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Probability Theory and Related Fields

Large deviations and exponential decay for the magnetization in a Gaussian random field

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Summary. We consider a continuous model for transverse magnetization of spins diffusing in a homogeneous Gaussian random longitudinal field $\{\lambda V(x);\, x \in \mathbb{R}^d\}$, where λ is the coupling constant giving the intensity of the random field. In this setting, the transverse magnetization is given by the formula $M(t) = \mathbb{E}\exp\{-\lambda^2 \int_0^t \int_0^t K(B_r - B_s)\; dsdr\}$, where $\{B_t;\, t \geq 0\}$ is the standard process of Brownian motion and $K(x)$ is the covariance function of the original random field $V(x)$. We use large deviation techniques to show that the limit $S(\lambda) = \lim_{t\to\infty} \frac{1}{t} \ln M(t)$ exists. We also determine the small λ behavior of the rate $S(\lambda)$ and show that it is indeed decaying as conjectured in the physics literature.

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1 Introduction

Random partial differential equations have been used by physicists to model various important problems of fluid and solid mechanics with great success. One such model concerns the relaxation of the transverse magnetization of magnetic moments due to diffusion in an imhomogeneous magnetic field. Though the relaxation of the transverse magnetization of spins diffusing in an imhomogeneous magnetic field is an old problems in physics, it has drawn much attention recently. Mitra P. P. and Doussal P.L. gave in [17] quite convincing arguments suggesting several results on the large time behavior of the transverse magnetization. The goal of this paper is to justify rigorously some of these results.

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We first describe the model. We denote by $\mathbb{R}^d$ the d-dimensional Euclidean space and we let $\{V(x);\, x \in \mathbb{R}^d\}$ be a homogeneous mean zero Gaussian random field with variance 1. Its statistical properpty is completely determined by its covariance function $K(x)$ or equivalently by its spectral measure $\nu(d\xi)$. Recall that the latter are related by:

$$K(x) = \int_{\mathbb{R}^d} e^{ixz}\,\nu(dz),$$

The density $m(t,x)$ of transverse magnetization of spins diffusing in the random longitunal field $\mathbf{H} = \lambda V(x)\mathbf{k}$, after suitable rescaling in space and time, is governed by the following equation:

$$\frac{\partial m(t,x)}{\partial t} = \frac{1}{2} \triangle m(t,x) + i\lambda V m(t,x), \qquad m(0,x) \equiv 1 \tag{1}$$

where $i = \sqrt{-1}$. See [17] for details. If we denote by $< \cdot >$ the average with respect to the *disorder* $\{V(x);\, x \in \mathbb{R}^d\}$, then a quantity of interest is the magnetization:

$$M(t) = < m(x,t) >$$

which is independent of x because of the shift invariance of the distribution of $V(x)$. The classical Feymann-Kac formula gives the representation:

$$M(t) = \mathbb{E}\exp[-\lambda^2 \int_0^t \int_0^t K(B_r - B_s)\, ds dr], \tag{2}$$

where $\{B_t;\, t \geq 0\}$ is a standard process of Brownian motion and $\mathbb{E}$ denotes the expectation with respect to this Brownian motion. The large time behavior of similar expectations for the pinned down random walk on the one dimensional lattice $\mathbb{Z}$ where studied in [16] (see especially Section 2) in the context of the analysis of random discrete Schrödinger operators. We study the long time behavior of $M(t)$ by means of large deviation techniques.

Before we state our main results, we introduce some more notations. We shall denote by $\mathscr{X}$ the space $C_0([0,\infty), \mathbb{R}^d)$ of all the continuous function from $[0,\infty)$ into $\mathbb{R}^d$ with initial value zero equipped with the topology of the local uniform convergence. Similarly we shall use the notation $\mathscr{X}_t$ for the restrictions to the interval $[0,t]$ of the elements of $\mathscr{X}$. the notation $C_0([0,t])$ will be used. $\mathscr{X}$ and $\mathscr{X}_t$ are Polish spaces. A typical element of $\mathscr{X}$ will be written as $\{\omega(t);\, 0 \leq t < \infty\}$. We denote by $\mathscr{P}_{si}(\mathscr{X})$ the set of the all the probability measures on $\mathscr{X}$ such that the coordinate process of $\mathscr{X}$ is a process with stationary increments. $\mathscr{P}_{si}(\mathscr{X})$ is a Polish space if endowed with the topology of the weak convergence of probability measures. Let Q_0 denote the law of the process $\{B_t;\, t \geq 0\}$ of standard Brownian motion. Obviously $Q_0 \in \mathscr{P}_{si}(\mathscr{X})$. For any $P \in \mathscr{P}_{si}(\mathscr{X})$, we denote by P^t the restriction of P to the σ-field generated by the coordinate functions $s \hookrightarrow \omega(s)$ for $0 \leq s \leq t$ (or equivalently the restriction of P to $\mathscr{X}_t$). For any $P \in \mathscr{P}_{si}(\mathscr{X})$ and $t > 0$, we define the relative entropy of P^t with respect to Q_0^t by

$$H(P^t|Q_0^t) = \sup_f \{ \int f \; dP - \ln[\int \exp(f) \; dQ_0] \},$$

where the supremum is taken over all the bounded continuous functions f on $\mathscr{X}_t$. $H(P^t|Q_0^t)$ so-defined is superadditive in t and we can define:

$$H(P|Q_0) = \lim_{t\to\infty} \frac{1}{t} H(P^t|Q_0^t) \tag{3}$$

for any $P \in \mathscr{P}_{si}(\mathscr{X})$. We are now in a position to state our main results.

