of mass, hence substantially complicating the above argument. We believe that our program could be carried out for the Gaussian case as well but we stop short of it also in view of the fact that its application to hydrodynamics is unclear (see point (d) in the following discussion). As in

[OVY] we consider only stationary measures having finite entropy density with respect to a grancanonical Gibbs measure. This condition seems to characterize a nice class of regular measures[1]. To complete our argument, various extra assumptions are necessary:

(a) The range of the stochastic interaction is finite but must be strictly larger than the one of the deterministic potential.

(b) The invariant measures considered must have sufficiently high particles density. More precisely, we need to be sure that, for almost any configuration, any sufficiently large box contains at least two particles interacting stochastically. The bound on the density we ask here is very rough, and we believe it can be substantially improved by using a more refined argument. Alternatively one can assume that the average potential energy is positive, which implies that in a box large enough at last two particle interact deterministically, therefore stochastically. Unfortunately potential energy is not a conserved quantity, so usually one does not have any information about its average value, that is why we prefer a condition on the density, which is stricter but easier to use.

(c) We assume that the measures considered are separately invariant for the deterministic and the stochastic dynamics. Furthermore, they must be reversible for the stochastic dynamics alone. The reversibility with respect to the global stochastic dynamics is a general condition which implies (cf. Proposition 2.2) the conditions previously used to derive the hydrodynamic limit: the invariance for each local stochastic dynamics (cf. [OVY]).

(d) In order to apply our results to obtain hydrodynamic limits following [OVY] we consider kinetic energies that are not quadratic, since [OVY] does not apply to the quadratic case. Nonetheless, we must assume a mild restriction on the kinetic energy function: the local dynamics cannot have undesired invariant (like the center of mass in the Gaussian case, see Lemma 2.5). We provide examples of kinetic energy functions that satisfy both our condition and the ones assumed in [OVY] (cf. Appendix 1).

As a consequence of our result, the hydrodynamic limit obtained in [OVY][2] is extended to the Hamiltonian dynamics with stochastic perturbation considered in the present paper.

For lattice systems the problem of ergodicity is solved in [FFL] in a more satisfactory way. In fact, there it is not needed condition (c), i.e., only the invariance with respect to the total dynamics (deterministic + stochastic) is required.

Concerning condition (c), notice that we could have asked the invariance for the finite stochastic dynamic in each finite box. We prove indeed that this is equivalent to the global reversibility (cf. Proposition 2.2). Proposition 2.2 has an interest in itself: it says that if a stochastic dynamics on a lattice in finite dimension is hypoelliptic then for the corresponding infinite dynamics all the reversible measures are given by Gibbs measures. This generalize a result of M. Zhu (cf.[Z]) to the "hypoelliptic" situation.

The next section contains a more precise description of the results outlined here, together with the plan of the paper.

[1] It is intended that we are always considering *translation invariant* measures

[2] See Theorem 2.1 of [OVY]

## 1. Notations and results

### Sample space

A point of $\mathbb{R}^3 \times \mathbb{R}^3$ will be denoted by $(q, p)$ and the sample space $\Omega$ will consist of points $\omega = \{(q_\alpha, p_\alpha)\}$. Any bounded region $B$ in $\mathbb{R}^3$ will contain only a finite number of particles, with positions $q_\alpha$, in addition one can think of $p_\alpha$ as tags, and consider the corresponding finite configuration in $B \times \mathbb{R}^3$.

### Interaction

We consider a radial repelling finite range smooth pair potential $V(q_\alpha - q_\beta) \not\equiv 0$ such that:

i) $V(x) = 0 \quad |x| > R_0$ (finite range)

ii) $V$ is superstable (i.e. it satisfies the superstability inequality: *there exists* $B > 0$ *and* $A > 0$ *such that for any finite box* $\Lambda$ *and any configuration we have*:

$$\sum_{q_\alpha \in \Lambda} \sum_{q_\beta \in \Lambda} V(q_\alpha - q_\beta) \geqq \frac{A}{|\Lambda|} |\omega_\Lambda|^2 - B|\omega_\Lambda| \tag{1.1}$$

((see [R])).

iii) $\langle x, \nabla V(x) \rangle \equiv \sum_i^3 x_i \frac{\partial V}{\partial x_i}(x) \le 0 \ \forall x \in \boldsymbol{R}^3$ (repelling interaction)

The repelling condition (iii) is of a technical nature and it should be possible to remove it by a more accurate analysis. However, by removing or weakening (iii), condition (ii) would become essential, while at present it is a consequence of (i) and (iii). In fact, (i) and (iii) force $V \geqq 0$ and $V(x) > 0$ for each $x$ in a neighborhood of 0, which implies a stronger inequality than (1.1).

### Kinetic energy

It is given by a convex function $\phi \in C^\infty(\mathbb{R}^3)$. One must distinguish between two cases:

(G) $\phi(p)$ is a quadratic function of $p$, which is the classical *Gaussian* case.

(NG) $\phi(p) = \sum_{i=1}^3 \varphi(p^i)$ with $\varphi$ a strictly convex smooth positive function on $\mathbb{R}$ with

(i) $\varphi''(x) \neq 0$ for each $x \in \mathbb{R}^3$.

(ii) $\frac{1}{2} \frac{d^2}{dx^2} (\varphi''(x))^2 = \varphi'''(x)^2 + \varphi^{iv}(x)\varphi''(x) \neq 0$ apart from, at most, finitely many points.

In addition, we require the invariance for reflections, i.e. $\varphi(x) = \varphi(-x)$. We will refer to this case as the *non-Gaussian* case.

Notice that, if $\frac{d^2}{dx^2}(\varphi''(x))^2 = 0$ for each $x$ the condition $\varphi(x) = \varphi(-x)$ implies $\varphi''(x)$ =constant, i.e., we have the Gaussian case. This shows that, morally, our classification covers all the possible cases; yet, it could be interesting to carry out a more detailed investigation.

In this paper we will consider only the non-Gaussian case, yet some results concerning the Gaussian case will be derived (Lemma 2.5, 2.6) in order to

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[3] This strengthened convexity will be needed only in the proof of Lemma 3.2 and could be weakened, by adding some unpleasant estimates, to saying that $\varphi''$ can be zero only at isolated points and its zeroes must have finite multiplicity

enlighten the difficulties (due to a peculiar degeneration) arising in such a case[4].

As already mentioned, our main motivation to treat the case (NG) is to apply the present results to the derivation of the hydrodynamic limit. To do so, the kinetic energy function must satisfy the conditions:

$$\left|\frac{\partial\phi}{\partial p^j}\right| \leqq C', \qquad \left|\frac{\partial^2\phi}{\partial p^j \partial p^i}\right| \leqq C'' \quad \forall p \in \mathbb{R}^3, \tag{1.2}$$

which are clearly not satisfied by the classical case (G).

**Hamiltonian dynamics**

The Hamiltonian is defined by the formal expression:

$$\mathscr{H}(\omega) = \sum_{\alpha} \phi(p_\alpha) + \frac{1}{2}\sum_{\alpha}\sum_{\beta\neq\alpha} V(q_\alpha - q_\beta)$$

and the Liouville operator by

$$L = \sum_{\alpha}\sum_{i=1}^{3}\left[\partial_{p_\alpha^i}\mathscr{H}\ \partial_{q_\alpha^i} - \partial_{q_\alpha^i}\mathscr{H}\ \partial_{p_\alpha^i}\right].$$

In this paper, we are not concerned with the existence of the dynamics generated by $L$ or its stochastic perturbations. Our aim is simply to characterize the probability measures on $\Omega$ that are 'formally' invariant (see Theorem 1.1). For a more detailed description of the above objects see [*AGGLM*, Sect.2].

**Stochastic perturbation of the dynamics**

We will use the notation $v_\alpha^i \equiv \phi_i(p_\alpha) \equiv \partial_{p_\alpha^i}\phi$. In the following, smooth will mean always differentiable infinitely many times.

For each smooth function $\eta_{\alpha\beta} : \mathbb{R}^6 \to \mathbb{R}^3$, (i.e. $\eta_{\alpha\beta} = \eta_{\alpha\beta}(p_\alpha, p_\beta)$) we define the vector field

$$X(\eta_{\alpha\beta}) = \langle \eta_{\alpha\beta}, D_{\alpha\beta}\rangle \equiv \sum_{i=1}^{3} \eta_{\alpha\beta}^i D_{\alpha\beta}^i$$

where $D_{\alpha\beta} = \partial_{p_\alpha} - \partial_{p_\beta}$.

We are interested in vector fields with null divergence, i.e.,

$$\mathrm{div}(X(\eta_{\alpha\beta})) = \langle D_{\alpha\beta}, \eta_{\alpha\beta}\rangle = \sum_{i=1}^{3} D_{\alpha\beta}^i \eta_{\alpha\beta}^i = 0. \tag{1.3}$$

Furthermore, we ask that $X(\eta_{\alpha\beta})$ is tangent to the surfaces, in $\mathbb{R}^3 \times \mathbb{R}^3$,

$$\begin{cases} p_\alpha^i + p_\beta^i = c^i, \quad i = 1,2,3, \\ \phi(p_\alpha) + \phi(p_\beta) = c^0. \end{cases}$$

[4] Actually, all what we do up to Lemma 2.5 holds for both cases

This is insured by the orthogonality relation

$$\langle \eta_{\alpha\beta}, D_{\alpha\beta}(\phi(p_\alpha) + \phi(p_\beta)) \rangle = 0 \tag{1.4}$$

(equivalently, $\langle \eta_{\alpha\beta}, v_\alpha \rangle = \langle \eta_{\alpha\beta}, v_\beta \rangle$), which will imply the conservation of energy and momenta with respect to the stochastic dynamics.

In addition, we require a further "Condition on the Noise" that boils down to a genericity condition and will be spelled out in detail in Sect. 2, before Lemma 2.7[5].

Let $X(\eta_{\alpha\beta})^*$ be the adjoint of $X(\eta_{\alpha\beta})$ with respect to the measures

$$e^{\lambda_4(-\phi(p_\alpha)-\phi(p_\beta))+\lambda\cdot(p_\alpha+p_\beta)}\, d\, p_\alpha\, d\, p_\beta$$

for any $\lambda_4 > 0$ and $\lambda = (\lambda_1, \lambda_2, \lambda_3)$, with the restriction that

$$\int \exp(-\lambda_4 \phi(p) + \lambda \cdot p)\, d\, p < +\infty.$$

We have, because the null divergence and the orthogonality property, that $X(\eta_{\alpha\beta})^* = -X(\eta_{\alpha\beta})$.

We use the previous vector fields to define an operator of the second order that will be the generator of the stochastic perturbation. Consider a finite number $K \geqq 3$ of vectors $\{\eta^\theta_{\alpha\beta}\}$ with the properties above. We define the operator

$$\hat{L}_{\alpha\beta} = -\frac{1}{2} \sum_{\theta=1}^{K} X(\eta^\theta_{\alpha\beta})^* X(\eta^\theta_{\alpha\beta}) \;=\; \frac{1}{2} \sum_{\theta} X(\eta^\theta_{\alpha\beta})^2.$$

Moreover, we require that, at each point, the linear combination of $\{\eta^\theta_{\alpha\beta}\}$ spans a two dimensional subspace of $\mathbb{R}^3$ (the maximum compatible with (1.4)), eventually apart from a set $\widetilde{\Sigma}^s_{\alpha\beta}$ consisting of the finite union of codimension–two manifolds. Therefore, $\hat{L}_{\alpha\beta}$ is selfadjoint, and elliptic outside $\widetilde{\Sigma}^s_{\alpha\beta}$. For later purposes, we define

$$\widetilde{\Sigma}_{\alpha\beta} = \widetilde{\Sigma}^s_{\alpha\beta} \cup \{(p_\alpha,\ p_\beta) \mid v_\alpha = v_\beta\}, \tag{1.5}$$

by convexity follows that $\widetilde{\Sigma}_{\alpha\beta}$ is the finite union of smooth manifold with codimension two as well.

Let $\sigma(q)$ be a radial smooth function on $\mathbb{R}^3$, such that $\sigma(q) > 0$ for each $\|q\| < R_1$, and $\sigma(q) = 0$ for each $\|q\| \geq R_1 > 4R_0$. Then we consider the operator

$$\hat{L} = \sum_{\alpha,\beta} \sigma(q_\alpha - q_\beta) \hat{L}_{\alpha\beta}.$$

Note that we can assume, without loss of generality, $\eta_{\alpha\beta} = \eta_{\beta\alpha}$.

[5] In Appendix I we show explicit examples that satisfy such condition, but we believe that it is "generically" satisfied and hence very general

In the following considerations it will be important that $\sigma$ is strictly positive for a radius $R_1$ strictly greater than $4R_0$, i.e. that the range of the stochastic interaction is larger than the one of the 'deterministic' interaction[6].

**Gibbs measures**

Let $\Lambda \subset \mathbb{R}^3$. Each configuration $\omega \in \Omega$ can be written as $\omega = \{\omega_\Lambda, \omega_{\Lambda^c}\}$ where $\omega_\Lambda = \{(q_\alpha, p_\alpha) \in \omega \,|\, q_\alpha \in \Lambda\}$.

Let $\mathbb{P}$ be a probability measure on $\Omega$. If the $\mathbb{P}$-conditional distribution of $\omega_\Lambda$, given the configuration outside $\omega_{\Lambda^c}$, is proportional to

$$\frac{1}{n!} \exp \left[ \lambda_0 n + \sum_{\alpha=1}^{n} \sum_{i=1}^{3} \lambda_i p_\alpha^i - \lambda_4 \mathscr{H}_{\Lambda,n}(\omega_\Lambda, \omega_{\Lambda^c}) \right],$$

then $\mathbb{P}$ is called Gibbs Measure (or grandcanonical Gibbs measure). In the above expression $n$ is the number of particles in $\Lambda$ (that we will denote by $|\omega_\Lambda|$) and the local Hamiltonian is defined by

$$\begin{aligned} &\mathscr{H}_{\Lambda,n}(\omega_\Lambda, \omega_{\Lambda^c}) \\ &\quad = \sum_{q_\alpha \in \omega_\Lambda} \left[ \phi(p_\alpha) + \frac{1}{2} \sum_{q_\beta \in \omega_\Lambda;\, \alpha \neq \beta} V(q_\alpha - q_\beta) + \sum_{q_\beta \in \omega_{\Lambda^c}} V(q_\alpha - q_\beta) \right]. \end{aligned}$$

**Statement of the result**

Let $Q$ and $P$ be two probability measures on $\Omega$, and let $Q_\Lambda$ and $P_\Lambda$ be their restrictions on a finite box $\Lambda$. The relative entropy of $Q_\Lambda$ with respect to $P_\Lambda$ is defined by[7]

$$H_\Lambda(Q|P) \;=\; \sup_{F \in \mathscr{F}_\Lambda} \left\{ \mathbb{E}^Q(F) - \log \, \mathbb{E}^P(\exp(F)) \right\} \tag{1.6}$$

where $\mathscr{F}_\Lambda$ are the smooth functions localized in $\Lambda$. For the properties of $H_\Lambda$ see, for example, [OVY]. In the following $Q$ will be the translation invariant measure under consideration, while $\mathbb{P}$ will be any grancanonical Gibbs measure for the interaction $V$.

**Lemma 1.1 .** *If there exists a constant $C$ such that for each box $\Lambda$,*

$$H_\Lambda(Q|\mathbb{P}) \leqq C|\Lambda|$$

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[6] The factor 4 is due to technical reasons, plays a role only in Sect. 4 and is certainly not optimal

[7] By $\mathbb{E}^Q$ we mean the expectation with respect to the measure $Q$

then,

(i) $\mathbb{E}^{Q}\left(|\Lambda|^{-2}|\omega_\Lambda|^2\right) \leqq C_1 < \infty,$

(ii) $\mathbb{E}^{Q}\left(|\Lambda|^{-1}\sum_{q_\alpha\in\Lambda}\|p_a\|\right) \leqq C_2 < \infty,$

(iii) $\mathbb{E}^{Q}\left(|\Lambda|^{-1}\sum_{q_\alpha\in\Lambda}\left[\phi(p_\alpha)+\sum_{q_\beta\in\omega}V(q_\alpha-q_\beta)\right]\right) \leqq C_3 < \infty,$

where $C_1$, $C_2$, $C_3$ are constants independent on $\Lambda$.

*Proof*. The inequalities (ii) and (iii) are consequences of the following entropy inequality:

$$\mathbb{E}^{Q}(F) \leqq \frac{1}{\beta}\log\mathbb{E}^{\mathbb{P}}\left(\exp(\beta F)\right) + \frac{1}{\beta}H(Q|\mathbb{P}), \tag{1.7}$$

which is valid for any local function $F$ and any constant $\beta > 0$. Inequality (1.7) follows directly from the definition (1.6). Then for sufficiently small $\beta$, we have

$$\mathbb{E}^{Q}\left(|\Lambda|^{-1}\sum_{q_\alpha\in\Lambda}\|p_a\|\right) \leqq \frac{1}{\beta|\Lambda|}\log\,\mathbb{E}^{\mathbb{P}}\left(\exp\left(\beta\sum_{q_\alpha\in\Lambda}\|p_a\|\right)\right)+\frac{1}{\beta}C \leqq \frac{1}{\beta}C'$$

In a similar way one can prove (iii).[8] While (i) follows by the same argument and the superstability inequality (1.1). □

Define

$$\begin{aligned}\rho(\omega) &= \lim_{|\Lambda|\to\infty}|\Lambda|^{-1}|\omega_\Lambda|,\\ \pi(\omega) &= \lim_{|\Lambda|\to\infty}|\Lambda|^{-1}\sum_{q_\alpha\in\Lambda}p_a \equiv \lim_{|\Lambda|\to\infty}|\Lambda|^{-1}\Pi_\Lambda,\\ e(\omega) &= \lim_{|\Lambda|\to\infty}|\Lambda|^{-1}\sum_{q_\alpha\in\Lambda}\Bigg[\phi(p_\alpha)+\frac{1}{2}\sum_{q_\beta\in\Lambda;\,\alpha\neq\beta}V(q_\alpha-q_\beta)\\ &\qquad +\sum_{q_\beta\notin\Lambda}V(q_\alpha-q_\beta)\Bigg]\\ &\equiv \lim_{|\Lambda|\to\infty}|\Lambda|^{-1}E_\Lambda.\end{aligned}$$

The above Lemma 1.1, and the translation invariance of Q, ensures that the limits $\rho(\omega)$, $e(\omega)$, $\pi(\omega)$ exist $Q$-almost everywhere[9].

