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Exponential estimate for reaction diffusion models

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Summary. We consider the superposition of a speeded up symmetric simple exclusion process with a Glauber dynamics, which leads to a reaction diffusion equation. Using a method introduced in [Y] based on the study of the time evolution of the H_{-1} norm, we prove that the mean density of particles on microscopic boxes of size N^{α}, for any $12/13 < \alpha < 1$, converges to the solution of the hydrodynamic equation for times up to exponential order in N, provided the initial state is in the basin of attraction of some stable equilibrium of the reaction–diffusion equation. From this result we obtain a lower bound for the escape time of a domain in the basin of attraction of the stable equilibrium point.

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Introduction

The major problem in the theory of hydrodynamic limit of interacting particle systems consists in describing the macroscopic time evolution of a gas from the microscopic interaction between molecules. Although physically well understood, this passage from the microscopic dynamics to macroscopic behaviour still presents in the general case some difficult mathematical problems.

The interacting particle systems introduced by Spitzer constitute a class of stochastic models with one macroscopic variable, the density, complex enough, one the one hand, to present interesting macroscopic behaviour and relatively simple, on the other hand, to allow rigorous mathematical proofs.

Until the breakthrough of Guo, Papanicolaou and Varadhan [GPV], where the intensive use of large deviation techniques led to a robust proof of the hydrodynamic behaviour of a large class of gradient systems with one

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conserved quantity, most methods to derive the hydrodynamic limit relied on specific properties of each model.

Investigating the time evolution of the entropy of the state of the process with respect to some reference equilibrium measure, Guo, Papanicolaou and Varadhan proved a law of large numbers for the empirical measure, the measure obtained assigning mass N^{-d} to each particle, where parameter N^{-1} represents the interdistance between particles and d the dimension. In other words, they proved that the density of particles in a small macroscopic neighborhood of a space point a at time t converges in probability to $\rho(t,a)$, where ρ is the solution of a differential equation, the so called hydrodynamic equation.

In this article we study a class of reaction diffusion models introduced by De Masi, Ferrari and Lebowitz in [DFL] and obtained by superposition of a speeded up stirring process and a Glauber dynamics in infinite volume, the lattice $\mathbb{Z}^d$. The authors proved, for this class of systems, propagation of chaos, a strong version of the hydrodynamic behaviour. They showed that under diffusive rescaling the time evolution of the density is described by solutions of reaction–diffusion equations of the type

$$\partial_t \rho = \frac{1}{2}\Delta\rho + G(\rho)\ . \tag{0.1}$$

Later, [JLV], following the ideas of [DV] and [KOV], proved large deviations from the hydrodynamic limit in infinite volume.

More recently, Yau in [Y], studying the time evolution of the H_{-1} norm of continuous spin systems associated to reaction–diffusion equations in finite volume, proved another strong version of the hydrodynamic limit. He showed that the mean density of particles on microscopic boxes of length N^{α}, for some $0 < \alpha < 1$, around a at time t converges in probability to $\rho(t,a)$ provided G is a one well potential. This statement being correct for exponentially, with respect to the parameter N, large time t.

In this article, exploiting the technique introduced by Yau, we prove an exponential estimate for the H_{-1} norm of the process in infinite volume. This is the content of the first main theorem.

On the contrary to large deviations estimates obtained in [JLV] the present method allows to go beyond the hydrodynamic scale provided the initial data is in the basin of attraction of some stable equilibrium of the hydrodynamic equation. It provides also information about the empirical measure in a more refined scale than the hydrodynamic one.

More precisely, as an application of the basic exponential estimate stated in Theorem 1.1, if the initial state is, in a sense to be specified later, in the basin of attraction of some stable equilibrium m^* of the ordinary differential equation $m'(t) = G(m(t))$, then the empirical density of particles in boxes of microscopic size of order N^{α}, for any $12/13 < \alpha < 1$, around a macroscopic point a at time t converges in probability to $\rho(t,a)$, if ρ denotes the solution of the hydrodynamic equation (0.1) and t is of order at most exponential. It should be stressed that we do not require m^* to be the unique stable equilibrium of equation (0.1). We do not reach α close to 0 because the hydrodynamic equation is not scale invariant.

From this estimate we get a lower bound for the escape time from a domain in the basin of attraction of a stable equilibrium m^*. This lower bound is exponential but is certainly not optimal.

Several articles have been devoted to the understanding of long time behaviour of interacting particle systems associated to reaction–diffusion equations when propagation of chaos is broken. Several situations have also been examined concerning the escape from certain unstable equilibrium points of equation (0.1), which requires a logarithmical or polynomial time (in the scaling parameter). Such analysis relies on estimates of truncated correlation functions which, in the case of a superposition of a speeded up stirring process and a Glauber dynamics, were developed exploiting the self duality of the stirring process. (cf. [CPPV, DP, DPPV, G] and references therein). In [DOPT] such an analysis is carried out for Glauber dynamics in Kac model and the authors also study the empirical density in boxes of intermediate space scale.

1 Notation and results

We consider a family of Markov processes on $\mathscr{X} = \{0,1\}^{\mathbb{Z}}$, whose generator is $L_N = N^2 L_0 + L_G$ given by

$$(L_0 f)(\eta) = \frac{1}{2}\sum_x (f(\eta^{x,x+1}) - f(\eta)) ,$$

$$(L_G f)(\eta) = \sum_x r(x,\eta)(f(\eta^x) - f(\eta)) .$$

In this formula, for an integer x and a configuration η, $\eta^{x,x+1}$ and η^x stand, respectively, for the configuration obtained from η by interchanging the occupation variables $\eta(x)$ and $\eta(x+1)$ and by flipping $\eta(x)$:

$$\eta^{x,x+1}(z) = \begin{cases} \eta(z) & \text{if } z \neq x, y \\ \eta(x+1) & \text{if } z = x \\ \eta(x) & \text{if } z = x+1 \end{cases}$$

$$\eta^x(z) = \begin{cases} \eta(z) & \text{if } z \neq x \\ 1-\eta(z) & \text{if } z = x \end{cases}$$

and $r(x,\eta) = r(\tau_x \eta)$ for some positive cylinder function r. Here $\tau_x \eta$ represents the translation of η by x: $\tau_x \eta(y) = \eta(x+y)$ for all $y \in \mathbb{Z}$.

For a configuration η, P_η^N will denote the law on the path space $D([0,\infty),\mathscr{X})$ of the Markov process with generator L_N when it starts from η and E_η^N the expectation with respect to P_η^N. Sometimes we will omit superscript N and use the same symbol to denote the restriction of the measure to some $D([0,t_0],\mathscr{X})$. Moreover, if v is a measure on $\mathscr{X}$ then P_v^N denotes the measure corresponding to the process starting from v: $P_v^N(\,\cdot\,) = \int P_\eta^N(\,\cdot\,) v(d\eta)$.

For $0 \leqq \rho \leqq 1$, let v_ρ be the translation invariant product measure on $\mathscr{X}$ with marginals given by

$$v_\rho\{\eta, \eta(x) = 1\} = \rho .$$

It is well known (cf. [DFL, DP]; see also [JLV] for another approach of the "kinetic limit" based on the behaviour of empirical measures) that in the limit $N \uparrow \infty$ there is propagation of chaos:

$$\lim_{N\to\infty} \sup_{x_1,\ldots,x_k} \left| E_\eta^N \left[\prod_{i=1}^k \eta_t(x_i) \right] - \prod_{i=1}^k E_\eta^N[\eta_t(x_i)] \right| = 0$$

for any configuration η. In this formula the supremum is taken over all distinct sites $x_1, \ldots, x_k$. Moreover

$$\lim_{N\to\infty} |E_\eta^N[\eta_t(x_i)] - u_t^N(x_i)| = 0$$

where $u = u^N$ is the solution to the discretized version of the hydrodynamic equation (0.1):

$$\begin{cases} \partial_t u_t^N(x) = \frac{N^2}{2}(\Delta u_t^N)(x) + G(u_t^N(x)) \\ u_0^N(x) = \eta(x) \end{cases} \tag{1.1}$$

for $x \in \mathbb{Z}$ and $t \geqq 0$. Here Δ stands for the discrete Laplacian:

$$(\Delta f)(x) = f(x+1) + f(x-1) - 2f(x), \quad x \in \mathbb{Z}\ ,$$

and $G(\rho)$ for the algebraic expected value, under the product measure ν_ρ, of the rate at which particles are flipped:

$$G(\rho) = \nu_\rho[(1 - 2\eta(0))r(\eta)]\ .$$

It follows from this result that if $(\mu^N)_{N \geqq 1}$ is a sequence of product measures associated to some smooth profile $\rho : \mathbb{R} \to \mathbb{R}_+ (\mu^N[\eta(x)] = \rho(x/N)$ for $x \in \mathbb{Z})$, then

$$\lim_{N\to\infty} \sup_{x,\ldots,x_k} \left| E_\mu^N \left(\prod^k \eta_t(x_i) \right) - \prod^k \rho(t, x_i/N) \right| = 0\ ,$$

where ρ is the solution of equation (0.1) and the supremum is taken over all distinct sites $x_1, \ldots, x_k$.

These results state that η_t should be close to the solution u_t^N of the differential equation (1.1). Our main theorem gives an exponential estimate for the H_{-1} norm of the difference $\eta_t - u_t$. To state this theorem we need to introduce some notation. For a cylinder function ψ we denote by $\widetilde{\psi}(\rho)$ the expected value of ψ under ν_ρ:

$$\widetilde{\psi}(\rho) = \nu_\rho[\psi]\ .$$

Denote by $\ell^2(\mathbb{Z})$ the space of functions $f : \mathbb{Z} \to \mathbb{R}$ with summable square. For two functions f and g in $\ell^2(\mathbb{Z})$, $f * g$ represents the convolution of f with g. Denote by K_N the kernel associated to the operator $(I - N^2\Delta)^{-1}$ in $\ell^2(\mathbb{Z})$ in the sense that $(I - N^2\Delta)^{-1} f = K_N * f$ for all functions f in $\ell^2(\mathbb{Z})$. The exact expression and basic properties of the kernel K_N are given in the Appendix for the sake of completeness, even if this is a simple calculation with Fourier transforms.

Fix a smooth function $\theta : \mathbb{R} \to \mathbb{R}_+$ that coincides with the absolute value outside the interval $[-1, 1]$: $\theta(a) = |a|$ for $|a| \geqq 1$ and denote by $K_{N,\theta}(x, y)$ the kernel defined by

$$K_{N,\theta}(x, y) = \exp\{-\theta(x/N)\} K_N(x, y) \exp\{-\theta(y/N)\} .$$

For a bounded function $f : \mathbb{Z} \to \mathbb{R}$, we represent by $\|f\|_{-1}$ the H_{-1} norm of f defined by

$$\|f\|_{-1}^2 = \langle K_{N,\theta} f, f \rangle ,$$

where $\langle f, g \rangle$ is the inner product on $\ell_2(\mathbb{Z})$ given by $\langle f, g \rangle = N^{-1} \sum_x f(x)g(x)$.

The two main results of this article may be stated as follows.

Theorem 1.1 *There exists an universal constant* C_0 *and* $\sigma = \sigma(\theta, G)$ *such that for all* $\delta < 2/13$ *we have*:

$$\sup_\eta E_\eta \left[\sup_{s \leqq t} \exp\{N^\delta \|\eta_s - u_s^N\|_{-1}^2 e^{-\sigma s}\} \right] \leqq C_0 e^{C_0 t}$$

for any $t \geqq 0$, *where* u^N *is the solution of equation* (1.1).