Theorem 1.1 *We assume that the function $K(x)$ is nonnegative and that it is integrable if $d = 1$ or $d = 2$ and that it satisfies the condition:*

$$\int \frac{\nu(d\xi)}{\|\xi\|^2} < \infty \tag{4}$$

if $d \geq 3$. Here and in the following $\|\xi\|$ denotes the Euclidean norm of $\xi \in \mathbb{R}^d$. Then the limit:

$$S(\lambda) = \lim_{t\to\infty} \frac{1}{t} \ln M(t) \tag{5}$$

exists and is given by the formula:

$$S(\lambda) = - \inf_{P \in \mathscr{P}_{si}(\mathscr{X})} \left[\mathbb{E}^P 2\lambda^2 \int_0^\infty K(\omega(t)) \; dt + H(P|Q_0) \right], \tag{6}$$

where $\mathbb{E}^P$ denotes the expectation with respect to the measure P.

Notice that the existence of $S(\lambda)$ is an easy consequence of the subadditivity property of $\frac{1}{t} \ln M(t)$. We do not know if the nonnegativity of the correlation of the random field $V(x)$ is necessary. The variational formula that we derive here is very convenient when it comes to analyzing the small λ behavior of $S(\lambda)$ and to show that the results conjectured in the physics literature (see for example [17]) do indeed hold. More precisely we prove:

Theorem 1.2 *The small λ behavior of $S(\lambda)$ is given by:*

$$\frac{S(\lambda)}{\lambda^{4/3}} = O(1) \text{ if } d = 1, \qquad \lim_{\lambda \searrow 0} \frac{S(\lambda)}{\lambda^2 \log(1/\lambda)} = \frac{2}{\pi} \int K(x) dx \text{ if } d = 2, \tag{7}$$

and:

$$\lim_{\lambda \searrow 0} \frac{S(\lambda)}{\lambda^2} = 4 \int \frac{\nu(d\xi)}{\|\xi\|^2}, \qquad \text{if } d \geq 3 \tag{8}$$

Remark on Theorem 1.2: As pointed out by the referee, the estimates (7) and (8) can be derived directly without appealing to the large deviation principle for white noise. We refrained from doing so because the calculations would not be significantly shorter. Moreover we believe the large deviation principle given in Theorem 2.1 is of independent interest, Finally, the use of this principle makes our proof more transparent and more in tune with the physicsal intuition presented in [17].

The above result was extended to α - stable fields on the lattice in [19]. A variational principle similar to (6) was derived in [10] in the case of the Wiener sausage with drift.

The main ingredient of the proof of Theorem 1.1 is large deviation techniques. It is not hard to see that the "paths", which contribute most in (2), are paths with a linear drift. These paths are not typical path for the process of Brownian motion. We deal here with events of exponentially small probabilities. That is the reason why large deviation theory plays an inportant role in this problem. Heuristically:

$$\lim_{t\to\infty}\frac{1}{t}\ln M(t)\approx\lim_{t\to\infty}\frac{1}{t}\ln\mathbb{E}\exp[-2\lambda^2\int_0^t(\int_0^T K(B(r+s)-B(r))\,ds)\,dr]$$

when T is large and consequently, large deviation techniques allow us to pin down:

$$\begin{aligned}\lim_{t\to\infty}\frac{1}{t}\ln\mathbb{E}\exp[-2\lambda^2\int_0^t(\int_0^T K(B(r+s)-B(r))\,ds)\,dr]\\ =-\inf_P\left[\mathbb{E}^P 2\lambda^2\int_0^T K(\omega(t))\,dt+H(P|Q_0)\right].\end{aligned}$$

The close relation between long time behavior of certain Wiener functionals (many partition functions of physical systems are of such a type) and large deviation theory is well known. The remarkable works of Donsker and Varadhan (see [7, 9] for example) are the prime reason and our work is should bring another example to demonstrate this relationship.

The rest of paper is organized as follow. In Section 2, we establish the facts from the large deviation theory for white noise which we need. In Section 3, we prove Theorem 1.1. In Section 4, we will derive Theorem 1.2.

For earlier related works in the physics literature we refer to [4, 3] and to the references quoted therein. See also [6] for an excellent survey. The probabilistic analysis of the fine structure of one-dimensional polymer measures can be found in the recent articles [2] and [12]. The interested reader can also consult the references quoted therein for earlier related works.