[8] Although the integrand (iii) is not localized in $\Lambda$, due to the compact support of $V$, it is localized in a slightly larger volume, which is as well for our present purposes

[9] The case of $e(\omega)$ is less obvious because of the presence of the boundary terms. Yet, consider a small box $\Delta$ and *Footnote 9 continued*

The aim of this paper is to prove the following:

**Theorem 1.2 .** *Let $Q$ be a translation invariant probability measure on $\Omega$, if*

(i) *There exists a constant $C$ such that for each box $\Lambda$, $H_\Lambda(Q|\mathbb{P}) \leqq C|\Lambda|$;*
(ii) *$\exists \rho_0 > \rho_*$ such that $Q(\{\omega \in \Omega \mid \rho(\omega) > \rho_0\}) = 1$, where $\rho_* = 3/(4R_1^3\pi)$;*
(iii) *$Q$ is invariant w.r.t. the dynamics generated by $L$ (the deterministic part), in the sense that, for any smooth local function $F_\Lambda(\omega_\Lambda)$,*

$$\mathbb{E}^Q(LF_\Lambda) \;=\; 0\;;$$

(iv) *$Q$ is reversible with respect to $\hat{L}$ (the stochastic perturbation), i.e., for any two smooth local functions $\varphi$ and $\psi$ holds*

$$\mathbb{E}^Q(\psi\hat{L}\varphi) \;=\; \mathbb{E}^Q(\varphi\hat{L}\psi)\;;$$

*then $Q$ belongs to the closed convex hull of the (gran canonical) Gibbs Measures.*

*Remark 1.3*. Condition (ii) on the density is a sufficient condition in order to always find at least two particles interacting stochastically. Since the range of the deterministic interaction is smaller than the one of the stochastic interaction, this condition may be replaced by ensuring that the average potential energy is strictly positive, i. e. if we define

$$u(\omega) \;=\; \lim_{|\Lambda|\to\infty} |\Lambda|^{-1} U_\Lambda \;\equiv\; \lim_{|\Lambda|\to\infty} |\Lambda|^{-1} \sum_{q_\alpha, q_\beta \in \Lambda} V(q_\alpha - q_\beta)$$

then the condition reads

$$Q(\{u(\omega) \geqq \varepsilon > 0\}) \;=\; 1.$$

Unfortunately, this condition is not practical because $u(\omega)$ does not correspond to a conserved quantity. It would be of no use for the application to hydrodynamics (cf.[OVY]), where we cannot have such information on Q.

The proof of Theorem 1.1 will be carried out in four parts. In the next section we will construct a multitude of local dynamics that leave the finite dimensional restrictions of the measure $Q$ invariant. Section 3 is dedicated to

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$$\widetilde{E}_\Delta = \sum_{q_\alpha \in \Delta} \left[\phi(p_\alpha) + \tfrac{1}{2} \sum_{q_\beta;\, \alpha\neq\beta} V(q_\alpha - q_\beta)\right],$$

then $\Lambda$ can be obtained by space translations $\tau_j$ of $\Delta$ ($\Lambda = \cup_j \tau_j \Delta$), with $\tau_i\Delta \cap \tau_j\Delta \subset \partial(\tau_i\Delta) \cup \partial(\tau_j\Delta)$ for $i \neq j$. If we introduce a box $\Lambda'$ concentrical to $\Lambda$ but $R_0$ larger than $\Lambda$, we can obtain $\Lambda'$ as well from space translations of $\Delta$. Then, thanks to the positivity of the potential,

$$\tfrac{1}{|\Lambda|} \sum_j \tau_j \widetilde{E}_\Delta \leq \tfrac{1}{|\Lambda|} E_\Lambda \leq \tfrac{|\Lambda'|}{|\Lambda|} \tfrac{1}{|\Lambda'|} {\sum_j}' \tau_j \widetilde{E}_\Delta,$$

where by $\sum_j$ we mean a sum over the space translations necessary to cover $\Lambda$, while by ${\sum_j}'$ the sum over the space translations needed to cover $\Lambda'$. Notice that if $\Lambda$ is a box of size $L$, then $|\Lambda| = L^3$ while $|\Lambda'| = (L + R_0)^3$, so $\lim_{|\Lambda|\to\infty} \frac{|\Lambda'|}{|\Lambda|} = 1$. Hence, $e(\omega)$ can be obtained by ergodic limit, with respect to space translations, as well.

the characterization of typical configurations for the class of measures $Q$ under consideration. In Sect. 4 we show that the above mentioned dynamics give a local characterization of $Q$ weaker than the one implied by DLR equations, but sufficient to claim that the global distribution of the momenta, conditioned to the positions, is given by a convex combination of "Maxwellian" (corresponding to the proper $\phi$). We conclude the argument in Sect. 5, along the line of [OVY], by proving that in the infinite limit the kinetic energy is "invariant" for the deterministic dynamic generated by $L$. Thus, each component of the convex combination is invariant for $L$. A classic argument (cf.[GV] and [OV]) shows that invariant distributions for $L$ that have distribution of the momenta conditioned to the position given by a Maxwellian are canonical Gibbs measures.

## 2. Clusters and local dynamics

Given a configuration $\omega$, we call "connected" two particles that are sufficiently close to interact stochastically ($\alpha$ and $\beta$ are connected if $\sigma(q_\alpha - q_\beta) > 0$, i.e. $|q_\alpha - q_\beta| < R_1$). We call "cluster" a set of particles such that any two can be joined by a chain of connected ones. We call "isolated cluster" a cluster such that the particles outside it are not connected to any particle in the cluster.

Any configuration $\omega$ can be grouped in many isolated clusters $\{\Gamma_i(\omega_q)\}$. These clusters may be finite or infinite. Also, the restriction of any configuration to a finite region $\omega_\Lambda$ is grouped into finite clusters $\{\Gamma_i^\Lambda(\omega_q)\}$, where we have overlooked the connections with the particles outside $\Lambda$.

To simplify notations, in this section we denote by $\widetilde{\mathbb{E}}$ the expectation of Q conditioned to a configuration of positions $\omega_q$.

Consider a cluster $\Gamma_i^\Lambda$, and let $n$ be the number of particles in it. Then $\hat{L}_{\Gamma_i^\Lambda} = \sum_{\alpha\beta\in\Gamma_i^\Lambda} \sigma(q_\alpha - q_\beta)\hat{L}_{\alpha\beta}$ is an operator on $\mathbb{R}^{3n}$ and it conserves the quantities

$$\sum_{q_\alpha\in\Gamma_i^\Lambda} p_\alpha^1, \quad \sum_{q_\alpha\in\Gamma_i^\Lambda} p_\alpha^2, \quad \sum_{q_\alpha\in\Gamma_i^\Lambda} p_\alpha^3, \quad \sum_{q_\alpha\in\Gamma_i^\Lambda} \phi(p_\alpha).$$

Let us consider the corresponding connected hypersurfaces of dimension $3n - 4$:

$$\Sigma_c \equiv \left\{ (p_1, \dots, p_n) \,\middle|\, \sum_\alpha p_\alpha^1 = c^1, \sum_\alpha p_\alpha^2 = c^2, \sum_\alpha p_\alpha^3 = c^3, \right.$$
$$\left. T = \sum_\alpha \phi(p_\alpha) = c^4 \right\}, \tag{2.1}$$

and the sets

$$\widetilde{\Sigma}_c \equiv \left\{ (p_1, \dots, p_n) \in \Sigma_c \,\middle|\, (p_\alpha, \ p_\beta) \in \widetilde{\Sigma}_{\alpha\beta} \quad \text{for some } \alpha, \beta \in \{1, \dots, n\} \right\}$$

where $\widetilde{\Sigma}_{\alpha\beta}$ has been defined by (1.5). Note that $\widetilde{\Sigma}_c$ is the finite union of smooth submanifolds in $\Sigma_c$ of codimension, at least, two; hence $\Sigma_c \setminus \widetilde{\Sigma}_c$ is connected.

**Lemma 2.1.** $L_{\Gamma_i^\Lambda}$ *is an hypoelliptic operator on* $\Sigma_c \backslash \widetilde{\Sigma}_c$*, i.e., the Lie algebra generated by*

$$\left\{X^{\theta}_{\alpha\beta} \mid \alpha, \beta \in \Gamma_i\right\}$$

*spans the tangent space of* $\Sigma_c$ *at each point in* $\Sigma_c \setminus \widetilde{\Sigma}_c$.

*Proof.* Let us fix our attention on an arbitrary point in $\Sigma_c \backslash \widetilde{\Sigma}_c$.

We can represent a cluster of $n$ particles by a graph $\widetilde{\mathscr{G}}_n$ with $n$ vertices. Each vertex of the graph corresponds to a particle; two vertices are joined by a bond (edge) if and only if they correspond to particles close enough to interact stochastically. In a cluster of $n$ particles there are at least $n-1$ bonds:

$$\sharp\{(\alpha, \beta) \mid \sigma(q_\alpha - q_\beta) < R_1\} \geq \; n - 1.$$

We consider then a minimal acyclical (i.e., the edges do not form any loop) connected sub-graph $\mathscr{G}_n$ of $\widetilde{\mathscr{G}}_n$. This amounts to choose $n-1$ bonds. We choose a set of $2n-2$ linearly independent vector fields $X^{\theta_i}_{\alpha\beta}$, $i \in \{1, 2\}$, $(\alpha\beta)$ being a bond in $\mathscr{G}_n$.

Two bonds are said to be contiguous if they have one vertex in common. There are at least $n-2$ couples of contiguous bonds. If we compute the commutator between two contiguous bonds $\{\alpha\beta\}$ and $\{\gamma\beta\}$ we obtain

$$\left[X^{\theta_i}_{\alpha\beta}, X^{\theta_j}_{\gamma\beta}\right] \; = \left\langle \eta^{\theta_i}_{\alpha\beta}, A_{\theta_j} D_{\gamma\beta} \right\rangle - \left\langle \eta^{\theta_j}_{\gamma\beta}, B_{\theta_i} D_{\alpha\beta} \right\rangle$$

where $A_{\theta_j}$ is the matrix whose element $kl$ is given by $D^k_{\alpha\beta}(\eta^{\theta_j}_{\gamma\beta})_l$ (that is, the derivative, with respect to the $k$ component of $p_\alpha - p_\beta$, of the $l$ component of $\eta^{\theta_j}_{\gamma\beta}$) and $B_{\theta_i}$ is the matrix $D^k_{\gamma\beta}(\eta^{\theta_i}_{\alpha\beta})_l$.

Let $\mathscr{G}_k$, $k < n$ be a sequence of connected sub-graphs of $\mathscr{G}_n$ containing $k$ vertices, $\mathscr{G}_k \subset \mathscr{G}_{k+1}$, and $\mathscr{A}_k$ the Lie algebra generated by the vector fields $X^{\theta_i}_{\alpha\beta}$, $(\alpha, \beta) \in \mathscr{G}_k$. We will show by induction that $\mathscr{A}_n$ contains $3n-4$ linearly independent vector fields.

The fact is clear for $\mathscr{A}_2$. Let us suppose that it is true for $\mathscr{A}_k$ and let us show that $\mathscr{A}_{k+1}$ contains three extra vector fields linearly independent from all the previous ones.

We start by noticing that $\mathscr{G}_{k+1}$ is obtained by adding a particle (vertex), say $\alpha$, to $\mathscr{G}_k$ and that such particle is connected with the rest of the graph by only one bond. Let us call $\beta$ the particle to which $\alpha$ is connected and $\gamma$ a particle, in $\mathscr{G}_k$, connected to $\beta$. $\mathscr{A}_{k+1}$ is larger (or equal) than the set of vector fields generated by all the vectors in $\mathscr{A}_k$, $X^{\theta_i}_{\alpha\beta}$ and $[X^{\theta_i}_{\alpha\beta}, X^{\theta_j}_{\gamma\beta}]$. If our inductive hypothesis is false, then there exist $\lambda_i \in \mathbb{R}$, $\mu_{ij} \in \mathbb{R}$ and $Y_{ij} \in \mathscr{A}_k$ such that

$$0 = \lambda_1 X^{\theta_1}_{\alpha\beta} + \lambda_2 X^{\theta_2}_{\alpha\beta} + \mu_{ij}\left[X^{\theta_i}_{\alpha\beta}, X^{\theta_j}_{\gamma\beta}\right] + Y_{ij}$$

for each $i, j \in \{1, 2\}$. To see that this is impossible we apply the above tangent vector to $p_\alpha$

$$0 = \lambda_1 \eta^{\theta_1}_{\alpha\beta} + \lambda_2 \eta^{\theta_2}_{\alpha\beta} - \mu_{ij}\left(B_{\theta_i}\right)^T \eta^{\theta_j}_{\gamma\beta}.$$

By multiplying this relation by $D_{\alpha\beta}T$, where $T$ is the kinetic energy defined inside (2.1), and by using the fact that, for points not in $\widetilde{\Sigma}_c$, $D_{\alpha\beta}T \neq 0$, follows:

$$0 = \mu_{ij}\left\langle B_{\theta_i} D_{\alpha\beta} T,\, \eta_{\gamma\beta}^{\theta_j}\right\rangle.$$

To simplify the previous expression we recall that, by definition, $\langle \eta_{\alpha\beta}^{\theta_j}, D_{\alpha\beta}T\rangle = 0$ and applying the operator $D_{\gamma\beta}$ to such an equality yields

$$0 = B_{\theta_i} D_{\alpha\beta} T + H_\beta \eta_{\alpha\beta}^{\theta_i},$$

where $H_\beta$ is the matrix $\partial_{p_\beta}\partial_{p_\beta} T$. It is essential to notice that the Hessian $H_\beta$ is positive definite, given the convexity of $\phi$.

By collecting the previous relations follows:

$$0 = \mu_{ij}\left\langle H_\beta \eta_{\alpha\beta}^{\theta_i},\, \eta_{\gamma\beta}^{\theta_j}\right\rangle.$$

To conclude the proof it is enough to show that at least one of the scalar products $\langle H_\beta \eta_{\alpha\beta}^{\theta_i}, \eta_{\gamma\beta}^{\theta_j}\rangle$ is different from zero. Notice that $\{\eta_{\alpha\beta}^{\theta_i}\}$ and $\{\eta_{\gamma\beta}^{\theta_j}\}$ span two two-dimensional planes in $\mathbb{R}^3$. Such planes must intersect, at least, at a line. Let $\zeta \neq 0$ be a vector belonging to such a line. Clearly there exists $\tau_i$ $\tau_i'$ such that $\sum_l \tau_l \eta_{\alpha\beta}^{\theta_l} = \zeta$ and $\sum_l \tau_l' \eta_{\alpha\beta}^{\theta_l} = \zeta$. This yields to the contradiction $0 = \langle H_\beta \zeta, \zeta\rangle$. That is, the vectors $X_{\alpha\beta}^{\theta_1}$, $X_{\alpha\beta}^{\theta_2}$, $\sum_{ij} \tau_i \tau_j' [X_{\alpha\beta}^{\theta_i}, X_{\gamma\beta}^{\theta_j}]$ are linearly independent with respect to themselves and with respect to the ones in the algebra $\mathscr{A}_k$. □

**Proposition 2.2.** *The condition (iv) of reversibility of the measure $Q$ implies that, for any bond corresponding to connected particles $b = \{\alpha, \beta\}$ in the configuration $\omega_q$ and any smooth local function $\phi$, we have*

$$\mathbb{E}^Q(X_{\alpha\beta}^{\theta}\phi \mid \omega_q) \equiv \widetilde{\mathbb{E}}\left(X_{\alpha\beta}^{\theta}\phi\right) = 0$$

*provided that $supp(\phi) \cap \widetilde{\Sigma}_c = \emptyset$.*

*Proof.* Let $\varphi$ and $\psi$ be arbitrary smooth local functions with support in a finite region $\Lambda'$, and $\Lambda$ a region containing $\Lambda'$ and so large that particles inside $\Lambda'$ cannot interact with particles outside $\Lambda$. Since the size of the support of a test function can be assumed arbitrarily small, without loss of generality, we will carry out a local argument; namely we will assume that the support of $\psi$ is contained in a conveniently small neighborhood of an arbitrary configuration not belonging to $\widetilde{\Sigma}_c$ (i.e., the support must be so small as not to intersect $\widetilde{\Sigma}_c$). If the configuration $\omega_q$ does not contain any cluster in $\Lambda$, then the proposition is obviously true. Next, we will assume that only one isolated cluster[10] $\Gamma_\Lambda$ is present (the case in which several isolated clusters are present can be treated in the same way, as we will remark at the end of the proof). The graph associated

[10] Remember that an "isolated cluster" in $\Lambda$ can still have interactions with particles outside $\Lambda$

to $\Gamma_\Lambda$ is finite and it will contain $M$ bonds. The reversibility condition (iv) implies:

$$-\widetilde{\mathbb{E}}\left(\psi \hat{L} \varphi\right) = \sum_{b \in \Gamma_\Lambda} \sum_{\theta} \sigma_b \widetilde{\mathbb{E}}\left(X_b^\theta \psi \; X_b^\theta \varphi\right)$$

where $b = (\alpha, \beta)$ is a generic bond in $\Gamma_\Lambda$, i.e., $\sigma_b = \sigma(q_\alpha - q_\beta) > 0$. Note that the operators $X_{\alpha\beta}$ with $q_\alpha$ or $q_\beta$ not in $\Lambda$ do not appear in the right hand side of the above equation, although it is possible that $\sigma(q_\alpha - q_\beta) \neq 0$; this is due to the fact that, since the test functions depend only on the particles in $\Lambda'$, if $q_\alpha \notin \Lambda$ and $\sigma(q_\alpha - q_\beta) \neq 0$, then $q_\beta \notin \Lambda'$ which implies $X_{\alpha\beta}\psi = 0 = X_{\alpha\beta}\varphi$.

