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Cyclically stationary Brownian local time processes

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Summary. Local time processes parameterized by a circle, defined by the occupation density up to time T of Brownian motion with constant drift on the circle, are studied for various random times T. While such processes are typically non-Markovian, their Laplace functionals are expressed by series formulae related to similar formulae for the Markovian local time processes subject to the Ray–Knight theorems for BM on the line, and for squares of Bessel processes and their bridges. For T the time that BM on the circle first returns to its starting point after a complete loop around the circle, the local time process is cyclically stationary, with same two-dimensional distributions, but not the same three-dimensional distributions, as the sum of squares of two i.i.d. cyclically stationary Gaussian processes. This local time process is the infinitely divisible sum of a Poisson point process of local time processes derived from Brownian excursions. The corresponding intensity measure on path space, and similar Lévy measures derived from squares of Bessel processes, are described in terms of a 4-dimensional Bessel bridge by Williams' decomposition of Itô's law of Brownian excursions.

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1 Introduction

Let P_δ denote the probability distribution and associated expectation operator governing a one-dimensional Brownian motion $(B_t,\ t \geqq 0)$ started at $B_0 = 0$, with drift δ. So the P_δ distribution of B_t is Gaussian with $P_\delta B_t = \delta t$ and $P_\delta[(B_t - \delta t)^2] = t$. Let $(\mathring{B}_t,\ t \geqq 0)$ be the BM on a circle of unit circumference obtained as $\mathring{B}_t = B_t \bmod 1$, where the circle is identified with $[0, 1)$.

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Let $(L_t^x,\ x \in \mathbb{R},\ t \geqq 0)$ be the usual bicontinuous local time process of B, normalized as occupation density relative to Lebesgue measure. The corresponding local time process for $\overset{\circ}{B}$ is $(\overset{\circ}{L}{}_t^u = \sum_{z \in \mathbb{Z}} L_t^{u+z},\ 0 \leqq u < 1)$ where $\mathbb{Z}$ is the set of integers. For a subinterval I of $\mathbb{R}$, let $C^+(I)$ denote the space of non-negative continuous paths with domain I. For a random time T, set $\overset{\circ}{L}_T = (\overset{\circ}{L}{}_T^u,\ 0 \leqq u < 1)$, and view $\overset{\circ}{L}_T$ as a $C^+[0,1)$ valued random path. This paper describes the P_δ distribution of $\overset{\circ}{L}_T$ on $C^+[0,1)$, for various random times T, by a combination of three methods:

1) decomposition of the Brownian path by excursion theory;

2) the Ray–Knight description of various linear local time processes in terms of squares of Bessel processes;

3) application of series formulae for the Laplace functionals of squares of Bessel processes.

Following Williams [66–68], methods 1) and 2) have been developed and applied by several authors. See for instance [51, 61, 40], and further work cited in [61]. Method 3), which is described in Sect. 2, is a substitute for the traditional approach to computing Laplace transforms of additive functionals of BM via solutions of a Sturm–Liouville equation, as presented for example in [23, 34, 24, 51, 6]. Series formulae for solutions of Sturm–Liouville equations are well known to analysts [42, 25, 9]. But this method has been neglected by probabilists, even though it greatly simplifies the computation of moment generating functions of stopped additive functionals of one-dimensional diffusions. Such applications, indicated briefly in Sects. 2 and 6 of this paper, will be treated in more detail elsewhere [53]. As indicated in [50], these techniques can also be applied to analyse local time processes defined by diffusions on a network as considered in [3]. The circle is the simplest example where the Markovian properties of linear local time processes are lost due to the feedback effect of a loop [12].

For a constant time t, Bolthausen [5] showed that as $t \to \infty$ the P_0 distribution of $(\overset{\circ}{L}_t - t)\sqrt{t}$ on $C[0,1)$ converges weakly to a cyclically stationary Gaussian process $(2b_u - 2\int_0^1 b_v\,dv,\ 0 \leqq u \leqq 1)$ where b is a standard Brownian bridge. Leuridan [40] used methods 1) and 2), as developed in [51], to describe the P_0 distribution of $\overset{\circ}{L}_T$ for T a hitting time or an inverse local time of $\overset{\circ}{B}$, and to recover Bolthausen's Gaussian limit. The process $\overset{\circ}{L}_T$ is not cyclically stationary for a fixed time T, nor for any of the random T's considered by Leuridan. A central result of this paper is the following:

Proposition 1 *Let* $T_\pm = \inf\{t\colon |B_t| = 1\}$, *the time when* $\overset{\circ}{B}$ *first returns to* 0 *by a complete loop around the circle, so* $\overset{\circ}{L}{}_{T_\pm}^u = L_{T_\pm}^u + L_{T_\pm}^{u-1}$, $0 \leqq u < 1$. *For each* $\delta \in \mathbb{R}$, *the* P_δ *distribution of* $\overset{\circ}{L}_{T_\pm}$ *on* $C^+[0,1)$ *is cyclically stationary, reversible, and infinitely divisible, with exponential marginals.*

Proposition 1, which is proved in Sect. 3, is a circular analog of the following result for linear BM:

Proposition 2 *For each* $\delta > 0$, *the* P_δ *distribution of* $(L^u_\infty,\ 0 \leqq u < \infty)$ *on* $C[0,\infty)$ *is stationary, reversible, and infinitely divisible, with exponential marginals.*

See [44, 51, 46] for similar variations of the Ray–Knight theorems from which Proposition 2 is easily obtained along with this more precise description:

$$(L^u_\infty,\ 0 \leqq u < \infty;\ P_\delta) \overset{d}{=} (Y^2(u) + Z^2(u),\ 0 \leqq u < \infty;\ \tilde{P}_\delta) \tag{1}$$

where $\overset{d}{=}$ denotes equality in distribution of processes on $C[0,\infty)$, and $\tilde{P}_\delta$ governs $(Y(u),\ u \geqq 0)$ and $(Z(u),\ u \geqq 0)$ as two i.i.d. stationary Ornstein–Uhlenbeck processes which are centered Gaussian with covariance function $\tilde{P}_\delta[Y(u)Y(v)] = (2\delta)^{-1}e^{-\delta|v-u|}$. For a vector of non-negative random variables $(V_1, \dots, V_n)$ defined on some probability space $(\Omega, \mathscr{F}, P)$, call the distribution of $(V_1, \dots, V_n)$ *multivariate* χ^2 *with d degrees of freedom* if it is the distribution of the sum of squares of d independent copies of a vector of centered jointly Gaussian variables, say $(Z_1, \dots, Z_n)$, for some $d = 1, 2, \dots$. In particular, say that the distribution of (V_1, V_2) is $\chi^2(d, \mu, \rho^2)$ if V_1 and V_2 have a bivariate χ^2 distribution with d degrees of freedom, and common mean μ and correlation ρ^2. Then Z_1 and Z_2 have common variance μ/d and correlation ρ. In terms of Laplace transforms, the P distribution of (V_1, V_2) is $\chi^2(d, \mu, \rho^2)$ iff for $\alpha_i \geqq 0$

$$P\exp(-\alpha_1 V_1 - \alpha_2 V_2) = (1 + \alpha_1\mu + \alpha_2\mu + (1-\rho^2)\mu^2\alpha_1\alpha_2)^{-d/2} . \tag{2}$$

See for instance [28, 8]. According to (1) for $0 \leqq u < v < \infty$

$$\textit{the } P_\delta \textit{ distribution of } (L^u_\infty,\ L^v_\infty) \textit{ is } \chi^2(2,\ \delta^{-1},\ e^{-2\delta(v-u)}) . \tag{3}$$

Proposition 2 implies the bivariate χ^2 distribution is infinitely divisible for all choices of the parameters, a result found analytically by Vere–Jones [64], who gave formulae for the corresponding density and Lévy measure. See also [27, 41] for related derivations of the multivariate χ^2 distribution from occupation times of birth and death processes, and [18] regarding conditions for infinite divisibility of the multivariate χ^2 distribution. In view of the close parallel between Propositions 1 and 2, it is natural to expect a χ^2 representation like (1) for the circular local time process $\overset{\circ}{L}_{T_\pm}$. It will be shown that for all $\delta \in \mathbb{R}$ and $0 \leqq u \leqq v < 1$

$$\textit{the } P_\delta \textit{ distribution of } (\overset{\circ}{L}{}^u_{T_\pm},\ \overset{\circ}{L}{}^v_{T_\pm}) \textit{ is } \chi^2(2,\ \mu_\delta,\ \rho^2_\delta(v-u)) \tag{4}$$

where $\mu_0 = 1$, $\rho^2_0(p) = 1 - 2p\bar{p}$ with $\bar{p} = 1 - p$, and for $\delta \neq 0$

$$\mu_\delta = \delta^{-1}\tanh\delta\ ; \qquad \rho^2_\delta(p) = 1 - \frac{2\sinh(p\delta)\sinh(\bar{p}\delta)}{\cosh(\delta)\tanh^2(\delta)} . \tag{5}$$

But the parallel stops here. It turns out that for each δ

$$\textit{the } P_\delta \textit{ trivariate distributions of } \mathring{L}_{T_\pm} \textit{ are not trivariate } \chi^2 . \tag{6}$$

This follows by comparison of the well known determinant formula for the Laplace transform of the multivariate χ^2 distribution with the Laplace transform of the finite-dimensional distributions of $\mathring{L}_{T_\pm}$, which can be described as follows (Corollary 10 and Proposition 11): for every finite subset F of $[0,1)$, and $\alpha_u \geqq 0$

$$P_0 \exp\left(-\sum_{u\in F} \alpha_u \mathring{L}^u_{T_\pm}\right) = \left(1 + \frac{1}{2} \sum_{A\subseteq F} \mathring{\Pi}(A) \prod_{u\in A} (2\alpha_u)\right)^{-1} \tag{7}$$

where $\sum_{A\subseteq F}$ is a sum over all non-empty subsets A of F, and $\mathring{\Pi}(A)$ is the product of the spacings around the circle between points of A:

$$\mathring{\Pi}(\{u_1, \dots, u_n\}) = \prod_{k=1}^{n} (u_k - u_{k-1}) \quad (0 \leqq u_1 < \cdots < u_n < 1) \tag{8}$$

where $u_0 = u_n - 1$. The cyclic stationarity of $\mathring{L}_{T_\pm}$ under P_0 is evident in this formula by the invariance of $\mathring{\Pi}(A)$ under cyclic shifts of A. For $\delta \neq 0$ the corresponding formula for P_δ is obtained by the following modification of the right side of (7): replace the $\frac{1}{2}$ by $(2\cosh\delta)^{-1}$, and modify the definition (8) of $\mathring{\Pi}(A)$ by replacing each factor $(u_k - u_{k-1})$ by $\delta^{-1}\sinh(u_k - u_{k-1})\delta$.