In light of this result we study the behaviour of the process η_t in the basin of attraction of a stable equilibrium point m^* of the ordinary differential equation $m'(t) = G(m(t))$. Starting from a configuration in this region, we prove a strong version of the hydrodynamic behaviour of η_t in the following sense. We show that the empirical density on boxes of size N^α, for some $0 < \alpha < 1$ and for a time interval $[0, \exp\{N^\gamma\}]$, $\gamma > 0$, converge to the solution $\rho(t, a)$ of the hydrodynamic equation (0.1).

For a positive integer ℓ, denote by $\eta^\ell(x)$ the mean density of particles in a box of length ℓ around x:

$$\eta^\ell(x) = (2\ell + 1)^{-1} \sum_{|y-x| \leqq \ell} \eta(y) .$$

Denote by m^* a stable equilibrium of the ordinary differential equation $m'(t) = G(m(t)) : G(m^*) = 0$ and $G'(m^*) < 0$. Let $d > 0$ so that $[m^* - d, m^* + d]$ is contained in the basin of attraction of m^* in the following sense: $\sup_{|m-m^*| \leqq d} G'(m) < 0$. In particular, there exists a constant $E_0(d, G)$ so that for every m in $[m^* - d, m^* + d]$,

$$|m(t) - m^*| \leqq |m - m^*| e^{-E_0 t} \tag{1.2}$$

if $m(t)$ is the solution of $m'(t) = G(m(t))$ with initial data equal to m.

Theorem 1.2 *Let* δ *be given by Theorem* 1.1 *and fix* $1 - (\delta/2) < \alpha < 1$ *and denote* N^α *by* M. *Let* $\rho_0 : \mathbb{R} \to [0, 1]$ *be a smooth profile in the basin of attraction of* m^*:

$$\sup_{u \in \mathbb{R}} |\rho_0(u) - m^*| < d .$$

Assume that $\eta_{(N)}$ *is a sequence of configurations on* $\mathcal{X}$ *associated to* ρ_0:

$$\lim_{N \to \infty} \sup_{x \in \mathbb{Z}} |\eta_{(N)}^M(x) - \rho_0(x/N)| = 0 .$$

Let $\delta > 0$ be given by Theorem 1.1. *Then, for all $\varepsilon > 0$, and all $\gamma < \delta/2 = 1/13$,*

$$\lim_{N\to\infty} P_{\eta(N)}\left[\sup_{t\leqq e^{N^\gamma}} \sup_{|x|\leqq e^{N^\gamma}} |\eta_t^M(x) - \rho(t,x/N)| > \varepsilon\right] = 0 .$$

The condition on the sequence of initial data can be relaxed. We need only the sequence to be close to the profile ρ_0 in an exponential box of size $C\exp\{N^\gamma\}$ for some finite constant C:

$$\lim_{N\to\infty} \sup_{|x|\leqq Ce^{N^\gamma}} |\eta_{(N)}^M(x) - \rho_0(x/N)| = 0 .$$

It follows from Theorem 1.2 that the time needed to leave the domain of attraction of a stable equilibrium point m^* is at least of order $\exp\{N^\gamma\}$ for $\gamma < 1/13$.

This article is divided as follows. In Sect. 2 we reduce the proof of Theorem 1.1 to the proof of a N^α-block estimate, for some $0 < \alpha < 1$, in the now classical terminology of entropy methods for derivation of hydrodynamic equations of interacting particle systems. In Sect. 3 we prove the N^α-block estimate. In Sect. 4 we prove Theorem 1.2. The last two sections contain some technical results used in the article.

2 An exponential estimate

The proof of Theorem 1.1 will be based on the analysis of some exponential martingales which we now consider and the "N^α-block estimate" proven in the next section.

For each $z \in \mathbb{Z}$, denote by $J_t^{z,z+1}$ the number of times in $[0,t]$ that the occupation variables $\eta(z)$ and $\eta(z+1)$ are interchanged and by J_t^z the number of times that the variable $\eta(z)$ is flipped:

$$J_t^{z,z+1} := \#\{s \leqq t:\ \eta_s = \eta_{s-}^{z,z+1},\ \eta_s \neq \eta_{s-}\}$$
$$J_t^z := \#\{s \leqq t:\ \eta_s = \eta_{s-}^z\} .$$

Recall from the statement of the theorem the definition of σ and denote by $F(\eta,t)$ the corrected H_{-1} norm of the difference $\eta_t - u_t$:

$$F(\eta,t) := e^{-\sigma t}\|\eta_t - u_t\|_{-1}^2 .$$

$F(\eta_t,t)$ can be expressed in terms of the generalized Poisson processes $\{J_t^{z,z+1};\ z \in \mathbb{Z}\}$ and $\{J_t^z;\ z \in \mathbb{Z}\}$:

$$F(\eta_t,t) - F(\eta_0,0) - \int_0^t \partial_s F(\eta_s,s)\,ds$$
$$= \sum_z \int_0^t (F(\eta_s^{z,z+1},s) - F(\eta_s,s))\,dJ_s^{z,z+1} + \sum_z \int_0^t (F(\eta_s^z,s) - F(\eta_s,s))\,dJ_s^z .$$

Recall that $\tilde{J}_t^{z,z+1} = J_t^{z,z+1} - (1/2)N^2\int_0^t ds\,[\eta_s(z) - \eta_s(z+1)]^2$ and $\tilde{J}_t^z = J_t^z - \int_0^t r(z,\eta_s)\,ds$ are mean zero martingales (cf. [KL]). Since they have no common jumps, they are orthogonal. In particular, for any $\gamma \in \mathbb{R}$, the process $M^\gamma(t)$ defined by:

$$M^\gamma(t) = \exp\left\{\gamma F(\eta_t,t) - \gamma F(\eta_0,0) - \gamma\int_0^t \partial_s F(\eta_s,s)\,ds\right.$$
$$- \frac{N^2}{2}\sum_z\int_0^t (e^{\gamma\nabla_{z,z+1}F(\eta_s,s)} - 1)[\eta_s(z) - \eta_s(z+1)]^2\,ds$$
$$\left.- \sum_z\int_0^t r(z,\eta_s)(e^{\gamma\nabla_z F(\eta_s,s)} - 1)\,ds\right\}$$

is a mean 1 positive local martingale. Here we used the notation

$$\nabla_{z,z+1}F(\eta_s,s) = F(\eta_s^{z,z+1},s) - F(\eta_s,s)\,,$$
$$\nabla_z F(\eta_s,s) = F(\eta_s^z,s) - F(\eta_s,s)\,.$$

Notice that $\nabla_{z,z+1}F(\eta_s,s)$ vanishes if $\eta(z) = \eta(z+1)$. In particular, we may remove $[\eta_s(z) - \eta_s(z+1)]^2$ in the second time integral in the definition of $M^\gamma(t)$. On the other hand, the H_{-1} norm of the difference $\eta_t - u_t$ vanishes at time $t = 0$ by definition of u_t: $F(\eta_0,0) = 0$. Thus, from the definition of the martingale $M^\gamma(t)$ and computing the time derivative of the H_{-1} norm, we obtain that

$$\exp\gamma\left\{F(\eta_t,t) + \int_0^t \sigma F(\eta_s,s)\,ds\right\}$$
$$= M^\gamma(t)\,\exp\left\{\int_0^t \gamma e^{-\sigma s}\,\partial_s\|\eta - u_s\|_{-1}^2\Big|_{\eta=\eta_s}\,ds\right\}$$
$$\times\exp\sum_z\int_0^t ds\left\{\frac{N^2}{2}(e^{\gamma\nabla_{z,z+1}F(\eta_s,s)} - 1) + r(z,\eta_s)(e^{\gamma\nabla_z F(\eta_s,s)} - 1)\right\}\,. \tag{2.1}$$

Set, once for all, γ to be equal to N^δ for some $0 < \delta$. To prove a slightly weaker version of the theorem, one that does not include a supremum over time, we just need to show that the right hand side of the last expression has bounded expected value. The following lemma is the first step in this direction. Its proof is an easy consequence of the explicit form of the kernel $K_N(\,\cdot\,)$ associated to the H_{-1} norm. For this reason we postponed it to Sect. 5 at the end of this article.

Lemma 2.1 *There exists an universal constant B_0 so that*

$$|\nabla_{z,z+1}F(\eta,s)| \leqq \frac{B_0}{N^2}e^{-\theta(z/N)} \quad and \quad |\nabla_z F(\eta,s)| \leqq \frac{B_0}{N}e^{-\theta(z/N)}\,.$$

From the elementary inequality $|e^x - 1 - x| \leqq \frac{x^2}{2} e^{|x|}$ and Lemma 2.1 we get that the third line of (2.1) is bounded above by

$$\exp\{C_1(r,\theta)N^{2\delta-1}\, t\}$$
$$\times \exp N^\delta \left\{ \frac{N^2}{2} \sum_z \int_0^t \nabla_{z,z+1} F(\eta_s, s)\, ds + \sum_z \int_0^t r(z, \eta_s) \nabla_z F(\eta_s, s)\, ds \right\}$$

for some constant $C_1(r,\theta)$ that depends only on $\|r\|_\infty$ and θ. We are now left to bound the expected value of the product of the first term on the right hand side of (2.1) with this last exponential. A simple computation, presented in Sect. 5 (cf. Lemma 5.1) for the sake of completeness, shows that

$$\partial_s \|\eta_s - u_s\|_{-1}^2 + (1/2)N^2 \sum_z \nabla_{z,z+1} \|\eta_s - u_s\|_{-1}^2 + \sum_z \tau_z r(\eta_s) \nabla_z \|\eta_s - u_s\|_{-1}^2$$
$$= N^{-2} \sum_{x,y} [\eta_s(x) - u_s(x)] K_{N,\theta}(x,y) [N^2 \Delta[\eta_s(\,\cdot\,) - u_s(\,\cdot\,)]](y)$$
$$+ 2N^{-2} \sum_{x,y} [\eta_s(x) - u_s(x)] K_{N,\theta}(x-y) [\tau_y r_0(\eta_s) - G(u_s(y))]$$
$$+ (1/2)N^{-1} \sum_x (\eta_s(x+1) - \eta_s(x))^2$$
$$\times N[K_{N,\theta}(x+1, x+1) + K_{N,\theta}(x,x) - 2K_{N,\theta}(x, x+1)]$$
$$+ N^{-2} \sum_x K_{N,\theta}(x,x) \tau_x r(\eta)\,. \tag{2.2}$$

In Lemma 5.2 we shall prove that this expression is bounded above by

$$C(r,\theta)\frac{\sqrt{\ell}}{\sqrt{N}} + \sigma(r,\theta)\|\eta_s - u_s\|_{-1}^2 + C(r)N^{-1} \sum_x |\tau_x V_{r_0,\ell}(\eta_s)| e^{-2\theta(x/N)}$$
$$+ N^{-1} \sum_x |\tau_x V_{W,\ell}(\eta_s)| e^{-2\theta(x/N)} + C(\theta)\frac{\ell}{N} N^{-1} \sum_x |(N\nabla u)(x)| e^{-2\theta(x/N)} \tag{2.3}$$

for every positive ℓ and some finite constant $\sigma(r,\theta)$. In this formula, $r_0(\eta)$ stands for the cylinder function $[1 - 2\eta(0)]r(\eta)$ and $W(\eta)$ for $\eta(0)\eta(1)$. Moreover, for a cylinder function ψ and a positive integer ℓ, we denote by $V_{\psi,\ell}$ the corrected average of ψ on a box of length ℓ around the origin:

$$V_{\psi,\ell}(\eta) = (2\ell' + 1)^{-1} \sum_{|x| \leqq \ell'} \{\tau_x \psi(\eta) - \tilde{\psi}(\eta^\ell(x))\}\,. \tag{2.4}$$

Here $\ell' = \ell(\psi) = \ell - s(\psi)$ where $s(\psi)$ is defined as the support of the cylinder function ψ: $s(\psi) = \min\{n \in \mathbb{N};\ \psi$ is a function of $\{\eta(-n), \ldots, \eta(n)\}\}$. Notice that ℓ' is defined so that $V_{\psi,\ell}$ depends on the configuration η only through $\eta(x)$ for x in $\{-\ell, \ldots, \ell\}$. We shall let ℓ depend on N later on.