A technical obstacle to our proof is that, in general, the vector fields $\{X_{b_1}^\theta, \ldots, X_{b_M}^\theta\}$ are neither linearly independent nor their linear combinations span all the Lie algebra that they generate. Typically, only $L \leqq KM$ such vector fields will be linearly independent[11], while the Lie algebra will be $N \geq L$ dimensional. To overcome such problem we choose, among $\{X_{b_1}^\theta, \ldots, X_{b_M}^\theta\}$ and their commutators, a subset of linearly independent vector fields $\{X_1, \ldots, X_N\}$ that form a base of the Lie algebra[12]. In addition, we require

$$\{X_1, \ldots, X_L\} \subset \{X_{b_1}^\theta, \ldots, X_{b_M}^\theta\}.$$

Thus, the original $KM$ vector fields can be expressed as linear combinations of the independent vector fields $\{X_1, \ldots, X_L\}$:

$$X_{b_j}^\theta = \sum_{i=1}^{L} v_{ji}^\theta \; X_i, \quad j = 1, \ldots, KM.$$

Since,

$$\begin{aligned}\left[X_{b_j}^\theta, X_{b_k}^{\theta'}\right] &= \sum_{l,\, p} \left[v_{jl}^\theta X_l, \; v_{kp}^{\theta'} X_p\right] \\ &= \sum_{l,\, p} \left\{ v_{jl}^\theta (X_l v_{kp}^{\theta'}) X_p - v_{kp}^{\theta'} (X_p v_{jl}^\theta) X_l + v_{jl}^\theta v_{kp}^{\theta'} [X_l, X_p] \right\}\end{aligned}$$

it is clear that $\{X_1, \ldots, X_L\}$ generates the complete Lie algebra under consideration.

Let $A$ be the $L \times L$ matrix with elements defined by

$$a_{i,k} = \sum_{j=1}^{M} \sum_{\theta} \sigma_{b_j} v_{ji}^\theta v_{jk}^\theta$$

then

$$\sum_{b \in \Gamma_\Lambda} \sum_{\theta} \sigma_b \widetilde{\mathbb{E}}\left(X_b^\theta \psi \; X_b^\theta \varphi\right) = \sum_{i,k=1}^{L} \widetilde{\mathbb{E}}\left(a_{ik} X_i \psi \; X_k \varphi\right).$$

[11] KM is the cardinality of $\{X_{b_1}^\theta, \ldots, X_{b_M}^\theta\}$; remember that $\theta \in \{1, \ldots, K\}$

[12] This is possible provided the support of $\psi$ is sufficiently small

It is easy to check that the matrix A is positive defined, and therefore invertible.

According to Lemma 2.1 $\{X_1, \dots, X_N\}$ span the tangent space of $\Sigma_c$ (the surfaces associated to the cluster $\Gamma_\Lambda$). Since such surfaces foliate the phase space of the particles contained in $\Gamma_\Lambda$, we can choose coordinates $(c, y)$ such that the vector fields $\{Y_i\}_1^N$, associated to the coordinates $\{y_i\}_1^N$, generates the tangent space of $\Sigma_c$ (i.e., for each $c$, $\{y\}$ is a system of coordinates for $\Sigma_c$). This implies, $\forall i, j$,

$$[Y_i, Y_j] = 0 \quad \text{and} \quad Y_i y_j = \delta_{ij}.$$

In addition, there exists an invertible $N \times N$ matrix $\Lambda$, such that

$$X_i = \sum_{j=1}^{N} \Lambda_{ij} Y_j \ .$$

Let us choose as function $\varphi$ a coordinate function $y_j$ multiplied by a smooth function with value one on the support of $\psi$, which, consequently, can be ignored. Applying $\hat{L}$ we have

$$\begin{aligned} \hat{L}\ y_j &= \sum_{k,i} X_k\ a_{ki}\ X_i\ y_j \\ &= \sum_{k,i} X_k\ a_{ki}\ \Lambda_{ij}, \end{aligned}$$

where we have used

$$X_i\ y_j \ = \ \sum_l \Lambda_{il}\ Y_l\ y_j \ = \ \Lambda_{ij}.$$

The reversibility relation then gives us:

$$-\sum_{k,i} \widetilde{\mathbb{E}}\left(\psi\ X_k\ \left(a_{ki}\ \Lambda_{ij}\right)\right) \ = \ \sum_{k,i} \widetilde{\mathbb{E}}\left(a_{ki}\ \Lambda_{ij}\ X_k\ \psi\right),$$

which is equivalent to

$$\sum_{k,i} \widetilde{\mathbb{E}}\left(X_k\left(\psi\ a_{ki}\ \Lambda_{ij}\right)\right) \ = \ 0.$$

Let $V = \Lambda^{-1}\mathbb{R}^L \subset \mathbb{R}^N$[13], then $A\Lambda : V \to \mathbb{R}^L$ is one to one and onto. Which means that for each $e^k \in \mathbb{R}^L$, $e^k = (0, \dots, 1, \dots, 0)$, there exists $\alpha^k \in V \subset \mathbb{R}^N$ such that $A\Lambda\alpha^k = e^k$. Moreover, in some small neighborhood of any configuration, $\alpha^k$ will vary smoothly.

We can make the following $L^2$ different choices of $\psi$

$$\psi_{jh} \ = \ \alpha_j^h \phi,$$

[13] By $\mathbb{R}^L$, here we mean $\{v \in \mathbb{R}^N \mid v_i = 0\ \forall i > L\}$.

where $\phi$ is a function with sufficiently small support around the configuration we are considering.

Summing over $j$ we obtain

$$0 = \sum_{i,j,k} \widetilde{\mathbb{E}}\left( X_k(\alpha_j^h \ \Lambda_{ij} a_{ki} \ \phi)\right) = \sum_k \widetilde{\mathbb{E}}\left(X_k e_k^h \phi\right),$$

that is to say

$$\widetilde{\mathbb{E}}(X_h \ \phi) \ = \ 0 \quad \forall\, h \in \{0, \dots, L\},$$

which implies our thesis.

The generalization to the situation where many clusters appear in the region $\Lambda$ is straightforward since, in the above argument, the coordinate functions $y_j$ are localized on the particular cluster we are considering. Hence, the argument simply factors over the different clusters. □

Up to now we have seen that the measure is invariant with respect to vector fields that generate the tangent space to the surfaces of the momenta of the clusters $\Gamma_i^\Lambda$ with constant kinetic energy and momentum. This was done only by using the reversibility of the stochastic dynamics. If $\Lambda$ contains a unique cluster (like in the case with infinite range stochastic interaction), then this would imply that the measure on the momenta conditioned on the position is Microcanonical, i.e. we would have directly Lemma 5.1 below[14]. Unfortunately, in our case we cannot ignore the existence of isolated clusters. So what we can conclude at this point is that, conditioned on the positions, the distribution of the velocities in each cluster is Microcanonical. In order to attain the statement of Lemma 5.1, we need to somehow exchange the particles between clusters. The only way to do this is to generate, with the help of the Hamiltonian dynamics, other dynamics for which the measure is invariant and that permit such exchanges of particles among clusters. In the rest of the section we will define these dynamics and prove their local properties, and in Sect. 4 we will use them to move particles among clusters.

We start by studying the Lie algebra generated by $\{X^\theta_{\alpha\beta};\ [X^\theta_{\alpha\beta}, L]\}_{q_\alpha, q_\beta \in \Lambda}$.

**Lemma 2.3.** *For each region $\Lambda$, and for each local smooth function $\varphi$ localized in $\Lambda$, calling $\mathscr{A}_\Lambda$ the Lie algebra generated by the operators $\{X^\theta_{\alpha\beta};\ [X^\theta_{\alpha\beta}, L]\}_{\substack{q_\alpha, q_\beta \in \Lambda \\ \|q_\alpha - q_\beta\| < R_1}}$, we have*

$$\mathbb{E}^Q\left(X\varphi \;\middle|\; |\omega_\Lambda| = n;\ \omega_{\Lambda^c}\right) = 0$$

*for each $X \in \mathscr{A}_\Lambda$.*

*Proof.* The difficulties arise because $L$ does not conserve the number of particles in a finite region. We need to use here the stationarity of $Q$ with respect to $L$.

[14] See the arguments after Proposition 4.1 for details

Let $\chi_\varepsilon(q)$ be a smooth function equal to one if $q \in \Lambda$, and equal to 0 if the distance between $q$ and $\Lambda$ is larger than $\varepsilon$. We can then define $N_\varepsilon \equiv \sum_\alpha \chi_\varepsilon(q_\alpha)$ to be an approximation of the number of particles in $\Lambda$. Clearly, when $\varepsilon$ goes to zero, $N_\varepsilon$ tends to the number of particles contained in the closure of $\Lambda$, which, since $Q$ is locally absolutely continuous, equals almost everywhere the number of particles contained in the interior.

Let $h$ be a smooth function on $\mathbb{R}^+$ with compact support and $\varphi$ any smooth local function with support contained in the interior of $\Lambda$; in addition, we consider arbitrary smooth functions $\psi^\theta_{\alpha\beta} : \mathbb{R}^6 \to \mathbb{R}$ with support in $\Lambda \times \Lambda$ and we use them to define the local operators $X(\psi) \equiv \sum_{\alpha\beta\theta} \psi^\theta_{\alpha\beta}(q_\alpha, q_\beta)\sigma(q_\alpha - q_\beta)X^\theta_{\alpha\beta}$. By definition all these operators are part of the Lie algebra $\mathscr{A}_\Lambda$; in addition, $X(\psi)$, $[X(\psi), L]$ generate $\mathscr{A}_\Lambda$. Using the previous definitions, since $X(\psi)h(N_\varepsilon) = 0$, we have,

$$0 = \mathbb{E}([L, X(\psi)]\varphi h(N_\varepsilon)) = \mathbb{E}(h[L, X(\psi)]\varphi) - \mathbb{E}(\varphi X(\psi) Lh).$$

Since $Lh(N_\varepsilon) = h'(N_\varepsilon) \sum_{\gamma \in \Lambda^c} \langle p_\gamma, \nabla \chi_\varepsilon(q_\gamma)\rangle$, we have that $X(\psi)Lh = 0$. So we conclude that

$$0 = \mathbb{E}(h(N_\varepsilon)[L, X(\psi)]\varphi).$$

Letting $\varepsilon \to 0$ proves that it is possible to condition with respect to the number of particles in $\Lambda$; a similar computation shows that it is possible to condition with respect to the configuration outside $\Lambda$ as well. $\square$

Lemma 2.3 shows that $\mathscr{A}_\Lambda$ has interesting local properties, these are further clarified by the following Lemma. Consider configurations with $n$ particles in $\Lambda$ and define $\Pi_\Lambda$, $E_\Lambda$ like in the equations above Theorem 1.2.

**Lemma 2.4.** *The Lie Algebra $\mathscr{A}_\Lambda$ is tangent to the surface $\Pi_\Lambda = \text{constant}$, $E_\Lambda = \text{constant}$, and acts only on observables depending on the coordinates of the particles inside $\Lambda$.*

*Proof.* Given two particles $\alpha, \beta \in \Gamma$ we have

$$X^\theta_{\alpha\beta}\Pi_\Lambda = 0,$$

$$X^\theta_{\alpha\beta}E_\Lambda = 0,$$

$$\left[X^\theta_{\alpha\beta}, L\right]\Pi_\Lambda = X^\theta_{\alpha\beta}\sum_\gamma \frac{\partial T}{\partial q_\gamma} = 0,$$

$$\left[X^\theta_{\alpha\beta}, L\right]E_\Lambda = X^\theta_{\alpha\beta}\left[\sum_\gamma \left\langle \frac{\partial \mathscr{H}}{\partial p_\gamma}, \frac{\partial E_\Lambda}{\partial q_\gamma}\right\rangle - \left\langle \frac{\partial \mathscr{H}}{\partial q_\gamma}, \frac{\partial E_\Lambda}{\partial p_\gamma}\right\rangle\right].$$

Letting $\Delta = \mathscr{H} - E_\Lambda = \sum_{\gamma \notin \Lambda} \phi(p_\gamma) + \frac{1}{2} \sum_{q_\gamma \notin \Lambda} \sum_{q_\delta \notin \Lambda;\, \delta \neq \gamma} V(q_\gamma - q_\delta)$, and $H_\gamma = \left( \frac{\partial^2 \phi(p_\gamma)}{\partial p_\gamma^i \partial p_\gamma^j} \right)$, we can rewrite the last equation as

$$\begin{aligned} \left[X^\theta_{\alpha\beta}, L\right] E_\Lambda &= X^\theta_{\alpha\beta} \left[ \sum_\gamma \left\langle \frac{\partial \Delta}{\partial p_\gamma}, \frac{\partial E_\Lambda}{\partial q_\gamma} \right\rangle - \left\langle \frac{\partial \Delta}{\partial q_\gamma}, \frac{\partial E_\Lambda}{\partial p_\gamma} \right\rangle \right] \\ &= - \left\langle \frac{\partial \Delta}{\partial q_\alpha}, H_\alpha \eta^\theta_{\alpha\beta} \right\rangle + \left\langle \frac{\partial \Delta}{\partial q_\beta}, H_\beta \eta^\theta_{\alpha\beta} \right\rangle = 0, \end{aligned}$$

since $\Delta$ does not depend on $q_\alpha$ or $q_\beta$.

Similarly, a direct computation shows that, if $q_\gamma \notin \Lambda$, then

$$\begin{aligned} \left[X^\theta_{\alpha\beta}, L\right] q_\gamma &= 0, \\ \left[X^\theta_{\alpha\beta}, L\right] p_\gamma &= 0. \qquad \square \end{aligned}$$

At this point we have to distinguish between the Gaussian and the non-Gaussian case. The difference is that in the Gaussian case the center of mass is always conserved. Define

$$\Theta_\Lambda = \sum_{q_\alpha \in \Lambda} q_\alpha \ .$$

**Lemma 2.5.** *If $\phi$ is quadratic, the Lie Algebra $\mathscr{A}_\Lambda$ is tangent to the surface* $\Pi_\Lambda = \text{constant}$*,* $E_\Lambda = \text{constant}$*, and* $\Theta_\Lambda = \text{constant}$.

*Proof.* All we need to compute is

$$\begin{aligned} &X^\theta_{\alpha\beta} Q_\Lambda = 0, \\ &\left[X^\theta_{\alpha\beta}, L\right] Q_\Lambda = X^\theta_{\alpha\beta} \sum_{\gamma \in \Lambda} \frac{\partial \phi}{\partial p_\gamma} = (H_\alpha - H_\beta) \eta^\theta_{\alpha\beta} = 0, \end{aligned}$$

since, in the present case, $H_\alpha = H_\beta = \text{constant}$. $\square$

This means that, in the Gaussian case, the vector fields we are considering preserve the center of mass, even if this is not conserved by $L$; accordingly, the Lie Algebra generated by $\{X^\theta_{\alpha\beta}, [X^\theta_{\alpha\beta}, L]\}$, for some $\alpha, \beta \in \Gamma$ ($\Gamma$ being some cluster in $\Lambda$), can be at most five dimensional[15]. It is interesting to notice that the algebra has indeed the largest possible dimension.

**Lemma 2.6.** *If $\phi$ is quadratic, and $\alpha, \beta \in \Gamma$ are connected, then the Lie Algebra generated by $\{X^\theta_{\alpha\beta}; [X^\theta_{\alpha\beta}, L]\}$ is five dimensional.*

[15] Here and in the following, for dimension of a Lie Algebra we mean its minimal dimension when restricted to the tangent spaces at different points

*Proof.* Applying the vector fields to $q_\alpha$ we have

$$\begin{aligned}\left[X^{\theta}_{\alpha\beta}, L\right] q_\alpha &= H_\alpha \eta^{\theta}_{\alpha\beta},\\ \left[X^{\theta}_{\alpha\beta}, \left[X^{\theta}_{\alpha\beta}, L\right]\right] q_\alpha &= H_\alpha D_{\alpha\beta}\left(\eta^{\theta}_{\alpha\beta}\right)\eta^{\theta}_{\alpha\beta}.\end{aligned}$$

This vectors span a three dimensional vector space and are linearly independent with respect with the vectors $X^{\theta}_{\alpha\beta}$. To see this, it is sufficient to consider a generic linear combination, equal it to 0, and multiply it by $H_\alpha^{-1} D_{\alpha\beta} E$, then

$$0 = \sum_i \mu_i \left\langle D_{\alpha\beta}E, \eta^{\theta_i}_{\alpha\beta}\right\rangle + v\left\langle D_{\alpha\beta}E, D_{\alpha\beta}\left(\eta^{\theta_1}_{\alpha\beta}\right)\eta^{\theta_1}_{\alpha\beta}\right\rangle. \tag{2.2}$$

Next, remember that $\langle D_{\alpha\beta}E, \eta^{\theta}_{\alpha\beta}\rangle = 0$, differentiating such an expression by $D_{\alpha\beta}$ one gets

$$(H_\alpha + H_\beta)\eta^{\theta}_{\alpha\beta} + D_{\alpha\beta}\left(\eta^{\theta}_{\alpha\beta}\right)^T D_{\alpha\beta}E = 0$$

and, multiplying it by $\eta^{\theta}_{\alpha\beta}$,

$$\left\langle \eta^{\theta}_{\alpha\beta}, (H_\alpha + H_\beta)\eta^{\theta}_{\alpha\beta}\right\rangle = -\langle D_{\alpha\beta}E, D_{\alpha\beta}\left(\eta^{\theta}_{\alpha\beta}\right)\eta^{\theta}_{\alpha\beta}\rangle.$$

Substituting the above equality in (2.2) we obtain

$$v\left\langle \eta^{\theta_1}_{\alpha\beta}, (H_\alpha + H_\beta)\eta^{\theta_1}_{\alpha\beta}\right\rangle = 0$$

that is $v = 0$. From this follows $\mu_i = 0$. □

The existence of such an unexpected conserved quantity is the reason why the Gaussian case is much harder to study than the non-Gaussian one. From now on we will consider the non-Gaussian case only.

In the non-Gaussian case the center of mass is not conserved by the vector fields we are considering, and we have no other obvious conserved quantity. We need a condition on the noise to ensure that there are no ("hidden") conserved quantities, beside those considered in Lemma 2.4; this amounts to require the analogous of Lemma 2.6. More precisely we require the following.