The existence of a cyclically stationary Brownian local time process was suggested by a problem about random mappings posed by Steve Evans, and the Brownian bridge asymptotics for random mappings of Aldous–Pitman [1]. See [2] for details. The process that arises in this setting is $\mathring{L}_{T_{-1}}$ where T_{-1} is the hitting time of -1 by B governed by P_0. Section 5 considers the distribution of $\mathring{L}_T$ for various random times T including T_{-1}. It appears that none of these local time processes $\mathring{L}_T$ has the two-sided Markov property. Eisenbaum–Kaspi [12] show this for one particular T, and similar arguments apply to the various other T's considered here. See also [19, 29] regarding the circular Ornstein–Uhlenbeck process, which is the two-sided Markov cyclically stationary Gaussian process with covariance function of (u,v) equal to $(2\delta(1-e^{-\delta}))^{-1}(e^{-\delta p} + e^{-\delta(1-p)})$ for $p = |v-u|$. It is curious that this process does not seem to arise in the description of circular Brownian local times. From (4) one can construct a cyclically stationary Gaussian process with continuous paths, the sum of squares of two i.i.d. copies of which has the same two-dimensional distributions as $\mathring{L}_{T_\pm}$. But even this process is not the circular Ornstein–Uhlenbeck process. As an immediate consequence of Proposition 1 there is the following:

Corollary 3 *For each $\delta \geqq 0$ there is a different one parameter family of infinitely divisible distributions on $C^+[0,1)$, denoted $(\mathring{Q}^\kappa_\delta,\ \kappa > 0)$, such that $\mathring{Q}^1_\delta$ is the P_δ distribution of the normalized circular local time process*

$(\delta \coth \delta)\ \mathring{L}_{T_\pm}$ *with mean* 1. *Under* $\mathring{Q}^\kappa_\delta$ *the coordinate process* $(X_u,\ 0 \leqq u < 1)$ *is cyclically stationary and reversible with gamma* $(\kappa, 1)$ *marginals.*

To illustrate, replacing the power -1 by $-\kappa$ in (7) gives the $\mathring{Q}^\kappa_0$ joint Laplace transform of $(X_u,\ u \in F)$. The structure of the infinitely divisible family $(\mathring{Q}^\kappa_0,\ \kappa > 0)$ is exposed in Sect. 4 by an explicit description of the corresponding Lévy measure on $C^+[0,1)$. The basic idea is that a process with distribution $\mathring{Q}^\kappa_0$ can be represented as an infinite sum of random pulses where a *pulse* is a continuous function on the circle which is strictly positive on some open interval and vanishes on the complement of this interval. The random pulses are the points of a Poisson point process on $C^+[0,1)$ with intensity measure $\kappa\mathring{M}$ for a σ-finite Lévy measure $\mathring{M}$ on $C^+[0,1)$ which is concentrated on pulses. A similar description can be given for any δ, by following the method of [51], where the Lévy measure corresponding to the Ornstein–Uhlenbeck process in Proposition 2 is described. To be more precise, make the following definition:

Definition 4 *Say that a* $C^+(I)$ *valued random variable* $Z = (Z^u,\ u \in I)$ *admits a* ***strong Lévy–Itô*** (Λ) ***representation*** *if* $Z^u = \sum_i Z^u_i$ *for all* $u \in I$ *almost surely, where the* Z_i *are the points of a Poisson process on* $C^+(I)$ *with mean measure* Λ, *defined with* Z *on some common probability space* $(\Omega, \mathscr{F}, P)$.

The distribution Q of Z on $C^+(I)$ is then the infinitely divisible distribution determined by the Lévy–Khintchine formula:

$$P \exp(-mZ) = Q \exp(-mX) = \exp(\Lambda(1 - e^{-mX})) \tag{9}$$

where m is a bounded positive measure on I, and for $W = X$ or Z, $mW = \int_I W_u\, m(du)$. In Sect. 4, after some development of results of [51] concerning the Lévy–Itô representation of squares of Bessel processes, it is shown that under P_0 the circular local time process $\mathring{L}_{T_\pm}$ admits a strong Lévy–Itô $(\mathring{M})$ representation for a Lévy measure $\mathring{M}$ which is described explicitly in terms of 4-dimensional Bessel bridges. In the Poisson $(\mathring{M})$ point process of pulses whose sum is $\mathring{L}_{T_\pm}$, each pulse is an increment $\mathring{L}_S - \mathring{L}_R$ of the $C[0,1)$ valued local time process, derived from an *excursion interval* (R,S) of the basic Brownian motion B, that is an interval such that $B_R = B_S = y$ for some y, and $B_t \neq y$ for $t \in (R,S)$. These excursion intervals are defined to be flat intervals of the past maximum process of B if $B_{T_\pm} = 1$, and flat intervals of the past minimum process of B if $B_{T_\pm} = -1$. Call a pulse *long* if it is strictly positive over the whole circle, and otherwise call it *short*. Ignoring events of probability zero, the pulse associated with an excursion interval (R,S) is long if $\max_{R \leqq t \leqq S} B_t - \min_{R \leqq t \leqq S} B_t \geqq 1$, that is if $\mathring{B}$ visits every point on the circle during the interval $[R,S]$. Summing the pulses of the local time process over *long* and *short* excursions yields an interesting decomposition of $\mathring{L}_{T_\pm}$ into two independent infinitely divisible cyclically stationary processes: $\overset{\circ}{L}_{T_\pm} = \overset{\circ}{L}_{\text{short}} + \overset{\circ}{L}_{\text{long}}$. To illustrate, the Laplace transform of the exponential distribution of $\overset{\circ}{L}{}^{u}_{T_\pm}$ admits the factorization $(1+\alpha)^{-1} = \Phi_{\text{short}}(\alpha)\Phi_{\text{long}}(\alpha)$ where $\Phi_{\text{short}}(\alpha) = P_0 \exp(-\alpha \overset{\circ}{L}{}^{u}_{\text{short}})$ is given by the formula

$$\Phi_{\text{short}}(\alpha) = e\left(\frac{\sqrt{2+\alpha}-\sqrt{\alpha}}{\sqrt{2+\alpha}+\sqrt{\alpha}}\right)^{(1+\alpha)/(\sqrt{\alpha}\sqrt{2+\alpha})} . \tag{10}$$

The density of the corresponding Lévy measure is $K_1(x)e^{-x}$ where $K_1(x)$ is the modified Bessel function. The decomposition of $T_\pm$ into time spent during long and short excursions yields some novel infinitely divisible laws on $(0,\infty)$ with Laplace transforms involving hyperbolic functions.

Finally, some open problems are mentioned in Sect. 7.

2 Squares of Bessel processes

For $d = 1,2,\ldots$ a process $(R_t,\ t \geqq 0)$ is a *d-dimensional Bessel process started at r* or BES_r^d for short, if $(R_t^2,\ t \geqq 0)$ is the sum of squares of d independent Brownian motions started at points $x_1,\ldots,x_d$ with $\sum_i x_i^2 = r^2$. For r with $r^2 = x$, the process $(R_t^2,\ t \geqq 0)$ is then a *squared d-dimensional Bessel process started at x*, or $BESQ_x^d$. The distribution of a $BESQ_x^d$ process on the space $C^+[0,\infty)$ of continuous non-negative paths is denoted by Q_x^d. Following Shiga–Watanabe [60], the definition of Q_x^d extends to all real $d \geqq 0$ via the infinite divisibility properties of the two parameter family $Q_x^d,\ x \geqq 0,\ d \geqq 0$. See also [51, 56]. Let $(X_u,\ u \geqq 0)$ denote the co-ordinate process on $C^+[0,\infty)$. As shown by Pitman–Yor [51, 52], for a positive measure m on $(0,\infty)$

$$Q_x^d \exp\left(-\int_0^\infty X_u m(du)\right) = \Psi_1^{-d/2} \exp\left(-\frac{x\Psi_0}{2\Psi_1}\right) \tag{11}$$

where $\Psi_0 = \Psi_0(m)$ and $\Psi_1 = \Psi_1(m)$ can be expressed in terms of the unique solution ϕ_m of the Sturm–Liouville equation

$$\frac{1}{2}\phi'' = m \cdot \phi \qquad \text{on } (0,\infty) \text{ with } \phi(0) = 1, \qquad 0 \leqq \phi \leqq 1\ ; \tag{12}$$

to be precise,

$$\Psi_1 = \frac{1}{\phi_m(\infty)}\ ; \qquad \frac{\Psi_0}{\Psi_1} = -\phi_m'(0) \tag{13}$$

where ϕ_m' is the right derivative of ϕ_m, and $\phi_m(\infty)$ is the limit of ϕ_m at ∞. It is known to analysts [25, 9] that under mild conditions on m solutions of Sturm–Liouville equations such as (12) can be expressed as infinite series of terms obtained from appropriate iterated intergrals with respect to m. See [53] for discussion of such formulas and their relation to the series for the $\Psi_i(m)$ presented in the following proposition:

Proposition 5 *For each positive measure m on* $[0,\infty)$ *such that*

$$m[0,\infty) < \infty \quad \textit{and} \quad \int_0^\infty x m(dx) < \infty \tag{14}$$

formula (11) *holds with* Ψ_i *as follows for* $i = 0$ *or* 1:

$$\Psi_i(m) = i + \sum_{n=1}^{\infty} m_{in} 2^n \tag{15}$$

where

$$m_{in} = \int\limits_{0 \leqq u_1 < \cdots} m(du_1) \cdots \int\limits_{\cdots < u_n < \infty} m(du_n) \quad u_1^i \prod_{k=2}^{n} (u_k - u_{k-1}) \,. \tag{16}$$

For $n = 1$ the empty product in (16) equals 1. So the first few m_{in} are

$$m_{01} = m[0,\infty); \qquad m_{02} = \int_0^\infty m(du) \int_u^\infty (v - u) m(dv) \,,$$

$$m_{11} = \int_0^\infty u m(du); \qquad m_{12} = \int_0^\infty m(du) \int_u^\infty u(v - u) m(dv)$$

Proof. Take (15) as the definition of the $\Psi_i(m)$. It can be shown directly that (11) holds, without consideration of the Sturm–Liouville equation. Note first that it is enough to show (11) for $x = 0$ and some $d > 0$, and for $d = 0$ and some $x > 0$. For a discrete measure $m = \sum_{u \in F} \alpha_u \varepsilon_u$, where F is a finite subset of $[0,\infty)$ and ε_u is a unit mass at u, the Ψ_i defined by (15) reduce to

$$\Psi_i\left(\sum_{u \in F} \alpha_u \varepsilon_u\right) = i + \sum_{A \subseteq F} \Pi_i(A) \prod_{u \in A} 2\alpha_u \tag{17}$$

where $\sum_{A \subseteq F}$ is a sum over all non-empty subsets A of F, and

$$\Pi_i(\{u_1, \ldots, u_n\}) = u_1^i \prod_{k=2}^{n} (u_k - u_{k-1}) \quad (0 \leqq u_1 < \cdots < u_n < \infty) \,.$$

The special case of (11) for $x = 0$, $d = 2$, and such a discrete m, appears in Problems 5 and 6 of Sect. 2.8 of Itô–McKean [23], solutions of which appear in Sect. 6.4B of [26]. The discrete form of (11) with $d = 0$, $x > 0$ can be established by the method of [23], that is induction on the number of elements of F, using the recursion derived from the Markov property of Q_x^d that appears in formulae (1.20) and (1.21) of Shiga–Watanabe [60], or see formula (2.j) of [51] (which should be corrected as follows: on the second last line of page 431, λ_{i+1} should be $\tilde{\lambda}_{i+1}$). Formula (11) for a bounded positive measure m with finite first moment is obtained from the discrete case by straightforward approximation. In particular, elementary estimates show that the series for Ψ_i converge rapidly provided m has a finite first moment. (cf. [9, Sect. 5.4, Exercises 1–3]). □

The Ray–Knight Theorems

The solution of the problems of [23] cited above for a discrete m yields also the following Laplace transform, where $T_1 = \inf\{t : B_t = 1\}$: for every bounded

m with support contained in $[0,1]$, and $\alpha > 0$,

$$P_0 \exp\left(-\alpha \int_0^1 L_{\tau_1}^{1-u} m(du)\right) = \frac{1}{\Psi_1(\alpha m)} . \tag{18}$$

Combined with (11) for $x = 0$, $d = 2$, this amounts to the theorem of Ray–Knight [30, 54] that

$$(L_{\tau_1}^{1-u},\ 0 \leqq u \leqq 1;\ P_0) \overset{d}{=} (X_u,\ 0 \leqq u \leqq 1;\ Q_0^2) \tag{19}$$

where $\overset{d}{=}$ denotes equality of distributions on $C^+[0,1]$. Closing up the gaps between positive excursions of B to obtain a reflecting BM (see [23, Sect. 2.11] or [58, Sect. III. 22]) yields the result of Knight [31] that also

$$(L_{T_\pm}^{1-u} + L_{T_\pm}^{u-1},\ 0 \leqq u \leqq 1;\ P_0) \overset{d}{=} (X_u,\ 0 \leqq u \leqq 1;\ Q_0^2) . \tag{20}$$

Consequently,

formula (18) *holds also with* $L_{T_\pm}^{1-u} + L_{T_\pm}^{u-1}$ *instead of* $L_{T_1}^{1-u}$. (21)

Let $(\tau_\ell,\ \ell \geqq 0)$ be the right-continuous inverse of the process $(L_t^0,\ t \geqq 0)$ of local times of B at zero. Using the formula of Williams [66] which is derived in Sect. 6.4C of [26], an argument parallel to the derivation of (18) shows that for every bounded positive measure m on $[0,\infty)$ with finite first moment, and $\alpha > 0$,

$$P_0 \exp\left(-\alpha \int_0^\infty L_{\tau_\ell}^x m(dx)\right) = \exp\left(-\frac{\ell}{2} \frac{\Psi_0(\alpha m)}{\Psi_1(\alpha m)}\right) . \tag{22}$$

Combined with (11) for $d = 0$, $x = \ell$, this amounts to the Ray–Knight theorem that

the P_0 *distribution of* $(L_{\tau_\ell}^u,\ u \geqq 0)$ *is* Q_ℓ^0 . (23)

Some further applications of these formulae are indicated briefly in Sect. 6. See also [68, 43–46, 48, 59, 56, 71, 61, 65] for other approaches to the Ray–Knight theorems and related connections between squared Bessel processes and Brownian local times.