2 Large deviations for white noise

In this section, we establish a large deviation principle for white noise at the *process level*. We use the notations introduced in the previous section. For any $t>0$ and $\omega(\cdot)\in\mathscr{X}$, we define the new path

$$\omega_t(s)=\begin{cases}\omega(s) & \text{if } 0\le s\le t\\ k\omega(t)+\omega(r) & \text{if } s=kt+r,\ \text{with } r,k\in\mathbb{Z}_+,\ 0\le r<t.\end{cases}$$

It is not hard to see that the *formal* derivative of $\omega_t(s)$ with respect to s is the *periodic extension* of the *formal derivative* of $\omega(s)$ with period t. For any

$A \subset \mathscr{X}$, we define the empirical measure for the increment of the coordinate process:

$$L(A,t,\omega) = \frac{1}{t}\int_0^t \mathbf{1}_A(\omega_t(s+\cdot) - \omega_t(s))\,ds\,, \tag{9}$$

where $\mathbf{1}_A$ denotes the characteristic function, also called the indicator function of the set A. Obviously $L(d\bar{\omega},t,\omega) \in \mathscr{P}_{si}(\mathscr{X})$. Our first main result in this section is the following large deviation principle for $L(\cdot,t,\omega)$ under Q_0.

Theorem 2.1 *The law of $L(d\bar{\omega},t,\omega)$ under Q_0 satisfies the large deviation principle on $\mathscr{P}_{si}(\mathscr{X})$ as $t \to \infty$ with the rate function $H(\cdot|Q_0)$.*

Recall that this means that:

$$\limsup_{t\to\infty} \frac{1}{t} \ln Q_0\{\omega : L(\cdot,t,\omega) \in C\} \le -\inf_{P\in C} H(P|Q_0), \tag{10}$$

for any closed set $C \subset \mathscr{P}_{si}(\mathscr{X})$; and

$$\liminf_{t\to\infty} \frac{1}{t} \ln Q_0\{\omega : L(\cdot,t,\omega) \in O\} \ge -\inf_{P\in O} H(P|Q_0), \tag{11}$$

for all the open set $O \subset \mathscr{P}_{si}(\mathscr{X})$.

Remarks on Theorem 2.1 :

1. In fact, the above large deviation principle is the large deviation principle for the empirical measure of white noise, which *lives* on the Schwartz space of generalized functions. From this point of view, it is the extension of the well known *level 3* large deviation principle for i.i.d random sequence [11]. An attempt in this direction was made earlier in [14] where a weaker (though very often equivalent) form of the large deviation principle (10-11) is formulated. The latter was used in the subsequent paper [15] where asymptotics for one-dimensional polymers were derived. It is quite possible that the part $d = 1$ of (7) could be derived from the estimates of this last work.
2. This type of large deviation principle was originally studied in [8]. Although the above result is not included in the framework of their paper, a slight modification of their argument for the upper bound (especially the exponentially tightness) suffices to show our result. We learned through private discussions with S.R.S. Varadhan that the result was known to him. We include it for the sake of completeness but instead of giving a detailed proof, we merely provide the nesssary modifications to the argument of [8]. These modifications will be organized into several steps.
3. The fact that $H(P|Q_0)$ is a rate function was established in [8]. (i) It is lower semi-continuous in P; (ii) $\{P : H(P|Q_0) \le a\}$ is a compact set for any a. More surprisingly, $H(P|Q_0)$ is *linear* in P (see Section 3 in [8]).

Following lemma, which provides lower bound(11), follows directly from the same argument in Sect. 5 of [9].

Lemma 2.1 *Let $P \in \mathscr{P}_{si}(\mathscr{X})$ be such that $H(P|Q_0) < \infty$ and U be the any neighborhood of P; then*

$$\lim_{t\to\infty} \frac{1}{t} \ln Q_0\{\omega : L(\cdot, t, \omega) \in U\} \geq -H(P|Q_0). \tag{12}$$

We start to give the modification for arguments of lower bound.

Lemma 2.2 *Let F be a continuous function on $\mathscr{X}$ depending on paths up to time T such that $\mathbb{E}^{Q_0} e^F \leq 1$. Then for any $t > 0$:*

$$\mathbb{E}^{Q_0} \exp\left[\frac{1}{T}\int_0^t F(\omega(\cdot + s) - \omega(s))\, ds\right] \leq 1$$

Proof. The method is a slight modication of the argument of Lemma 4.1 in [8]. We define

$$G_s(\omega) = \sum_{\substack{k:k\geq 0\\ s+kT\leq t}} F(\omega(\cdot + s + kT) - \omega(s + kT)).$$

So:

$$\begin{aligned}\mathbb{E}^{Q_0} \exp\left[\frac{1}{T}\int_0^t F(\omega(\cdot + s) - \omega(s))\, ds\right] &= \mathbb{E}^{Q_0} \exp\left[\frac{1}{T}\int_0^T G_s(\omega)\, ds\right]\\ &\leq \frac{1}{T}\int_0^T \mathbb{E}^{Q_0} \exp\{G_s(\omega)\}\, ds.\end{aligned}$$

As an immediate consequence of the independence of the increments of Brownian motion, we have:

$$\mathbb{E}^{Q_0} \exp\{G_s(\omega)\} \leq 1.$$

This completes the proof. □

Lemma 2.3 *Let F be a bounded continuous function on $\mathscr{X}$ depending on paths up to time T such that $\mathbb{E}^{Q_0} e^F \leq 1$. Then for all $t > 0$,*

$$\mathbb{E}^{Q_0} \exp\left[\frac{t}{T}\mathbb{E}^{L(\cdot,t,\omega)} F\right] \leq \exp[2 \sup_{\omega\in\mathscr{X}} |F|]. \tag{13}$$