**Condition on the Noise.** *For each two particles $\alpha, \beta$, interacting stochastically, we require that the Lie algebra generated by the vectors $X^{\theta}_{\alpha\beta}$ and $[X^{\theta}_{\alpha\beta}, L]$ is eighth dimensional at each point of every surface with fixed total energy and total momentum except, at most, for the finite union of smooth manifolds of codimension two $\widetilde{\Sigma}_{\alpha\beta}$.*

In Appendix I we show that if $\phi$ satisfies (NG) then there exists choices of $\eta_{\alpha\beta}$ for which the above condition is fulfilled (in fact, probably the condition is fulfilled for almost all possible choices).

We introduce a family of surfaces in $\mathbb{R}^{6n}$,

$$\Xi(n, \Pi, E, \omega_c) = \Bigg\{ (q,\ p) \in \mathbb{R}^{6n} \ \Bigg| \ \sum_\alpha p_\alpha = \Pi;$$
$$\sum_\alpha \phi(p_\alpha) + \frac{1}{2} \sum_{\alpha,\beta} V(q_\alpha - q_\beta)$$
$$+ \sum_\alpha \sum_{\beta\in\omega_c} V(q_\alpha - q_\beta) = E \Bigg\}$$

and let $\widetilde{\widetilde{\Xi}}$ be the union of the sets for which $(p_\alpha,\ p_\beta) \in \widetilde{\widetilde{\Sigma}}_{(\alpha,\beta)}$, for some $\alpha \neq \beta$. By hypotheses $\widetilde{\widetilde{\Xi}}$ has at least codimension two in $\Xi$, in additions it has zero Lebesgue measure.

**Lemma 2.7.** *For all $n \in \mathbb{N}$, for almost all $\Pi, E$, for each $X \in \mathscr{A}_\Lambda$, and any local function $\varphi$ with support contained in $\Lambda$ and disjoint from $\widetilde{\widetilde{\Xi}}$*

$$\mathbb{E}(X\varphi \mid \omega_\Lambda \in \Xi, \omega_{\Lambda^c}) = 0.$$

*In addition, for configurations for which $\Lambda$ contains a unique cluster the Lie algebra $\mathscr{A}_\Lambda$ spans all the tangent space of $\Xi$ at each point of $\Xi \backslash \widetilde{\widetilde{\Xi}}$.*

*Proof.* The first condition follows from Lemma 2.3 and Lemma 2.4. To address the second part of the lemma we start an induction similar to the one used in Lemma 2.1.[16] We want to generate a $6n - 4$ dimensional Lie algebra. Hence, we need six new independent vector fields at each step of the argument (i.e., for every new particle $\alpha$ that we add to the cluster).

From the proof of Lemma 2.1 we have already three independent vector fields generated by $\{X_{\alpha\beta}^{\theta^i}, X_{\beta\gamma}^{\theta^j}\}$. All these are acting only in the direction of the momenta, so all we need is to look at the action of the new vector fields on the position directions to establish their linear independence.

Define

$$\widetilde{L}_{\alpha\beta}^{\theta_k} = \left[ X_{\alpha\beta}^{\theta_k}, L \right],$$

$$\overline{L}_{\alpha\beta\gamma}^{\theta_k\theta_l} = \left[ X_{\beta\gamma}^{\theta_l}, \widetilde{L}_{\alpha\beta}^{\theta_k} \right].$$

Applying these vector fields to $q_\alpha$ we have:

$$\widetilde{L}_{\alpha\beta}^{\theta_k} q_\alpha = X_{\alpha\beta}^{\theta_k} L q_\alpha = H_\alpha \eta_{\alpha\beta}^{\theta_k}$$
$$\overline{L}_{\alpha\beta\gamma}^{\theta_k\theta_l} q_\alpha = X_{\beta\gamma}^{\theta_l} H_\alpha \eta_{\alpha\beta}^{\theta_k} = H_\alpha \left( D_\beta \eta_{\alpha\beta}^{\theta'} \right)^T \eta_{\beta\gamma}^{\theta}.$$

It is then enough to prove that the vectors $X_{\alpha\beta}^{\theta_i}$, $[X_{\alpha\beta}^{\theta_i}, X_{\gamma\beta}^{\theta_j}]$, $\widetilde{L}_{\alpha\beta}^{\theta_i}$, $\sum_{ij} \xi_{ij} \overline{L}_{\alpha\beta\gamma}^{\theta_i\theta_j}$, for some choice of $\xi_{ij}$, and $Y$ (where $Y$ belongs to the lie algebra generated

[16] Again, we can restrict ourselves to a local argument. In particular, we can assume that the support of $\varphi$ be so small that the cluster structure in it is constant

by the vectors already considered during the induction procedure) are linearly independent. Again we assume that it is not so, i.e.,

$$0 = \sum_{k=1}^{2} \Lambda_k X_{\alpha\beta}^{\theta_k} + \mu_{ij} \left[ X_{\alpha\beta}^{\theta_i}, X_{\gamma\beta}^{\theta_j} \right] + \sum_{k=1}^{2} v_k \widetilde{L}_{\alpha\beta}^{\theta_k} + \tau \sum_{kl} \xi_{kl} \overline{L}_{\alpha\beta\gamma}^{\theta_k\theta_l} + Y,$$

for some $\Lambda_i$, $\mu_{ij}$, $v_i$, $\tau$, $Y$. We apply the previous expression to, $q_\alpha$ and obtain

$$0 = \sum_{k=1}^{2} v_k H_\alpha \eta_{\alpha\beta}^{\theta_k} + \tau \sum_{kl} \xi_{kl} H_\alpha \left( D_\beta \eta_{\alpha\beta}^{\theta_l} \right)^T \eta_{\gamma\beta}^{\theta_k}.$$

If we multiply by $H_\alpha^{-1} D_{\alpha\beta} E$, recalling the properties of $\eta$, we obtain

$$0 = \tau \sum_{kl} \xi_{kl} \left\langle H_\alpha \eta_{\alpha\beta}^{\theta_k}, \eta_{\gamma\beta}^{\theta_l} \right\rangle.$$

Which shows that, out of $\widetilde{\Sigma}$, it is always possible to choose $\xi_{kl}$ such that the sum is different from zero. This implies $\tau = 0$ and allows us to conclude the proof in complete analogy with Lemma 2.1. □

As promised, we have found a bundle of local dynamics preserving the measure $Q$ (or, more precisely, its local conditional measures), i.e. the dynamics generated by the vector fields in the Lie algebra $\mathscr{A}_\Lambda$.

## 3. Conditioning to typical configurations

Using the entropy bound and large deviations estimates, we will show here that certain configurations have probability 0 for any measure $Q$ satisfying our hypotheses. We will need to exclude these configurations from the considerations of the next section.

First of all, we want to disregard configurations with locally big barriers of potential, so we are going to analyze those configurations displaying high local density.

**Lemma 3.1.** *Let $\Lambda \subset \mathbb{R}^3$ and $\Delta \subset \Lambda$ be a box of size $R_1$, consider the following configurations*

$$\Omega_\Lambda^\varepsilon = \{ \omega \mid \exists \Delta \subset \Lambda : \, |\omega_\Delta| \geqq \varepsilon^{-1} |\Lambda|^{\frac{1}{2}} \}. \tag{3.1}$$

*If $Q$ satisfies condition (i) of Theorem 1.2 (entropy bound), then there exists $C > 0$ such that:*

$$Q(\Omega_\Lambda^\varepsilon) \leqq C\varepsilon^2.$$

*Proof.* By the entropy inequality

$$Q(\Omega_\Lambda^\varepsilon) \leqq \frac{\log 2 + H_\Lambda(Q|\mathbb{P})}{\log\left(1 + \mathbb{P}(\Omega_\Lambda^\varepsilon)^{-1}\right)}$$

(which is a consequence of (1.7)), and condition (i) of Theorem 1.2, we need only to prove that for a given grancanonical measure $\mathbb{P}$

$$\mathbb{P}(\Omega_\Lambda^\varepsilon) \leqq C_2|\Lambda| \exp(-C_1|\Lambda|\varepsilon^{-2})$$

for some constants $C_1, C_2 > 0$ independent from $\varepsilon$ and $\Lambda$.

Since the measure $\mathbb{P}$ is translation invariant

$$\mathbb{P}(\Omega_\Lambda^\varepsilon) \leqq \frac{C_1}{R_1}|\Lambda|\mathbb{P}(\{|\omega_\Delta| > \varepsilon^{-1}|\Lambda|^{\frac{1}{2}}\}),$$

for some fixed box $\Delta \subset \Lambda$. Accordingly, (setting $\Gamma = \int_{\mathbb{R}^3} e^{-\lambda_4\phi(p)+\sum_{i=1}^3 \lambda_i p_i} dp$ and $V_\Delta = \frac{1}{2}\sum_{i,j=1}^n V(q_i - q_j) + \sum_{i=1}^n \sum_{q_\beta \in \omega_\Delta^c} V(q_i - q_\beta)$)[17]

$$\begin{aligned}
&\mathbb{P}(|\omega_\Delta| > \varepsilon^{-1}|\Lambda|^{1/2}) \\
&\quad = Z_\Delta^{-1} \sum_{n \geqq \varepsilon^{-1}|\Lambda|^{1/2}}^{\infty} \frac{e^{\lambda_0 n}\Gamma^n}{n!} \int_{\Delta^n} e^{-\lambda_4 V_\Delta} \\
&\quad \leqq Z_\Delta^{-1} \sum_{n \geqq \varepsilon^{-1}|\Lambda|^{1/2}}^{\infty} \frac{e^{\lambda_0 n}\Gamma^n}{n!} \exp\left[-\lambda_4 A \frac{\varepsilon^{-2}|\Lambda|}{|\Delta|} + \lambda_4 B \varepsilon^{-1}|\Lambda|^{1/2}\right] |\Delta|^n \\
&\quad \leqq C_2 e^{-C_3\varepsilon^{-2}|\Lambda|},
\end{aligned}$$

where we have used the explicit form of the grand canonical measures, the positivity and the superstability of the potential. □

Another information needed in the following arguments is a bound on the total kinetic energy shared by a large number of particles with respect to some partition. In fact, in Sect. 4 we will construct explicitly a special way of grouping particles in sets, larger than the clusters given by the interaction, and we will need that many of such groups have "enough" kinetic energy. The exact formulation of the estimate needed is a bit technical and need some preliminary notation. Let $\underline{\mathscr{P}}_n$ be the collection of all possible partitions of the set $\{1, ..., n\}$. Suppose that there exists $\mathscr{P} : \Omega_\Lambda \to \cup_{n\in\mathbb{N}}\underline{\mathscr{P}}_n$ such that $\mathscr{P}(\omega_\Lambda) \in \underline{\mathscr{P}}_{|\omega_\Lambda|}$ and depends only on the positions of the particles (one such function is obtained by associating to each configuration $\omega_\Lambda$ the partition in clusters; another one, to which the following lemma will be applied, is constructed in Sect. 4). In addition, suppose that it is defined[18] $\widehat{\mathscr{P}}(\omega_\Lambda) \subset \{P \in \mathscr{P}(\omega_\Lambda) \,|\, \#P \geqq L/B\}$ (remember that $L$ is the linear size of the box $\Lambda$, while $B > 0$ is a fixed constant that will be chosen before Definition 4.9), in other words we consider only the elements that are sufficiently large. Set $\widehat{P}(\omega_\Lambda) = \bigcup_{\hat{P}\in\hat{\mathscr{P}}(\omega_\Lambda)} \hat{P}$. Next, we want

[17] The result presented here holds in much greater generality (cf. [R]) . Yet, using explicitly the positivity of the potential, the proof boils down to the following three lines

[18] Here and in the following, given a set $A$ by $\#A$ we mean the cardinality of $A$

to consider

$$\underline{\widehat{\mathscr{P}}}_{\omega_\Lambda} = \{\mathscr{P} \in \underline{\mathscr{P}}_{|\omega_\Lambda|} \mid \forall P \in \mathscr{P} \text{ if } P \cap \widehat{P}(\omega_\Lambda) = \emptyset \text{ then } P \in \mathscr{P}(\omega_\Lambda);\\ \text{if } P \cap \widehat{P}(\omega_\Lambda) \neq \emptyset \text{ then } P \subset \widehat{P}(\omega_\Lambda) \text{ and } \forall \hat{P} \in \mathscr{P}(\omega_\Lambda)\\ \text{either } P \cap \hat{P} = \emptyset \text{ or } \hat{P} \subset P\}.$$

In essence, $\underline{\widehat{\mathscr{P}}}_{\omega_\Lambda}$ is the collection of all the partitions that agree with $\mathscr{P}(\omega_\Lambda)$ on the "small" elements, while the "large" elements are obtained by joining "large" elements of $\mathscr{P}(\omega_\Lambda)$. The following lemma claims that for each such a partitions the probability that each "large" element has very little energy is very low.

**Lemma 3.2.** *Let $a > 0$, and $\varepsilon > 0$ sufficiently small, $\Lambda \subset \mathbb{R}^3$ and consider the following configurations*

$$\widetilde{\Omega}^\varepsilon_\Lambda = \left\{ \omega \,\middle|\, \sum_{P \in \hat{\mathscr{P}}(\omega_\Lambda)} \#P \geqq \frac{a}{8}|\Lambda|;\ \forall \mathscr{P} \in \underline{\hat{\mathscr{P}}}_{\omega_\Lambda} \text{ and } \forall P \in \mathscr{P}, P \cap \widehat{P}(\omega_\Lambda) \neq \emptyset,\right.\\ \left. \sum_{\alpha \in P} \left[ \phi(p_\alpha) - \phi\left( \frac{1}{\#P} \sum_{\beta \in P} p_\beta \right)\right] < \varepsilon \#P \right\}. \tag{3.2}$$

*If $Q$ satisfies condition $(i)$ of Theorem 2.1, then*

$$\lim_{\varepsilon \to 0} \lim_{|\Lambda| \to \infty} Q(\widetilde{\Omega}^\varepsilon_\Lambda) = 0.$$

*Proof.* We will use the same entropy inequality as in the previous lemma. In order to simplify notations, we choose a grancanonical measure $\mathbb{P}$ corresponding to the parameters $\lambda_1 = \lambda_2 = \lambda_3 = 0$ and $\lambda_4$ such that $\Gamma_\Lambda = \int e^{-\lambda_4 \phi(p)} dp = 1$.

Let us define

$$Y_m = \frac{1}{m} \sum_{i=1}^m \phi(p_i) - \phi\left( \frac{1}{m} \sum_{i=1}^m p_i \right)$$

and observe that since $\phi$ is convex $Y_m$ is non–negative. Then we have

$$\mathbb{P}(\widetilde{\Omega}^\varepsilon_\Lambda) \leqq Z_\Lambda^{-1} \sum_{n \geqq a|\Lambda|} \frac{e^{n\lambda_0}}{n!} \int_{\Lambda^n} e^{-\lambda_4 V_\Lambda} \sum_{\mathscr{P} \in \underline{\widehat{\mathscr{P}}}_q} \prod_{\substack{P \in \mathscr{P} \\ P \cap \hat{P}(\omega_\Lambda) \neq \emptyset}} J_{\#P},$$

where

$$J_m = \int_{Y_m < \varepsilon} e^{-\lambda_4 \sum_{i=1}^m \phi(p_i)} d^m p.$$

Obviously the first step is to study $J_m$ in the limit $m \to \infty$.

Define $\pi_m = \frac{1}{m}\sum_{i=1}^m p_i$ then

$$\int_{Y_m<\varepsilon} e^{-\lambda_4 \sum_{i=1}^m \phi(p_i)}\, d^m p \leqq \int_{Y_m<\varepsilon;|\pi_m|\leqq h} e^{-\lambda_4 \sum_{i=1}^m \phi(p_i)}\, d^m p + \int_{|\pi_m|\geqq h} e^{-\lambda_4 \sum_{i=1}^m \phi(p_i)}\, d^m p.$$

By standard Large Deviations (cf. [V]) for independent variables we have

$$\lim_{h\to\infty}\lim_{m\to\infty}\frac{1}{m}\log\int_{|\pi_m|\geqq h} e^{-\lambda_4 \sum_{i=1}^m \phi(p_i)}\, d^m p = -\infty,$$

while for the first term we use the exponential Chebicheff inequality. For any $\beta > 0$

$$\int_{Y_m<\varepsilon;|\pi_m|\leqq h} e^{-\lambda_4 \sum_{i=1}^m \phi(p_i)}\, d^m p \leqq e^{\beta\varepsilon m}\int_{|\pi_m|\leqq h} e^{-m\beta Y_m} e^{-\lambda_4 \sum_{i=1}^m \phi(p_i)}\, d^m p.$$

Hence, we need to estimate

$$\lim_{m\to\infty}\frac{1}{m}\log\int_{|\pi_m|\leqq h} e^{-m\varepsilon Y_m} e^{-\lambda_4 \sum_{i=1}^m \phi(p_i)}\, d^m p \leqq \sup_{\mu;|\bar{\mu}|\leqq h}\left\{\beta\left[\phi(\bar{\mu}) - \widehat{\phi}(\mu)\right] - I(\mu)\right\},$$

where $\mu(p)$ are probability densities on $\mathbb{R}^3$ (with respect to $e^{-\lambda_4\phi}dp$),

$$\bar{\mu} = \int p\mu(p)e^{-\lambda_4\phi}\, dp, \qquad \widehat{\phi}(\mu) = \int \phi(p)\mu(p)e^{-\lambda_4\phi}\, dp$$

and

$$I(\mu) = \int \mu(p)\log(\mu(p))\, e^{-\lambda_4\phi(p)} dp\ .$$

Next,

$$\sup_{\mu;|\bar{\mu}|\leqq h}\left\{\beta\left[\phi(\bar{\mu}) - \widehat{\phi}(\mu)\right] - I(\mu)\right\} \leqq \sup_{\mu;|\bar{\mu}|\leqq h}\left\{\beta C_\phi(h) Var(\mu) - I(\mu)\right\},$$

where $\mathrm{Var}(\mu) = \int (p-\bar{\mu})^2\mu(p)e^{\lambda\phi(p)}$, and $C_\phi(h) = \inf_{|\xi|\leqq h}\phi''(\xi) > 0$ by hypothesis.