Examples. For $m(dy) = f(y)dy$ write $\Psi_i(f)$ and f_{in} instead of $\Psi_i(m)$ and m_{in}. Set $f(y) = 0$ if $y < 0$. For $a, b, c > 0$, let $f_{a,b,c} : x \to af((x-c)/b)$. Then for $i = 0$ or 1 and $n = 1, 2, \ldots$, (15) and (16) imply

$$(f_{a,b,c})_{in} = a^n b^{2n-i}(f_{in} + icf_{0n}) , \tag{24}$$

$$\Psi_i(f_{a,b,c}) = b^{-i}\Psi_i(ab^2 f) + ic\Psi_0(ab^2 f) . \tag{25}$$

For m the uniform distribution on $(0,1)$ with density $1_{(0,1)}$, the integrals (16) and series (15) are easily evaluated as follows

$$(1_{(0,1)})_{0n} = \frac{1}{(2n-1)!} ; \qquad (1_{(0,1)})_{1n} = \frac{1}{(2n)!}$$

$$\Psi_0(\alpha 1_{(0,1)}) = \sqrt{2\alpha}\sinh\sqrt{2\alpha} ; \qquad \Psi_1(\alpha 1_{(0,1)}) = \cosh\sqrt{2\alpha} . \tag{26}$$

For the indicator of an interval (c,d), say $1_{(c,d)}(x) = 1_{(0,1)}(x-c)/(d-c)$, the formula (25) yields

$$\Psi_0(\alpha 1_{(c,d)}) = \sqrt{2\alpha}\sinh(\sqrt{2\alpha}(d-c))\,, \tag{27}$$

$$\Psi_1(\alpha 1_{(c,d)}) = \cosh(\sqrt{2\alpha}(d-c)) + c\Psi_0(2\alpha 1_{(c,d)})\,. \tag{28}$$

Substituting these expressions in (18) and (22) yields formulae for the Laplace transform of the time spent by B in (c,d) up to time T_1 for $0 \leqq c < d \leqq 1$, or up to time τ_ℓ for $0 \leqq c < d < \infty$. Similar formulae can be obtained with one or both of c and d negative. Another variation is obtained with (21). See [69, 32, 34, 51, 13] for instances of these formulae, and further variations which can be recovered by the same method. Formula (36) in the next section gives an application on the circle. As a general rule, any explicit solution of a Sturm–Liouville problem like (12), of which a great many are known, (see e.g. [9, Exercise 5.4.15; 51, 53]), typically yields an evaluation of one or both of the basic functions $\Psi_i(m)$ for some m. Such Ψ_i can then be transformed to obtain other Ψ_i as above, without any further discussion of boundary conditions for the Sturm–Liouville equation. See also [9, Sect. 6.9], for some more sophisticated transformations related to Krein's theory of strings.

Formulae for Bessel bridges

For $x, y, d \geqq 0$ let $Q^d_{x\to y}$ denote the distribution on $C^+[0,1]$ or $C^+[0,1)$ of the $BESQ^d$ bridge obtained from the Q^d_x conditional distribution of $(X_u,\ 0 \leqq u \leqq 1)$ given $X_1 = y$. According to [51, 52], for m with support contained in $[0,1]$

$$Q^d_{x\to 0}\exp\left(-\int_0^1 X_u m(du)\right) = \Psi^{-d/2}\exp\left(-\frac{x}{2}\left(\frac{\hat{\Psi}_1}{\Psi} - 1\right)\right) \tag{29}$$

where $\hat{\Psi}_1 = \hat{\Psi}_1(m) = \Psi_1(\hat{m})$ for $\hat{m}$ the image of m via the map $u \to 1-u$, and $\Psi = (\Psi_1\hat{\Psi}_1 - 1)/\Psi_0$. It can also be shown that

$$\hat{\Psi}_1(m) = 1 + \sum_{n=1}^{\infty}\hat{m}_n 2^n\,; \qquad \Psi(m) = 1 + \sum_{n=1}^{\infty} m_n 2^n \tag{30}$$

where both $\hat{m}_n$ and m_n are given by expressions like (16). To be precise, $\hat{m}_n = m_{01n}$ and $m_n = m_{11n}$ where, for $i = 0$ or 1, m_{i1n} is defined like m_{in} in (16) but with an extra factor $(1-u_n)$ in the integrand. In particular, to complement (26),

$$\hat{\Psi}_1(\alpha 1_{(0,1)}) = \cosh\sqrt{2\alpha}\,; \qquad \Psi(\alpha 1_{(0,1)}) = \sinh\sqrt{2\alpha}/\sqrt{2\alpha}\,. \tag{31}$$

3 The circular local time process at $T_\pm$

The following lemma is a key ingredient in the proof of Proposition 1.

Lemma 6 (Knight [31]). *Let G be the time of the last zero of B before time $T_{\pm}$. Under P_0 governing B as a Brownian motion with zero drift,*

(i) *$L^0_{T_{\pm}}$ has standard exponential distribution*: $P_0(L^0_{T_{\pm}} \in d\ell) = e^{-\ell} d\ell, \ell > 0$.

(ii) *Given $L^0_{T_{\pm}} = \ell$ the processes $(L^u_G,\ 0 \leqq u \leqq 1)$ and $(L^{-u}_G,\ 0 \leqq u \leqq 1)$ are independent with identical distribution $Q^0_{\ell \to 0}$.*

(iii) *The process $(L^{|u|}_{T_{\pm}} - L^{|u|}_G,\ 0 \leqq u \leqq 1)$ has distribution $Q^2_{0 \to 0}$.*

(iv) *The two processes $(L^v_G,\ -1 \leqq v \leqq 1)$ and $(L^{|u|}_{T_{\pm}} - L^{|u|}_G,\ 0 \leqq u \leqq 1)$ and the random sign $B_{T_{\pm}} \in \{-1, +1\}$ are mutually independent.*

Remark 7. The process in (iii) is the process of occupation densities of the path fragment $(|B|_{G+s},\ 0 \leqq s \leqq T_{\pm} - G)$. As shown by Williams [66–68], this fragment has the distribution of a BES^3_0 process stopped at its first hit of 1.

Translating Lemma 6 into terms of the circular local time process yields the next lemma. See also Proposition 21 for a generalization derived by excursion theory. Note that G is also the last zero of $\mathring{B}$ before time $T_{\pm}$, and that $\mathring{L}^0_{T_{\pm}} = L^0_{T_{\pm}}$. Let $P * Q$ denote convolution of two distributions on $C^+[0,1)$, that is the distribution of $Y + Z$ for independent random elements Y and Z with distributions P and Q.

Lemma 8 *Under P_0*

(i) *The distribution of $\mathring{L}_G$ on $C^+[0,1)$ is $\int_0^\infty Q^0_{\ell \to 0} * \hat{Q}^0_{\ell \to 0} e^{-\ell}\, d\ell$ where $\hat{Q}^0_{\ell \to 0}$ is image of $Q^0_{\ell \to 0}$ via time reversal.*

(ii) *The distribution of $\mathring{L}_{T_{\pm}} - \mathring{L}_G$ on $C^+[0,1)$ is $Q^2_{0 \to 0}$.*

(iii) *The two processes $\mathring{L}_G$ and $\mathring{L}_{T_{\pm}} - \mathring{L}_G$ and the random sign $B_{T_{\pm}}$ are mutually independent.*

(iv) *The distribution of $\mathring{L}_{T_{\pm}}$ is $Q^2_{0 \to 0} * (\int_0^\infty Q^0_{\ell \to 0} * \hat{Q}^0_{\ell \to 0} e^{-\ell}\, d\ell)$.*

(v) *The process $\mathring{L}_{T_{\pm}}$ and the random sign $B_{T_{\pm}}$ are independent.*

Proof. Part (i) follows from parts (i) and (ii) of Lemma 6 by conditioning on $L^0_{T_{\pm}}$. Parts (ii) and (iii) follow from (ii) and (iii) of Lemma 6 and reversibility of $Q^2_{0 \to 0}$. Parts (iv) and (v) follow from parts (i), (ii) and (iii). □

Notation. For the rest of this section, let m denote an arbitrary bounded positive measure on $[0,1)$, and let $\Psi_0, \Psi_1, \hat{\Psi}_1, \Psi$ be defined in terms of m as in (15) and (30). For a process $(X_u,\ 0 \leqq u < 1)$ let $mX = \int_0^1 m(du) X_u$.

Proof of Proposition 1. Consider first the case $\delta = 0$. Part (iv) of Lemma 8 combined with (29) allows the following computation:

$$P_0 \exp(-m\overset{\circ}{L}_{T_\pm}) = (Q^2_{0\to 0} e^{-mX}) \int_0^\infty (Q^0_{\ell\to 0} e^{-mX})(\hat{Q}^0_{\ell\to 0} e^{-mX}) e^{-\ell}\, d\ell$$

$$= \Psi^{-1} \int_0^\infty d\ell \exp\left[-\frac{\ell}{2}\left(\frac{\hat{\Psi}_1}{\Psi} - 1\right) - \frac{\ell}{2}\left(\frac{\Psi_1}{\Psi} - 1\right) - \ell\right] = \frac{2}{\Psi_1 + \hat{\Psi}_1} \, .$$

Take m to be discrete and use (17). The result is (7), since for a finite subset A of $[0,1)$, $\overset{\circ}{\Pi}(A) = \Pi_1(A) + \Pi_1(\hat{A})$ where $\hat{A}$ is the reversal of A. The cyclic stationarity of $\overset{\circ}{L}_T$ is now apparent, and reversibility is obvious for $\delta = 0$. Infinite divisibility follows easily from the same decomposition, using standard ideas of subordination, and the infinite divisibility of the exponential distribution of $\overset{\circ}{L}{}^0_T$ and the various squared Bessel components. See formula (51) in the next section for the consequent expression for the Lévy measure. For $\delta \neq 0$ the Cameron–Martin formula (see e.g. [16, Sect. I.11]) combined with the independence of $\overset{\circ}{L}_{T_\pm}$ and $B_{T_\pm}$ yields

$$P_\delta \exp\left(-m\overset{\circ}{L}_{T_\pm}\right) = \cosh(\delta) P_0 \exp\left(-\left(m + \frac{1}{2}\delta^2\lambda\right)\overset{\circ}{L}_{T_\pm}\right) \tag{32}$$

where λ is Lebesgue measure on $[0,1)$. This formula and the cyclic stationarity of $\overset{\circ}{L}_{T_\pm}$ under P_0 imply that $\overset{\circ}{L}_{T_\pm}$ is cyclically stationary under P_δ too.

The same goes for reversibility. The P_δ distribution of $\overset{\circ}{L}_{T_\pm}$ can be shown to be infinitely divisible by using the Cameron–Martin formula to obtain a variation of Lemma 8 for P_δ. See also Remark 15 in Sect. 4. □

Definition 9 *For a measure* m *on* $[0,1)$ *define* $\overset{\circ}{\Psi} = \overset{\circ}{\Psi}(m)$ *by*

$$\overset{\circ}{\Psi} = \frac{1}{2}(\Psi_1 + \hat{\Psi}_1) = 1 + \frac{1}{2}\sum_{n=1}^\infty \overset{\circ}{m}_n 2^n \tag{33}$$

where $\overset{\circ}{m}_n = m_{1n} + \hat{m}_{1n}$, *that is*

$$\overset{\circ}{m}_n = \int_{0 \leqq u_1 < \cdots} m(du_1) \cdots \int_{\cdots < u_n < 1} m(du_n) \prod_{k=1}^n (u_k - u_{k-1}) \quad \text{where } u_0 = u_n - 1 \, . \tag{34}$$

From the proof of Proposition 1 and the formulae of Proposition 5, there is the following companion to the Ray–Knight formulae (18), (21) and (22):

Corollary 10

$$P_0 \exp\left(-\alpha m \overset{\circ}{L}_{T_\pm}\right) = \left(\overset{\circ}{\Psi}(\alpha m)\right)^{-1} = \left(1 + \frac{1}{2}\sum_{n=1}^\infty \overset{\circ}{m}_n (2\alpha)^n\right)^{-1} \, . \tag{35}$$

To illustrate, for $m(du) = f(u)du$, formula (35) gives the Laplace transform of $\int_0^{T_\pm} f(\overset{\circ}{B}_t)dt$. If $U_1, \ldots, U_n$ are i.i.d with density $f/\|f\|$, where $\|f\|$

$= \int_0^1 f(u)du$, then $\mathring{m}_n$ equals $\|f\|^n/n!$ times the expected product of the n spacings around the circle between points of the random set $\{U_1, \dots, U_n\}$.