Proof. Observe that

$$|t\,\mathbb{E}^{L(\cdot,t,\omega)} F - \int_0^t F(\omega(\cdot + s) - \omega(s))\, ds| \leq 2T \sup_{\omega\in\mathscr{X}} |F|.$$

In account of Lemma 2.2, our result is immediate. □

Lemma 2.4 *Let C be a* compact *set in $\mathscr{P}_{si}(\mathscr{X})$. Then*

$$\limsup_{t\to\infty} \frac{1}{t} \ln Q_0\{\omega : L(\cdot, t, \omega) \in C\} \leq -\inf_{P\in C} H(P|Q_0).$$

Proof. Let E_T be the set of all functions F as in Lemma 2.3. From (13), we have

$$Q_0\{\omega : L(\cdot, t, \omega) \in C\} \leq \exp\{2 \sup_{\omega \in \mathscr{X}} |F|\} \exp\{-\frac{t}{T} \inf_{P \in C} \mathbb{E}^P F\},$$

for any $F \in E_T$. Thus

$$\begin{aligned}\limsup_{t\to\infty} \frac{1}{t} \ln Q_0\{\omega : L(\cdot, t, \omega) \in C\} &\leq -\sup_{T>0} \sup_{F\in E_T} \inf_{P\in C} \frac{1}{T} \int_{\mathscr{X}} F(\omega)\, dP \\ &= -\inf_{P\in C} \sup_{T>0} \sup_{F\in E_T} \frac{1}{T} \int_{\mathscr{X}} F(\omega)\, dP \\ &= -\inf_{P\in C} H(P|Q_0).\end{aligned}$$

This completes the proof. □

The following lemma provides the *exponential tightness* of $L(\cdot, t, \omega)$. It is used to extend the lower bound estimate in Lemma 2.4 from compact sets to closed sets.

Lemma 2.5 *If we set:*

$$M(\omega) = \sum_{i=1}^{d} \sup_{0\leq r,s\leq 1} \frac{|\omega_i(r+s) - \omega_i(s)|}{r^{\frac{1}{4}}},$$

where ω_i stands for the i-th component of ω, then:

$$\limsup_{t\to\infty} \frac{1}{t} \ln \mathbb{E}^{Q_0} \exp\left[t\,\mathbb{E}^{L(\cdot,t,\omega)} M\right] < \infty. \tag{14}$$

Proof: The result follows immediately from the combination of the argument in Lemma 2.2 and the following Gaussian estimate:

$$\mathbb{E} \exp\left[\sup_{0\leq s,r\leq 1} \frac{|B(r+s) - B(r)|}{s^{1/4}}\right] < \infty,$$

which is an immediate consequence of Borell's inequality (see for example Theorem 2.1 of [1]). □

Notice that $\{P \in \mathscr{P}_{si}(\mathscr{X});\ \mathbb{E}^P M \leq a\}$ is a compact set for any a. The next lemma is an immediate consequence of combination of the previous lemma and Lemma 2.4.

Lemma 2.6 *Let C be a* closed *set in $\mathscr{P}_{si}(\mathscr{X})$. Then*

$$\limsup_{t\to\infty} \frac{1}{t} \ln Q_0\{\omega : L(\cdot, t, \omega) \in C\} \leq -\inf_{P\in C} H(P|Q_0).$$

Proof of Theorem 2.1. Indeed we have done all necessary steps to prove this theorem. Here we only need to point out that Lemma 2.1 gives us the (10) and Lemma 2.6 provides the (11). □

The following limit is one of the key steps in our proof of the main result of this paper. It is a consequence of the above large deviation principle and the so-called Varadhan's Lemma from the general theory of large deviations (see [20]).

Corollary 2.1 *For any $T > 0$, we have:*

$$\lim_{t\to\infty} \frac{1}{t} \ln \mathbb{E}^{Q_0} \exp\left[-\int_0^t \left(\int_0^T 2\lambda^2 K(\omega(s+r)-\omega(r))\, ds\right)\, dr\right]$$
$$= -\inf_{P\in\mathscr{P}_{si}(\mathscr{X})} \left[\mathbb{E}^P \int_0^T 2\lambda^2 K(\omega(s))\, ds + H(P|Q_0)\right]. \tag{15}$$

Proof. Notice that:

$$\int_0^t (\int_0^T K(\omega(s+r)-\omega(r))\, ds)\, dr = t\,\mathbb{E}^{L(\cdot,t,\omega)} \int_0^T K(\omega(s))\, ds + O(1),$$

where $O(1)$ is order 1 term in t. On the other hand, because of Varadhan's Lemma and Theorem 2.1 we have:

$$\lim_{t\to\infty} \frac{1}{t} \ln \mathbb{E}^{Q_0} \exp\left[-t\,\mathbb{E}^{L(\cdot,t,\omega)} \int_0^T K(\omega(s))\, ds\right]$$
$$= \sup_{P\in\mathscr{P}_{si}(\mathscr{X})} \left[-\mathbb{E}^P \int_0^T 2\lambda^2 K(\omega(s))\, ds - H(P|Q_0)\right]$$
$$= -\inf_{P\in\mathscr{P}_{si}(\mathscr{X})} \left[\mathbb{E}^P \int_0^T 2\lambda^2 K(\omega(s))\, ds + H(P|Q_0)\right].$$

Hence:

$$\lim_{t\to\infty} \frac{1}{t} \ln \mathbb{E}^{Q_0} \exp\left[-\int_0^t \left(\int_0^T 2\lambda^2 K(\omega(s+r)-\omega(r))\, ds\right)\, dr\right]$$
$$= -\inf_{P\in\mathscr{P}_{si}(\mathscr{X})} \left[\mathbb{E}^P \int_0^T 2\lambda^2 K(\omega(s))\, ds + H(P|Q_0)\right].$$

This completes the proof. □

3 Decay rate

In this section, we prove Theorem 1.1. We break the proofs into several steps. We establish the upper bound first.