The variational problem $\sup_{\mu;\bar{\mu}=v}\{\beta C_\phi(h)\mathrm{Var}(\mu) - I(\mu)\}$ can be explicitly solved and yields

$$\mu(p) = \frac{e^{-\beta C_\phi(h)(p-v)^2 + z(v)p}}{\int e^{-\beta C_\phi(h)(p-v)^2 + z(v)p - \lambda_4\phi(p)}\, dp},$$

with $z(0) = 0$ (due to the symmetry of $\phi$) and

$$z'(v) = \frac{1}{\mathrm{Var}(\mu)} + 2\beta C_\phi(h).$$

Substituting we obtain

$$\begin{aligned}
&\sup_{\bar\mu}\left\{-z(\bar\mu)\bar\mu+\log\int e^{-\beta C_\phi(h)(p-\bar\mu)^2+z(\bar\mu)p-\lambda_4\phi(p)}\,dp\right\}\\
&\quad\leqq \int e^{-\beta C_\phi(h)p^2-\lambda_4\phi(p)}\,dp\\
&\quad\leqq \log\int e^{-\beta C_\phi(h)p^2}\,dp=\frac{1}{2}\log\left(\frac{\pi}{\beta C_\phi(h)}\right).
\end{aligned}$$

Consequently,

$$\inf_{\beta\geqq 0}\left\{\beta\varepsilon+\frac{1}{2}\log\left(\frac{\pi}{\beta C_\phi(h)}\right)\right\}\leqq -C_1\log\left(\frac{C_\phi(h)}{\varepsilon}\right).$$

Finally, by choosing a sequence $h_\varepsilon\to\infty$ as $\varepsilon\to 0$ sufficiently slow so that $C_\phi(h_\varepsilon)/\varepsilon\to+\infty$, it follows

$$\lim_{\varepsilon\to 0}\lim_{m\to\infty}\frac{1}{m}\log J_m=-\infty.$$

Accordingly, since $J_m$ is an increasing function of $\varepsilon$, for each $M>0$ there exists $\Lambda_0\subset\mathbb{R}^3$ and $\varepsilon_0>0$ such that for each $\Lambda\supset\Lambda_0$ and $\varepsilon<\varepsilon_0$

$$J_m\leqq e^{-mM}$$

for each $m\geqq B^{-1}L$ ($L$ is the linear size of $\Lambda$). Thanks to such an estimate we can conclude the Lemma:

$$\mathbb{P}(\widetilde\Omega^\varepsilon_\Lambda)\leqq Z_\Lambda^{-1}\sum_{n\geqq a|\Lambda|}\frac{e^{n\lambda_0}}{n!}\int_{\Lambda^n}e^{-\lambda_4 V_\Lambda}e^{-\frac{a}{8}M|\Lambda|}\#\underline{\widehat{\mathscr{P}}}_q.$$

By construction $\#\underline{\widehat{\mathscr{P}}}_q$ is the number of different partitions of $\ell\equiv\#\underline{\widehat{\mathscr{P}}}(q)\leqq nB/L$ elements. That is,

$$\begin{aligned}
\#\underline{\widehat{\mathscr{P}}}_q&=\sum_{k=1}^{\ell}\sum_{\substack{\{j_i\}_{i=1}^k;\,j_i\geqq 1\\ \sum_{i=1}^k j_i=\ell}}\frac{\ell!}{k!\prod_{i=1}^k j_i!}\leqq\sum_{k=1}^{\ell}\sum_{\substack{\{j_i\}_{i=1}^k;\,j_i\geqq 1\\ \sum_{i=1}^k j_i=\ell}}\frac{\ell!}{\prod_{i=1}^k j_i!}\\
&=\sum_{\substack{\{j_i\}_{i=1}^\ell;\,j_i\geqq 0\\ \sum_{i=1}^\ell j_i=\ell}}\frac{\ell!}{\prod_{i=1}^\ell j_i!}=\ell^\ell\leqq\left(\frac{nB}{L}\right)^{nB/L},
\end{aligned}$$

where, in the last equality, we have used the Taylor expansion of $(\sum_{i=1}^\ell x_i)^\ell$.

By the superstability it follows that there exist constants $c_i$ such that

$$\mathbb{P}(\widetilde\Omega^\varepsilon_\Lambda)\leqq\sum_{n\geq a|\Lambda|}\frac{1}{n!}\exp\left[c_1 n-c_2\frac{n^2}{|\Lambda|}-\frac{a}{8}M|\Lambda|\right]n^{\frac{n}{L}}|\Lambda|^n\leqq e^{-c_3M|\Lambda|}.$$

Hence,

$$\lim_{\varepsilon\to 0}\lim_{\Lambda\to\infty}\frac{1}{|\Lambda|}\log \mathbb{P}(\widetilde{\Omega}^{\varepsilon}_{\Lambda})^{-1}=+\infty.$$

This last result implies the lemma, thanks to the same entropy inequality used in Lemma 3.1. □

## 4. Clustering

Before getting into the technicalities of the clusters deformations, let us pause here to explain our strategy.

As we already mentioned in Sect. 2, from Lemma 2.1 and 2.2 and the arguments after Proposition 4.1 will follow that the measure $Q$ on a box $\Lambda_0$ conditioned on the positions, on the total momentum and on total kinetic energy:

$$Q_{\Lambda_0}\left(d p_1,\dots,d p_n \middle| q_1,\dots,q_n;\sum_{\alpha=1}^{n} p_\alpha,\sum_{\alpha=1}^{n}\phi(p_\alpha)\right) \tag{4.1}$$

is Microcanonical only for the $p$'s corresponding to the particles in the same cluster. In particular, this measure is symmetric for exchange of momentum between particles of the same cluster (by "exchange of momentum" we mean any transfer of momentum among two particles that conserves the total momentum and the total kinetic energy). If we could show that a measure is symmetric for exchange of momentum between clusters, it would follow that such a measure is Microcanonical, i.e. Lemma 5.1 below. One way to achieve this could be to find a transformation on the phase space, for which the measure $Q$ is invariant, that brings a particle $\alpha$ from a cluster $\Gamma_1$ in "contact" to another cluster $\Gamma_2$, then exchanges the momenta with a particle $\beta$ of $\Gamma_2$, then brings back $\alpha$ to the initial position in the cluster $\Gamma_1$. We cannot do exactly this, but we will exchange momenta between the clusters performing more complicated transformations for which our measure $Q$ is still invariant.

Given a box $\Lambda_0$ and a configuration $\omega\in\Omega$, for each couple of label $\alpha$, $\beta$ and $\eta\in\mathbb{R}^3$, $\|\eta\|=1$, let $T^{\eta}_{\alpha,\beta}\omega$ the configuration obtained by exchanging momenta between the particle $\alpha$ and particle $\beta$ in the "direction" $\eta$ respecting the conservation laws (here $\alpha$ and $\beta$ are two particles with position in $\Lambda_0$)[19]. Observe that only momenta is exchanged while positions are unchanged. Furthermore, such operation does not change the total momenta in $\Pi_{\Lambda_0}$, nor the total kinetic energy $K_{\Lambda_0}$ in the region $\Lambda_0$. As already mentioned, all we need to prove, to show that $Q_{\Lambda_0}$ is Microcanonical, is that

$$\int \sum_{q_\alpha,\, q_\beta\in\Lambda_0}\left[F(T^{\eta_{\alpha\beta}}_{\alpha,\beta}\omega)-\ F(\omega)\right]\, dQ(\omega)\ =\ 0 \tag{4.2}$$

[19] That is, calling $p'_\alpha$, $p'_\beta$ the new momenta in the configuration $T^{\eta}_{\alpha\beta}\omega$, set $p'_\alpha=p_\alpha+\lambda\eta$ and $p'_\beta=p_\beta-\lambda\eta$. Since we require $p_\alpha+p_\beta=p'_\alpha+p'_\beta$ and $\phi(p_\alpha)+\phi(p_\beta)=\phi(p'_\alpha)+\phi(p'_\beta)$, it follows that $\lambda$ is uniquely determined

for any local smooth function $F(\omega)$ and any arbitrary choice of the vectors $\eta_{\alpha\beta}$ (from now on we will set $T_{\alpha,\beta} \equiv T_{\alpha,\beta}^{\eta_{\alpha\beta}}$).

It is very easy to see why (4.2) implies the symmetry of the measure on the momenta (4.1). Choose $F(\omega) = F_1(p_{\Lambda_0})F_2(q_{\Lambda_0}, \Pi_{\Lambda_0}, K_{\Lambda_0})$. Since $T_{\alpha,\beta}$ leaves invariant $F_2$, one can condition the relation (4.2) on the quantities on which $F_2$ depends and obtain

$$\int \left[F_1(T_{\alpha,\beta}\, p_{\Lambda_0}) - F_1(p_{\Lambda_0})\right] \, dQ(p_{\Lambda_0} \big| q_{\Lambda_0}, \Pi_{\Lambda_0}, K_{\Lambda_0}) \; = \; 0, \tag{4.3}$$

i.e. that the measure defined by (4.1) is invariant for exchange of momenta between particles.

What we already know is that (4.3) is true if $\alpha$ and $\beta$ are in the same cluster (defined by the configuration $q_{\Lambda_0}$ on which we have conditioned).

By condition (ii)[20] of our main theorem, we can choose $a > 0$ such that $\rho(\omega) > \rho_* + 2a$ with $Q$–probability 1. For any $\varepsilon > 0$ small enough, and $\Lambda \supset \Lambda_0$ large enough, with linear size $L$, define the set of good configurations

$$\widehat{\Omega}_{\Lambda,\varepsilon} = \left\{ \omega \;\Big|\; \left| \frac{|\omega_{\Lambda_1}|}{|\Lambda_1|} - \rho(\omega) \right| \leqq a \;;\; \left| \frac{|\omega_{\Lambda}|}{|\Lambda|} - \rho(\omega) \right| \leqq a \right\} \cap (\Omega_\Lambda^\varepsilon)^c \cap (\widetilde{\Omega}_\Lambda^\varepsilon)^c,$$

where $\Lambda_1$ is a box concentric to $\Lambda$ and of linear size $L/2$. Then by Lemmas 3.1 and 3.2

$$\lim_{\varepsilon \to 0} \lim_{|\Lambda| \to \infty} Q\left( (\widehat{\Omega}_{\Lambda,\varepsilon})^c \right) \; = \; 0 \; .$$

So it is enough to show that, for any $\varepsilon > 0$, we can find $\Lambda$ large enough such that

$$\int_{\widehat{\Omega}_{\Lambda,\varepsilon}} \sum_{q_\alpha, q_\beta \in \Lambda_0} \left[ F(T_{\alpha,\beta}\omega) - \; F(\omega) \right] \, dQ(\omega) \; = \; 0 \tag{4.4}$$

for any bounded function F localized in $\Lambda_0$.

Let $\Xi_\Lambda(n, \Pi, E, \omega_c) \subset \mathbb{R}^{6n}$ be the surface on which $n$ particles have positions in $\Lambda$, total momentum $\Pi$, and total energy $E$ (note that the total energy inside $\Lambda$ is affected by $\omega_c$). Because of the boundaries $\omega_c$, this surface may have many different connected components $\Xi_\Lambda^j(n, \Pi, E, \omega_c) \subset \mathbb{R}^{6n}$.

**Proposition 4.1.** *For any $\varepsilon > 0$ there exists $\Lambda$ large enough such that the measure $Q$ restricted to $\Xi_\Lambda^j(n, \Pi, E, \omega_c) \cap \widehat{\Omega}^{\Lambda,\varepsilon}$, is proportional to the Microcanonical measure*[21] *for almost all $\Pi$, $E$ and $\omega_c$.*

[20] If, as noted in Remark 1.3, the condition is on the potential energy, just substitute the definition of $\widehat{\Omega}^{\Lambda,\varepsilon}$ with

$$\widehat{\Omega}^{\Lambda,\varepsilon} = \left\{ \omega : U_{\Lambda_1} \, > a|\Lambda_1|, U_\Lambda \, > a|\Lambda| \right\} \cap (\Omega_\Lambda^\varepsilon)^c \cap (\widetilde{\Omega}_\Lambda^\varepsilon)^c.$$

and the rest of the argument of this section will remain essentially unchanged

[21] To define the microcanonical measure consider that $(\Lambda \times \mathbb{R}^3)^n$ is foliated by the surfaces $\Xi(E, \Pi)$ when varying $E$ and $\Pi$. Accordingly, it is possible to define the conditioning of the Lebesgue measure on $(\Lambda \times \mathbb{R}^3)^n$ to almost all the above mentioned surfaces. Such a

It is easy to see that (4.4), and therefore (4.2), follows from Proposition 4.1. In fact, $\omega$ and $T_{\alpha,\beta}\omega$ belong always to the same connected component (connected components can be distinguished only by the positions $q$'s), and Microcanonical measures are invariant for exchanges of momenta between the particles. The rest of the section is dedicated to the proof of Proposition 4.1. The proof will be complete only at the very end of Sect. 4.

We now fix the box $\Lambda$, and we drop the index $\Lambda$ when this will not create confusion; moreover, in the rest of the paragraph we will drop the index $j$ and $\Xi$ will refer to a fixed connected component.

What we have proven in the previous section is that our Lie algebra $\mathscr{A}$ generates the tangent space of $\Xi(n, \Pi, E, \omega_c)$ only at those points corresponding to a unique cluster.

Let us call $d\mu_{n,\Pi,E,\omega_c}(q, p)$ the measure Q conditioned on surface $\Xi(n, \Pi, E, \omega_c)$ i.e.

$$\int_{\Xi(n,\Pi,E,\omega_c)} f(q, p)\, d\mu_{n,\Pi,E,\omega_c}(q, p) = \mathbb{E}^{Q}\left(f(\omega_\Lambda) \,\middle|\, |\omega_\Lambda| = n,\, \Pi_\Lambda(\omega) = \Pi,\, E_\Lambda(\omega) = E,\, \omega_c\right).$$

Since all the quantities we have conditioned on, in the definition of $\mu$, are conserved by the vector fields of the Lie subalgebra generated only by the particles in $\Lambda$, the conditional measure $d\mu$ is invariant for such a subalgebra (Lemma 2.7); moreover, the subalgebra is composed by null divergence vector fields. This implies that, in a sufficiently small neighborhood $B$ of a point corresponding to a configuration with a unique cluster, the measure $d\mu_n$ is proportional to the Microcanonical measure. More precisely consider an open set $B \subset \Xi$ with a constant cluster structure and let $\chi$ be the characteristic function of such a set. If all the configurations in $B$ have a *unique* cluster and $v_i \neq v_j$ for every $i$, $j$, it follows

$$\int_{\Xi(n,\Pi,E,\omega_c)} \chi(q, p) F_\Lambda(q, p)\, d\mu_{n,\Pi,E,\omega_c}(q, p) = Z(n, \Pi, E, \omega_c) \int_{\Xi(n,\Pi,E,\omega_c)} \chi(q, p) F_\Lambda(q, p)\, dM(q, p)$$

where $dM$ is the microcanonical measure on $\Xi$ and $Z$ is a normalization constant. To see this, notice that the microcanonical measure is invariant with respect to $\mathscr{A}_\Lambda$. Moreover, there exists vector fields $\{Y_i\}_{i=1}^m$ from $\mathscr{A}_\Lambda$ that span all the tangent space of $\Xi$ at each point of $B$ (provided $B$ is chosen small enough). Hence, $d\mu$ must be an invariant measure for the elliptic operator $\sum_{i=1}^m Y_i^* Y_i$. The claim follows since it is well known that such an elliptic operator has a unique invariant measure.

If in the configurations in $B$ are present several not interacting clusters $\{\Gamma_i\} = \widetilde{\Gamma}$, then from Sect. 2 follows that the Lie algebra $\mathscr{A}_\Lambda$, restricted to $\Xi$,

conditional measure is exactly the Microcanonical measure on $\Xi$. This Microcanonical measure is also the only one invariant for the action of every zero divergence vector field tangent to the surfaces $\Xi$. Notice that we use the name "Microcanonical" both for the measure just described and for its conditional to the positions does not necessarily span all the tangent space. Yet, for each $\Gamma_i \in \widetilde{\Gamma}$, we can consider the surface $\Xi(\Gamma_i)$ obtained by fixing the positions of the particles not in $\Gamma_i$[22]. From Sect. 2 follows then that the Lie Algebra $\mathscr{A}_\Lambda$, restricted to the surface $\Xi(\Gamma_i)$ spans all its tangent space. Thus, the simple application of the invariance with respect to the available vector fields yields the weaker result

$$\begin{aligned}&\int_{\Xi_j(n,\Pi,E,\omega_c)} \chi(q,\ p)F_\Lambda(q,\ p)\,d\mu_{n,\Pi,E,j,\omega_c}(q,\ p)\\ &= Z(n,\Pi,E,j,\omega_c)\int_{\Xi_j(n,\Pi,E,\omega_c)} \chi(q,\ p)F_\Lambda(q,\ p)\,dM_{\widetilde{\Gamma}}(q,\ p)\end{aligned}$$

where

$$M_{\widetilde{\Gamma}}(q,\ p)(\cdot \mid (q_j,\ p_j) \notin \Gamma_i) = M_{\Gamma_i}((q,\ p) \in \Gamma_i)$$

$M_{\Gamma_i}$ being the microcanonical measure for the particles belonging to $\Gamma_i$.

Yet, it is possible to use the dynamics generated by the vector fields in order to get a better result. We will show that one can construct maps, connected to cluster deformations, with the property of preserving both the measures $d\mu$ and $dM$. To be more concrete we need to define precisely what is meant by deforming a cluster.

Recall that $\widetilde{\Xi} = \{(q,\ p) \in \Xi \mid (p_i,\ p_j) \in \widetilde{\Sigma}_{ij}$ for some $i, j\}$. Moreover, given a partition $\mathscr{P}$ of the particles (i.e., $\cup_{P\in\mathscr{P}} P = \{1, ..., n\}$ for each $P_1, P_2 \in \mathscr{P}$, $P_1 \neq P_2$, $P_1 \cap P_2 = \emptyset$) we will say that a measure is microcanonical with respect to the partition $\mathscr{P}$ if for each $P \in \mathscr{P}$ conditioning the measure to all the particles not in $P$ one obtains the microcanonical measure for the particles in $P$. (From now on, with an evident abuse of notations, we will use $M_{\mathscr{P}}$ to designate any measure which is Microcanonical with respect to $\mathscr{P}$.)

Furthermore, by $\mathscr{A}_{\delta,\mathscr{P}}$ we will mean the Lie algebra generated by the vector fields associated to bonds in which the particles are closer than $R_1 - \delta$, for some fixed $\delta$ smaller than $R_1 - R_0$, and belongs to the same element of the partition $\mathscr{P}$; finally, by $\mathscr{A}_{\delta,\mathscr{P}}(\xi)$ we designate the restriction of $\mathscr{A}_{\delta,\mathscr{P}}$ at $\mathscr{T}_\xi\Xi$[23].