Occupation time of an interval on the circle

Consider the occupation time $A(I,t) = \int_0^t 1(\mathring{B}_s \in I)ds$ for an interval I on the circle. From (35), (27), (28), for every interval I of the circle of length p, the time $A(I, T_\pm)$ that $\mathring{B}$ spends in I up to time $T_\pm$ has the same infinitely divisible distribution with Laplace transform

$$P_0 \exp(-\alpha A(I, T_\pm)) = \left(\cosh(p\sqrt{2\alpha}) + \frac{1}{2}(1-p)\sqrt{2\alpha}\sinh(p\sqrt{2\alpha})\right)^{-1} . \quad (36)$$

According to Theorem 4.2.16 of Knight [34], which follows similarly from (21), (27) and (28), the P_0 Laplace transform of the time spent by $|B|$ in $[0,p]$ before time $T_\pm$ is given by the right side of (36) with the $\frac{1}{2}$ replaced by 1. That the $\frac{1}{2}$ is needed in (36) can be checked as follows: as $p \to 0$, $A([0,p], T_\pm)/p$ converges a.s. to $L^0_{T_\pm}$ with Laplace transform $1/(1+\alpha)$. But the time spent by $|B|$ in $[0,p]$ before time $T_\pm$ must be normalized by $2p$ instead of p to obtain the same limit.

The Laplace functional of the P_δ distribution of $\mathring{L}_{T_\pm}$

For the circular Brownian motion with drift, a first formula for the Laplace functional of the P_δ distribution of $\mathring{L}_{T_\pm}$ is obtained by combining (32) and (35). But there is a more interesting formula which lies a little deeper:

Proposition 11 *For* $\delta \neq 0$:

$$P_\delta \exp(-m\mathring{L}_{T_\pm}) = \left(1 + (2\cosh\delta)^{-1}\sum_{n=1}^{\infty} \mathring{m}_{n,\delta} 2^n\right)^{-1} \quad (37)$$

where

$$\mathring{m}_{n,\delta} = \delta^{-n} \int_{0 \leqq u_1 < \cdots} m(du_1) \cdots \int_{\cdots < u_n \leqq 1} m(du_n) \prod_{k=1}^{n} \sinh(u_k - u_{k-1})\delta \quad (38)$$

with $u_0 = u_n - 1$.

Proof. Formula (37) will be obtained by development of the right side of (32). Consider first $m = \alpha_0\varepsilon_0 + \alpha_u\varepsilon_u$ where ε_u is a unit mass at u. Let $\mu = \frac{1}{2}\delta^2\lambda$ where λ is Lebesgue measure on $[0,1)$. Then from (33)

$$P_0 \exp\left(-\alpha_0 \mathring{L}^0_{T_\pm} - \alpha_u \mathring{L}^u_{T_\pm} - \frac{1}{2}\delta^2 T_\pm\right) = \left(1 + \frac{1}{2}\sum_{n=1}^{\infty} (\alpha_0\varepsilon_0 + \alpha_u\varepsilon_u + \mu)^\circ_n \; 2^n\right)^{-1}$$

where $(\alpha_0\varepsilon_0 + \alpha_u\varepsilon_u + \mu)^\circ_n$ denotes the quantity $\mathring{m}_n$ in (33) for the measure $m = \alpha_0\varepsilon_0 + \alpha_u\varepsilon_u + \mu$. Let μ_n denote the coefficient m_n in (30) for $m = \mu$, and let $\bar{u} = 1 - u$. The quantity $(\alpha_0\varepsilon_0 + \alpha_u\varepsilon_u + \mu)^\circ_n$ can be evaluated as follows:

for $n = 1$: $\mathring{\mu}_1 + \alpha_0 + \alpha_u$

for $n = 2$: $\mathring{\mu}_2 + (\alpha_0 + \alpha_u)\mu_1 + \alpha_0\alpha_u \; u\bar{u}$

for $n = j + 2 \geqq 2$

$$\overset{\circ}{\mu}_n + (\alpha_0 + \alpha_u)\mu_{n-1} + \alpha_0\alpha_u u\bar{u} \sum_{k=0}^{j} u^{2k}\mu_k \bar{u}^{2j-2k}\mu_{j-k} \ .$$

The summation index k counts how many u_i in the repeated integral (34) for $(\alpha_0\varepsilon_0 + \alpha_u\varepsilon_u + \mu)_n^\circ$ fall in the interval $(0,u)$. The powers of u and $\bar{u}$ appear by making the appropriate linear changes of variables to replace each integral over $[0,u]$ or $[u,1]$ by an integral over $[0,1]$. Summing over n, the desired Laplace transform is found to be

$$\left(\overset{\circ}{\Psi}(\mu) + (\alpha_0 + \alpha_u)\Psi(\mu) + 2\alpha_0\alpha_u u\bar{u}\Psi(u^2\mu)\Psi(\bar{u}^2\mu)\right)^{-1} \ .$$

Combined with (26), (31) and (32), this yields the proposition for $m = \alpha_0\varepsilon_0 + \alpha_u\varepsilon_u$. A similar calculation yields the result for a general discrete m, and the argument is completed for an arbitrary finite measure m on $[0,1)$ by a routine weak approximation. □

Example 12 Let $m = \alpha\lambda$ for a positive scalar α and Lebesgue measure λ on $[0,1)$. From (32)

$$P_\delta \exp(-\alpha\lambda\overset{\circ}{L}_{T_\pm}) = P_\delta \exp(-\alpha T_\pm) = \frac{\cosh\delta}{\cosh\sqrt{2\alpha + \delta^2}}$$

Comparison with formula (38) yields the identity

$$\cosh\sqrt{2\alpha + \delta^2} = \cosh\delta + \tfrac{1}{2}\sum_{n=1}^{\infty} \frac{1}{n}\left(\frac{2\alpha}{\delta}\right)^n f_n(\delta) \qquad (39)$$

where $f_1(\delta) = \sinh(\delta)$ and for $n = 2, 3, \ldots$

$$f_n(\delta) = \int\limits_{v_i \geqq 0,} \cdots \int\limits_{\sum_1^n v_i = 1} \prod_{i=1}^{n} f_1(v_i\delta)dv_1 \cdots dv_{n-1} \ . \qquad (40)$$

For instance,

$$f_2(\delta) = \frac{1}{2\delta}(\delta\cosh\delta - \sinh\delta); \quad f_3(\delta) = \frac{1}{8\delta^2}(\delta^2\sinh\delta - 3\delta\cosh\delta + 3\sinh\delta) \ .$$

A generalization of the identity (39)

To check (39) directly, consider functions $f_n(\delta)$ defined by the integral formula (40), for an arbitrary continuous function f_1 defined on $[0,1]$ instead of $f_1(\delta) = \sinh\delta$. The $f_n(\delta)$ are then determined by

$$f_n(\delta) = \int_0^1 v^{n-2} f_{n-1}(v\delta) f_1(\bar{v}\delta)dv \qquad (41)$$

where $\bar{v} = 1 - v$. Let

$$F(\alpha, \delta) = \sum_{n=1}^{\infty} f_n(\delta)\alpha^n \ . \qquad (42)$$

Then easily from (41)

$$F(\alpha,\delta) = \alpha f_1(\delta) + \alpha \int_0^1 v^{-1} F(v\alpha, v\delta) f_1(\bar{v}\delta) dv \,. \tag{43}$$

Retracing this argument shows that if a function $F(\alpha,\delta)$ is of the form (42) for some sequence of continuous functions $f_n(\delta)$, and $F(\alpha,\delta)$ satisfies (43), then (41) and (40) hold. Returning to consideration of (39), differentiation with respect to α shows that (39) is equivalent to

$$F(\alpha,\delta) = \frac{\alpha\delta}{\sqrt{\alpha\delta+\delta^2}} \sinh\sqrt{\alpha\delta+\delta^2} \quad \text{for } f_1(\delta) = \sinh\delta \,. \tag{44}$$

This is verified by checking (43), which, after setting $\beta = \sqrt{\alpha\delta+\delta^2}$, reduces to the elementary formula

$$\int_0^1 \sinh(v\beta)\sinh(\bar{v}\delta)\,dv = \frac{\delta\sinh\beta - \beta\sinh\delta}{\beta^2-\delta^2} \,. \tag{45}$$

4 Lévy–Itô representations

If a probability measure Q and a σ-finite measure Λ on $C^+(I)$ are related by the Lévy–Khintchine formula (9), let us say simply that Q is *infinitely divisible with Lévy measure* Λ. Assume that Λ places zero mass on the path that is identically zero. Then Q and Λ determine each other uniquely. As shown in Pitman–Yor [51], it follows from (11) and (29) that

$$Q_x^d \textit{ is infinitely divisible with Lévy measure } xM + dN \tag{46}$$

$$Q_{x\to 0}^d \textit{ is infinitely divisible with Lévy measure } xM_0 + dN_0 \tag{47}$$

for some Lévy measure M, N on $C^+[0,\infty)$ and M_0 and N_0 on $C^+[0,1]$. These Lévy measures will now be described by a development of ideas from [51]. The following results involve the Ray–Knight descriptions of linear Brownian local times, and Williams' decomposition of a Brownian excursion, [70, Sect. II.67]. The basic idea can be stated informally as follows. When a Brownian local time process indexed by $v \in I$ is decomposed as a sum of pulses derived from various excursions, the pulse derived from *either* an excursion above x with maximum level y *or* an excursion below y with minimum level x, typically has the following distribution $P_{x,y}$:

Definition 13 *For a subinterval I of* $\mathbb{R}$, *and $x, y \in I$ with $x < y$, let $P_{x,y}$ be the probability distribution on $C^+(I)$ of a process $X_{x,y}$ that vanishes off the interval (x,y), and on (x,y) is a BESQ^4 bridge from* 0 *to* 0 *of length $(y-x)$:*

$$X_{x,y}(v) = (y-x)\,S_4\left(\frac{v-x}{y-x}\right) 1(x \leqq v \leqq y) \quad (v \in I) \tag{48}$$

where S_4 has distribution $Q_{0\to 0}^4$.

Proposition 14 *The Lévy measures defined by* (46) *and* (47) *are*

$$M = \tfrac{1}{2} \int_0^\infty dy \frac{P_{0,y}}{y^2}; \qquad N = \tfrac{1}{2} \int_0^\infty dx \int_x^\infty dy \frac{P_{x,y}}{(y-x)^2}\,, \tag{49}$$

$$M_0 = \tfrac{1}{2} \int_0^1 dy \frac{P_{0,y}}{y^2}; \qquad N_0 = \tfrac{1}{2} \int_0^1 dx \int_x^1 dy \frac{P_{x,y}}{(y-x)^2}\,. \tag{50}$$

Proof. As shown in [51], a strong Lévy–Itô (ℓM) representation of the $BESQ_\ell^0$ distributed process in (23) is obtained by decomposing the $C^+[0,\infty)$ valued local time process L_{τ_ℓ} as the sum of pulses derived from excursions of B from 0. Consequently (Theorem (4.2) of [51]), M is the distribution of the total local time pulse generated by a Brownian excursion (ε_t, $0 \leqq t \leqq \zeta$) distributed according to Itô's law for positive excursions of B from 0. Williams's description of (ε_t, $0 \leqq t \leqq \zeta$) given $\max_{0 \leqq t \leqq \zeta} \varepsilon_t = y$, in terms of pasting back to back two independent BES_0^3 processes (each run till it first hits y), implies the formula for M in (49) with $P_{0,y}$ the distribution on $C^+[0,\infty)$ of the total local time process derived from the two BES^3 fragments. By Remark 7 and Brownian scaling, each BES^3 fragment has a local time process on $[0,y]$ which is a $BESQ^2$ bridge from 0 to 0 of length y. Summing the two independent $BESQ^2$ bridges yields a $BESQ^4$ bridge. So $P_{0,y}$ is the distribution described by Definition 13 for $x=0$. This proves the formula for M in (49). The formula for N in (49) follows from the description of N obtained similarly in [51] using the other Ray–Knight theorem (19): $N = \int_0^\infty M_x \, dx$ where under M_x the path is identically zero up to time x and (X_{x+u}, $u \geqq 0$) has distribution M. To obtain the expressions (50), consider a process $Z = (Z(u),\ u \geqq 0)$ with strong Lévy–Itô (Λ) representation, for $\Lambda = M$ or N, and condition on the event on $Z(1) = 0$. □

Remark 15 As in [51], the results of Proposition 14 have straightforward extensions to the case with squares of Ornstein–Uhlenbeck processes instead of squares of Bessel processes. The connection with local time processes and excursions of BM with drift δ is provided by Proposition 2. But details of this case are left to the reader.