Lemma 3.1 *Assume that $K(x)$, λ and Q_0 are as in Theorem 1.1. Then:*

$$\limsup_{t\to\infty} \frac{1}{t} \ln M(t) \le -\inf_{P\in\mathscr{P}_{si}(\mathscr{X})} \left[\mathbb{E}^P 2\lambda^2 \int_0^\infty K(\omega(t))\, dt + H(P|Q_0)\right]. \tag{16}$$

Proof. Since $K(x) \geq 0$, then for any fixed T such that $t > T > 0$ we have:

$$
\begin{aligned}
&-\int_0^t \left(\int_0^{t-s} K(\omega(r+s)-\omega(s)\right) \, dr) \, ds \\
&\quad \leq -\int_0^{t-T} (\int_0^T K(\omega(r+s)-\omega(s)) \, dr) \, ds.
\end{aligned}
$$

and if we apply Corollary 2.1 to the right hand side of the above expression, we obtain:

$$
\begin{aligned}
&\limsup_{t\to\infty} \frac{1}{t} \ln M(t) \\
&\leq \limsup_{t\to\infty} \frac{1}{t} \ln \mathbb{E}^{Q_0} \exp\left[-\int_0^{t-T} \left(\int_0^T K(\omega(r+s)-\omega(s)) \, dr\right) \, ds\right] \\
&= -\inf_P \left[\mathbb{E}^P \int_0^T 2\lambda^2 K(\omega(s)) \, ds + H(P|Q_0)\right].
\end{aligned}
$$

Now we send $T \to \infty$ and we get:

$$
\limsup_{t\to\infty} \frac{1}{t} \ln M(t) \leq -\lim_{T\to\infty} \inf_P \left[\mathbb{E}^P \int_0^T 2\lambda^2 K(\omega(s)) \, ds + H(P|Q_0)\right].
$$

It only remains to check that:

$$
\begin{aligned}
&\lim_{T\to\infty} \inf_P \left[\mathbb{E}^P \int_0^T 2\lambda^2 K(\omega(s)) \, ds + H(P|Q_0)\right] \\
&\quad = \inf_P \{\mathbb{E}^P \int_0^\infty 2\lambda^2 K(\omega(s)) \, ds + H(P|Q_0)\}.
\end{aligned}
$$

This will be established in the next lemma. Modulo this, the proof is complete. □

The next lemma provides the properties of the decay rate, which we need.

Lemma 3.2 *For each fixed $\lambda > 0$, there exists a minimizer P_0 for the variational problem (6). It is ergodic and $H(P_0|Q_0) < \infty$. Moreover:*

$$
\lim_{T\to\infty} S^T = S(\lambda)
$$

provided we set:

$$
S^T(\lambda) = \inf_P \left[\mathbb{E}^P \int_0^T 2\lambda^2 K(\omega(s)) \, ds + H(P|Q_0)\right].
$$

Proof. Let us set:

$$
f(P) = \mathbb{E}^P \int_0^\infty 2\lambda^2 K(\omega(s)) \, ds + H(P|Q_0).
$$

Then $f(P) \geq 0$. Next, we consider the upper bound. For each element $\beta \in \mathbb{R}^d$ let us denote by Q_β the distribution of Brownian motion with drift β. In other words Q_β is the law of $B_t + \beta t$ if B_t is a standard Brownian motion. Obviously $Q_\beta \in \mathscr{P}_{si}(\mathscr{X})$ and it is well known that

$$H(Q_\beta|Q_0) = \frac{1}{2}\|\beta\|^2.$$

Using the distribution Q_β as a trial in the definition of $S(\lambda)$ we get:

$$\begin{aligned} S(\lambda) &= \inf_{P\in\mathscr{P}_{si}(\mathscr{X})}\left[2\lambda^2\mathbb{E}^P\int_0^\infty K(\omega(t))\,dt + H(P|Q_0)\right] \\ &\leq 2\lambda^2\mathbb{E}^{Q_\beta}\int_0^\infty K(\omega(t))\,dt + H(Q_\beta|Q_0)\} \\ &= 2\lambda^2\int_{\mathbb{R}^d}\frac{\nu(d\xi)}{\|\xi\|^2/2 + i\xi\cdot\beta} + \frac{1}{2}\|\beta\|^2, \end{aligned} \tag{17}$$

if we use (3) and:

$$\begin{aligned} \mathbb{E}^{Q_\beta}\int_0^\infty K(\omega(t))\,dt &= \int_0^\infty \mathbb{E}\{K(B_t + \beta t)\}dt \\ &= \int_0^\infty\int_{\mathbb{R}^d}\mathbb{E}\{e^{i\xi\cdot(B_t+\beta t)}\}\nu(d\xi)dt \\ &= \int_0^\infty\int_{\mathbb{R}^d}e^{t(\|\xi\|^2/2+i\xi\cdot\beta)}\}\nu(d\xi)dt \\ &= \int_{\mathbb{R}^d}\frac{\nu(d\xi)}{\|\xi\|^2/2 + i\xi\cdot\beta} \end{aligned}$$

which shows, provided we choose $\beta \neq 0$ when $d = 1$ or $d = 2$, that $S(\lambda) < \infty$ in all cases. Let P_n be a minimizing sequence that we select so that $f(P_n) < f(Q_\beta) + 1$. Therefore:

$$H(P_n|Q_0) < f(Q_\beta) + 1,$$

and as a consequence, $\{P_n\}_{n=1,2,\dots}$ is tight. Let P_0 be a limit point. The lower semicontinuity of $H(P|Q_0)$ and standard properties of the weak convergence of probability measures imply that P_0 is a minimizer. Moreover, the linearity of $H(P|Q_0)$ makes it possible to select an extremum which is ergodic. Obviously

$$S^T(\lambda) \leq S(\lambda),$$

for any $T > 0$ and if we let $T \to \infty$, we get:

$$\lim_{T\to\infty} S^T(\lambda) \leq S(\lambda).$$

In order to check that:

$$\lim_{T\to\infty} S^T(\lambda) \geq S(\lambda),$$

we choose for each $T > 0$, $P_T \in \mathscr{P}_{si}(\mathscr{X})$ such that:

$$S^T(\lambda) > \mathbb{E}^{P_T} \int_0^T 2\lambda^2 K(\omega(s))\, ds + H(P_T|Q_0)\} - \frac{1}{T+1}.$$

This implies that:

$$\sup_T H(P_T|Q_0) < \infty,$$

and therefore that the family $\{P_T\}_T$ is tight. Let $\bar{P}$ be a limit point. Then for any $T' > 0$, we have:

$$\lim_{T\to\infty} S^T \geq \mathbb{E}^{P_T} \int_0^{T'} K(\omega(t))\, dt + H(P_T|Q_0) + \frac{1}{T}.$$

Let T blows to ∞, then

$$\lim_{T\to\infty} S^T \geq \mathbb{E}^{\bar{P}} \int_0^{T'} K(\omega(t))\, dt + \limsup_{T\to\infty} H(P_T|Q_0).$$

Since $H(P|Q_0)$ is lower semicontinuous, therefore

$$\liminf_{T\to\infty} H(P_T|Q_0) \geq H(\bar{P}|Q_0).$$

At this point, we send $T' \to \infty$, we obtain

$$\lim_{T\to\infty} S^T \geq f(\bar{P}) \geq S(\lambda).$$

This completes our proof. □

The next lemma establishs the lower bound.

Lemma 3.3 *Assume that $K(x)$, λ and Q_0 are as in Theorem 1.1. Then*

$$\liminf_{t\to\infty} \frac{1}{t} \ln M(t) \geq - \inf_{P\in\mathscr{P}_{si}(\mathscr{X})} \left[2\lambda^2 \mathbb{E}^P \int_0^\infty K(\omega(t))\, dt + H(P|Q_0) \right]. \quad (18)$$

Proof. For any $P \in \mathscr{P}_{si}(\chi)$ such that $H(P|Q_0) < \infty$, using Jensen's inequality we get:

$$\begin{aligned}
&\frac{1}{t} \ln M(t) \\
&\quad = \frac{1}{t} \ln \mathbb{E}^{Q_0} \exp \left[-2\lambda^2 \int_0^t \left(\int_0^{t-s} K(\omega(r+s) - \omega(s))\, dr \right) ds \right] \\
&= \frac{1}{t} \ln \mathbb{E}^P \exp \left[-2\lambda^2 \int_0^t \left(\int_0^{t-s} K(\omega(r+s) - \omega(s))\, dr \right) ds - \log \frac{dP}{dQ_0}|_{\mathscr{F}_t} \right] \\
&= -\frac{2\lambda^2}{t} \mathbb{E}^P \int_0^t \left(\int_0^{t-s} K(\omega(r+s) - \omega(s))\, dr \right) ds - \frac{1}{t} \mathbb{E}^P \log \frac{dP}{dQ_0}|_{\mathscr{F}_t} \\
&\geq -\frac{2\lambda^2}{t} \mathbb{E}^P \int_0^t \left(\int_0^{\infty} K(\omega(r+s) - \omega(s))\, dr \right) ds - \frac{1}{t} \mathbb{E}^P \frac{dP}{dQ_0}|_{\mathscr{F}_t} \\
&= -2\lambda^2 \mathbb{E}^P \int_0^\infty K(\omega(r))\, dr - \frac{1}{t} H(P^t|Q_0^t).
\end{aligned}$$

In the above we used once more the nonnegativity of $K(x)$ and the stationarity of the increments under P. Next we send $t \to \infty$. In account of the fact that the second term in the above right hand side approaches $H(P|Q_0)$, we have:

$$\liminf_{t\to\infty} \frac{1}{t} \ln M(t) \geq -2\lambda^2 \mathbb{E}^P \int_0^\infty K(\omega(r))\, dr - H(P|Q_0).$$

This completes the proof. □

4 Small λ behavior of the rate $S(\lambda)$

This section is devoted to the proof of Theorem 1.2.