Thus, choosing σ appropriately, up to this point we proved that $\exp\{\gamma F(\eta_t, t)\}$ is bounded above by

$$
\begin{aligned}
&\exp\{C(r,\theta)[N^{2\delta-1} + \ell^{1/2} N^{\delta-(1/2)}]\, t\} M^{\gamma}(t) \\
&\times \exp N^{\delta} \int_0^t \left\{ C(r) N^{-1} \sum_x |\tau_x V_{r_0,\ell}(\eta_s)| e^{-2\theta(x/N)} \right. \\
&\left. + N^{-1} \sum_x |\tau_x V_{W,\ell}(\eta_s)| e^{-2\theta(x/N)} \right\} ds \\
&\times \exp N^{\delta} \left\{ \int_0^t \frac{\ell}{N} C(\theta) N^{-1} \sum_x |(N \nabla u_s)(x)| e^{-2\theta(x/N)}\, ds \right\}
\end{aligned}
$$

for some finite constants $C(r)$, $C(\theta)$ and $C(r,\theta)$.

We shall prove in Lemma (6.2) of the Appendix that the time integral of the L_1 norm of the discrete derivative of u is bounded. More precisely that there exists a constant $B_2 = B_2(G)$ such that

$$
\int_0^t N^{-1} \sum_x |(N\nabla u_s)(x)| e^{-2\theta(x/N)}\, ds \leqq B_2(1+t)\, .
$$

From this bound and Schwarz inequality we obtain that

$$
\begin{aligned}
&E_\eta \left[\sup_{t \leqq t_0} \exp\{\gamma F(\eta_t, t)\} \right] \\
&\leqq \exp\{C(r,\theta)[N^{2\delta-1} + \ell^{1/2} N^{\delta-(1/2)}]\, t_0\} \times E_\eta^{1/2} \left[\sup_{t \leqq t_0} (M^{\gamma}(t))^2 \right] \\
&\times E_\eta^{1/4} \left[\exp 4N^{\delta} C(r) \int_0^{t_0} N^{-1} \sum_x |\tau_x V_{r_0,\ell}(\eta_s) e^{-2\theta(x/N)}|\, ds \right] \\
&\times E_\eta^{1/4} \left[\exp 4N^{\delta} \int_0^{t_0} N^{-1} \sum_x |\tau_x V_{W,\ell}(\eta_s)| e^{-2\theta(x/N)}\, ds \right] .
\end{aligned}
$$

Notice that $(M^{\gamma}(t))^2$ is equal to

$$
\begin{aligned}
&M^{2\gamma}(t) \exp\left\{ \frac{N^2}{2} \sum_z \int_0^t [(e^{2\gamma \nabla_{z,z+1} F(\eta_s,s)} - 1) - 2(e^{\gamma \nabla_{z,z+1} F(\eta_s,s)} - 1)]\, ds \right\} \\
&\times \exp\left\{ \sum_z \int_0^t r(z,\eta_s) [(e^{2\gamma \nabla_z F(\eta_s,s)} - 1) - 2(e^{\gamma \nabla_z F(\eta_s,s)} - 1)]\, ds \right\} .
\end{aligned}
$$

Here we used again that $\nabla_{z,z+1} F(\eta_s, s)$ vanishes if $\eta_s(z) = \eta_s(z+1)$. Using the elementary inequality $|(e^{2x} - 1) - 2(e^x - 1)| \leqq 3x^2 e^{|2x|}$, from Lemma 2.1

we get that $(M^{\gamma}(t))^2$ is bounded above by

$$M^{2\gamma}(t)\exp\left[C(r)\left\{N^3\left(\frac{B_0N^{\delta}}{N^2}\right)^2 e^{2B_0N^{\delta-2}}\,t+N\left(\frac{B_0N^{\delta}}{N}\right)^2 e^{2B_0N^{\delta-1}}\,t\right\}\right]$$
$$\leqq M^{2\gamma}(t)\exp\{C(r)N^{2\delta-1}\,t\}$$

for some constant $C(r)$ depending only on r. In particular $E_\eta[\{\sup_{t\leqq t_0}M^{\gamma}(t)\}^2] < \infty$ and by Doob's maximal inequality

$$E_\eta\left[\left(\sup_{t\leqq t_0}M^{\gamma}(t)\right)^2\right]\leqq 2\,\exp\{C(r)N^{2\delta-1}\,t_0\}\,.$$

To conclude the proof of Theorem 1.1 it remains to show that, for $\delta<2/13$, there exists a sequence $\ell_N \ll N^{1-2\delta}$ such that for every cylinder function ψ,

$$\sup_\eta E_\eta\left[\exp N^{\delta}\left\{\int_0^t N^{-1}\sum_x|\tau_x V_{\psi,\ell_N}(\eta_s)|e^{-2\theta(x/N)}\,ds\right\}\right]\leqq e^{C(r,\theta,\psi)t}$$

for all $t\geqq 0$ and for some constant $C(r,\theta,\psi)$ depending on r, θ and ψ only. This is the content of the main result of next section.

3 N^α-block estimate

Proposition 3.1 *Fix a cylinder function ψ. Recall the definition of $V_{\psi,\ell}(\eta)$ given in* (2.4). *There exists a finite constant $C(r,\theta,\psi)$ such that*

$$\log\sup_\eta E_\eta\left[\exp N^{\delta}\int_0^t\frac{1}{N}\sum_x|\tau_x V_{\psi,\ell_N}(\eta_s)|e^{-2\theta(x/N)}\,ds\right]$$
$$\leqq C(r,\theta,\psi)t\left\{1+\frac{\sqrt{\ell+N^{1+a-(\delta/2)}}\,\ell}{N^{1-(3\delta/2)}}+\frac{N^{\delta}}{\sqrt{\ell}}\right\}\,. \tag{3.1}$$

for all $a>0$, $0<\delta<1$ and positive integer ℓ.

In the light of this statement it is easy to conclude the proof of Theorem 1.1.

Proof of Theorem 1.1 Recall from Sect. 2 the following restrictions on δ and ℓ: $\delta<1/2$, $\ell<N^{1-2\delta}$. From these inequalities it follows that $\ell<N^{1+a-(\delta/2)}$. In particular the right hand side of (3.1) is bounded by

$$C(r,\theta,\psi)t\left\{1+\frac{\ell}{N^{\{(1-a)/2\}-(5\delta/4)}}+\frac{N^{\delta}}{\sqrt{\ell}}\right\}\,.$$

The best choice of ℓ is $\ell=N^{\{(1-a)/3\}-(\delta/6)}$. With this choice we obtain that the last expression is bounded by

$$C(r,\theta,\psi)t\{1+N^{(13\delta/12)-\{(1-a)/6\}}\}$$

which is bounded by $C(r,\theta,\psi)t$ for $\delta < 2/13$ and a sufficiently small. Moreover, with such δ, we have that $\ell = \ell_N$ is indeed bounded by $N^{1-2\delta}$. □

We turn now to the proof of Proposition 3.1.

Proof of Proposition 3.1 Fix a cylinder function ψ and a positive integer ℓ. To keep notation as simple as possible denote the positive function $N^{\delta-1}\sum_x |\tau_x V_{\psi,\ell_N}(\eta)| e^{-2\theta(x/N)}$ simply by $V_\delta(\eta)$. From the Markov property, for any positive integer M we have that

$$\sup_\eta E_\eta\left[\exp\int_0^t V_\delta(\eta_s)\,ds\right] \leqq \left(\sup_\eta E_\eta\left[\exp\int_0^{t/M} V_\delta(\eta_s)\,ds\right]\right)^M .$$

Expanding the exponential we obtain that the logarithm of the right hand side is equal to

$$M\log\sup_\eta\sum_{k\geqq 0}\ \int\cdots\int_{0<t_1<\cdots<t_k<t/M} E_\eta\left[\prod_{i=1}^k V_\delta(\eta_{t_i})\right] dt_1\cdots dt_k .$$

Since V_δ is positive, we apply again Markov property to bound this expression by

$$M\log\sum_{k\geqq 0}\left(\sup_\eta E_\eta\left[\int_0^{t/M} V_\delta(\eta_s)\,ds\right]\right)^k = M\log\left(1-\sup_\eta E_\eta\left[\int_0^{t/M} V_\delta(\eta_s)\,ds\right]\right)^{-1}$$

provided that

$$\sup_\eta E_\eta\left[\int_0^{t/M} V_\delta(\eta_s)\,ds\right] < 1 . \tag{3.2}$$

From the definition of V_δ we see that it is a positive function bounded above by $C(\theta)N^\delta\|\psi\|_\infty$. In particular the left hand side of (3.2) is bounded by $1/2$ if $M > C'(\theta)N^\delta t\,\|\psi\|_\infty$. Choosing such M and since $\log\{(1-A)^{-1}\} \leqq A/(1-A) \leqq 2A$ if $A \leqq 1/2$, we obtain that the left hand side of (3.1) is bounded above by

$$2M\sup_\eta E_\eta\left[\int_0^{t/M} V_\delta(\eta_s)\,ds\right]$$

provided $M > C'(\theta)N^\delta t\,\|\psi\|_\infty$.

Recall the definition of V_δ. The above expectation is bounded by

$$\begin{aligned} &2MN^{-1+\delta}\sum_x e^{-2\theta(x/N)}\sup_\eta E_\eta\left[\int_0^{t/M}\tau_x V_{\psi,\ell}(\eta_s)\,ds\right] \\ &\quad\leqq C(\theta)MN^\delta\sup_\eta E_\eta\left[\int_0^{t/M} V_{\psi,\ell}(\eta_s)\,ds\right] \end{aligned} \tag{3.3}$$

since the dynamics is translation invariant. To conclude the proof of the proposition it remains to show that the above expression is bounded by the right hand side of (3.1). This is done following the standard method of reducing an estimate of a non equilibrium expectation to an equilibrium large deviation estimate by means of the entropy inequality.

Since the invariant measures of the process are not explicitly known we are forced to use the space–time entropy to compare our process with one that has a different dynamics. The natural reference process is the one where particle flips at a constant rate since for this one the Bernoulli product measure with density 1/2 is invariant.

Consider thus $\widetilde{P}_\alpha$ the law of the Markov process on $D([0,t],\mathscr{X})$ with generator

$$\widetilde{L}_N = N^2 L_0 + \sum_x (f(\eta^x) - f(\eta))$$

and initial measure ν_α, the Bernoulli product measure with density α. Notice that since flip rates are constant $\nu_{1/2}$ is reversible for this process.

Of course, since we are in infinite volume P_η is not absolutely continuous with respect to $\widetilde{P}_{1/2}$ and we need to perform a cutoff to restrict the original problem to a finite volume problem. This is the content of the first lemma. To state it we need to introduce some notation. For a positive integer n and a measure μ on $\mathscr{X}$, denote by $\widetilde{P}^n_\mu$ the probability on the path space $D([0,t],\mathscr{X})$ of the process evolving according to the generator

$$\widetilde{L}_{N,n} = N^2 L_0 + \sum_{x\in\Lambda_n} r(x,\eta)[f(\eta^x) - f(\eta)] + \sum_{x\notin\Lambda_n} [f(\eta^x) - f(\eta)]$$

and starting from μ. Here and below, for a positive integer m, $\Lambda_m = \{-m,\dots,m\}$. Moreover, denote by $\widetilde{E}^n_\mu$ expectation with respect to $\widetilde{P}^n_\mu$. Thus, for the dynamic $\widetilde{L}_{N,n}$, on Λ_n particles flip as in the original process and outside Λ_n they flip independently at a constant rate. It is natural to believe that the expectation of a local function under P_η and under $\widetilde{P}^n_\eta$ do not differ too much if n is large. This is the content of the next lemma.