Circular Lévy–Itô representations

By development of Proposition 14 and its relation to local times of linear BM there is the following result for circular BM. The discussion will be restricted to the case of zero drift. But similar results for non-zero drift can be obtained using Remark 15.

Proposition 16 *Under* P_0 *the local time process* $\mathring{L}_{T\pm}$, *whose distribution is determined by* (7), *admits a strong Lévy–Itô* ($\mathring{M}$) *representation, with*

$$\mathring{M} = \int_0^1 dy \int_{-1}^y \frac{\mathring{P}_{x,y}}{(y-x)^2} dx = 2N_0 + \int_0^\infty v^{-1} e^{-v} \, dv \, Q_{v\to 0}^0 * \hat{Q}_{v\to 0}^0 \tag{51}$$

where $\overset{\circ}{P}_{x,y}$ *is the image of* $P_{x,y}$ *after wrapping of the pulse around the circle, that is the probability distribution on* $C^+[0,1)$ *of*

$$(X_{x,y}(u) + X_{x,y}(u-1),\ 0 \leqq u < 1)$$

for $X_{x,y}$ *the random path in* $C^+[-1,1]$ *defined in* (48).

Notation For a random subset A of $[0,\infty)$, let L_A denote the process

$$L_A^x = \int_{t=0}^{\infty} 1(t \in A)\,dL_t^x \quad (x \in [-1,1])\,. \tag{52}$$

In particular, for a random interval, say $A = [R,S]$, $L_{[R,S]} = L_S - L_R$ for L_S and L_R as before, e.g. $L_S = L_{[0,S]}$. Put $\overset{\circ}{L}{}_A^u = L_a^u + L_a^{u-1}$, $0 \leqq u \leqq 1$ and $\overset{\circ}{L}_A = (\overset{\circ}{L}{}_A^u,\ 0 \leqq u \leqq 1)$. For any random interval A, and also for various other A's considered below which are countable unions of intervals, the processes L_A and $\overset{\circ}{L}_A$ have continuous paths. Then L_A and $\overset{\circ}{L}_A$ will be regarded as random paths in $C^+[-1,1]$ and $C^+[0,1)$ respectively.

Proof of Proposition 16. Due to the independence of $\overset{\circ}{L}_{T\pm}$ and $B_{T\pm}$ (Lemma 8(v)), it suffices to consider the process $\overset{\circ}{L}_{T\pm}$ conditionally given $B_{T\pm} = 1$. Let $T_y = \inf\{t\colon B_t > y\}$. As a consequence of Itô's theory of Brownian excursions, [22, 57, 56] conditionally given $B_{T\pm} = 1$, the $C^+[-1,1]$ valued point process of local time pulses $(L_{[T_{y-},T_y]},\ 0 \leqq y \leqq 1)$ is an inhomogeneous Poisson marked point process with intensity measure $dy\,\mu_y(d\xi)$, $0 \leqq y \leqq 1$, $\xi \in C^+[-1,1]$ where $\mu_y = \int_{-1}^{y} (y-x)^{-2} P_{x,y}\,dx$. So given $B_{T\pm} = 1$, the $C^+[0,1)$ valued point process

$$(\overset{\circ}{L}_{[T_{y-},T_y]},\ 0 \leqq y \leqq 1) \tag{53}$$

is also inhomogeneous Poisson, with intensity measure the $dy\,\mu_y(d\xi)$ distribution of $(\xi_u + \xi_{u-1},\ 0 \leqq u \leqq 1)$. This observation, and the decomposition $\overset{\circ}{L}_{T\pm} = \sum_{0<y<1} \overset{\circ}{L}_{[T_{y-},T_y]}$ conditionally given $B_{T\pm} = 1$, imply all the assertions of the Proposition, apart from the second equality in (51). But this follows easily from Lemma 8. (See the proof of Proposition 17 for some details.) □

Decompositions of the circular local time process

Various decompositions of $\overset{\circ}{L}_{T\pm}$, can now be described by splitting the Poisson point process of pulses (53) into independent components. As a preliminary, observe that given $B_{T\pm} = 1$, the $C^+[-1,1]$ valued local time process $L_{T\pm}$ decomposes as the sum of three independent components $L_{T\pm} = L_{\text{short}+} + L_{\text{short}-} + L_{\text{long}}$ obtained by classifying the pulses into the following three categories, where y and x represent the levels of the maximum and minimum of the excursion associated with a pulse:

$$\begin{array}{ll} \text{short}+ & \text{if } 0 < x < y \\ \text{short}- & \text{if } y - 1 < x \leqq 0 < y \\ \text{long} & \text{if } x \leqq y - 1 \end{array}$$

Thus a pulse (or its corresponding excursion) is called as *short* or *long* according to whether the range $y-x$ of the excursion is less than 1 or at least 1. Each short pulse is further classified as $+$ if its support is entirely contained in $(0,1)$ and $-$ if its support intersects $[-1,0]$. By wrapping around the circle, there is a corresponding decomposition of $\overset{\circ}{L}_{T\pm}$ into three independent infinitely divisible components

$$\overset{\circ}{L}_{T\pm} = \overset{\circ}{L}_{\text{short}+} + \overset{\circ}{L}_{\text{short}-} + \overset{\circ}{L}_{\text{long}} \tag{54}$$

which holds also without conditioning on $B_{T\pm}$ provided the definitions are modified appropriately given $B_{T\pm} = -1$. Call a $C[0,1)$ valued process, or a measure on $C[0,1)$, *symmetric* if it is cyclically stationary and reversible.

Proposition 17 *The following statements hold under* P_0. *In the decomposition* (54) *of* $\overset{\circ}{L}_{T\pm}$, *the distribution of* $\overset{\circ}{L}_{\text{short}+}$ *is* $Q^2_{0\to 0}$ *with Lévy measure* $\overset{\circ}{M}_{\text{short}+} = 2N_0$. *The distribution of* $\overset{\circ}{L}_{\text{short}-} + \overset{\circ}{L}_{\text{long}}$ *is* $\int_0^\infty e^{-\ell}\, d\ell\, Q^0_{\ell\to 0} * \hat{Q}^0_{\ell\to 0}$, *with Lévy measure*

$$\overset{\circ}{M} - 2N_0 = \int\limits_0^\infty v^{-1} e^{-v}\, dv\, Q^0_{v\to 0} * \hat{Q}^0_{v\to 0}\,. \tag{55}$$

Let $\overset{\circ}{L}_{\text{short}} = \overset{\circ}{L}_{\text{short}+} + \overset{\circ}{L}_{\text{short}-}$. *The decomposition*

$$\overset{\circ}{L}_{T\pm} = \overset{\circ}{L}_{\text{short}} + \overset{\circ}{L}_{\text{long}} \tag{56}$$

expresses $\overset{\circ}{L}_{T\pm}$ *as the sum of two independent processes, each of which is infinitely divisible and symmetric. The corresponding Lévy measures* $\overset{\circ}{M}_{\text{short}}$ *and* $\overset{\circ}{M}_{\text{long}}$ *on* $C^+[0,1]$ *are*

$$\overset{\circ}{M}_{\text{short}} = \int\limits_0^1 dy \int\limits_{y-1}^{y} \frac{\overset{\circ}{P}_{x,y}}{(y-x)^2} dx; \qquad \overset{\circ}{M}_{\text{long}} = \int\limits_0^1 dy \int\limits_{-1}^{y-1} \frac{\overset{\circ}{P}_{x,y}}{(y-x)^2} dx\,. \tag{57}$$

Each of the measures $\overset{\circ}{M}_{\text{short}}$, $\overset{\circ}{M}_{\text{long}}$, *and* $\overset{\circ}{M}$ *is symmetric.*

Proof. These assertions follow directly from the preceding development. The identification $\overset{\circ}{M}_{\text{short}+} = 2N_0$ follows from (51) and (50), so the distribution of $\overset{\circ}{L}_{\text{short}+}$ is $Q^2_{0\to 0}$. Comparison with the last-exit decomposition in Lemma 8, that is

$$\overset{\circ}{L}_{T\pm} = \overset{\circ}{L}_G + \overset{\circ}{L}_{[G,T\pm]} \tag{58}$$

where G is the time of the last zero of B before time $T_\pm$, identifies the distribution of $\overset{\circ}{L}_{\text{short}-} + \overset{\circ}{L}_{\text{long}}$, and yields its Lévy measure, due to the infinite divisibility of the family $(Q^0_{\ell\to 0},\ \ell \geqq 0)$ and the well known formula $v^{-1}e^{-v}$ for the density at v of the Lévy measure of the standard exponential distribution of $L^0_{T\pm}$. (The identity (55) can also be derived using the relation between

$BESQ^0$ and $BESQ^4$ bridges described above (5.c) of [51]). The measure $\overset{\circ}{M}$ is symmetric by the symmetry of $\overset{\circ}{L}_{T\pm}$ and the Lévy–Khintchine formula (9). Since‘ $\overset{\circ}{M}_{\text{long}}$ is the restriction of $\overset{\circ}{M}$ to the symmetric subset $\{\inf_u \overset{\circ}{X}_u > 0\}$ of $C^+[0,1]$, this measure too is symmetric, and so is $\overset{\circ}{M}_{\text{short}} = \overset{\circ}{M} - \overset{\circ}{M}_{\text{long}}$. Again by the Lévy–Khintchine formula, the distributions of both $\overset{\circ}{L}_{\text{short}}$ and $\overset{\circ}{L}_{\text{long}}$ must be symmetric. □

A path transformation

According to the above proposition and Lemma 8(ii), the process $\overset{\circ}{L}_{\text{short}}$ has the same distribution as $\overset{\circ}{L}_{[G,T\pm]}$. There is the following pathwise explanation of this identity in distribution: given $B_{T\pm} = 1$, if the short+ excursions are strung together to form a process by closing up the gaps between these excursions, the resulting process has the same distribution as $(B_{G+v},\ 0 \leqq v \leqq T_\pm - G)$, as described in Remark 7. This follows from the identical Poisson character of the two excursion processes.

The Lévy measure for the short excursions

The symmetry of $\overset{\circ}{M}_{\text{short}}$ is made obvious by the following variations of (57):

$$\overset{\circ}{M}_{\text{short}} = \int_0^1 dy\,\Theta_y(\hat{M}_0) = \int_0^1 dy\,\Theta_y(M_0)$$

where $\Theta_y(K)$ denotes the image of the measure K on $C^+[0,1]$ after a cyclic shift by y, $\hat{M}_0 = \int_{-1}^0 \overset{\circ}{P}_{x,0} x^{-2}\,dx$ is the time reversal of M_0 in (50), and the expression with M_0 instead of $\hat{M}_0$ follows from the time reversibility of $Q^4_{0\to 0}$.