**Definition 4.2.** *By "allowed deformation" with respect to a partition $\mathscr{P}$ and a tolerance $\delta \in \mathbb{R}^+$, we mean a piecewise smooth curve $\gamma : [0, 1] \to \Xi\backslash\widetilde{\Xi}$ with the property that, for each $s \in [0, 1]$, $\gamma'(s) \in \mathscr{A}_{\delta,\mathscr{P}}(\gamma(s))$.*

Note that, in a given configuration, the clusters form a partition.

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[22] To be more precise, suppose that $\Gamma_i$ consists of $m$ particles. Fix the position and velocities of all the particles in $\Lambda$ not belonging to $\Gamma_i$ and call their total energy $E_1$ and their total momentum $\Pi_1$. Then, $\Xi(\Gamma_i)$ is the surface in $\mathbb{R}^m$ defined by $\sum_{\alpha\in\Gamma_i} p_\alpha = \Pi - \Pi_1 \equiv \Pi'$ and $\sum_{\alpha\in\Gamma_i}\phi(p_\alpha) + \frac{1}{2}\sum_{\alpha,\beta\in\Gamma_i} V(q_\alpha - q_\beta) + \sum_{\alpha\in\Gamma_i,\beta\notin\Gamma_i} V(q_\alpha - q_\beta) = E - E_1 \equiv E'$. Notice that we are not writing explicitly the dependence on $E'$ and $\Pi'$, since this does not create ambiguities

[23] Clearly $\mathscr{A}_{\delta,\mathscr{P}}(\xi)$ is a linear subspace of $\mathscr{T}_\xi\Xi$

**Definition 4.3.** *Given a set $B \in \Xi$ we call "$\mathscr{P}(B)$" the coarsest partition of $\{1, ..., n\}$ finer than (or equal to) the partitions produced by the isolated clusters at each $\xi$ in B.*

The cornerstone of our approach is given by the following proposition.

**Proposition 4.4**. *Given a configuration $\xi \equiv (q, p) \in \Xi \backslash \widetilde{\widetilde{\Xi}}$, let*

$$r = \sup_{\alpha,\beta:\ |q_\alpha - q_\beta| < R_1} |q_\alpha - q_\beta|$$

*and $\delta \leqq R_1 - r$, suppose that there exists a neighborhood B of $\xi$ in which the measure $\mu$ is microcanonical with respect to some partition $\mathscr{P}$ (coarser than $\mathscr{P}(\{\xi\})$) and an allowed deformation with respect to $\mathscr{P}$ and $\delta$ connecting $\xi$ to some other configuration $\xi_1 \in \Xi \backslash \widetilde{\widetilde{\Xi}}$, then there exists a neighborhood $B_1 \subset B$ of $\xi$ such that*[24]

$$\mu|_{B_1} = M_{\mathscr{P} \wedge \mathscr{P}(\{\xi_1\})}|_{B_1}.$$

The proof of Proposition 4.4 is the content of Subsection 4.1.

Proposition 4.4. shows how we can prove that the measure is microcanonical with respect to coarser and coarser partitions. Hence, we can have the wanted result, provided we can generate enough allowed deformation. In Subsect. 4.2, we will prove the existence of enough cluster deformations, more precisely we will prove Proposition 4.1.

### *4.1 Cluster deformation*

*Proof of Proposition 4.4*. The following is a useful auxiliary lemma.

**Lemma 4.5.** *If a measure is microcanonical with respect to a partition $\mathscr{P}_1$ and, at the same time, with respect to a partition $\mathscr{P}_2$, then it is microcanonical with respect to the partition $\mathscr{P}_1 \wedge \mathscr{P}_2$.*

*Proof*. Start with the following observation: let $P$ and $G$ two element respectively of $\mathscr{P}_1$ and $\mathscr{P}_2$ such that $P \cap Q \neq \emptyset$. Then since the measure is invariant for exchanges of momenta between particles inside $P$ and inside $Q$, then it is invariant for exchanges in $P \cup Q$[25]. This implies that it is microcanonical in $P \cup Q$. By definition of $\mathscr{P}_1 \wedge \mathscr{P}_2$, it follows that it is microcanonical with respect to this coarser partition. □

We start by constructing explicitly a map $\Upsilon$ that leaves invariant $\mu$ and such that $\Upsilon(\xi) = \xi_1$. The idea is to define a one parameter family of vector

[24] Given two partitions $\mathscr{P}_1, \mathscr{P}_2$ by $\mathscr{P}_1 \wedge \mathscr{P}_2$ we mean the finest partition coarser than both $\mathscr{P}_1$ and $\mathscr{P}_2$

[25] Since $P \cup Q$ has less constraints than $P$ and $Q$ separately (namely, only the total momenta and energy in $P \cup Q$), the last sentence is not completely obvious. The proof is a simplification of the one found in Appendix II. The argument in Appendix II is complicated by the necessity to realize the exchange of momenta via allowed deformations, which it is irrelevant in the present context

fields in a neighborhood of the allowed deformation $\gamma$; the wanted map will be obtained by integrating such a family in a neighborhood of the curve $\gamma$[26].

We parameterize $\gamma$ by arc-length. By definition, $\gamma \subset \Xi \setminus \widetilde{\widetilde{\Xi}}$. Let $L = \text{length}(\gamma)$ then $\gamma(0) = \xi$ and $\gamma(L) = \xi_1$. Moreover, for each $s \in [0, L]$, $\gamma'(s)$ can be written as $\sum_{i=1}^m a_i(s) X_i$ where $a_i(s)$ are smooth functions depending only on $s$ and $X_i$ are vector fields, in the Lie algebra $\mathscr{A}_{\delta, \mathscr{P}}(\gamma(s))$, that leave $Q$ invariant[27]; here $m$ is some integer depending only on $\gamma$. Then we define a time dependent vector field by

$$V(q,\ p, s) = \sum_{i=1}^{m} a_i(s) X_i(q,\ p)$$

and the corresponding flow $\Phi$ by[28]

$$\begin{aligned} &\frac{d}{ds}\Phi(q,\ p, s) = V(\Phi(q,\ p, s), s) \\ &\Phi(q,\ p, 0) = (q,\ p). \end{aligned}$$

By continuity there exists $\varepsilon$ such that for each $s \in [0, 1]$, setting $\gamma(s) = (q(s),\ p(s))$, if $\widetilde{q} \in \Lambda^n$, $\widetilde{p} \in \mathbb{R}^{3n}$, $\|\widetilde{q}_i - q_i(s)\| < \varepsilon$, then $V(\widetilde{q}, \widetilde{p}, s) \in \mathscr{A}_{0, \mathscr{P}}((\widetilde{q}, \widetilde{p}))$. By choosing the initial conditions in a sufficiently small neighborhood $B_0$ of $\xi$, we can ensure that the solutions of the differential equations, with initial conditions in $B_0$, are closer than $\frac{\varepsilon}{2}$ to the curve $\gamma$.

We define $\Upsilon$ on $B_0$ by

$$\Upsilon(\xi) = \Phi(\xi, L).$$

Since $V$ is always tangent to $\Xi$ it follows that $\Upsilon$ is a well defined function from $B_0 \subset \Xi \setminus \widetilde{\widetilde{\Xi}}$ to $\Xi \setminus \widetilde{\widetilde{\Xi}}$.

$\Upsilon(\xi) = \xi_1$ follows immediately from the construction ($\gamma(s)$ is a solution of the differential equation). In addition, for each smooth function $F$, with $\text{supp} F \subset B_0$, $\mathbb{E}^Q(F \circ \phi_s^{-1}) = \mathbb{E}^Q(F)$; that is $\mu$ is invariant with respect to $\Upsilon$.

Next, it is easy to check that the vector fields $X_i$ have zero divergence and are tangent to the surfaces of constant momentum and energy of the various elements in $\mathscr{P}$. Accordingly, any measure that is microcanonical with respect to a partition coarser than $\mathscr{P}$ will be left invariant by the flow; that is $M_{\mathscr{P}}$ is invariant with respect to $\Upsilon$.

We are now able to state a first helpful result.

**Lemma 4.6.** *Given a configuration $\xi \equiv (q,\ p) \in \Xi \setminus \widetilde{\widetilde{\Xi}}$ suppose that there exists a neighborhood $B$ of $\xi$ in which the measure $\mu$ is Microcanonical with respect*

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[26] Here we will consider only the case in which $\gamma$ is smooth, the generalization to piecewise smooth being trivial

[27] The fact that the $a_i$ can be chosen smooth follows from our requirement that $\gamma' \in \mathscr{A}_\delta(\gamma)$, and is indeed the reason of such a requirement

[28] Here we are abusing notations and using the same symbol to designate both the vector field and its coordinates with respect to the basis $\{\partial_q, \partial_p\}$

*to some partition $\mathscr{P}$ (coarser than $\mathscr{P}(\{\xi\})$) and an allowed deformation with respect to $\mathscr{P}$ connecting $\xi$ to some other configuration $\xi_1 \in \Xi \backslash \widetilde{\widetilde{\Xi}}$, then there exists a neighborhood $B_2$ of $\xi_1$ such that*

$$\mu|_{B_2} = M_{\mathscr{P}}|_{B_2}.$$

*Proof.* Choose a neighborhood $B_1 \subset B_0$ of $\xi$ such that $\mathscr{P}(\Upsilon B_1)$ is equal to $\mathscr{P}(\{\xi_1\})$. Let $\chi$ be the characteristic function of $\Upsilon B_1$ and $F_\Lambda$ a smooth local function with support disjoint from $\widetilde{\widetilde{\Xi}}$.

$$\begin{aligned}
&\int_{\Xi(n,\Pi,E,\omega_c)} \chi(q,\ p) F_\Lambda(q,\ p)\, d\mu_{n,\Pi,E,\omega_c}(q,\ p) \\
&= \int_{\Xi(n,\Pi,E,\omega_c)} \chi \circ \Upsilon(q,\ p) F_\Lambda \circ \Upsilon(q,\ p)\, d\mu_{n,\Pi,E,\omega_c}(q,\ p) \\
&= \int_{\Xi(n,\Pi,E,\omega_c)} \chi \circ \Upsilon(q,\ p) F_\Lambda \circ \Upsilon(q,\ p)\, dM_{\mathscr{P}}(q,\ p) \\
&= \int_{\Xi(n,\Pi,E,\omega_c)} \chi(q,\ p) F_\Lambda(q,\ p)\, dM_{\mathscr{P}}(q,\ p).
\end{aligned}$$

Since $M_{\mathscr{P}}(\widetilde{\widetilde{\Xi}}) = 0$, for each partition $\mathscr{P}$, and $\mu_{n,\Pi,E,\omega_c}$ is absolutely continuous with respect to $M$ for almost all $\Pi$, $E$, $\omega_c$, the equality holds true for each $F_\Lambda$. Note that we have used the invariance, with respect to the map $\Upsilon$, both of the measure $\mu$ and of the microcanonical measures $M_{\mathscr{P}}$.

The consequence of the above chain of equalities is that $\mu$, restricted to $\Upsilon B_1$, must be microcanonical with respect to $\mathscr{P}$. □

Let us conclude the proof of Proposition 4.4. According to Lemma 4.6 $\mu$ is microcanonical with respect to $\mathscr{P}$ in a neighborhood of $\xi_1$. In addition, we know from the considerations after proposition 4.1 that $\mu$ must be microcanonical with respect to $\mathscr{P}(\{\xi_1\})$ as well. Hence, by Lemma 4.5, it must be microcanonical with respect to $\mathscr{P} \wedge \mathscr{P}(\{\xi_1\})$ in $\Upsilon B_1$. Using again Lemma 4.6 (in the opposite direction) the result follows. □

Now that Proposition 4.4 is proven, let us put it to work. Our condition on the density insures that in the box $\Lambda$ there is, at least, a two particle cluster. If all the clusters in a given configuration could be connected, via an allowed deformation, to each nearby cluster, the argument would be easily concluded. Unfortunately, this is not always possible. The obstacle is that, in order to extract two particles from a cluster, may be needed more energy than it is available in the cluster itself[29]. Such clusters are "locked" and, in principle, it may be impossible to remove them.

It is then painfully clear that our argument can be concluded only via a discussion of such a pathological behavior; this the task of the next section.

[29] E.g., think of a cluster at a corner with a concave shape and zero kinetic energy; in addition the boundary condition may prevent the cluster from sliding along a side of the box

*4.2 Killing locked clusters*

This section is dedicated to concluding the proof of Proposition 4.1.

Let $\xi_0 \in (\Xi \backslash \widetilde{\Xi}) \bigcap \widehat{\Omega}_{\Lambda,\varepsilon}$, consider a neighborhood $B_0 \subset \Xi \backslash \widetilde{\Xi}$, contained in a ball of sufficiently small radius $\delta$ such that $\mathscr{P}_0 = \mathscr{P}(\{\xi_0\})$ is equal to $\mathscr{P}(B_0)$. Our task will be to show that there is an allowed deformation that, together with Proposition 4.4, can be used to show that $\mu$ is microcanonical in a neighborhood of $\xi_0$ contained in $B_0$.

We will start by showing that $\mu$ is microcanonical with respect to partitions coarser than $\mathscr{P}_0$.

Choose any element containing more than one particle, consider its convex hull and choose a particle $\alpha$ situated at an extremal point but further away than $R_1$ from the boundary (if such a particle exists). The chosen particle $\alpha$ can clearly be brought at a distance larger than $R_0$, but less than $R_1$, from the other particles in the cluster in such a way that along the deformation the total potential energy decreases (here we are using in an essential way the repellent nature of the potential). Since along an allowed deformation the total energy of the element must be conserved, one must change the kinetic energy of the particles to compensate the change in potential energy[30].

Select then the extremal particle in the convex hull of the element (with $\alpha$ removed), closest to $\alpha$. Such a particle can be extracted as well to a distance larger than $R_0$, but less than $R_1$. The two particles can now be brought to a distance less than $R_1$, but larger than $R_0$, if they are not so already. This constitutes a two particle cluster that can be moved to touch another element[31].

Of course, it could happen that a new element is touched by one of the two particles before the above process is completed; this would be as good, since our goal is to join different elements. In addition, if the touched element consists of only one particle, it is allowed to use such a particle to create a new two particle cluster (hence, disregarding one of the previous two) that can be moved to touch other elements.

In force of Proposition 4.4, this allows to show that, in a neighborhood of $\xi_0$, $\mu$ must be microcanonical with respect to coarser partitions than the one in cluster.

We can iterate the above strategy until we get the coarsest possible partition $\mathscr{P}_*$. Such a partition depends only on the positions in the configuration $\xi_0$ and is uniquely determined by them.

The partition $\mathscr{P}_*$ is the partition $\mathscr{P}(\xi_0)$ that we will use to apply Lemma 3.2.

As we have seen, to carry out the above scheme we need to exchange freely the momenta among particles in the same element[32]. More precisely,

[30] E.g., choose any two particles $\alpha'$, $\beta'$, with momenta $p_{\alpha'}$, $p_{\beta'}$, belonging to the element and change their momenta to $p_{\alpha'} + \lambda\eta$, $p_{\beta'} - \lambda\eta$, where $\eta \in \mathbb{R}^3$, $\|\eta\| = 1$, and $\lambda$ is chosen to ensure energy conservation

[31] By "touch" we always mean "closer than $R_1$ but further away than $R_0$"

[32] In reality, up to now it suffices the already discussed possibility to exchange freely momenta inside the same cluster (by properly choosing the particles in which to store the excess of energy), but in the future the extra freedom discussed below will be essential

given a configuration $\xi = (q, p)$ and a partition $\mathscr{P}$ let us define

$$\Pi(\xi, \mathscr{P}) = \Bigg\{ (q', p') \in \Xi \backslash \widetilde{\Xi} \;\Bigg|\; q' = q;\ \forall P \in \mathscr{P} \quad \sum_{j \in P} p_j = \sum_{j \in P} p'_j;\ \sum_{j \in P} \phi(p_j) = \sum_{j \in P} \phi(p'_j) \Bigg\}.$$

We will call $\mathscr{P}$-complete a configuration $\xi$ that can be connected via an allowed deformation with respect to $\mathscr{P}$ to each configuration in $\Pi(\xi, \mathscr{P})$[33]. In other words, if $\xi$ is $\mathscr{P}$ complete, then any transformation $T_{\alpha\beta}$ (cf. (4.2)), in which $\alpha$, $\beta$ belong to the same element of $\mathscr{P}$, can be realized by allowed transformations. By construction, $\xi_0$ is $\mathscr{P}_0$-complete.

Let us show that $\xi_0$ is $\mathscr{P}_*$-complete. This is done by exhibiting allowed deformations relative to $\mathscr{P}_*$ that connect $\xi_0$ to any other configuration in $\Pi(\xi_0, \mathscr{P}_*)$. We construct explicitly the needed deformations. To each deformation used to construct $\mathscr{P}_*$ one can associate the following: transfer to the two particles in question any needed amount of energy and momentum compatible with the conservation of total energy and momentum in the cluster. Extract the two particles from the cluster (the excess of energy so created is stored in extra kinetic energy of the other particles in the cluster) and move the two particles to touch the same neighbor cluster as before (without interacting deterministically); at this point energy and momentum can be exchanged. Finally, move back the two particles to their original position.

By eventually repeating such a procedure, we can exchange any amount of moments, compatible with the conservation laws, between the two clusters. Playing this game with all the various deformations used to construct $\mathscr{P}_*$ it follows that $\xi_0$ is $\mathscr{P}_*$-complete. Although this last fact seems quite intuitive the proof is not immediate (due to the limitations on the momentum that can be exchanged conserving the kinetic energy) and can be found in Appendix II.

If at this point we have obtained the trivial partition we are done. But sometimes elements may be separated by large clusters with little kinetic energy. At this point it is necessary to use the properties of the configurations in $\widehat{\Omega}_{\Lambda, \varepsilon}$.

We intend to produce a sequence of configurations $\xi_i$ and coarser and coarser partitions $\mathscr{P}_i$, where $\xi_i$ is connected to $\xi_{i+1}$ by an allowed deformation with respect to $\mathscr{P}_i$. Proposition 4.4 will be used to show that $\mu = M_{\mathscr{P}_i}$ in some neighborhood of $\xi_i$. From this it will follow that $\mu = M_{\mathscr{P}_i}$ also in some neighborhood of $\xi_0$ (by Lemma 4.6).