Lemma 3.2 *For each positive integer n, denote by $\mathscr{E}_n$ the set of pair of configurations (η,ξ) that coincide on $\{0,1\}^{\Lambda_n}$:*

$$\mathscr{E}_n = \{(\eta,\xi)\in\mathscr{X}^2;\ \eta(x) = \xi(x) \text{ for } x\in\Lambda_n\}\ .$$

Then,

$$\sup_{(\eta,\xi)\in\mathscr{E}_n} \left| E_\eta\left[\int_0^{t/M} V_{\psi,\ell}(\eta_s)\,ds\right] - \widetilde{E}^n_\xi\left[\int_0^{t/M} V_{\psi,\ell}(\eta_s)\,ds\right]\right|$$
$$\leqq C(\psi)e^{(2b+1)t/M}\,\frac{N^2 t}{[n-\ell]M}\exp\{-C_1[n-\ell]^2 M(N^2 t)^{-1}\} \qquad (3.4)$$

provided $n-\ell \ll N^2M^{-1}$. In this formula and below, b denotes the length of the support of $r(\cdot)$.

Proof The main problem is to control the information coming from infinity in finite time. Fix a pair of configurations (η,ξ) in $\mathscr{E}_n$. We shall couple both processes so that particles jump and flip together as much as possible. The absolute value of the difference of expectations is bounded above by

$2\|\psi\|_\infty$ times the probability of having a discrepancy among the two processes in $\{-\ell, \ldots, \ell\}$ before time t/M.

For $y \notin \Lambda_n$, denote by X_t^y the position at time t of the particle (or the hole) at y at time 0. X_t^y behaves as a symmetric random walk with jump rate N^2. Recall that we denote by b the support of the flip rate $r(\eta)$. Assume without loss of generality that r is bounded by 1. In particular, if there is a discrepancy at site z, at rate at most 1, a new discrepancy appears at site y for $|y - z| \leqq b$.

Let X_t be a branching random walk starting with one particle at the origin. Particles move as nearest neighbor symmetric exclusion random walks with rate N^2 and a particle at site x creates a new particle at site $x + y$ at rate 1 for $y \in \Lambda_b$. Thus for each t, X_t is a finite subset of $\mathbb{Z}$ that corresponds to the position of particles at time t. We shall denote by N_t the total number of particles for X_t: $N_t = |X_t|$.

With these notation, the absolute value of the difference of expectations in the statement of the lemma is bounded by

$$4\|\psi\|_\infty \sum_{y \notin \Lambda_n} P[\{X_s + y\} \cap \Lambda_\ell = \emptyset \text{ for all } 0 \leqq s \leqq t/M] \,.$$

The sum over y is equal to

$$\sum_{y \notin \Lambda_n} \sum_{k \geqq 1} P[\{X_s + y\} \cap \Lambda_\ell = \emptyset \text{ for all } 0 \leqq s \leqq t/M | N_{t/M} = k] P[N_{t/M} = k] \,.$$

Denote by Z_s a random walk starting from the origin with jump rates $N^2/2$ to the left and $(N^2/2) + b$ to the right. Given that there are k particles at time t/M and since the branching mechanism is independent from the particles jumps, the probability of $X_s + y$ intercepting Λ_ℓ is bounded above by $kP[\sup_{0 \leqq s \leqq t/M} Z_s \geqq y - \ell]$. Thus the last sum is bounded by

$$2E[N_{t/M}] \sum_{y \geqq n} P\left[\sup_{0 \leqq s \leqq t/M} Z_s \geqq y - \ell\right] \,.$$

These two quantities are not difficult to estimate. The expectation of the total number of particles is bounded by the expectation of the total population of a branching process with rate $B = 2b + 1$. It is therefore bounded by $e^{Bt/M}$.

On the other hand, since the random walk Z_s has a constant drift to the right, by the reflexion principle, the probability is bounded above by

$$2 \sum_{y \geqq n} P[Z_{t/M} \geqq y - \ell] \,.$$

By Tchebycheff exponential inequality and standard large deviations arguments, this probability is bounded above by $C_0([n - \ell]M/N^2 t)^{-1} \exp\{-C_1[n - \ell]^2 M(N^2 t)^{-1}\}$ for some universal constants C_0 and C_1 and provided $n - \ell \ll N^2 M^{-1}$. We have thus proved that the supremum in the statement of the lemma is bounded above by

$$C(\psi) e^{Bt/M} \frac{N^2 t}{[n - \ell]M} \exp\{-C_1[n - \ell]^2 M(N^2 t)^{-1}\}$$

provided $n - \ell \ll N^2 M^{-1}$. $\square$

We have now to choose n appropriately. The unique restriction on n is that the right hand side of (3.4) multiplied by $N^{\delta}M$ should be uniformly bounded on N. We set therefore once for all $n - \ell = N^{1+a}(t/M)^{1/2}$ for some small positive a. With this choice and averaging over configurations, we have that the expectation (3.4) is bounded above by

$$C(\theta)MN^{\delta} \sup_{\xi\in\mathscr{X}_n} \widetilde{E}^n_{\mu_{\xi,1/2}}\left[\int_0^{t/M} V_{\psi,\ell}(\eta_s)\,ds\right] + o_N(1)t\ . \tag{3.5}$$

In this formula, $\mathscr{X}_n$ denotes the space of configurations $\{0,1\}^{\Lambda_n}$ and $\mu_{\xi,1/2}$ the product measure with marginals equal to the Bernoulli measure for $x \notin \Lambda_n$ and equal to ξ for $x \in \Lambda_n$:

$$\mu_{\xi,1/2}\{\eta, \eta(x) = 1\} = \begin{cases} \xi(x) & \text{if } x \in \Lambda_n\ , \\ 1/2 & \text{otherwise}\ . \end{cases}$$

Now that we reduced our infinite volume problem to a finite volume one, we are in a position to apply the entropy inequality.

For any $\gamma > 0$ we have by the entropy inequality that

$$\begin{aligned} &E^n_{\mu_{\xi,1/2}}\left[\int_0^{t/M} V_{\psi,\ell}(\eta_s)\,ds\right] \\ &\leqq \frac{1}{\gamma}H(P^n_{\mu_{\xi,1/2}}|\widetilde{P}_{1/2}) + \frac{1}{\gamma}\log \widetilde{E}_{1/2}\left[\exp\left[\gamma\int_0^{t/M} V_{\psi,\ell}(\eta_s)\,ds\right]\right]\ , \end{aligned} \tag{3.6}$$

where $H(P^n_{\mu_{\xi,1/2}}|\widetilde{P}_{1/2})$ is the relative entropy of $P^n_{\mu_{\xi,1/2}}$ with respect to $\widetilde{P}_{1/2}$ on $D([0,t/M],\mathscr{X})$ and $\widetilde{E}_{1/2}$ denotes expectation with respect to $\widetilde{P}_{1/2}$. We need to estimate each term on the right hand side of (3.6). We consider first the relative entropy.

Lemma 3.3 *There exists a constant $C = C(r)$ so that for all $M \geqq t$*

$$H(P^n_{\mu_{\xi,1/2}}|\widetilde{P}_{1/2}) = \int \log\left(\frac{dP^n_{\mu_{\xi,1/2}}}{d\widetilde{P}_{1/2}}\right) dP^n_{\mu_{\xi,1/2}} \leqq C(2n+1)$$

uniformly over $\xi \in \mathscr{X}_n$.

Proof Since the Radon-Nikodym derivative of $P^n_{\mu_{\xi,1/2}}$ with respect to $\widetilde{P}_{1/2}$ is equal to

$$\frac{d\mu_{\xi,1/2}}{d\nu_{1/2}} \exp\left\{\sum_{x\in\Lambda_n}\int_0^{t/M} \log\tau_x r(\eta_s)\,dJ^x_s - \sum_{x\in\Lambda_n}\int_0^{t/M}\{\tau_x r(\eta_s) - 1\}\,ds\right\}$$

we have that the relative entropy of $P^n_{\mu_{\xi,1/2}}$ with respect to $\widetilde{P}_{1/2}$ is equal to

$$H(\mu_{\xi,1/2}|\nu_{1/2}) + E_{\mu_{\xi,1/2}}\left[\sum_{x\in\Lambda_n}\int_0^{t/M} \log\tau_x r(\eta_s)\,dJ^x_s - \sum_{x\in\Lambda_n}\int_0^{t/M}\{\tau_x r(\eta_s) - 1\}\,ds\right].$$

In this formula $H(\mu_{\xi,1/2}|\nu_{1/2})$ stands for the relative entropy of $\mu_{\xi,1/2}$ with respect to the product measure $\nu_{1/2}$. Since for every cylinder function χ and x in Λ_n, $\int_0^t \chi(\eta_s)\,dJ_s^x - \int_0^t \chi(\eta_s)(\tau_x r)(\eta_s)\,ds$ is a mean zero martingale, this last expectation is equal to

$$(2n+1)\log 2 + E_{\mu_{\xi,1/2}}\left[\sum_{x\in\Lambda_n}\int_0^{t/M}[\tau_x r(\eta_s)\log\tau_x r(\eta_s) - \tau_x r(\eta_s) + 1]\,ds\right]$$

$$\leqq (2n+1)\log 2 + \frac{(2n+1)t}{M}\sup_{0\leqq\theta\leqq\|r\|_\infty}\{\theta\log\theta - \theta + 1\} \leqq C(2n+1)$$

for all $M \geqq t$. The lemma is therefore proved with $C = \log 2 + \sup_{0\leqq\theta\leqq\|r\|_\infty}\{\theta\log\theta - \theta + 1\}$. □

We now turn back to the proof of Proposition 3.1 and consider the second term of the right hand side of (3.6).

Lemma 3.4 *There exists a finite constant $C(\psi)$ such that*

$$\gamma^{-1}\log\widetilde{E}_{1/2}\left[\exp\left[\gamma\int_0^{t/M} V_{\psi,\ell}(\eta_s)\,ds\right]\right] \leqq C(\psi)\,\frac{t}{M}\left\{\frac{\gamma\ell^2}{N^2} + \frac{1}{\ell^{1/2}}\right\}$$

for all $\gamma > 0$.