The Lévy measure for the long excursions

From (57), the measure $\overset{\circ}{M}_{\text{long}}$ on $C^+[0,1]$ has total mass

$$\int_0^1 dy \int_{-1}^{y-1} (y-x)^{-2}\,dx = 1 - \log 2.$$

So the number of long excursions up to time $T_\pm$, say $\#_{\text{long}}$, has Poisson distribution with mean $(1-\log 2)$. Given that $\#_{\text{long}} = n$, the local time pulses of these excursions, when presented in a random order independent of the excursions, form a sequence of n i.i.d random pulses with the distribution $\overset{\circ}{M}_{\text{long}}/(1-\log 2)$. (This is false if the randomized order is replaced by the natural time ordering of excursions: before wrapping, a pulse of range $r > 1$ cannot occur until the maximum process has reached at least $r-1$, so bigger pulses will tend to come later). To describe $\overset{\circ}{M}_{\text{long}}$ more explicitly, let (Y,Z) be picked at random from $[0,1]^2$ according to the probability density

$$P(Y \in dy, Z \in dz) = \frac{1(z+y \leqq 1)\,dz\,dy}{(1-\log 2)(y+z)^2}$$

and let S have distribution $Q^4_{0\to 0}$ independently of (Y,Z). Then, from (57), the random pulse

$$(Y+Z)\left[S_4\left(\frac{u+Z}{Y+Z}\right)1(u<Y)+S_4\left(\frac{u+Z-1}{Y+Z}\right)1(u>1-Z)\right] \quad (59)$$

where $0 \leqq u \leqq 1$.

has distribution $\mathring{M}_{\text{long}}/(1-\log 2)$. According to Proposition 17, this process is symmetric, something not at all obvious from the above construction.

Decomposition of the one-dimensional distributions

For $x>0$ let $\rho_{\text{short}}(x)$ and $\rho_{\text{long}}(x)$ denote the densities at x of the one-dimensional distributions of $\mathring{M}_{\text{short}}$ and $\mathring{M}_{\text{long}}$ respectively. Let $\Phi_{\text{short}}(\alpha)$ and $\Phi_{\text{long}}(\alpha)$ be the corresponding Laplace transforms of $\mathring{L}^u_{\text{short}}$ and $\mathring{L}^u_{\text{long}}$. Thus for every $0 \leqq u \leqq 1,\ \alpha > 0$

$$P_0\exp(-\alpha\mathring{L}^u_{\text{short}}) = \Phi_{\text{short}}(\alpha) = \exp\left(-\int_0^\infty (1-e^{-\alpha x})\rho_{\text{short}}(x)\,dx\right) \quad (60)$$

and similarly for *long* instead of *short*. The one-dimensional distribution of $\mathring{L}^u_{T\pm}$ is exponential with rate 1, with Laplace transform $(1+\alpha)^{-1}$ and Lévy density $x^{-1}e^{-x}$, $x>0$, So the independent decomposition (56) gives

$$\Phi_{\text{short}}(\alpha)\Phi_{\text{long}}(\alpha) = (1+\alpha)^{-1} \quad (\alpha > 0) \quad (61)$$

$$\rho_{\text{short}}(x)+\rho_{\text{long}}(x) = x^{-1}e^{-x} \quad (x>0) \quad (62)$$

Proposition 18

$$\rho_{\text{short}}(x) = \frac{1}{2}\int_0^1 u^{-2}\exp\left(\frac{-x}{2u(1-u)}\right)du = K_1(x)e^{-x} \quad (63)$$

where $K_1(x)$ *is the modified Bessel function,*

$$\Phi_{\text{short}}(\alpha) = \exp\left(\sum_{n=1}^\infty \frac{(n-1)!(n+1)!}{(2n+1)!}(-2\alpha)^n\right) \quad (64)$$

and there is the alternative expression (10) *for* $\Phi_{\text{short}}(\alpha)$.

Remark. 19 The coefficient of α in (64) shows that $\mathring{L}^u_{\text{short}}$ has mean 2/3. Consequently from (56), $\mathring{L}^u_{\text{long}}$ has mean 1/3. Integration over u shows that the mean total lengths of the short and long excursions are also 2/3 and 1/3 respectively. See (68) and (69) for the corresponding Laplace transforms.

Proof. By symmetry, it suffices to consider $u = 0$. From (51), for any non-negative function f vanishing at 0

$$\int_0^\infty f(x)\rho_{\text{short}}(x)\,dx = \int_0^1 dy \int_0^{1-y} \overset{\circ}{P}_{-z,y} f(X_0)(y+z)^{-2}dz$$

where the $\overset{\circ}{P}_{-z,y}$ distribution of X_0 is gamma with shape parameter 2 and rate $(y+z)/(2y)$, so

$$\overset{\circ}{P}_{-z,y} f(X_0) = \int_0^\infty f(x)\left(\frac{y+z}{2y}\right)^2 x\,\exp\left(-\frac{x(y+z)}{2y}\right)dx\,.$$

The first equality in (63) follows easily. A change of variables yields

$$\rho_{\text{short}}(x) = e^{-2x}\int_0^\infty (t+1)t^{-1/2}(t+2)^{-1/2}e^{-tx}\,dt\,.$$

Now the standard integral $\int_0^\infty t^{-1/2}(t+2)^{-1/2}e^{-tx}\,dt = e^x K_0(x)$ where K_0 is the usual modified Bessel function (see e.g. [47, p. 18, 2.48], where the right side should be corrected as follows: $e^{ap}K_0(ap)$ should be $e^{(1/2)ap}K_0((1/2)ap))$ allows the evaluation

$$\rho_{\text{short}}(x) = e^{-2x}\left(e^x K_0(x) - \frac{d}{dx}[e^x K_0(x)]\right) = e^{-x}K_1(x)\,.$$

Formulae (64) and (10) are obtained by substituting the middle expression in (63) into (60) and then switching the order of integration. □

Decomposition of the total time

From (54) the time $T_\pm$ is the sum of independent random times spent during various types of excursions, say $T_\pm = T_{\text{short}+} + T_{\text{short}-} + T_{\text{long}}$. As shown by Knight [31], the last-exit decomposition (58) implies that the Laplace transform $P_0\exp(-\alpha T_\pm) = (\cosh\theta)^{-1}$, where $\theta = \sqrt{2\alpha}$, factors as

$$\frac{1}{\cosh(\theta)} = \left(\frac{\theta}{\sinh\theta}\right)\left(\frac{\tanh\theta}{\theta}\right) \tag{65}$$

where the factors are the Laplace transforms of $T_\pm - G$ and G, as restated in the second equalities of (66) and (67) below. These equalities, and the second equality in (68), also due to Knight [31], follow from Lemma 6, (27)–(29) and (31). The remaining equalities in (66)–(70) follow immediately by Proposition 17. Using the notation $\theta = \sqrt{2\alpha}$, and writing simply P instead of P_0 governing B as a BM with no drift,

$$P\exp(-\alpha T_{\text{short}+}) = \frac{\theta}{\sinh\theta} = P\exp(-\alpha(T_\pm - G))\,, \tag{66}$$

$$P\exp(-\alpha(T_\pm - T_{\text{short}+})) = \frac{\tanh\theta}{\theta} = P\exp(-\alpha G)\,, \tag{67}$$

$$P\exp(-\alpha T_{\text{short}}) = \exp(1-\theta\coth\theta) = P\exp(-\alpha G|L^0_{T_\pm} = 1)\,, \tag{68}$$

$$P\exp(-\alpha T_{\mathrm{long}}) = \frac{\exp(\theta \coth \theta - 1)}{\cosh\theta}\,, \tag{69}$$

$$P\exp(-\alpha T_{\mathrm{short}-}) = \frac{\sinh\theta}{\theta}\exp(1-\theta\coth\theta)\,. \tag{70}$$

Of these formulae, the most interesting are (69) and (70), which present the Laplace transforms of two infinitely divisible distributions on $[0, \infty)$ that do not seem to have been encountered before. The Laplace transform (69) expands as

$$P\exp(-\alpha T_{\mathrm{long}}) = 1 - \frac{1}{3}\alpha + \frac{3}{10}\alpha^2 - \frac{1409}{5670}\alpha^3 + \cdots \tag{71}$$

confirming the result of Remark 19 that the mean of T_{long} is 1/3. In fact, each of the random variables $T_{\mathrm{short}+}, T_{\mathrm{short}-}$ and T_{long} has the same mean 1/3. Both $T_{\mathrm{short}+}$ and $T_{\mathrm{short}-}$ are strictly positive random variables with continuous distributions on $(0, \infty)$. However T_{long} has a compound Poisson distribution that has a continuous component on $(0,\infty)$ and an atom at 0 whose size may be found from (69):

$$P(T_{\mathrm{long}} = 0) = \lim_{\alpha\to\infty} P\exp(-\alpha T_{\mathrm{long}}) = 2/e\,. \tag{72}$$

As a check, from the discussion above (59), T_{long} is distributed as the sum of $\#_{\mathrm{long}}$ i.i.d. r.v's with continuous distribution on $(0,\infty)$, where $\#_{\mathrm{long}}$ is Poisson with mean $1-\log 2$. So $P(T_{\mathrm{long}} = 0) = \exp(\log 2 - 1) = 2/e$. The common distribution of the terms in this sum have density $\rho(x)/(1-\log 2)$ where (69) yields

$$\int_0^\infty x\rho(x)e^{-\alpha x}\,dx = \left(\frac{1}{\sinh\theta}\right)^2 - \frac{1}{\theta\sinh\theta\cosh\theta}\,. \tag{73}$$

5 Results for other random times

The Laplace functional of $\overset{\circ}{L}_T$ for many random times T besides $T_\pm$ can be obtained by variations of the method of Sect. 3. Throughout this section, let $P = P_0$ govern B as a BM with zero drift. Extensions to P_δ for $\delta \neq 0$ are straightforward, as in Sect. 3.

A class $\mathscr{T}$ of random times T such that $\overset{\circ}{B}_T = 0$ will now be defined. This class $\mathscr{T}$ includes $T_\pm$, the inverse local time of B at zero $\tau_\ell = \inf\{t : L_t^0 > \ell\}$, and the inverse local time of $\overset{\circ}{B}$ at zero $\overset{\circ}{\tau}_\ell = \inf\{t : \overset{\circ}{L}{}_t^0 > \ell\}$.

Definition 20 *Let $\mathscr{T}$ be the collection of random times T of the form either $T = \overset{\circ}{\tau}_R$ or $T = \overset{\circ}{\tau}_{R-}$ where R is a positive measurable function of the time-changed process $(B_{\overset{\circ}{\tau}_\ell},\ \ell \geqq 0)$.*

The process $(B_{\overset{\circ}{\tau}_\ell},\ \ell \geqq 0)$ is a continuous time symmetric random walk on the integers, with i.i.d. exponential (1) holds independent of i.i.d. Bernoulli(1/2) jumps of ± 1. Note that if $T \in \mathscr{T}$ then $\overset{\circ}{B}_T = 0$ and $\overset{\circ}{L}{}_T^0 = R$. Let N_T be the number of loops (of either sign) completed by $\overset{\circ}{B}$ up to time T. That is to say N_T is the number of jumps of $(B_{\overset{\circ}{\tau}_\ell},\ 0 \leqq \ell \leqq \overset{\circ}{L}{}_T^0)$, where a jump if any at local time $\overset{\circ}{L}{}^0_T = R$ is counted if $T = \overset{\circ}{\tau}_R$ but not if $T = \overset{\circ}{\tau}_{R-}$. The following proposition generalizes Lemma 8 and the similar decomposition of $\overset{\circ}{L}_{\overset{\circ}{\tau}_\ell}$ given in Leuridan [40]. See also [50] where this proposition is generalized to Brownian motion on a network. Recall that $*$ denotes convolution of distributions on $C[0,1)$.