We will also ensure that the configurations $\xi_i$ that we are going to construct are all $\mathscr{P}_i$-complete and that $\xi_i \in \widehat{\Omega}_{\Lambda, \varepsilon}$.

The general strategy to produce the wanted deformations will be the following: first we will see that $\mathscr{P}_*$ contains many particles belonging to "large" elements, then we will repeatedly apply Lemma 3.2 to show that there is always at least one element with enough energy to allow the extraction of a two

[33] The reason why we are interested in this concept is that, in some sense, it is a generalization of hypoellilticity; in fact, thanks to Proposition 4.4, it allows to conclude that the measure, restricted to $\Pi(\xi, \mathscr{P})$ is microcanonical with respect to $\mathscr{P}$

particle cluster. We will so obtain a partition with only one "large" element, and Lemma 3.2 will imply that we can create a "free two particles cluster." That is, two particles with so much energy that can be moved anywhere in the box $\Lambda$ (remember that $\xi_0 \in \widehat{\Omega}_{\Lambda,\varepsilon}$, so it enjoys a maximal local density in $\Lambda$, hence there is a maximal potential barrier to moving a particle that grows only like $|\Lambda|^{1/2}$).

Let us start by seeing that indeed the creation of a "free two particles cluster" would conclude the argument.

**Definition 4.7.** *We will call "free two particle cluster" any two particles at a distance less than $R_1$ such that their "available kinetic energy"*[34] *is larger than $\varepsilon^{-1}|\Lambda|^{1/2}V_0$.*

The justification of the name lies in the proof of the following Lemma.

**Lemma 4.8.** *If a configuration $\xi \in \Xi \cap \widehat{\Omega}_{\Lambda,\varepsilon}$ contains a free two particles cluster, then, in a neighborhood of $\xi$, $\mu$ is microcanonical.*

*Proof.* Since $\xi \in \widehat{\Omega}_{\Lambda,\varepsilon}$ it follows that a free two particles cluster has enough available energy to be moved anywhere in the box $\Lambda$. Accordingly we can produce allowed deformations that bring the free two particle cluster in contact with any given particle in $\Lambda$ and the result follows from the repeated use of Proposition 4.4. $\square$

We are left with the task of producing a free two particle cluster. As already mentioned we will do so by repeated use of Lemma 3.2.

To make the above considerations precise a little geometry is needed, the geometric ingredient will take the form of a sequence of concentrical boxes. Consider a sequence of concentrical boxes $\Lambda_k$, $k \geq 1$, such that $\Lambda_{k+1} \supset \Lambda_k$ and $(B-1)R_1 <\mathrm{dist}(\partial\Lambda_k, \partial\Lambda_{k+1}) < BR_1$ for some fixed $B$ large enough[35]. Without loss of generality, we can assume that there exists a constant $C$ such that $\Lambda_{CL} = \Lambda$.

**Definition 4.9.** *Given a partition $\mathscr{P}$, we will call an element "large" if it contains at least one particle belonging to $\Lambda_{CL}\backslash\Lambda_{CL-1}$ (i.e., it touches the boundary) and one belonging to $\Lambda_1$.*

We now choose $\widehat{\mathscr{P}}(\xi_0)$ to be exactly the set of large elements of $\mathscr{P}_*$ ($\mathscr{P}$ $(\xi_0)$). For the application of Lemma 3.2 it is essential to know how many particles belong to $\widehat{\mathscr{P}}(\xi_0)$ (if they are less than $\frac{a}{8}|\Lambda|$, then Lemma 3.2 is empty).

[34] By "available energy" of a group of $m$ particles, with total momentum $\Pi$, we mean the maximal amount of kinetic energy that can be liberated and transferred to other particles or converted in potential energy, i.e. $\sum_\gamma \phi(p_\gamma) - m\phi(\frac{1}{m}\Pi)$

[35] For example $B > 6R_0$ would do: it allows to move a two particle cluster in the corridor between two consecutive boxes being sure that the two particles do no interact deterministically with the ones outside the corridor. As we will see shortly this is exactly what we will need to do

This is easily estimated since, as a result of our previous construction, we can restrict ourselves to a situation in which all the particles belonging to $\Lambda_1$ (in the configuration $\xi_0$) must either be part of a large element of the final partition $\mathscr{P}_*$ or constitute a one particle cluster.

In order to see this, suppose that there is an element, not large, containing more than two particles, with one particle in $\Lambda_1$. For it is not large, it does not touch the boundary: so it is possible to extract a two particle cluster from its convex hull. Since we have already played this game to exhaustion it must be that there are no particles outside the convex hull of the element, otherwise one could obtain a coarser partition by using the above mentioned strategy. Accordingly, any other particle must be contained in the convex hull of such an element. Suppose now that there exists another element containing more than one particle in $\Lambda_1$. Such an element cannot touch $\partial\Lambda$, otherwise it would have particles outside the convex hull of the element previously considered. By the same argument as before all the particles must be contained in the convex hull of the new element. Hence, the convex hull of this two elements are the same. Therefore the extremal points must be in common contrary to the assumption that the two elements are different.

The only alternative is that all the other elements consist of only one particle, but our bound on the density would imply that our element contains at least $a(\frac{L}{2})^3$ particles, and therefore (provided $L > (8a^{-1}V_0\varepsilon^{-1})^{2/3}$) enough available kinetic energy to form a free two particle cluster (see Lemma 3.2).

Therefore, we need consider only the configurations $\xi_0$ for which all the elements containing more than one particle, of whose at least one in $\Lambda_1$, are large in $\mathscr{P}(\xi_0)$. Hence, at least $a(\frac{L}{2})^3$ of the particles belong to large elements, i.e. to $\widehat{\mathscr{P}}(\xi_0)$, (the other $\rho_*(\frac{L}{2})^3$ particles belonging to $\Lambda_1$ may be isolated one particles clusters). In other words, all the configurations that we are left to study satisfy the first condition of $\widetilde{\Omega}^{\varepsilon}_{\Lambda}$.

Another interesting property of large elements is that we can assume that they contain at least one particle in each region $\Delta_k \equiv \Lambda_k\backslash\Lambda_{k-1}$. Otherwise, one can extract a two particle cluster from the element restricted to $\Lambda_{k-1}$ (more precisely the extremal particles closer to $\Delta_k$) and move the two particles into the region $\Delta_k$. Since the partition is already the coarsest obtainable with such deformations it is not possible that the two particles can be brought in contact with a new particle, i.e. no particle can be contained in $\Delta_k$. This implies immediately that no other cluster with more than two particles can be present in $\Lambda_{k-1}$ (again, if this would not be the case, one could extract two particles from the other element and move them to the element into consideration). Hence, such an element would contain at least $a(\frac{L}{2})^3$ particles and have enough energy to create a free two particle cluster.

According to Lemma 3.2, there must be at least one (large) element $P$ of $\widehat{\mathscr{P}}(\xi_0)$ such that $\sum_{\alpha\in P}[\phi(p_\alpha) - \phi(\frac{1}{\#P}\sum_{\beta\in P} p_\beta)] \geq \varepsilon\#P$. Let us consider such an element. Clearly, there must be a region $\Delta_k$ in which it has less than $(B\#P)/L$ particles. We can assume, without loss of generality, that in such a region there is a two particle cluster (if not it can be created by an allowed deformation). Since we can always choose $L > \varepsilon^{-1}BV_0$, it follows that the element can transfer to the two particle cluster enough energy to let it move anywhere in the region $\{\zeta \in \Delta_k \mid \mathrm{dist}(\zeta, \partial\Delta_k) \geqq R_0\}$ in order to touch another element. If the two particle cluster cannot enter in contact with any other element, then it would be the only large element, hence, by Lemma 3.2, it would have at least $\varepsilon a(L/2)^3$ available energy, enough to create a free two particle cluster.

For the remaining configurations we have obtained a new partition $\mathscr{P}_1 \in \widehat{\underline{\mathscr{P}}}_{\xi_0}$ coarser than $\mathscr{P}_*$; yet $\xi_0$ could not be $\mathscr{P}_1$-complete[36]. To overcome this we consider a slightly different configuration $\xi_1 \in \Omega_{\Lambda,\varepsilon}$ that is $\mathscr{P}_1$-complete. Simply move the two particles cluster in the region $\{\xi \in \Delta_k \mid \mathrm{dist}(\xi, \partial\Delta_k) \geq R_0\}$ at a distance larger than $R_0$ from the other particles in the same element. This new configuration $\xi_1$ has lower local density, the same density in $\Lambda$ and in $\Lambda_1$, higher available energy, and the same large elements; finally, by the arguments already mentioned it is $\mathscr{P}_1$ complete.

We can then apply Lemma 3.2 to $\mathscr{P}_1$ and obtain a new large element from which, in the configuration $\xi_1$, can be extracted a two particle cluster, whereby obtaining a new, coarser, partition $\mathscr{P}_2 \in \widehat{\underline{\mathscr{P}}}_{\xi_0}$ and a new configuration $\xi_2 \in \widehat{\Omega}_{\Lambda,\varepsilon}$ that is $\mathscr{P}_2$-complete.

This process will stop only with the creation of a free two particle cluster or when it will be left just one large element. Since such an element, again by Lemma 3.2, will have enough energy to create a free two particle cluster we have shown that $\mu$ is Microcanonical in some neighborhood of $\xi_0$.

**Conclusion of the proof of Proposition 4.1**

Up to now, we have shown that for each $\xi \in (\Xi \backslash \widetilde{\Xi}) \cap \widehat{\Omega}_{\Lambda,\varepsilon}$ there exists a neighborhood $B \subset \Xi$ of $\xi$, such that $\mu|_B = M|_B$.

To prove this we have used Proposition 4.4 to show that, if we can construct allowed deformations that bring particles of different clusters together, then in a neighborhood of $\xi$, $\mu$ must be microcanonical with respect to a partition coarser than the one in clusters. The technical obstacle to this program has been the possibility of very low energy clusters from which no particle can be extracted[37]. To overcome such an obstacle we have shown that, if we consider only elements that contain at least one particle close to the center of the box, then some locked element must be fairly large and therefore, since $\xi \in \widehat{\Omega}_{\Lambda,\varepsilon}$, must contain sufficient energy to extract a particle. We continue the process until we are able to extract two particles with so much energy that they can be brought into contact with any other particle.

It suffices then to apply the previous discussion to each point in $(\Xi \backslash \widetilde{\Xi}) \cap \widehat{\Omega}_{\Lambda,\varepsilon}$. Accordingly, in the neighborhood of each point, $\mu$ is proportional to the microcanonical measure. This implies immediately that $\mu$ is proportional to the

[36] Think of the case in which the goal is to transfer all the available energy of the element, to which the two particles belong, to the other element (the one that the two particles are brought into contact with). It could easily happen that not enough energy is left in the original element to take the two particles back to their original position

[37] Indeed, we extract first all the particles that can be extracted without paying any energy; after that, if more than one element is still present, we need to worry about how much energy is at our disposal.

microcanonical measure on all $\Xi \cap \widehat{\Omega}_{\Lambda,\varepsilon}$. Hence, it follows (4.4) and we can assert that $Q_{\Lambda_0}$ is microcanonical.

## 5. Proof of Theorem 1

The conclusion of the previous section is summarized by the following lemma:

**Lemma 5.1.** *For almost every configuration of the positions $\omega_q$ and any $\Lambda_0$, the conditional measure $Q$ on $p_{\Lambda_0}$ given $\omega_q$ and*

$$\sum_{q_j \in \Lambda} \phi(p_j) = const$$

$$\sum_{q_j \in \Lambda} p_j = const$$

*is the microcanonical measure on the corresponding surface.*

At this point we are in the same situation as in [OVY] (after Lemma 4.5 there). In fact, as a consequence of the previous lemma, the distribution of the momentum conditioned on the positions is given by a convex combination of measures of the form[38]

$$\pi(dp \mid \Lambda) = \prod_{\alpha} \frac{\exp\left[\sum_{i=1}^{3} \lambda_i p_\alpha^i - \lambda_4 \phi(p_\alpha)\right]}{\text{Normalization}}\, dp_\alpha .$$

**Lemma 5.2.** *For any configuration $\omega = \{(q_\alpha, p_\alpha)\}$, let $\vec{z}(\omega)$ be the density, momenta and kinetic energy associated with the configuration defined by*

$$z^0(\omega) = \lim_{\delta\to 0} z^0_{\chi,\delta}(\omega) = \lim_{\delta\to 0} \delta^3 \sum_{q_\alpha \in \omega} \chi(\delta q_\alpha),$$

$$z^\mu(\omega) = \lim_{\delta\to 0} z^\mu_{\chi,\delta}(\omega) = \lim_{\delta\to 0} \delta^3 \sum_{q_\alpha \in \omega} \chi(\delta q_\alpha)\, p^\mu_\alpha(\omega), \quad \mu = 1,2,3,$$

$$z^4(\omega) = \lim_{\delta\to 0} z^4_{\chi,\delta}(\omega) = \lim_{\delta\to 0} \delta^3 \sum_{q_\alpha \in \omega} \chi(\delta q_\alpha)\, \phi(p_\alpha)\ .$$

*Here $\chi$ is a cutoff function of total integral one, $\vec{z}^{\,\mu}(\omega)$ exist almost everywhere and are independent of the cutoff $\chi$. Furthermore, $\vec{z}(\omega)$ are constants of the motion for $L$ in the sense that*

$$\int h(\vec{z}(\omega))\, LF(\omega)\, dQ = 0,$$

*for all local smooth functions $F$ and all smooth functions $h$ with compact support.*

*Proof.* This was proven in [OVY] for bounded $\phi'$. For completeness, we present here the proof for unbounded $\phi'$.

[38] This is a consequence of the Hewitt–Savage law, cf. [HS]

By the same argument used immediately after Lemma (1.1) these limits clearly exist and are independent of the cutoff $\chi$.

By condition (iii) in Theorem (1.1)

$$
\begin{aligned}
0 &= \int L(Fh(z^{\mu}_{\chi,\delta}(\omega)))\,dQ \\
&= \int (LF)\,h(z^{\mu}_{\chi,\delta}(\omega))\,dQ + \int F\,L\,h(z^{\mu}_{\chi,\delta}(\omega))\,dQ.
\end{aligned}
$$

The first term converges to $\int h(z^{\mu}_{\chi}(\omega))LF\,dQ$ as $\delta \to 0$. We only have to show that the second term converges to zero as $\delta \to 0$. Clearly, it suffices to show that as $\delta \to 0$

$$
\int |Lz^{\mu}_{\chi,\delta}|\,dQ \to 0\ , \quad \mu = 0,\dots,4. \tag{5.1}
$$

This is easy to show for $\mu = 0, 1, 2, 3$ (as in [OVY p. 544]).

For $\mu = 4$ we have

$$
\begin{aligned}
\mathbb{E}^{Q}\left(\left|Lz^{4}_{\chi,\delta}\right|\right) =& \mathbb{E}^{Q}\left(\left|\delta\,\delta^{3}\sum_{i,\alpha}\chi_i(\delta q_\alpha)\,\phi_i(p_\alpha)\,\phi(p_\alpha)\right|\right) \\
&+ \mathbb{E}^{Q}\left(\left|\delta^{3}\sum_{\alpha\neq\beta}\sum_{i}\chi(\delta q_\alpha)\,\phi_i(p_\alpha)\,V_i(q_\alpha - q_\beta)\right|\right).
\end{aligned}
$$

Only the second term of the right end side present difficulties. Let $w_i(\vec{z})$ and $\sigma_i(\vec{z})$ denote the expectation and variance of $\phi_i(p_\alpha)$ with respect to $Q$ conditioned on $\vec{z}$. These can be computed explicitly by using the characterization of the conditional measure given $\omega_q$ and $\vec{z}$. We can bound the second term of the RHS of the above expression by

$$
\begin{aligned}
&\mathbb{E}^{Q}\left(|\delta^{3}\sum_{i}\sum_{\alpha\neq\beta}\chi(\delta q_\alpha)\,\phi_i(p_\alpha)\,V_i(q_\alpha - q_\beta)|\right) \\
&= \mathbb{E}^{Q}\left(|\delta^{3}\sum_{i}\sum_{\alpha}\chi(\delta q_\alpha)\big[\phi_i(p_\alpha) - w_i\big]\sum_{\beta\neq\alpha}V_i(q_\alpha - q_\beta)|\right) \\
&\quad + \mathbb{E}^{Q}\left(|\delta^{3}\sum_{\alpha\neq\beta}\sum_{i}\chi(\delta q_\alpha)\,V_i(q_\alpha - q_\beta)\,w_i|\right).
\end{aligned}
$$

The second term of the RHS (third line above) can be bounded as before. Using the Schwarz inequality the first term can be bounded by

$$\sum_i \mathbb{E}^Q\left(\mathbb{E}^Q\left(\left[\delta^3\sum_\alpha \chi(\delta q_\alpha)\left[\phi_i(p_\alpha)-w_i\right]\sum_{\beta\neq\alpha}V_i(q_\alpha-q_\beta)\right]^2\,\middle|\,\vec{z}\right)^{1/2}\right)$$

$$=\sum_i \mathbb{E}^Q\left(\sqrt{\sigma_i(\vec{z})}\delta^3\,\mathbb{E}^Q\left(\sum_\alpha \chi(\delta q_\alpha)^2\left(\sum_\beta V_i(q_\alpha-q_\beta)\right)^2\,\middle|\,\vec{z}\right)^{1/2}\right)$$

$$\leqq \sum_i \mathbb{E}^Q(\sigma_i(\vec{z}))^{1/2}\delta^3\mathbb{E}^Q\left(\sum_\alpha \chi(\delta q_\alpha)^2\left(\sum_{\beta\neq\alpha}V_i(q_\alpha-q_\beta)\right)^2\right)^{1/2}.$$

By the condition on $\phi$ and the entropy argument we have that $\mathbb{E}^Q(\sigma_i(\vec{z}))$ is finite. To bound the second expectation, let us divide the set $\left\{x\,|\,|x|\leqq 2\delta^{-1}\right\}$ into boxes of size $2R_0$ ($R_0$ is the range of $V$). Let $\sigma$ index the boxes and let $N_\sigma$ be the number of particles in the $\sigma$ box.

$$\delta^3\sum_i \mathbb{E}^Q\left(\sum_\alpha \chi(\delta q_\alpha)^2\left[\sum_\beta V_i(q_\alpha-q_\beta)\right]^2\right)^{1/2}\leq \text{const. }\delta^3\,\mathbb{E}^Q\left(\sum_\sigma N_\sigma^3\right)^{1/2}.$$

By convexity and the inequality $(\sum_\sigma N_\sigma^3)^{1/3}\leqq(\sum_\sigma N_\sigma^2)^{1/2}$ we see that the above expression is bounded by

$$\text{const. }\delta^3\mathbb{E}^Q\left(\left[\sum_\sigma N_\sigma^2\right]^{3/4}\right)\leqq\ \text{const. }\delta^3\left[\mathbb{E}^Q\left(\sum_\sigma N_\sigma^2\right)\right]^{3/4}.$$

By Lemma (1.2)(i) and the translation invariance, $\mathbb{E}^Q(\sum_\sigma N_\sigma^2)$ is bounded by $\delta^{-3}$; hence, the quantity under consideration is bounded by const. $\delta^{3/4}$. This concludes the proof of the Lemma 4.2. □

By the previous lemma $Q$ conditioned on $\vec{z}(\omega)$ is still invariant for $L$. Since we assume that Q is translation invariant, we can apply Lemma 4.10 in [OVY] and obtain that these conditioned distributions are given by grancanonical Gibbs measures, concluding our proof.