Proof By Feymann-Kac formula and reversibility of $\nu_{1/2}$ we have

$$\gamma^{-1}\log\widetilde{E}_{1/2}\left[\exp\left[\gamma\int_0^{t/M} V_{\psi,\ell}(\eta_s)\,ds\right]\right]$$

$$\leqq \frac{t}{M}\sup_f\left\{\int_{\mathscr{X}} V_{\psi,\ell}\,f\,d\nu_{1/2} - \gamma^{-1}\widetilde{D}_N(f)\right\}, \tag{3.7}$$

where the supremum is carried over all densities f with respect to $\nu_{1/2}$ ($f \geqq 0$ and $\int f\,d\nu_{1/2} = 1$) and where $\widetilde{D}_N$ is the Dirichlet form associated to the generator $\widetilde{L}_N$:

$$\widetilde{D}_N(f) = -\langle\sqrt{f},\widetilde{L}_N\sqrt{f}\rangle_{\nu_{1/2}}$$

$$= -N^2\langle\sqrt{f},L_0\sqrt{f}\rangle_{\nu_{1/2}} + \frac{1}{2}\int\sum_{i\in\mathbb{Z}}(\sqrt{f(\eta^i)} - \sqrt{f(\eta)})^2\,d\nu_{1/2}\,.$$

Here $\langle\,\cdot\,,\,\cdot\,\rangle_{1/2}$ represents the usual inner product in $L^2(\nu_{1/2})$. Denote by D_N the accelerated part of the Dirichlet form $\widetilde{D}_N$:

$$D_N(f) = -\langle\sqrt{f},L_0\sqrt{f}\rangle_{\nu_{1/2}} = \sum_{i\in\mathbb{Z}} I_{i,i+1}(f)$$

where

$$I_{i,i+1}(f) = \frac{1}{4}\int(\sqrt{f(\eta^{i,i+1})} - \sqrt{f(\eta)})^2\,d\nu_{1/2}$$

so that $N^2 D_N(f) \leqq \widetilde{D}_N(f)$. In particular the right hand side of (3.7) is bounded above by

$$\frac{t}{M} \sup_f \left\{ \int_{\mathscr{X}} V_{\psi,\ell}\, f\, d\nu_{1/2} - N^2 \gamma^{-1} D_N(f) \right\} . \tag{3.8}$$

For a positive integer k, denote by $\nu^k_{1/2}$ the product measure on $\mathscr{X}_k = \{0,1\}^{\Lambda_k}$ with marginals equal to the marginals of $\nu_{1/2}$ and by $\mathscr{D}_k$ the Dirichlet form D_N restricted to $\mathscr{X}_k$:

$$\mathscr{D}_k(f) = \sum_{i=-k}^{k-1} I^k_{i,i+1}(f)$$

where

$$I^k_{i,i+1}(f) = \frac{1}{4} \int (\sqrt{f(\eta^{i,i+1})} - \sqrt{f(\eta)})^2\, d\nu^k_{1/2} \tag{3.9}$$

for all densities f with respect to $\nu^k_{1/2}$. Moreover, for a density f with respect to $\nu_{1/2}$, denote by f_k the conditional expectation of f given $\{\eta(x);\ |x| \leqq k\}$:

$$f_k(\xi) = E_{\nu_{1/2}}[f \,|\, \eta(x) = \xi(x) \text{ for } |x| \leqq k] \quad \text{for all } \xi \in X_k .$$

Recall from the previous section that the cylinder function $V_{\psi,\ell}$ depends on η only through the coordinates $\{\eta(x),\ |x| \leqq \ell\}$. In particular

$$\int_{\mathscr{X}} V_{\psi,\ell}\, f\, d\nu_{1/2} = \int_{\mathscr{X}_\ell} V_{\psi,\ell}\, f_\ell\, d\nu^\ell_{1/2}$$

On the other hand, by the convexity of the Dirichlet form, we have that $\mathscr{D}_\ell(f_\ell) \leqq D_N(f)$ since for each $-\ell \leqq i \leqq \ell - 1$, $I^\ell_{i,i+1}(f_\ell) \leqq I_{i,i+1}(f)$.

Therefore the supremum (3.8) is bounded above by

$$\frac{t}{M} \sup_f \left\{ \int_{\mathscr{X}_\ell} N^\delta V_{\psi,\ell}\, f\, d\nu^\ell_{1/2} - N^2 \gamma^{-1} \mathscr{D}_\ell(f) \right\} .$$

Here the supremum is carried over all densities with respect to the product measure $\nu^\ell_{1/2}$ and $\mathscr{D}_\ell$ is the Dirichlet form defined in (3.9).

The second step consists in projecting the density over hyperplanes with fixed total number of particles since it is on these hyperplanes that the process is ergodic. For each integer $0 \leqq j \leqq 2\ell + 1$, denote by $\nu^{\ell,j}$ the restriction of the measure $\nu^\ell_{1/2}$ on the hyperplane $\mathscr{X}_{\ell,j}$ of configurations of $\mathscr{X}_\ell$ with j particles:

$$\mathscr{X}_{\ell,j} = \left\{ \eta \in \mathscr{X}_\ell;\ \sum_{|x| \leqq \ell} \eta(x) = j \right\}$$

$$\nu^{\ell,j}(\xi) = \frac{\nu^\ell_{1/2}(\xi)}{\nu^\ell_{1/2}(\eta;\ \eta \in \mathscr{X}_{\ell,j})} \quad \text{for all } \xi \in \mathscr{X}_{\ell,j} .$$

Notice that $\nu^{\ell,j}$ is the uniform measure on $\mathscr{X}_{\ell,j}$. For each density f on $\mathscr{X}_\ell$, denote by f_j the projection of f on the hyperplane $\mathscr{X}_{\ell,j}$:

$$f_j(\xi) = \frac{f(\xi)}{E_{\nu_{1/2}}\left[f\middle|\sum_{|x|\leqq\ell}\eta(x) = j\right]} = \frac{f(\xi)}{\int f(\eta)\nu^{\ell,j}(d\eta)} \ .$$

With this notation, we may rewrite the expression inside the last supremum as

$$\sum_{j=0}^{2\ell+1} C_j(f)\left\{\int_{\mathscr{X}_{\ell,j}} V_{\psi,\ell}\, f_j\, d\nu^{\ell,j} - N^2\gamma^{-1}\mathscr{D}_{\ell,j}(f_j)\right\} \ . \tag{3.10}$$

Here $C_j(f) = \int \mathbf{1}_{\{\mathscr{X}_{\ell,j}\}} f\, d\nu^{\ell}_{1/2}$ and $\mathscr{D}_{\ell,j}$ is the Dirichlet form $\mathscr{D}_\ell$ restricted to the hyperplane $\mathscr{X}_{\ell,j}$:

$$\mathscr{D}_{\ell,j}(f) = (1/4)\sum_{i=-\ell}^{\ell-1}\int (\sqrt{f(\eta^{i,i+1})} - \sqrt{f(\eta)})^2\, d\nu^{\ell,j}$$

for all densities f with respect to $\nu^{\ell,j}$. Notice that $\sum_j C_j(f) = 1$.

We shall decompose the sum (3.10) in two terms:

$$\sum_{j=0}^{2\ell+1} C_j(f)\int_{\mathscr{X}_{\ell,j}} V_{\psi,\ell}\, d\nu^{\ell,j}$$
$$+ \sum_{j=0}^{2\ell+1} C_j(f)\left\{\int_{\mathscr{X}_{\ell,j}} V_{\psi,\ell}\,(f_j - 1)\, d\nu^{\ell,j} - N^2\gamma^{-1}\mathscr{D}_{\ell,j}(f)\right\} \ . \tag{3.11}$$

Recall the definition of the cylinder function $V_{\psi,\ell}$. By Schwarz inequality, for each fixed j, the integral appearing on the first line is bounded above by

$$\left\{\frac{1}{(2\ell+1)^2}\sum_{y,x}\nu^{\ell,j}[\{\tau_y\,\psi - \widetilde{\psi}(j/2\ell+1)\}\{\tau_x\,\psi - \widetilde{\psi}(j/2\ell+1)\}]\right\}^{1/2} .$$

Here summation is carried over all integers x and y such that $|x| \leqq \ell'$ and $|y| \leqq \ell'$ and $\ell' = \ell(\psi)$ is defined just after (2.4) in the previous section. For a cylinder function χ, by the local central limit theorem (cf. [DF]), there exists a constant $C(\chi)$ such that

$$\sup_{0\leqq j\leqq 2\ell+1} |\nu^{\ell,j}(\chi) - \nu_{j/2\ell+1}(\chi)| \leqq C(\chi)\ell^{-1} \ .$$

In particular the last expression is bounded above by

$$\left\{\frac{1}{(2\ell+1)^2}\sum_{y,x}\nu_{(j/2\ell+1)}[\{\tau_y\,\psi - \widetilde{\psi}(j/2\ell+1)\}\right.$$
$$\left.\times\{\tau_x\,\psi - \widetilde{\psi}(j/2\ell+1)\}] + \frac{C(\psi)}{\ell}\right\}^{1/2} .$$

Since $\sum_j C_j(f) = 1$, since ν_α is a product measure and since $\tilde{\psi}(\alpha)$ is the expectation of ψ with respect to ν_α, this expression is bounded above by $C(\psi)\ell^{-1/2}$.

On the other hand, since $\|V_{\psi,\ell}\|_\infty \leqq 2\|\psi\|_\infty$ and since f_j is a density with respect to the measure $\nu^{\ell,j}$, the expression inside braces in the second line of (3.11) is bounded above by

$$C(\psi) \int_{\mathscr{X}_{\ell,j}} \left| f_j(\xi) - \int_{\mathscr{X}_{\ell,j}} f_j(\eta)\, \nu^{\ell,j}(d\eta) \right| \nu^{\ell,j}(d\xi) - N^2\gamma^{-1}\mathscr{D}_{\ell,j}(f)$$

$$\leqq C(\psi) \int\int \nu^{\ell,j}(d\xi)\, \nu^{\ell,j}(d\eta)\, |\sqrt{f_j(\eta)} - \sqrt{f_j(\xi)}|\; |\sqrt{f_j(\eta)} + \sqrt{f_j(\xi)}|$$

$$- N^2\gamma^{-1}\mathscr{D}_{\ell,j}(f)\,.$$

From now on, to keep the notation simple, we omit the subscript j of the density f_j. Moreover, until the end of the proof of the lemma, the value of the constant $C(\psi)$ may change from line to line. By Schwarz inequality and since f is a density this expression is bounded above by

$$C(\psi) \left(\int\int \nu^{\ell,j}(d\xi)\, \nu^{\ell,j}(d\eta)\, |\sqrt{f(\eta)} - \sqrt{f(\xi)}|^2 \right)^{1/2} - N^2\gamma^{-1}\mathscr{D}_{\ell,j}(f)\,.$$

Notice that the integral of the last formula is equal to two times the variance of $\sqrt{f}$. By the spectral gap for symmetric simple exclusion processes in finite volume (cf. [Q]), the last expression is bounded above by

$$C(\psi)\ell\sqrt{\mathscr{D}_{\ell,j}(f)} - N^2\gamma^{-1}\mathscr{D}_{\ell,j}(f) \leqq C(\psi)\frac{\ell^2\gamma}{N^2}$$

since $\sup_x\{\beta x - \tilde{\beta}x^2\} \leqq \beta^2/(4\tilde{\beta})$.

In conclusion, since $\sum_j C_j(f) = 1$, we proved that the right hand side of (3.7) is bounded above by

$$C(\psi)\, \frac{t}{M} \left\{ \frac{\gamma\ell^2}{N^2} + \frac{1}{\ell^{1/2}} \right\}\,. \qquad \square$$

We are now ready to conclude the proof of Proposition 3.1. From Lemmas 3.3 and 3.4 it follows that the left hand side of (3.6) is bounded above by

$$C(\psi, r) \left\{ \frac{n}{\gamma} + \frac{t}{M}\frac{\gamma\ell^2}{N^2} + \frac{t}{M}\frac{1}{\ell^{1/2}} \right\}$$

for all positive γ. Optimizing in γ we obtain that this last expression is bounded by

$$C(r,\psi) \left\{ \sqrt{\frac{t}{M}}\frac{n^{1/2}\ell}{N} + \frac{t}{M}\frac{1}{\ell^{1/2}} \right\}\,.$$

From this bound and in view of (3.5), we have that (3.3) is bounded above by

$$C(r,\theta,\psi)N^{\delta}M\left\{\sqrt{\frac{t}{M}}\frac{\{\ell+N^{1+a}\sqrt{t/M}\}^{1/2}\ell}{N}+\frac{t}{M}\frac{1}{\ell^{1/2}}\right\}+o_N(1)t\ .$$

Notice that this expression is increasing in M and recall that the unique restriction imposed on M is that $M \geqq C(\theta,\psi)N^{\delta}t$. We shall therefore choose such M and obtain that this expression is bounded above by

$$C(r,\theta,\psi)tN^{2\delta}\left\{\frac{\{\ell+N^{1+a-(\delta/2)}\}^{1/2}\ell}{N^{1+(\delta/2)}}+\frac{1}{N^{\delta}\ell^{1/2}}\right\}+o_N(1)t\ .$$

This concludes the proof of the proposition. □

4 Long time behaviour

The proof of Theorem 1.2 is divided in few lemmas. We need first to compare the discrete solution u_t^N defined in (1.1) with the continuous solution ρ.