Proposition 21 *For* $T \in \mathscr{T}$ *the conditional distribution of* $\overset{\circ}{L}_T$ *given* $N_T = n$ *and* $\overset{\circ}{L}{}^0_T = \ell$ *is* $Q^0_{\ell\to 0} * \hat{Q}^0_{\ell\to 0} * Q^{2n}_{0\to 0}$. *That is to say, the distribution of* $\overset{\circ}{L}_T$ *is*

$$\int_0^\infty \sum_{n=0}^\infty P(N_T = n, \overset{\circ}{L}{}^0_T \in d\ell) Q^0_{\ell\to 0} * \hat{Q}^0_{\ell\to 0} * Q^{2n}_{0\to 0}\,. \tag{74}$$

Proof. Following the style of argument in Sect. 5 of [51], decompose $\overset{\circ}{L}_T$ as the sum of pulses derived from individual excursions $\overset{\circ}{\varepsilon}$ of $\overset{\circ}{B}$ away from 0. Call $\overset{\circ}{\varepsilon}$ a *loop* if $\overset{\circ}{\varepsilon}$ returns to 0 on the opposite side from which it starts. Otherwise call $\overset{\circ}{\varepsilon}$ a *non-loop*. According to Itô's [22] excursion theory, when the pulses are viewed as a $C^+[0,1)$ valued point process parameterized by local time of $\overset{\circ}{B}$ at 0, the pulses of loops and the pulses of non-loops form independent Poisson processes. The point process of pulses of loops is defined by the sequence of i.i.d. exponential spacings between loops on the local time scale, and the i.i.d. sequence of $C^+[0,1)$ valued pulses. By Lemma 8 the pulse of each loop has distribution $Q^2_{0\to 0}$, independently of the signs of all the loops. The distribution of the sum of n such pulses is therefore $Q^{2n}_{0\to 0}$ by the additivity of squares of Bessel bridges. Similarly, if non-loops are classified in the obvious way as either positive or negative, for each fixed ℓ the local time process $\overset{\circ}{L}_{\overset{\circ}{\tau}_\ell}$ contains a contribution from pulses of positive non-loops with distribution $Q^0_{\ell\to 0}$, and an independent contribution from pulses of negative non-loops with distribution $\hat{Q}^0_{\ell\to 0}$. By definition, $T \in \mathscr{T}$ has the property that $(N_T, \overset{\circ}{L}{}^0_T)$ is a measurable function of $(B_{\overset{\circ}{\tau}_\ell}, \ell \geqq 0)$, that is a function of the i.i.d exponential spacings between loops on the local time scale and the i.i.d. sequence of signs of the loops. So $(N_T, \overset{\circ}{L}{}^0_T)$ is independent of both the i.i.d. sequence of pulses of the loops, and of the Poisson point process of pulses of non-loops. Since $\overset{\circ}{B}_T = 0$ the process $\overset{\circ}{L}_T$ decomposes as the sum of pulses from N_T loops, and the sum of pulses of the non-loops up to local time $\overset{\circ}{L}{}^0_T$ and the conclusion follows. □

Corollary 22 *For* $T \in \mathscr{T}$,

$$P \exp(-m\overset{\circ}{L}_T) = \int_0^\infty \sum_{n=0}^\infty P(N_T = n, \overset{\circ}{L}{}^0_T \in d\ell) \Psi^{-n} \exp(-\ell(\overset{\circ}{\Psi} - 1)) \tag{75}$$

for $\overset{\circ}{\Psi}$ *and* Ψ *as in* (33) *and* (30).

Proof. Apply the previous proposition and (29). □

Example 23 Leuridan [40] obtained (74) for $\mathring{\tau}_\ell = \inf\{t : \mathring{L}^0_t > \ell\}$. Then $\mathring{L}^0_{\mathring{\tau}_\ell} = \ell$ and $N_{\mathring{\tau}_\ell}$ has Poisson distribution with mean ℓ. So (75) yields

$$P\exp(-m\mathring{L}_{\mathring{\tau}_\ell}) = \exp\left(-\ell(\mathring{\Psi} - 1)/\Psi\right) . \tag{76}$$

The calculation of features of the one- and two-dimensional distributions of $\mathring{L}_{\mathring{\tau}_\ell}$, as undertaken in [40], is simplified by application of this formula.

Corollary 24 *Let $T_{(0)} = 0$ and let $T_{(1)}, T_{(2)}, \ldots$ be the successive times that $\mathring{B}$ returns to 0 after complete loops around the circle. Let $S_n = B_{T_{(n)}}$, so $(S_0, S_1, \ldots)$ is the usual embedding of a symmetric random walk in Brownian motion. Let N be a non-negative integer valued r.v. which is conditionally independent of $(B_t,\ t \geqq 0)$ given $(S_1, S_2, \ldots)$. Let*

$$G(z) = \sum_n P(N = n) z^n .$$

Then the circular local time process $\mathring{L}_{T_{(N)}}$ is cyclically stationary, with

$$P\exp(-m\mathring{L}_{T_{(N)}}) = G(1/\mathring{\Psi}) .$$

Proof. By the strong Markov property of $\mathring{B}$, the sequence of circular local time processes $(\mathring{L}_{T_{(n)}} - \mathring{L}_{T_{(n-1)}},\ n = 1, 2, \ldots)$ is a sequence of i.i.d. copies of $\mathring{L}_{T(1)} - \mathring{L}_{T(0)} = \mathring{L}_{T\pm}$. By Lemma 8, this i.i.d. sequence is independent of the i.i.d. sequence of signs of the successive loops of $\mathring{B}$ that determine the random walk (S_n). Corollary 24 now follows from (35). □

To illustrate, Corollary 24 shows that $P\exp(\alpha\mathring{L}^u_{T(N)}) = G(1/(1+\alpha))$ for $0 \leqq u < 1$, and with (36) gives the Laplace transform of the time spent by $\mathring{B}$ in an interval of length p up to time $T_{(N)}$. If the distribution of N is infinitely divisible, then so is the distribution of $\mathring{L}_{T_{(N)}}$, by a standard subordination argument. The following example shows that the distribution $\mathring{L}_{T_{(N)}}$ may be infinitely divisible even if that of N is not:

Example 25 Let T_a be the first time B_t hits a. Then $T_1 = T_{(N)}$ where N is the hitting time of 1 for the walk, with $G(z) = z^{-1}(1 - \sqrt{1 - z^2})$. So

$$P\exp(-m\mathring{L}_{T_1}) = \mathring{\Psi}\left(1 - \sqrt{1 - (\mathring{\Psi})^{-2}}\right) . \tag{77}$$

For example

$$P\exp(\alpha\mathring{L}^u_{T_1}) = 1 + \alpha - \sqrt{2\alpha + \alpha^2} \quad (0 \leqq u < 1) . \tag{78}$$

For $u = 0$, $\mathring{L}^0_{T_1} = \inf\{\ell : B_{\mathring{\tau}_\ell} = 1\}$ is the hitting time of 1 by a continuous time symmetric random walk on the integers. Formula (78) then agrees with the standard expression [15, formula (3.10)] for the Laplace transform of this hitting time. The fact that the distribution of $\overset{\circ}{L}_{T_1}$ is cyclically stationary can be seen directly as follows. For $0 < a < 1$ the distribution of B is preserved by the path transformation which exchanges the segments of path of $\overset{\circ}{B}$ on $[0, T_a]$ and $[T_a, T_1]$. This remark, combined with the observation that $\overset{\circ}{L}_{T_1}$ is the sum of N i.i.d. copies of $\overset{\circ}{L}_{T_\pm}$, yields an elementary proof of the cyclic stationarity of $\overset{\circ}{L}_{T_\pm}$. In this example, the possible values of N are $\{1, 3, 5, \ldots\}$, so the distribution of N is not infinitely divisible. However, by consideration of excursions below the maximum, much as in Sect. 4, it is clear that the distribution of $\overset{\circ}{L}_{T_1}$ is infinitely divisible, with Lévy measure Λ on $C^+[0,1)$ that may be obtained as follows from M on $C[0,\infty)$ as in (49): $\Lambda = 2\int_0^1 \overset{\circ}{M}_u \, du$ where $\overset{\circ}{M}_u$ is image of $\overset{\circ}{M}_0$ after a cyclic shift by u, and $\overset{\circ}{M}_0$ is the M distribution of $(\sum_{n=0}^{\infty} X_{n+v},\ 0 \leqq v < 1)$. The identity obtained by inserting this description of Λ and (77) into the Lévy–Khintchine formula (9) seems quite non-trivial.

The cover time

Let T_{cover} be the *cover time* for the circular Brownian motion, that is the inf of times t such that the range of $(\overset{\circ}{B}_s,\ 0 \leqq s \leqq t)$ equals [0,1). Put another way, $T_{\text{cover}} = \inf\{t : R_t = 1\}$ where $R_t = \max_{0 \leqq s \leqq t} B_s - \min_{0 \leqq s \leqq t} B_s$. It is known that

$$P(B_{T_{\text{cover}}} \in dx) = |x| dx \quad (-1 \leqq x \leqq 1) \tag{79}$$

which implies $\overset{\circ}{B}_{T\text{cover}}$ has uniform distribution on [0,1). Let $\tilde{T}$ be the first time that $\overset{\circ}{B}$ reaches the point $\overset{\circ}{B}_{T\text{cover}}$. There is the following Williams decomposition at time $\tilde{T}$ which is a variation of results of Imhof [20, 21] and Vallois [62, 63]: *Conditionally given* $B_{T\text{cover}} = x > 0$, *the processes* $(x - B_t,\ 0 \leqq t \leqq \tilde{T})$ *and* $(B_{\tilde{T}+s},\ 0 \leqq s \leqq T_{\text{cover}} - \tilde{T})$ *are independent, the first a* BES^3_{1-x} *run till its hitting time of* 1, *and the second a* BES^3_0 *run till its hitting time of* 1. This decomposition and Remark 7 yield a formula for the Laplace functional of $\overset{\circ}{L}_{T\text{cover}}$:

$$P\exp(-m\overset{\circ}{L}_{T\text{cover}}) = \int_0^1 \frac{x\Psi(xm_{0x}) + \bar{x}\Psi(\bar{x}m_{x1})}{\Psi^2(m_x)} dx \tag{80}$$

where Ψ is defined by (29), $\bar{x} = 1 - x$, and for a measure m on [0,1) and $x \in [0,1)$ the measures m_x, m_{0x} and m_{x1} on [0,1) are defined as follows:

m_x is the image of m via the map $u \to u - x \bmod 1$;
m_{0x} is the image of the restriction of m to $[0,x)$ via the map $u \to u/x$;
m_{x1} is the image of the restriction of m to $[x,1)$ via $u \to (u-x)/(1-x)$.

In particular, given $B_{T\text{cover}} = x > 0$ the local time $\overset{\circ}{L}{}^0_{T\text{cover}} = L^0_{T\text{cover}}$ decomposes as the sum of two i.i.d. exponentials with rates $(2x\bar{x})^{-1}$, and (80) yields

$$P\exp(-\alpha\overset{\circ}{L}{}^0_{T\text{cover}}) = \int_0^1 \frac{dx}{(1 + 2\alpha x\bar{x})^2} = \frac{1}{2+\alpha} + \frac{2\operatorname{arctanh}\sqrt{\frac{\alpha}{2+\alpha}}}{\sqrt{\alpha}(2+\alpha)^{3/2}}\ . \tag{81}$$

A similar but more complicated expression can be obtained from (80) for the Laplace transform of $\overset{\circ}{L}{}^u_{T\text{cover}}$ for all $0 < u < 1$. The transform (81) can be explicitly inverted by noting that $P(\overset{\circ}{L}{}^0_{T\text{cover}} \geqq \ell) = P(X_\ell + Y_\ell \leqq 1)$ where $X_\ell = \max_{0 \leqq s \leqq \tau_\ell} B_s$ and $Y_\ell = -\min_{0 \leqq s \leqq \tau_\ell} B_s$, and τ_ℓ is the inverse local time process of B at zero. It is well known that X_ℓ and Y_ℓ are i.i.d. with $P(X_\ell \leqq x) = \exp(-\ell/2x)$, and the convolution integral can be evaluated using formulae around (63) to give

$$P(\overset{\circ}{L}{}^0_{T\text{cover}} \geqq \ell) = \ell K_1(\ell) e^{-\ell} \tag{82}$$

where K_1 is the modified Bessel function. The inequality $\overset{\circ}{L}{}^0_{T\text{cover}} \leqq L^0_{T_\pm}$ and the exponential distribution of $L^0_{T_\pm}$ imply $\ell K_1(\ell) \leqq 1$, as is easily verified analytically. As another example, taking $m = \alpha\lambda$ in (80) for Lebesgue measure λ on [0,1) recovers the formula $P\exp(-\alpha T_{\text{cover}}) = \operatorname{sech}^2(\sqrt{\alpha/2})$ obtained in [20].

Let $U = \overset{\circ}{B}_{T\text{cover}}$. Note that U is the a.s. unique zero of the process $\overset{\circ}{L}_{T\text{cover}}$. From the Williams decomposition and Remark 7, U has uniform distribution on [0,1), and independently of U

$$\textit{the process}\ (\overset{\circ}{L}{}^{U+s}_{T\text{cover}} - \overset{\circ}{L}{}^{U+s}_{\tilde{T}},\ 0 \leqq s < 1)\ \textit{has distribution}\ Q^2_{0\to 0} \tag{83}$$

where $U + s$ is understood mod 1. So the process $\overset{\circ}{L}_{T\text{cover}} - \overset{\circ}{L}_{\tilde{T}}$ is stationary, with Laplace functional

$$P\exp[-m(\overset{\circ}{L}_{T\text{cover}} - \overset{\circ}{L}_{\tilde{T}})] = \int_0^1 \frac{dx}{\Psi(m_x)}\ . \tag{84}$$

But neither the processes $\overset{\circ}{L}_{T\text{cover}}$ and $\overset{\circ}{L}_{\tilde{T}}$ is stationary, due to (85) below.