**Appendix 1**

To show that our conditions on the noise (in particular the one imposed in Sect. 2) and the kinetic energy in the non-Gaussian case are far from empty, we give here an example of stochastic perturbation that satisfies such condition. This is the only point where we use our requirement on the form of the kinetic energy $\phi$.

**Lemma A.1.** *If* $\{\eta^{\theta}_{\alpha\beta}\} = \{e_1 \wedge D_{\alpha\beta}E, e_2 \wedge D_{\alpha\beta}E, e_3 \wedge D_{\alpha\beta}E\}$ *and* $\phi(p_a) = \sum_{i=1}^{3} \varphi(p^i_\alpha)$, *with* $(\varphi''')^2 + \varphi^{iv}\varphi'' = 0$ *at most at finitely many points, then condition on the noise is satisfied.*

*Proof.* A simple computation shows

$$\begin{aligned}
\left[X^i_{\alpha\beta}, L\right] q_\alpha &= H_\alpha \eta^i_{\alpha\beta}, \\
\left[X^i_{\alpha\beta}, L\right] q_\beta &= -H_\beta \eta^i_{\alpha\beta}, \\
\left[X^j_{\alpha\beta}, \left[X^i_{\alpha\beta}, L\right]\right] q^l_\alpha &= \left(\eta^j_{\alpha\beta}\right)_l \left(\eta^i_{\alpha\beta}\right)_l H'_{\alpha ll} + H_{\alpha ll}\left(e^i \wedge (H_\alpha + H_\beta)\eta^j_{\alpha\beta}\right)_l, \\
\left[X^i_{\alpha\beta}, \left[X^j_{\alpha\beta}, L\right]\right] q^l_\beta &= \left(\eta^j_{\alpha\beta}\right)_l \left(\eta^i_{\alpha\beta}\right)_l H'_{\beta ll} - H_{\beta ll}\left(e^i \wedge (H_\alpha + H_\beta)\eta^j_{\alpha\beta}\right)_l,
\end{aligned}$$

where $H_{\alpha ll}$ stand for the element $ll$ of the diagonal matrix $H_\alpha$. The matrix $H'_\alpha$ is the derivative of the matrix $H_\alpha$; $(\cdot)_l$ stands for the $l$th component of the corresponding vectors. Now, let us take the six vectors obtained by letting $i$, $j$ vary only in $\{1, 2\}$. We define the vectors $w^{ij}$ by

$$w^{ij}_l = \left(\eta^i_{\alpha\beta}\right)_l \left(\eta^j_{\alpha\beta}\right)_l.$$

Let us consider

$$\sum_{i=1}^{2} \mu_i \left[X^i_{\alpha\beta}, L\right] + \sum_{i,j=1}^{2} v_{ij}\left[X^j_{\alpha\beta}, \left[X^i_{\alpha\beta}, L\right]\right] = 0.$$

Applying the above vector fields to $q_\alpha$, $q_\beta$, we have

$$\begin{aligned}
0 &= \sum_{i=1}^{2} \mu_i H_\alpha \eta^i_{\alpha\beta} + \sum_{i,j=1}^{2} v_{ij}\left[H'_\alpha w^{ij} + H_\alpha\left(e^i \wedge (H_\alpha + H_\beta)\eta^j_{\alpha\beta}\right)\right], \\
0 &= -\sum_{i=1}^{2} \mu_i H_\beta \eta^i_{\alpha\beta} + \sum_{i,j=1}^{2} v_{ij}\left[H'_\beta w^{ij} - H_\beta\left(e^i \wedge (H_\alpha + H_\beta)\eta^j_{\alpha\beta}\right)\right].
\end{aligned}$$

If we multiply the first by $(H_\alpha)^{-1}$, the second by $(H_\beta)^{-1}$, and add one to the other, then we get

$$0 = \sum_{i,j=1}^{2} v_{ij} A w^{ij},$$

where $A = 1/2\{H'_\alpha H^{-1}_\alpha + H'_\beta H^{-1}_\beta\}$. Notice that $A$ is invertible out of a set of codimension 1 (see later for more details), consequently

$$\begin{aligned}
0 &= \sum_{i,j=1}^{2} v_{ij} w^{ij}, \\
0 &= \sum_{i=1}^{2} \mu_i \eta^i_{\alpha\beta} + \sum_{i,j=1}^{2} v_{ij} e^i \wedge (H_\alpha + H_\beta)\eta^j_{\alpha\beta}.
\end{aligned}$$

To conclude we need an explicit representations of the vectors involved in the previous equations. Let $D^i_{\alpha\beta}E = \zeta_i$, $h_i = H_{\alpha ii} + H_{\beta ii}$, then a direct computation yields

$$\begin{aligned}
&\eta^1 = (0, -\zeta_3, \zeta_2),\\
&\eta^2 = (\zeta_3, 0, -\zeta_1),\\
&w^{11} = (0, \zeta_3^2, \zeta_2^2),\\
&w^{12} = w^{21} = (0, 0, -\zeta_1\zeta_2),\\
&w^{22} = (\zeta_3^2, 0, \zeta_1^2),\\
&e^1 \wedge (H_\alpha + H_\beta)\eta^1_{\alpha\beta} = (0, -\zeta_2 h_3, -\zeta_3 h_2),\\
&e^1 \wedge (H_\alpha + H_\beta)\eta^2_{\alpha\beta} = (0, \zeta_1 h_3, 0),\\
&e^2 \wedge (H_\alpha + H_\beta)\eta^1_{\alpha\beta} = (\zeta_2 h_3, 0, 0),\\
&e^2 \wedge (H_\alpha + H_\beta)\eta^2_{\alpha\beta} = (-\zeta_1 h_3, 0, -\zeta_3 h_1).
\end{aligned}$$

Immediately follows $v_{22} = v_{11} = 0$ and $v_{12} = -v_{21}$, which, substituted in the remaining equations, yields

$$\Omega \begin{pmatrix} \mu_1 \\ \mu_2 \\ v_{12} \end{pmatrix} = 0.$$

For some matrix $\Omega$ with $\det(\Omega) = \zeta_1\zeta_2\zeta_3(h_2 + h_3)$. Since the determinant is equal zero on a set of codimension one, we have that the vector are linearly independent, out of a set of codimension one.

This set of codimension one consists of $\cup_i\{p \mid \varphi'''(p^i_\alpha)\varphi''(p^i_\alpha) = -\varphi'''(p^i_\beta)\varphi''(p^i_\beta)\}$, where the matrix $A$ is not invertible[39], and $\cup_i\{p^i_\alpha = p^i_\beta\}$, where the matrix $\Omega$ is not invertible. To get codimension two we have to analyze all the different cases one by one, since they are treated all in the same way we will consider only the points on the set $\{p^1_\alpha = p^1_\beta\}$, and we will leave the rest to the skeptical reader. We can clearly ignore points of the above set that also belong to some other singular set: they belong to a set of codimension two. For points in the set under consider we will have $\zeta_1 = 0$, while all the other components will be different from zero. This implies that $w^{12} = w^{21} = e^1 \wedge (H_\alpha + H_\beta)\eta^2_{\alpha\beta} = 0$, we need then to produce more vectors, i.e., compute more commutators. It turns out to be sufficient to compute

$$\begin{aligned}
&\left[X^2, \left[X^1_{\alpha\beta}, \left[X^2_{\alpha\beta}, L\right]\right]\right] q_\alpha = H_\alpha v_1 + H'_\alpha v_2,\\
&\left[X^2, \left[X^1_{\alpha\beta}, \left[X^2_{\alpha\beta}, L\right]\right]\right] q_\beta = -H_\beta v_1 + H'_\beta v_2,\\
&\left[X^2, \left[X^2_{\alpha\beta}, \left[X^1_{\alpha\beta}, L\right]\right]\right] q_\alpha = H'_\alpha v_2,\\
&\left[X^2, \left[X^2_{\alpha\beta}, \left[X^1_{\alpha\beta}, L\right]\right]\right] q_\beta = H'_\beta v_2,
\end{aligned}$$

[39] The condition of the hypothesis ensure that such set is a smooth codimension one manifold unless $\varphi'''(p^i_\alpha)^2 + \varphi^{iv}(p^i_\alpha)\varphi''(p^i_\alpha) = \varphi'''(p^i_\beta)^2 + \varphi^{iv}(p^i_\beta)\varphi''(p^i_\beta) = 0$, which can happen only on a set of codimension two

where $v_1 = (0, \zeta_3 h_1 h_3, 0)$, and $v_2 = (0, 0, -\zeta_2\zeta_1 h_1)$. We have then to study the linear combination

$$\sum_{i=1}^{2} \mu_i \left[X^i_{\alpha\beta}, L\right] + \sum_{i,j=1}^{2} \nu_{ij} \left[X^j_{\alpha\beta}, \left[X^i_{\alpha\beta}, L\right]\right] + \varepsilon_1 \left[X^2, \left[X^1_{\alpha\beta}, \left[X^2_{\alpha\beta}, L\right]\right]\right] + \varepsilon_2 \left[X^2, \left[X^2_{\alpha\beta}, \left[X^1_{\alpha\beta}, L\right]\right]\right] = 0$$

where $\nu_{12}$, $\nu_{21}$ are taken to be zero since the corresponding commutators, when restricted to the $q_\alpha$, $q_\beta$ space, would not contribute anything of interest. As before, we apply the vectors to the coordinates $q_\alpha$, $q_\beta$, we multiply by $H_\alpha^{-1}$ and $H_\beta^{-1}$ and add the corresponding equations, in so doing we obtain

$$\sum_{i,j=1}^{2} \nu_{ij} w^{ij} + (\varepsilon_1 + \varepsilon_2) v_2 = 0$$

from this follows immediately $\nu_{11} = \nu_{22} = 0$, $\varepsilon_1 = -\varepsilon_2$. Substituting in the original equation we get

$$0 = \sum_{i=1}^{2} \mu_i H_\alpha \eta^i_{\alpha\beta} + \varepsilon_1 H_\alpha v_1,$$

which implies $\mu_i = \varepsilon_i = 0$ on a set of codimension two. □

## Appendix II

We will prove here that if $\xi$ is $\mathscr{P}$–complete, and two particle can be extracted from an element $P_1$ to join $P_2$ (or viceversa), then $\xi$ is complete for the partition $\mathscr{P}_*$ obtained from $\mathscr{P}$ joining $P_1$ and $P_2$.
Choose $\eta \in \Pi(\xi, \mathscr{P}_*)$. Call $\alpha$, $\beta$ the two particles that are allowed to move along $\gamma$.

The rough idea is to transfer energy and momentum between the elements[40] $P_1$ and $P_2$ by using the particles $\alpha, \beta$. Unfortunately, there are limits to how much momentum or energy we can transfer to the particles, due to the necessity to conserve the total energy and momentum of the clusters. To overcome this we will show that each $\eta \in \Pi(\xi, \mathscr{P}_*)$ can be deformed into the special configuration $\zeta \in \Pi(\xi, \mathscr{P}_*)$ defined by,[41]

$$\begin{aligned} p_\sigma &= \frac{\pi(\xi, P_1 \cup P_2)}{\#(P_1 \cup P_2)} \quad \forall \sigma \notin \{\alpha, \beta\}, \\ p_\alpha &= \frac{\pi(\xi, P_1 \cup P_2)}{\#(P_1 \cup P_2)} + \lambda v, \\ p_\beta &= \frac{\pi(\xi, P_1 \cup P_2)}{\#(P_1 \cup P_2)} - \lambda v, \end{aligned}$$

[40] Note that $\{\alpha, \beta\} \subset P_1$ and that in the configuration $\xi$ $P_1$ still form an element

[41] For each $P \subset \{1, \ldots, n\}$, by $\pi(\xi, P)$ and $K(\xi, P)$ we mean, respectively, the total momentum and kinetic energy, in the configuration $\xi$, of the particles belonging to $P$; by $\#P$ we mean, as usual, the cardinality of the set $P$

with some fixed $v \in \mathbb{R}^3$, $\|v\| = 1$, and $\lambda$ determined by

$$K(\xi, P_1 \cup P_2) = [\#(P_1 \cup P_2) - 2]\phi\left(\frac{\pi(\xi, P_1 \cup P_2)}{\#(P_1 \cup P_2)}\right) + \phi(p_\alpha) + \phi(p_\beta).$$

The desired allowed transformation will then be obtained by deforming $\xi$ into $\zeta$ and then by running backward the allowed transformation that connects $\eta$ to $\zeta$ (since the reverse of an allowed transformation it is still an allowed transformation).

Since, by convexity,

$$K(\xi, P_1 \cup P_2) \geqq \#(P_1 \cup P_2)\phi\left(\frac{\pi(\xi, P_1 \cup P_2)}{\#(P_1 \cup P_2)}\right),$$

if $\lambda = 0$ then $\Pi(\xi, \mathscr{P}_*)$, restricted to the particles in $P_1 \cup P_2$ consists of only the point $\xi$ and we have nothing to prove. Otherwise we proceed as follows: we make an allowed deformation that set all the moments in $P_1 \backslash \{\alpha, \beta\}$ equal to $(\#P_1)^{-1}\pi(\xi, P_1)$ while $p_\alpha = (\#P_1)^{-1}\pi(\xi, P_1) + v_1 v$ and $p_\beta = (\#P_1)^{-1}\pi(\xi, P_1) + v_1 v$, and $v_1$ is determined by the conservation of $K(\xi, P_1)$. Then we move the coordinates of the particles $\alpha, \beta$ accordingly to $\gamma$ but without changing their momenta. Once they get in touch with $P_2$ we change the momenta of the particles in $P_2$ to

$$p_* = \frac{1}{\#P_2 + 2}\left(\pi(\xi, P_2) + \frac{2}{\#P_1}\pi(\xi, P_1)\right),$$

apart from $p_\alpha = p_* + v_2 v$ and $p_\beta = p_* - v_2 v$, again $v_2$ is determined by the conservation of the kinetic energy of the new cluster $P_2 \cup \{\alpha, \beta\}$. Finally, we move back the particles $\alpha$, $\beta$ to their original position in the configuration $\xi$ and share again their momentum among all the particles in $P_1$ as we have done at the beginning. Let us call $\xi_{1,1}$ the configuration reached in such a way. Calling $\delta_0 = (1/\#P_1)\pi(\xi, P_1) - (1/\#P_2)\pi(\xi, P_2)$ and $\delta_1 = (1/\#P_1)\pi(\xi_{1,1}, P_1) - (1/\#P_2)\pi(\xi_{1,1}, P_2)$ a direct computation shows that

$$\delta_1 = \left(1 - \frac{2\#(P_1 \cup P_2)}{\#P_1(\#P_2 + 2)}\right)\delta_0.$$

If we iterate further the procedure just described we see that the difference between the average momentum in $P_1$ and $P_2$ goes to zero, this shows that we are getting closer and closer to the configuration $\zeta$; unfortunately only asymptotically. Nevertheless, after a finite number of iterations we will get to a configuration $\zeta_0$ for which

$$2\phi\left(\frac{\pi(\zeta_0, P_1)}{2} - \frac{(\#P_1 - 2)\pi(\xi, P_1 \cup P_2)}{2\#(P_1 \cup P_2)}\right) + (\#P_1 - 2)\phi\left(\frac{\pi(\xi, P_1 \cup P_2)}{\#(P_1 \cup P_2)}\right) < K(\zeta_0, P_1). \tag{A2.1}$$

Let $p_\sigma(\eta)$ be the momentum of the particle $\sigma$ in the configuration $\eta$. We deform $\zeta_0$ into $\zeta_1$ defined by

$$p_\sigma(\zeta_1) = \frac{\pi(\xi, P_1 \cup P_2)}{\#(P_1 \cup P_2)} \quad \text{for } \sigma \in P_1 \backslash \{\alpha, \beta\},$$

$$p_\alpha(\zeta_1) = \frac{\pi(\xi, P_1 \cup P_2)}{\#(P_1 \cup P_2)} + \frac{1}{2}\left[\pi(\zeta_0, P_1) - \#P_1 \frac{\pi(\xi, P_1 \cup P_2)}{\#(P_1 \cup P_2)}\right] + vv,$$

$$p_\beta(\zeta_1) = p_\alpha(\zeta_1) - 2vv,$$

where $v$ is defined by $K(\zeta_1, P_1) = K(\zeta_0, P_1)$. All this is possible provided (A2.1) is satisfied; in fact, (A2.1) express simply that there is sufficient energy to deform the momenta of the particles in $P_1$ to the above values. After achieving the configuration $\zeta_1$, to obtain the configuration $\zeta$ it suffices to take the particles $\alpha, \beta$ to $P_2$, adjust the momenta of the particles of $P_2$ to the value $\pi(\xi, P_1 \cup P_2)/\#(P_1 \cup P_2)$, which will make all the momenta agree with the ones in the configuration $\zeta$ and take $\{\alpha, \beta\}$ back to their original position in $\xi$.

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