Lemma 4.1 *For each configuration η on $\mathcal{X}$, denote by ρ^{η} the solution of the hydrodynamic equation* (0.1) *with initial data given by $\rho_0(a)=\eta([aN])$, for a in $\mathbb{R}$. There exists a constant C_0 depending only on $\|G\|_\infty$ and $\|G'\|_\infty$ so that*

$$\sup_{\eta}\sup_{x}|u_t^{N,\eta}(x)-\rho_t^{\eta}(x/N)|\ \leqq\ \frac{C_0}{N}e^{C_0t}\ .$$

Proof Fix a configuration η. Since η shall remain fixed, we omit in this proof indices η of u_t^{η} and ρ_t^{η}. This lemma is a simple consequence of the integral representation of the differential equations for u and ρ:

$$u_t(x)=\sum_y p_t(y,x)\eta(y)+\int_0^t ds\sum_y p_{t-s}(y,x)G(u_s(y))\ ,$$

$$\rho_t(a)=\int W_t(a-b)\eta([bN])\,db+\int_0^t ds\int W_{t-s}(a-b)G(\rho_s(b))\,db\ ,$$

where $W_t:\mathbb{R}\to\mathbb{R}_+$ stands for the gaussian kernel: $W_t(a)=(2\pi t)^{-1/2}\exp\{-a^2/2t\}$ and p_t for the transition probability of a symmetric nearest neighbor random walk on $\mathbb{Z}$ accelerated by N^2.

From this representation and since $\int db\,|W_t(b-a)-W_t(b-a')|\ \leqq\ C_1|a-a'|t^{-1/2}$ for some universal constant C_1, we have that

$$|\rho_t(a)-\rho_t(a')|\ \leqq\ C(\|G\|_\infty)\{t^{-1/2}+t^{1/2}\}|a-a'|\ .$$

On the other hand, by the local central limit theorem, there exists an universal constant C_2 such that

$$\sum_y\left|p_t(y-x)-\int_{y/N}^{(y+1)/N}db\,W_t(b-(x/N))\right|\ \leqq\ \frac{C_2}{N\sqrt{t}}\ .$$

In view of the previous two estimates and the integral representation of u and ρ, it is not difficult to conclude the proof of the lemma with a Gronwall type argument. □

If the initial configuration η is close to a stable equilibrium point m^* of equation $m'(t) = G(m(t))$, we would expect the solution u_t^η to remain close to the stable equilibrium m^*. This is the content of the next result.

Lemma 4.2 *Recall the definition of d given just before Theorem* 1.2. *Consider a sequence $\eta_{(N)}$ of configurations such that*

$$\lim_{N\to\infty} \sup_x |\eta^N_{(N)}(x) - m^*| < d \ .$$

Then, for all $\varepsilon > 0$, there exists $t_0 = t_0(\varepsilon)$, such that for all sufficiently large N,

$$\sup_{t \geqq t_0} \sup_{x\in\mathbb{Z}} |u_t^{\eta(N)}(x) - m^*| \leqq \varepsilon \ .$$

In particular, since the H_{-1} norm is bounded above by the L_2 norm and $u_t^{\eta(N)}$ is positive and bounded by 1, $\sup_{t\geqq t_0}\sup_x \|\tau_x u_t^{\eta(N)} - m^*\|_{-1}^2 \leqq \varepsilon D_1(\theta)$ where

$$D_1(\theta) = \max\left\{1, \sup_N N^{-1} \sum_y e^{-2\theta(y/N)}\right\} \ . \tag{4.1}$$

Proof Fix $\varepsilon \leqq d$ and $\alpha > 0$. By assumption, there exists N_0 so that $|\eta^N_{(N)}(x) - m^*| \leqq d - \alpha$ for all x in $\mathbb{Z}$ and $N \geqq N_0$. Fix $0 < \beta < 1$. It follows from last bound, the integral representation for $\rho_t^{\eta(N)}$ and some simple computations that

$$\sup_a |\rho^{\eta(N)}_{N^{-\beta}}(a) - m^*| \leqq d - (\alpha/2) \tag{4.2}$$

for all $N \geqq N(\alpha,\beta)$. We shall prove this claim at the end of the lemma.

By Lemma 4.1, we obtain that

$$\sup_x |u^{\eta(N)}_{N^{-\beta}}(x) - m^*| \leqq d$$

for N sufficiently large.

Recall from Sect. 1 the definition of d and E_0. Let $u^\pm : \mathbb{Z} \to \mathbb{R}$ be defined by $u^\pm(x) = m^* \pm d$ for all x in $\mathbb{Z}$. The solution $u_t^\pm$ of the discrete hydrodynamic equation (1.1) with initial data $u^\pm$ is constant in space and $u_t^\pm(0) = m_t$, where m_t is the solution of the ordinary differential equation $m'(t) = G(m(t))$. In particular, by (1.2), $|u_t^\pm(x) - m^*| \leqq d \exp\{-E_0 t\}$.

On the other hand, by the maximum principle, solutions of the discrete hydrodynamic equation (1.1) are monotone in the sense that $u^1 \leqq u^2$ implies that $u_t^1 \leqq u_t^2$ for all $t \geqq 0$. Therefore, since $u^- \leqq u_{N^{-\beta}} \leqq u^+$,

$$\sup_{t\geqq 0} \sup_x |u^{\eta(N)}_{t+N^{-\beta}}(x) - m^*| \leqq d e^{-E_0 t}$$

for all $N \geqq N(\beta, \varepsilon, G)$. This concludes the proof of the lemma.

We now return to the proof of estimate (4.2). By the integral representation of $\rho_t^{\eta(N)}$ and since the gaussian kernel is an even function, $|\rho_t^{\eta(N)}(a) - m^*|$ is bounded above by

$$\sum_z \left| H_t(z/N - a) - (2N+1)^{-1} \sum_{|y-z| \leqq N} H_t(y/N - a) \right|$$
$$+ \sum_z H_t(z/N - a)|\eta^N_{(N)}(z) - m^*| + \|G\|_\infty t\ .$$

In this formula, $H_t(a)$ stands for $\int_a^{a+(1/N)} W_t(b)\,db$. Recall the definition of N_0 given in the beginning of the proof. For $N \geqq N_0$, the second term is bounded by $d - \alpha$. A simple Taylor's expansion shows that the first term is bounded by $C(tN)^{-1}$ for some finite universal constant C. In particular, for any $0 < \beta < 1$, there exists $N(\alpha, \beta)$ so that $|\rho^{\eta(N)}_{N-\beta}(a) - m^*| \leqq d - (\alpha/2)$ for $N \geqq N(\alpha, \beta)$. □

It is easy to extend the proof of Lemma 4.2 to the case where $\rho_t^{\eta(N)}$ replaces $u_t^{\eta(N)}$. In particular, under the assumption of Lemma 4.2, in view of Lemma 4.1, for every $\varepsilon > 0$, there exists $N_0(\varepsilon)$ so that

$$\sup_{t \geqq 0} \sup_x |u_t^{\eta(N)}(x) - \rho_t^{\eta(N)}(x/N)| \leqq \varepsilon \tag{4.3}$$

for all $N \geqq N_0(\varepsilon)$.

The next step in the proof consists in showing that if the initial profile is close in a compact set to the stable equilibrium m^*, it shall remain close to the equilibrium at later times in smaller compacts. More precisely, we have the following result.

Lemma 4.3 *For every $\varepsilon > 0$, there exists $t_0 = t_0(\varepsilon)$ with the following property. For all $t_1 \geqq t_0$ and $t_2 > 0$, there exists and $B = B(\varepsilon, t_1, t_2)$ such that for each R:*

$$\sup_{|a| \leqq R+B} |\rho_0(a) - m^*| \leqq d \Rightarrow \sup_{t_1 \leqq t \leqq t_1 + t_2} \sup_{|a| \leqq R} |\rho_t(a) - m^*| \leqq \varepsilon\ .$$

Proof Fix $\varepsilon > 0$ and $t_2 > 0$. Let $B > 0$ to be appropriately chosen later. Denote, respectively, by λ_t and $\widetilde{\rho}_t$ the solutions of the hydrodynamic equation (0.1) with initial profile λ_0 and $\widetilde{\rho}_0$ given by

$$\lambda_0(a) = m^* + d, \qquad \widetilde{\rho}_0(a) = (m^* + d)\mathbf{1}\{|a| \leqq R + B\} + \mathbf{1}\{|a| > R + B\}\ .$$

Since and $\rho_0 \leqq \widetilde{\rho}_0$ we have that $\rho_t \leqq \widetilde{\rho}_t$ for all $t \geqq 0$. On the other hand, by (1.2), $\sup_a \lambda_t(a) \leqq m^* + d \exp\{-E_0 t\}$.

Denote by $A(t,a)$ the difference $\widetilde{\rho}_t(a) - \lambda_t(a)$. By the integral representation of solutions of the hydrodynamic equation, we have that

$$A(t,a) \leqq \int_{|b| \geqq R+B} W_t(a-b)\,db + \|G'\|_\infty \int_0^t ds \int db\, W_{t-s}(a-b) A(s,b)\ .$$

By Gronwall inequality and some simple computation, if we denote by $F(t,a)$ the function $\int_{|b|\geqq R+B} W_t(a-b)\,db$, we have that

$$A(t,a) \leqq F(t,a) + C\int_0^t e^{C(t-s)}[W_{t-s} * F(s,\cdot)](a)\,ds \leqq \{1+Ce^{Ct}\}F(t,a)\,,$$

where $C = \|G'\|_\infty$.

Therefore, since ρ_t is bounded above by $\widetilde{\rho}_t$, and $A_t = \widetilde{\rho}_t - \lambda_t$, from the previous estimate on λ_t and on A_t, we obtain that for all a in $[-R,R]$,

$$\begin{aligned}\rho(t,a) &\leqq m^* + de^{-E_0 t} + \{1+Ce^{Ct}\}\int_{|b|\geqq R+B} W_t(a-b)\,db\\ &\leqq m^* + de^{-E_0 t} + C_2 e^{Ct}\exp\{-Bt^{-1/2}\}\end{aligned}$$

for some constants C and C_2 depending only on G.

Let t_0 be chosen so that $de^{-C_0 t} \leqq \varepsilon/2$. For every $t_1 \geqq t_0$, we may find $B = B(\varepsilon, t_1, t_2)$ so that $\sup_{t_1\leqq t\leqq t_1+t_2} C_2\exp\{Ct - Bt^{-1/2}\} \leqq \varepsilon/2$. □

Lemma 4.4 *There exists $\varepsilon_0 = \varepsilon(\theta, G)$, such that for every $\varepsilon < \varepsilon_0$ there exists $t_0 = t_0(\varepsilon)$ with the following property. For all $t_1 \geqq t_0$ and $t_2 > 0$ there exists $K = K(\varepsilon, t_1, t_2)$ such that*

$$\sup_{|x|\leqq(R+K)N} \|\tau_x\eta - m^*\|_{-1}^2 \leqq \varepsilon \;\Rightarrow\; \sup_{t_1\leqq t\leqq t_1+t_2}\;\sup_{|x|\leqq RN} \|\tau_x u_t^\eta - m^*\|_{-1}^2 \leqq \varepsilon/2 \tag{4.4}$$

for all $N \geqq N(\varepsilon)$.