The first zero after the cover time

Let T be a stopping time of B, and $0 \leqq c < \infty$. An argument using Dynkin's formula shows that

$$P\overset{\circ}{L}{}^u_T = c \quad \textit{for all}\ 0 \leqq u < 1 \quad \textit{if and only if}\ PT = c \quad \textit{and}\ \overset{\circ}{B}_T = 0\ \textit{a.s.} \tag{85}$$

And it is easily seen that if $T > 0$ and $\overset{\circ}{L}_T$ is stationary then $T \geqq T_{\text{cover}}$ a.s.. See [14] for related results. Let T_* be the time of the first return of $\overset{\circ}{B}$ to 0 after time T_{cover}. Combining the above observations shows that

$$\textit{if}\ T > 0 \quad \textit{and}\ PT < \infty \quad \textit{and}\ \overset{\circ}{L}_T\ \textit{is stationary, then a.s.}\ B_T = 0 \quad \textit{and}\ T \geqq T_* \tag{86}$$

So the following question arises:

Question 26 Is the process $\overset{\circ}{L}_{T*}$ stationary ?

The Williams decomposition used to obtain (80) yields the following expression for the Laplace functional $\overset{\circ}{L}_{T_*}$:

$$P\exp(-m\overset{\circ}{L}_{T_*}) = \frac{1}{\Psi(m)} \int_0^1 \left(\frac{x\Psi(xm_{0x}) + \bar{x}\Psi(\bar{x}m_{x1})}{\Psi(m_x)} \right)^2 dx\ . \tag{87}$$

So the question is whether this expression is invariant under cyclic shifts of m. Consider the following two special cases:

(i) m is concentrated on at most two points.
(ii) m is a multiple of uniform distribution on a subinterval of the circle.

Formula (87) in case (i) gives the joint Laplace transform of $\mathring{L}^u_{T_*}$ and $\mathring{L}^v_{T_*}$ for arbitrary u and v in [0,1), and in case (ii) gives the Laplace transform of the occupation time of a subinterval of the circle up to time T_*. In both cases it is possible to simplify the right side of (87) by calculus. In separate calculations for the two cases using *Mathematica*, some remarkable simplifications occur. It is found that in both these cases the Laplace functional can be expressed as follows:

$$P\exp(-m\mathring{L}_{T_*}) = \frac{\Phi}{1+\Phi}\left(1 + \frac{\Phi}{\sqrt{1-\Phi^2}}\operatorname{arctanh}\sqrt{1-\Phi^2}\right) \tag{88}$$

where $\Phi = \Phi(m) = P\exp(-m\mathring{L}_{T_\pm}) = 1/\mathring{\Psi}(m)$ as in (33), and

$$\operatorname{arctanh}(x) = x + \frac{1}{3}x^3 + \frac{1}{5}x^5 + \cdots \quad (x^2 < 1)$$

so the right side of (88), call it Φ_*, expands as

$$\Phi_* = \Phi + (1-\Phi)\Phi^2\left(\frac{1}{3} + \frac{1}{5}(1-\Phi^2) + \frac{1}{7}(1-\Phi^2)^2 + \cdots\right). \tag{89}$$

Because $\Phi(m)$ is invariant under cyclic shifts of m, so is $\Phi_*(m)$. So (88) in case (i) shows that the two-dimensional distributions of $\mathring{L}_{T_*}$ are invariant under cyclic shifts, and in case (ii) that the distribution of the occupation time of a sub-interval of the circle up to time T_* depends only on the length of the interval. Note that for m a point mass at 0, $\mathring{L}^0_{T_*} = \mathring{L}^0_{T\text{cover}} = L^0_{T\text{cover}}$, and (88) then reduces to (81). So for every $u \in [0,1)$ the distribution of $\mathring{L}^u_{T_*}$ is identical to the distribution of $\mathring{L}^0_{T\text{cover}}$ described by formula (82).

Using (88) for two-point distributions, it can be checked for arbitrary m that the two sides of (88) with αm instead of m, viewed as power series in α, have the same coefficients of $1, \alpha$ and α^2, namely 1, $-P(m\mathring{L}_{T_*}) = -(2/3)\mathring{m}_1$, and

$$\frac{1}{2}P((m\mathring{L}_{T_*})^2) = \frac{2}{5}\mathring{m}_1^2 - \frac{4}{3}\mathring{m}_2 \tag{90}$$

where the $\mathring{m}_n$ are defined by (34). So there is much evidence for the following conjecture, which would imply an affirmative answer to question (26):

Conjecture 27 *Formula* (88) *holds for all finite measures m on* [0, 1).

In connection with this conjecture, it turns out that for m a point mass or Lebesgue measure, the expression $(\Phi/\sqrt{1-\Phi^2})\operatorname{arctanh}\sqrt{1-\Phi^2}$ appearing in (88) is identical to the Laplace functional in (84). While it should be easier to resolve whether or not this identity extends to all measures m, the relation between this coincidence and (88) is not clear.

6 Further applications of the series formulae for Bessel processes

This section points out a number of applications of Proposition 5 to one-dimensional diffusion processes. See [53] for further details and developments.

Assumption *Throughout this section suppose as in* (14) *that the measure* m *on* $[0,\infty)$ *has finite total mass and finite first moment.*

Corollary 28 *Let* m_{in} *be as in* (16)*. The functions* $\Psi_0(\alpha m)$ *and* $\Psi_1(\alpha m)$ *are entire functions of* α *defined by the power series*

$$\Psi_i(\alpha m) = i + \sum_{n=1}^{\infty} m_{in}(2\alpha)^n \quad (i = 0 \text{ or } 1) \tag{91}$$

Consequently, (11), (18) *and* (22) *hold for all* $\alpha > -\varepsilon_m$ *for some* $\varepsilon_m > 0$.

Thus (11), (18), (22) and other such formulae for Laplace transforms involving the Ψ_i yield moment generating functions, from which moments can be read by formal manipulation of power series. For example:

Corollary 29 *All positive integer moments of the* Q_x^d *distribution of* $\int_0^\infty X_u m(du)$ *are finite, and given by polynomials in* x, d *and the* m_{in} *obtained from formal power series manipulations on* (11) *with* $\Psi_i(\alpha m)$ *instead of* Ψ_i.

There is an alternative expression for the Laplace transform in (22) which is well known (see e.g. [23, Sect. 6.1–6.2; 33; 35, Sect. 4]). Let $A_m(t) = \int_0^\infty L_t^u m(du)$. Then

$$P_0 \exp(-\alpha A_m(\tau_\ell)) = \exp\left(\frac{-\ell}{g_m(\alpha, 0, 0)}\right) \tag{92}$$

where $g_m(\alpha, x, y)$ is the Green function of the quasi-diffusion X_m defined by $X_m(u) = B(T_{m,u})$ where $(T_{m,u},\ u \geqq 0)$ is the right-continuous inverse of the additive functional $(A_m(t),\ t \geqq 0)$ of B. As shown in [33, 35], the function $\alpha \to g_m(\alpha, 0, 0)$ is the function known in Krein's theory of vibrating strings [36, 37, 25, 35] as the *characteristic function* of the mass distribution $2m$, for which many different expressions are known. Combining (22) and (92) yields a particularly simple one that does not seem to appear in the literature:

Corollary 30

$$g_m(\alpha, 0, 0) = 2\Psi_1(\alpha m)/\Psi_0(\alpha m) \tag{93}$$

where the $\Psi_i(\alpha m)$ *are the entire functions defined by the series* (91).

According to a remarkable result of Krein, the mass distribution $2m$ can be recovered from its characteristic function. As a consequence:

Corollary 31 *The measure* m *can be recovered from the two positive sequences* $(m_{01}, m_{11}, \ldots)$ *and* $(m_{11}, m_{21}, \ldots)$ *defined by* (16).

By considering variations of the functions Ψ_i like Ψ in (30) with an arbitrary endpoint x instead of 1, both the increasing and decreasing solutions of the Sturm–Liouville equation $\frac{1}{2}\phi'' = \alpha m \cdot \phi$, hence the Green function $g_m(\alpha, x, y)$, can be expressed by explicit series formulae involving iterated integrals with respect to m (cf. [9, Sect. 5.4; 25, Sect. 2.3]). Such formulae have numerous applications to the computation of quantities of probabilistic interest, by classical applications of the Green function [23]. To illustrate, assume now for simplicity that $m\{0\} = 0$. Differentiation of the exponent in (92) yields:

Corollary 32 *The Lévy measure* Λ_m *of the subordinator* $(A_m(\tau_\ell), \ell \geqq 0)$, *which is the inverse of the local time of process of* X *at zero, is given by*

$$\int_0^\infty y\Lambda_m(dy)\,e^{-\alpha y} = -\frac{d}{d\alpha}\,\frac{\Psi_0(\alpha m)}{2\Psi_1(\alpha m)}\,. \tag{94}$$

Consequently, all moments of Λ_m *are finite, and these moments are polynomials in* $(m_{01}, m_{11}, \dots)$ *and* $(m_{11}, m_{21}, \dots)$ *with rational coefficients obtained from* (91) *and* (94) *by formal power series manipulations.*

In connection with formula (94), by combination of standard renewal theory [15] and the theory of excursions for the stationary version of the quasi-diffusion X_m, for which see [49], the measure

$$F_m(dy) = \left(\int_0^\infty x\Lambda_m(dx)\right)^{-1} y\Lambda_m(dy)$$

has the following probabilistic interpretation. Let $G_{m,u}$ be the last zero of X_m before time u and $D_{m,u}$ the first zero of X_m after time u. Then F_m is the limiting distribution of $D_{m,u} - G_{m,u}$ as $u \to \infty$.

For some recent applications of Krein's theory of strings to probabilistic problems, and references to earlier work, see [4, 7, 39, 38].

7 Open Problems

1. See Question 26 and Conjecture 27.
2. Provide some criteria for when expressions like (7) and the inverse of (17) for $i = 1$ generate multivariate Laplace transforms. The structure of the expression with the sum over subsets gives consistency of corresponding f.d.d.'s if they exist. So this is a natural way to generate processes with exponential marginals. The question is what sort of function of A is an acceptable substitute for the product $\overset{\circ}{\Pi}(A)$ in (7) or $\Pi_1(A)$ in (17)? See e.g. [55] for background on related questions. What about other parameter sets besides the line or a circle? If there are more such processes, are they continuous? infinitely divisible?
3. For $\overset{\circ}{Q}{}^\kappa_\delta$ as in Corollary 3, find the distribution of $\max_{0 \leqq u < 1} X_u$ and/or $\min_{0 \leqq u < 1} X_u$. It is easy to see that $\operatorname{argmax}_{0 \leqq u < 1} X_u$ is $\overset{\circ}{Q}{}^\kappa_\delta$ a.s. unique for all $\delta \geqq 0$, $\kappa > 0$, hence uniformly distributed on $[0, 1)$ by cyclic stationarity. From the local time representation for $\kappa = 1$ it is clear that $\overset{\circ}{Q}{}^\kappa_\delta(\min_{0 \leqq u \leqq 1} X_u > 0) = 1$ for $\kappa \geqq 1$, and then $\operatorname{argmin}_{0 \leqq u < 1} X_u$ will be $\overset{\circ}{Q}{}^\kappa_\delta$ a.s. unique and uniform on $[0, 1)$. But for $0 < \kappa < 1$ the Lévy–Itô representation and the recurrence of state 0 for BES_0^d with $d < 2$ imply that $\overset{\circ}{Q}{}^\kappa_\delta(\min_{0 \leqq u \leqq 1} X_u = 0)$ is strictly between 0 and 1, and given this event X will have lots of zeros. A finite dimensional integral for the probability of this event can be given using results of Sect. 4 and excursion theory. See Eisenbaum [11] regarding related questions

for linear Brownian local times and references to earlier work of Borodin and others on this topic.
4. It is known that squares of Bessel processes arise as the total mass process of measure-valued branching process. Le Gall [17] established deep connections between such superprocesses and the theory of Brownian excursions. Is there a superprocess analog of Proposition 5? If so, how does it relate to Dynkin's [10] formulae for moments of the random field generated by a superprocess?

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