We first prove the lower bound. In order to do so we fix $t > 0$ and we apply the entropy inequality:

$$\mathbb{E}^P f \leq \log \mathbb{E}^{Q_0} e^f + H(P|Q_0)$$

to the function:

$$f = -2\lambda^2 t \int_0^t K(\omega(s))\, ds$$

and to the restrictions P^t and Q_0^t of P and Q_0 to the σ-field $\mathscr{F}_t$. We get:

$$-\log \mathbb{E}^{Q_0} e^{-2\lambda^2 t \int_0^t K(\omega(s))\, ds} \leq 2\lambda^2 t \mathbb{E}^P \int_0^t K(\omega(t))\, dt + H(P^t|Q_0^t).$$

Using the superadditivity of the entropy appearing in the right hand side we get (after dividing both sides by t and using once more the nonnegativity of the function K):

$$-\frac{1}{t} \log \mathbb{E}^{Q_0} e^{-2\lambda^2 t \int_0^t K(\omega(s))\, ds} \leq 2\lambda^2 \mathbb{E}^P \int_0^\infty K(\omega(t))\, dt + H(P|Q_0) \qquad (19)$$

which shows that the left hand side is a lower bound for $S(\lambda)$. Such a bound can be derived directly using arguments simular those in Lemma 2.2. But such a derivation would not be very constructive, so we refrained from using it. We now consider separately the various cases. If $d = 1$ we choose $t = \lambda^{-4/3}$ and the lower bound becomes:

$$S(\lambda) \geq -\lambda^{4/3} \log \mathbb{E}^{Q_0} e^{-2\lambda^{2/3} \int_0^{\lambda^{-4/3}} K(\omega(s))\, ds}$$

and the Kallianpur-Robbins law for Brownian motion (see for example [13] p.229) implies that the right hand side is equivalent to $c\lambda^{4/3}$ for some $c > 0$. The integrability of the covariance function K was also used. This completes the proof of the lower bound in one dimension. The Kallianpur-Robbins law can also be used in the case $d = 2$ provided we use the substitution $t = a/\lambda^2 \log 1/\lambda$ for some $a > 0$ to be chosen later. The lower bound (19) becomes:

$$S(\lambda) \geq -\frac{\lambda^2 \log 1/\lambda}{a} \log \mathbb{E}^{Q_0} e^{-(2a/\log 1/\lambda) \int_0^{a/\lambda^2 \log 1/\lambda} K(\omega(s))\, ds}$$

and consequently:

$$\frac{S(\lambda)}{\lambda^2 \log 1/\lambda} \geq -\frac{1}{a} \log \mathbb{E}^{Q_0} \exp\left[-\frac{2a(\int K(x)dx)}{\pi} e_\lambda\right]$$

where, according to the Kallianpur-Robbins law, the random variable e_λ converges in distribution to a exponential distribution with parameter 1. Consequently:

$$\liminf_{\lambda \searrow 0} \frac{S(\lambda)}{\lambda^2 \log 1/\lambda} \geq -\frac{1}{a} \log \frac{1}{1 + \frac{2a}{\pi} \int K(x)dx}$$

and letting the parameter a go to zero we get:

$$\liminf_{\lambda \searrow 0} \frac{S(\lambda)}{\lambda^2 \log 1/\lambda} \geq \frac{2}{\pi} \int K(x)dx.$$

We now consider the case $d \geq 3$. Assumption (4) implies that the random variable $\int_0^\infty K(\omega(s))ds$ exists and is finite Q_0 almost surely. Let us fix $\delta > 0$ and $T > 0$ momentarily. Substituting $t = \delta\lambda^{-2}$ in the lower bound (19) gives:

$$S(\lambda) \geq -\frac{\lambda^2}{\delta} \log \mathbb{E}^{Q_0} e^{-2\delta \int_0^{\delta\lambda^{-2}} K(\omega(s)\, ds}$$

and consequently (by restricting λ so that $\delta\lambda^{-2} > T$):

$$\liminf_{\lambda \searrow 0} \frac{S(\lambda)}{\lambda^2} \geq -\frac{1}{\delta} \log \mathbb{E}^{Q_0} e^{-2\delta \int_0^T K(\omega(s)\, ds}.$$

Letting $\delta \searrow 0$ in the right hand side we get (see for example Exercise 5.b if Chapter 3 of [18]):

$$\liminf_{\lambda \searrow 0} \frac{S(\lambda)}{\lambda^2} \geq 2\mathbb{E}^{Q_0} \int_0^T K(\omega(s)\, ds$$

and we conclude the proof of the lower bound by letting $T \nearrow \infty$.