Proof The proof is divided in several steps. We first define K. Recall the definition of $D_1(\theta)$ given in (4.1). Let $D_2(\varepsilon,\theta)$ be large enough so that $N^{-1}\sum_{|y|\geqq D_2 N}\exp\{-2\theta(y/N)\} \leqq \varepsilon/6$. Set $K(\varepsilon,t_1,t_2) = B(\varepsilon/(4D_1(\theta)), t_1 + t_d, t_2) + D_2(\varepsilon,\theta)$, where $B(\varepsilon,t_1,t_2)$ is the constant given by Lemma 4.3 and $t_d = d/(2\|G\|_\infty)$.

We now show that we may bound the empirical density around the origin by the H_{-1} norm. More precisely, let $H : \mathbb{R} \to \mathbb{R}_+$ be a smooth positive function such that $N^{-1}\sum_x H(x/N) = 1$. Denote by $\eta^H(x)$ the integral of H with respect to the empirical measure translated by x: $\eta^H(x) = N^{-1}\sum_y H(y/N)\eta(x+y)$. By Schwarz inequality,

$$|\eta^H(x) - m^*| \leqq \|He^{2\theta(\cdot)}\|_1 \|\tau_x\eta - m^*\|_{-1}\,. \tag{4.5}$$

Therefore, by assumption,

$$\sup_{|x|\leqq(R+K)N} |\eta^H(x) - m^*| \leqq \|He^{2\theta(\cdot)}\|_1 \varepsilon^{1/2}\,.$$

The third step consists in proving that the hydrodynamic solution ρ_t^η is uniformly close to m^* in the compact $[-R-K,\ R+K]$ at time $t_d = d/(2\|G\|_\infty)$.

By the integral representation for ρ_t^η and with the notation just introduced, $|\rho_t^\eta(a) - m^*|$ is bounded above by $|\eta^{H_t^{a,N}}([aN]) - m^*| + \|G\|_\infty t$. Here $H_t^{a,N}$ is defined by $H_t^{a,N}(b) = NH_t([aN]N^{-1} - a + b)$ and H_t is the function introduced in the proof of Lemma 4.2. By (4.5), $|\eta^{H_t^{a,N}}([aN]) - m^*|$ is bounded by

$\|H_t^{a,N} e^{2\theta(\cdot)}\|_1 \, \|\tau_{[aN]}\eta - m^*\|_{-1}$. On the one hand, a simple computation shows that $\sup_{|b| \leqq 1} \|NH_t(b - \cdot)e^{2\theta(\cdot)}\|_1^2$ is bounded by $C(\theta)(1 + t^{-2})t^{-1/2}\exp\{t/2\}$. On the other hand, by assumption, for $|a| \leqq R + K$, $\|\tau_{[aN]}\eta - m^*\|_{-1}^2 \leqq \varepsilon$. In conclusion, we showed that

$$\sup_{|a| \leqq R+K} |\rho_t^\eta(a) - m^*| \leqq \{C(\theta)(1 + t^{-2})t^{-1/2}e^{t/2}\varepsilon\}^{1/2} + \|G\|_\infty t \ .$$

Therefore, if we set $t_d = d/(2\|G\|_\infty)$ and choose $\varepsilon_0(\theta, G)$ so that $C(\theta)(1 + t_d^{-2})t_d^{-1/2}\exp\{t_d/2\}\varepsilon_0$ is bounded by $d^2/4$, for configurations η satisfying assumption (4.4), with $\varepsilon < \varepsilon_0$ we have that $\sup_{|a| \leqq R+K} |\rho_{t_d}^\eta(a) - m^*| \leqq d$.

Fix such $\varepsilon < \varepsilon_0$. By Lemma 4.3 there exists $s_0 = s_0(\varepsilon)$ such that for all $s_1 \geqq s_0$ and $t_2 > 0$,

$$\sup_{s_1 \leqq t \leqq s_1 + t_2} \ \sup_{|a| \leqq R+K-B} |\rho_{t_d+t}^\eta(a) - m^*| \leqq \frac{\varepsilon}{4D_1(\theta)}$$

for $B = B(\varepsilon/(4D_1(\theta)), s_1, s_2)$. In particular, if we set $t_0(\varepsilon) = t_d + s_0(\varepsilon)$, for all $t_1 \geqq t_0$ and $t_2 > 0$,

$$\sup_{t_1 \leqq t \leqq t_1 + t_2} \ \sup_{|a| \leqq R+K-B} |\rho_t^\eta(a) - m^*| \leqq \frac{\varepsilon}{4D_1(\theta)}$$

for $B = B(\varepsilon/(4D_1(\theta)), t_1 + t_d, t_2)$.

By Lemma 4.1, for $N \geqq N(\varepsilon)$,

$$\sup_{t_1 \leqq t \leqq t_1 + t_2} \ \sup_{|x| \leqq (R+K-B)N} |u_t^\eta(a) - m^*| \leqq \frac{\varepsilon}{3D_1(\theta)} \ . \tag{4.6}$$

To conclude the proof of the lemma it remains to bound the H_{-1} norm of $u_{t_d+t}^\eta - m^*$ by its L_1 norm. This is not difficult. Recall the definition of $D_1(\theta)$ and $D_2(\theta, \varepsilon)$ given in the beginning of the proof. Since the H_{-1} norm is bounded by the L_2 norm and since u_t^η and m^* are positive and bounded by 1,

$$\|\tau_x u_t^\eta - m^*\|_{-1}^2 \leqq D_1(\theta) \sup_{|y| \leqq D_2 N} |u_t^\eta(x + y) - m^*| + \varepsilon/6 \ .$$

Thus for $|x| \leqq RN$ and $t_1 \leqq t \leqq t_1 + t_2$,

$$\|\tau_x u_t^\eta - m^*\|_{-1}^2 \leqq D_1(\theta) \sup_{|y| \leqq (R+D_2)N} |u_t^\eta(y) - m^*| + \varepsilon/6 \leqq \varepsilon/2 \ .$$

Last inequality follows from (4.6) and because we set $K = B + D_2$. □

We are now ready to prove a result concerning the long time behaviour of (η_t).

Proposition 4.5 *Let $\eta_{(N)}$ be a sequence of configurations in the basin of attraction of the stable equilibrium m^* in the following sense*:

$$\lim_{N\to\infty} \sup_x |\eta_{(N)}^N(x) - m^*| < d \ .$$

Then, for every $\varepsilon > 0$ *and every* $\gamma < \delta/2$,

$$\lim_{N\to\infty} P_{\eta(N)}\left[\sup_{t\leqq\exp\{N^\gamma\}}\max_{|x|\leqq\exp\{N^\gamma\}}\|\tau_x\eta_t - \tau_x u_t^{\eta(N)}\|_{-1}^2 > \varepsilon\right] = 0 \;.$$

In fact, it follows from the proof of this proposition that the probability converges exponentially fast to 0.

Proof To distinguish among t_0 given by Lemmas 4.2 and 4.4, we shall denote the first one by t_0^2 and the second one by t_0^4. Recall the definition of ε_0 given in Lemma 4.4. Fix $0 < \varepsilon < \varepsilon_0$ and $T > 0$ to be chosen later. Denote by $\widetilde{u}_t$ the function defined by

$$\widetilde{u}_t = \begin{cases} u_t^{\eta(N)} & \text{for } 0 \leqq t < T, \\ u_{t-kT+T/2}^{\eta_{kT-T/2}} & \text{for } kT \leqq t < (k+1)T \text{ and } k \geqq 1. \end{cases}$$

By Theorem 1.1, by Markov property and by definition of $\widetilde{u}_t$,

$$P_{\eta(N)}\left[\sup_{t\leqq\exp\{N^\gamma\}}\max_{|x|\leqq\exp\{N^\gamma\}}\|\tau_x\eta_t - \tau_x\widetilde{u}_t\|_{-1}^2 > \varepsilon/4\right]$$

converges exponentially fast to 0 because the dynamics is translation invariant, $\tau_x u_t^\eta = u_t^{\tau_x\eta}$ and $2\gamma < \delta$. Therefore, by the elementary inequality $(a+b)^2 \leqq 2a^2 + 2b^2$, to prove the proposition, it is enough to show that T can be chosen so that

$$P_{\eta(N)}\left[\sup_{t\leqq\exp\{N^\gamma\}}\max_{|x|\leqq\exp\{N^\gamma\}}\|\tau_x\widetilde{u}_t - \tau_x u_t^{\eta(N)}\|_{-1}^2 > \varepsilon/4\right]$$

vanishes as $N\uparrow\infty$. Since $\widetilde{u}_t$ and $u_t^{\eta(N)}$ coincide for $t \leqq T$, we may restrict the supremum to the interval $[T, \exp\{N^\gamma\}]$. If T is chosen larger than $t_0^2(\varepsilon/16D_1(\theta))$, by Lemma 4.2, for $t \geqq T$, $\|\tau_x u_t^{\eta(N)} - m^*\|_{-1}^2 \leqq \varepsilon/16$. In particular, $\|\tau_x\widetilde{u}_t - \tau_x u_t^{\eta(N)}\|_{-1}^2 \leqq 2\|\tau_x\widetilde{u}_t - m^*\|_{-1}^2 + \varepsilon/8$. Therefore, to prove the theorem, it is enough to show that

$$P_{\eta(N)}\left[\max_{1\leqq k\leqq T^{-1}\exp\{N^\gamma\}}\sup_{T/2\leqq t\leqq 3T/2}\max_{|x|\leqq\exp\{N^\gamma\}}\|\tau_x u_t^{\eta_{kT-T/2}} - m^*\|_{-1}^2 > \varepsilon/16\right]$$

vanishes as $N\uparrow\infty$. To keep notation simple we shall denote $\varepsilon/16$ by ε'.

Assume that $T \geqq 2t_0^4(\varepsilon')$ and that $K \geqq K(2\varepsilon', T/2, T)$ given by Lemma 4.4. Set $Z_N = 3T^{-1}KN\exp\{N^\gamma\}$. The above probability is bounded by

$$P_{\eta(N)}\left[\max_{1\leqq k\leqq T^{-1}\exp\{N^\gamma\}}\max_{|x|\leqq Z_N - kKN}\sup_{T/2\leqq t\leqq 3T/2}\|\tau_x u_t^{\eta_{kT-T/2}} - m^*\|_{-1}^2 > \varepsilon'\right] \;.$$

By Lemma 4.4 and our choice of K, $\max_{|x|\leqq Z_N-kKN}\sup_{T/2\leqq t\leqq 3T/2}\|\tau_x u_t^{\eta_{kT-T/2}}-m^*\|_{-1}^2\leqq\varepsilon'$ provided that we have $\max_{|x|\leqq Z_N-(k-1)KN}\|\tau_x\eta_{kT-T/2}-m^*\|_{-1}^2\leqq 2\varepsilon'$. Last probability is therefore bounded by

$$P_{\eta(N)}\left[\max_{1\leqq k\leqq T^{-1}\exp\{N^\gamma\}}\ \max_{|x|\leqq Z_N-(k-1)KN}\|\tau_x\eta_{kT-T/2}-m^*\|_{-1}^2>2\varepsilon'\right]\ .$$

Denote by $\mathscr{S}_k$ the set of configurations η such that $\max_{|x|\leqq B_n-kKN}\|\tau_x\eta-m^*\|_{-1}^2\leqq 2\varepsilon'$. Last expression is bounded above by

$$P_\eta\left[\max_{|x|\leqq Z_N}\|\tau_x\eta_{T/2}-m^*\|_{-1}^2>2\varepsilon'\right]+\sum_{k=2}^{T^{-1}\exp\{N^\gamma\}}\ \sup_{\eta\in\mathscr{S}_{k-2}}P_\eta[\eta_{T/2}\in\mathscr{S}_{k-1}^c]\ . \tag{4.7}$$

We shall estimate these two terms separately.