We first consider the case $d = 1$ and we use the upper bound (17) with $\beta \neq 0$. We get:

$$S(\lambda) \leq \lambda^2 \int_{-\infty}^{+\infty} \frac{\nu(d\xi)}{\xi^2/4 + \beta^2} + \frac{1}{2}\beta^2.$$

Using the substitution $\beta = \lambda^{2/3}$, the right hand side is easily shown to be controlled by the quantity:

$$2\lambda^{4/3}\left[\pi \int K(x)dx + \frac{1}{2}\right]$$

because of the de la Vallée Poussin theorem which says that:

$$\lim_{\epsilon \searrow 0} \epsilon \int \frac{\nu(d\xi)}{\xi^2 + \epsilon^2} = \pi \int K(x)\, dx.$$

See for example [5] Chapter I. This concludes the proof of the small λ asymptotic in the one dimensional case. Next we consider the case $d = 2$. Choosing β of the form $(r\cos\theta, r\sin\theta)$ in (17) and integrating out θ over the unit circle we obtain the upper bound:

$$S(\lambda) \leq 4\lambda^2 \int \frac{\nu(d\xi)}{\|\xi\|^2\sqrt{1+(2r/\|\xi\|)^2}} + \frac{1}{2}r^2.$$

Denoting by $n(\xi)$ density of the radial marginal of $\nu(d\xi)$ we get:

$$S(\lambda) \leq 4\lambda^2 \int_0^\infty \frac{n(\xi)d\xi}{\sqrt{\xi^2+4r^2}} + \frac{1}{2}r^2.$$

which proves that:

$$S(\lambda) \leq 8\pi n(0)\lambda^2 \log\frac{1}{\lambda}$$

if we make the substitution $r = \lambda$ in the above bound. This completes the proof of the upper bound and of the expected result in the case $d = 2$ because:

$$n(0) = \frac{1}{(2\pi)^2}\int K(x)\,dx\,.$$

Finally we consider the case $d \geq 3$. The upper bound is obtained from (17) by choosing $\beta = 0$ and using assumption (4). As in the case $d = 2$ the upper bound so obtained is equal to the lower bound.

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References

1. R. J. Adler (1990): An Introduction to Continuity, Extrema, and Related Topics for General Gaussian Processes. IMS Lect. Notes # 12. Hayward.
2. E. Bolthausen (1990): On self-repellent one dimensional random walks. Probab. Theory Relat. Fields **86**, 423–441
3. R.M. Cotts, R.C. Wayne (1966): Nuclear-Magnetic-Resonance Study of Self-Diffusion in a Bounded Medium, Physical Review **151**, 264–272
4. R. Czech, K.W. Kehr (1984): Spin Depolarization for One-Dimensional Random Walk, Physical Review Letters **53**, 1783–1786.
5. R. Carmona, J. Lacroix (1990): Spectral Theory of Random Schrödinger Operators. Birkhaüser, Boston.
6. P.L. Doussal (1992): Diffusion in Layered Random Flows, Polymers, Electrons in Random Potentials, and Spin Depolarization in Random Fields, (preprint).
7. M.D. Donsker, S.R.S. Varadhan (1975): Asympototics for the Wiener Sausage, Comm. Pure & App. Math. **28**, 525–565.
8. M.D. Donsker, S.R.S. Varadhan (19783): Asympototic Evaluation of Certain Markov Process Expectations for Large Time, IV Comm. Pure and Appl. Math. **36**, 183–212.
9. M.D. Donsker, S.R.S. Varadhan (1983): Asympototics for the Polaron, Comm. Pure & App. Math. **36**, 505–528

10. T. Eisele, R. Lang (1987): Asymptotics for the Wiener Sausage with Drift. Probab. Theory Relat. Fields **74**, 125–140.
11. R.S. Ellis (1985): Large deviations and statistical mechanics. Grundleheren der Mathematischen Wissenschaften vol.271. Springer Verlag, New York N.Y.
12. R. van der Hofstad, F. den Hollander (1995): Scaling for a Random Polymer. Commun. Math. Phys. **169**, 397–440.
13. K. Itô, H.P. McKean, Jr. (1974): Diffusion Processes and their Sample Paths. Springer Verlag, New York, N.Y.
14. S. Kusuoka (1983): The variational principle for stationary Gaussian Markov fields. in Theory and Applications of Random Fields, Proc. IFIP-WG 7/[, Bengalore, 1982, ed. G. Kallianpur. Lect. Notes in Control and Information **49**, 179–187.
15. S. Kusuoka (1984): Asymptotics of polymer measures in one dimension. in Proc. Bielefeld Meeting 'Stochastic Processes and Infinite Dimensional Analysis'
16. P. March, A.S. Sznitman (1987): Some connections between excursion theory and the discrete Schrödinger equation with random potentials. Probab. Theory Relat. Fields **75**, 11–53.
17. P.P. Mitra, P.L. Doussal (1991): Long-Time Magnetization Relaxation of Spins Diffusing in a Random Field, Physical Review B **44** , 1203–12038.
18. W. Rudin (1974): Real and Complex Analysis. 2nd Ed. Mc Graw-Hill, New York, N.Y.
19. S. Sethuraman, L. Xu (1995): The Problem of Spin Depolarization in a Symmetric α – Stable Fields on the Lattice. (preprint)
20. S.R.S. Varadhan (1984): Large deviations and applications, CBMS-NSF Regional Conference Series in Applied Mathematics, Vol. 46, SIAM.

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