For η in $\mathscr{S}_{k-2}$, $T>2t_0^4(\varepsilon')$ and $K\geqq K(\varepsilon',T/2,T)$, by Lemma 4.4, $\|\tau_x u_{T/2}^\eta-m^*\|_{-1}^2\leqq\varepsilon'/2$ for $|x|\leqq Z_N-(k-1)KN$. Therefore, for x in this range, $\|\tau_x\eta_{T/2}-m^*\|_{-1}^2\leqq 2\|\tau_x\eta_{T/2}-\tau_x u_{T/2}^\eta\|_{-1}^2+\varepsilon'$. In particular,

$$\sup_{\eta\in\mathscr{S}_{k-2}}P_\eta[\eta_{T/2}\in\mathscr{S}_{k-1}^c]\leqq\sup_{\eta\in\mathscr{S}_{k-2}}P_\eta\left[\max_{|x|\leqq Z_N-(k-1)KN}\|\tau_x\eta_{T/2}-\tau_x u_{T/2}^\eta\|_{-1}^2>\varepsilon'/2\right].$$

Since the process is translation invariant and $\tau_x u_t^\eta=u_t^{\tau_x\eta}$, by Theorem 1.1, this probability is bounded by

$$Z_N\sup_\eta P_\eta[\|\eta_{T/2}-u_{T/2}^\eta\|_{-1}^2>\varepsilon'/2]\leqq C(T)Z_N\exp\{-C(T)\varepsilon'N^\delta\}$$

for some finite constant $C(T)$ depending on T only. It follows from the definition of Z_N that the second term in (4.7) converges exponentially fast to 0 because, by assumption, $2\gamma<\delta$.

We now turn to the first term of (4.7). Notice that for $T/2\geqq t_0^2(\varepsilon'/2D_1(\theta))$, $\|\tau_x u_{T/2}^{\eta(N)}-m^*\|_{-1}^2\leqq\varepsilon'/2$. Thus $\|\tau_x\eta_{T/2}-m^*\|_{-1}^2\leqq 2\|\tau_x\eta_{T/2}-\tau_x u_{T/2}^{\eta(N)}\|_{-1}^2+\varepsilon'$. The first term in (4.7) is therefore bounded by

$$P_{\eta(N)}\left[\max_{|x|\leqq Z_N}\|\tau_x\eta_{T/2}-\tau_x u_{T/2}^{\eta(N)}\|_{-1}^2>\varepsilon'/2\right]\ .$$

The very same reasons invoked above to prove that the second term in (4.7) converges to 0, shows that this probability vanishes exponentially fast.

To conclude the proof of this proposition, it remains to recollect all restrictions on T and K and check that they are not self contradictory. Notice that $t_0^2(\cdot)$, $t_0^4(\cdot)$ and $K(\cdot,t_1,t_2)$ are increasing functions of ε^{-1}. In light of this, we impose T to be larger than $2t_0^2(\varepsilon/32D_1(\theta))$ and $2t_0^4(\varepsilon/16)$. For such fixed T, we asked K to be larger than $K(\varepsilon/16,T/2,T)$. □

We are now ready to prove Theorem 1.2. Fix a positive integer M and denote by ξ_t the Markov process with generator $N^2L_0+M^{-2}L_G$. Later on M shall be taken as $N^{1-\alpha}$ for $0<\alpha<1$. ξ_t may therefore be interpreted as a small perturbation of the symmetric simple exclusion process. For these fixed N and

M, and a configuration ξ in $\mathscr{X}$, denote by $w_t^{\eta,N,M}: \mathbb{Z} \to [0,1]$ the solution of the equation

$$\begin{cases} \partial_t w_t^{\xi,N,M}(x) = (N^2/2)(\Delta w_t^{\xi,N,M})(x) + M^{-2} G(w_t^{\xi,N,M}(x)) \\ w_0^{\xi,N,M}(x) = \xi(x) \, . \end{cases}$$

The reader should check that the same arguments presented in this section and in Sects. 2 and 3 apply to this new dynamics were the Glauber part is decelerated by M^2. In fact estimates of Sects. 2 and 3 are slightly better since this process is a small perturbation of the symmetric exclusion dynamics for which all estimates are simpler. For the results presented in this section, however, one should be careful since the relaxation times t_0 of Lemmas 4.2, 4.3 and 4.4 will, of course, depend on M since it is the Glauber part that makes the density converge to the equilibrium states of equation $m'(t) = G(m(t))$. More precisely, t_0 is of order M^2. This forces us to apply Theorem 1.1 for times of this order. In this case, in order to estimate probabilities by using Chebyshev inequality and Theorem 1.1, we have to take $M^2 \ll N^\delta$ if we want these probabilities to be exponentially small. This imposes $2(1-\alpha) < \delta$ since by definition $M = N^{1-\alpha}$.

By Proposition 4.5, for all sequences $\xi_{(N)}$ of configurations such that

$$\lim_{N\to\infty} \sup_x |\xi^N_{(N)}(x) - m^*| < d \, ,$$

every $\varepsilon > 0$ and every $\gamma < \delta/2$,

$$P^{N,M}_{\xi_{(N)}} \left[\sup_{t \leqq \exp\{N^\gamma\}} \max_{|x| \leqq \exp\{N^\gamma\}} \|\tau_x \xi_t - \tau_x w_t^{\xi_{(N)}}\|^2_{-1} > \varepsilon \right] \tag{4.8}$$

converges exponentially fast to 0. Here the superindices N, M of P indicates that ξ_t is evolving according to the generator $N^2 L_0 + M^{-2} L_G$.

Denote by ζ_t the process ξ_t accelerated by M^2. ζ_t evolves according to the generator $(NM)^2 L_0 + L_G$. Let $v_t^{\xi,N,M} = w_{tM^2}^{\xi,N,M}$. It is easy to check that v_t is the solution of

$$\begin{cases} \partial_t v_t^{\xi,N,M}(x) = ((NM)^2/2)(\Delta v_t^{\xi,N,M})(x) + G(v_t^{\xi,N,M}(x)) \\ v_0^{\xi,N,M}(x) = \xi(x) \, . \end{cases}$$

By (4.3), $v_t^{\xi,N,M}$ is close to the solution ρ_t^ξ of the hydrodynamic equation (0.1) with initial condition ξ: for all $\varepsilon > 0$, there exists N_0 such that

$$\sup_{t \geqq 0} \sup_x |v_t^{\xi,N,M}(x) - \rho_t^\xi(x/NM)| \leqq \varepsilon \tag{4.9}$$

for all N sufficiently large. Notice that in this last formula, for ρ_t, space is now rescaled by NM.

Since we are rescaling time by a polynomial factor in N and our estimates are exponential, by (4.8),

$$P^{N,M}_{\xi_{(N)}} \left[\sup_{t \leqq \exp\{N^\gamma\}} \max_{|x| \leqq \exp\{N^\gamma\}} \|\tau_x \xi_{tM^2} - \tau_x \omega_{tM^2}^{\xi_{(N)}}\|^2_{-1} > \varepsilon \right]$$

vanishes as $N \uparrow \infty$ for all $\varepsilon > 0$ provided $\gamma < \delta/2$.

Let H be a positive smooth function with compact support and such that $N^{-1}\sum_x H(x/N) = 1$. We have seen in Lemma 4.4 that for all functions ψ in $\ell_2(\mathbb{Z})$, $|N^{-1}\sum_x H(x/N)\tau_x\psi|$ is bounded by $C(H)\|\psi\|_{-1}^2$. In particular, taking $\psi = \tau_x\xi_{tM^2} - \tau_x\omega_{tM^2}^{\xi(N)}$ we obtain that

$$\left|\xi_{tM^2}^H(x) - N^{-1}\sum_y H(y/N)\omega_{tM^2}^{\xi(N)}(y+x)\right|^2 \leqq C'(H)\|\tau_x\xi_{tM^2} - \tau_x\omega_{tM^2}^{\xi(N)}\|_{-1}^2 .$$

Here we used the notation introduced in Lemma 4.4. In particular, for every $\varepsilon > 0$,

$$\lim_{N\to\infty} P_{\xi(N)}^{N,M}\left[\sup_{t\leqq\exp\{N^\gamma\}}\max_{|x|\leqq\exp\{N^\gamma\}}\left|\xi_{tM^2}^H(x) - N^{-1}\sum_y H(y/N)v_t^{\xi(N)}(y+x)\right| > \varepsilon\right] = 0.$$

By (4.9), we may replace in this probability $v_t^{\xi(N)}(y+x)$ by $\rho_t^{\xi(N)}((y+x)/MN)$. On the other hand, arguments similar to the ones of Lemma 4.1 and 4.2 show that if the sequence $\xi_{(N)}$ is associated to the profile $\rho_0 : \mathbb{R} \to [0,1]$ in the sense that

$$\lim_{N\to\infty}\sup_x |\xi_{(N)}^N(x) - \rho_0(x/MN)| = 0 .$$

then $\rho^{\xi(N)}$ is uniformly close to the solution of the hydrodynamic equation (0.1) with initial data ρ_0: for every $\varepsilon > 0$ and $t > 0$, there exists N_0 such that

$$\sup_{t\leqq t_0}\sup_a |\rho_t^{\xi(N)}(a) - \rho(t,a)| \leqq \varepsilon .$$

Moreover, if ρ_0 belongs to the basin of attraction of m^*: $\sup_a|\rho_0(a) - m^*| < d$, then we may iterate the previous argument to extend the inequality to all times:

$$\sup_{t\geqq 0}\sup_a |\rho_t^{\xi(N)}(a) - \rho(t,a)| \leqq \varepsilon$$

for all sufficiently large N. In this case we may replace in the previous probability $\rho_t^{\xi(N)}(a)$ by $\rho(t,a)$. Moreover, in the case where ρ_0 is smooth we get that $\rho(t,\cdot)$ is smooth as well. In particular $N^{-1}\sum_y H(y/N)\rho(t,(x+y)/MN)$ is close to $\rho(t,x/NM)$ since $N^{-1}\sum_y H(y/N) = 1$. In conclusion,

$$\lim_{N\to\infty} P_{\xi(N)}^{N,M}\left[\sup_{t\leqq\exp\{N^\gamma\}}\max_{|x|\leqq\exp\{N^\gamma\}}|\xi_{tM^2}^H(x) - \rho(t,x/MN)| > \varepsilon\right] = 0 .$$

To conclude the proof of Theorem 1.2, it remains to choose a sequence H_k of functions converging to $(1/2)\mathbf{1}\{|a| \leqq 1\}$ and set $\widetilde{N} = NM$.

5. Auxiliary lemmas

In this section we shall prove all bounds used in Sect. 3. We start proving Lemma 2.1.

Proof of Lemma 2.1 From the definition of $\nabla_{z,z+1}$ and ∇_z, it follows that

$$\begin{aligned}
&\nabla_{z,z+1}F(\eta,s)\\
&\quad = \frac{2e^{-\sigma s}}{N^2}[\eta(z+1)-\eta(z)]\sum_x[\eta(x)-u_s(x)][K_{N,\theta}(x,z)-K_{N,\theta}(x,z+1)]\\
&\qquad + \frac{e^{-\sigma s}}{N^2}[\eta(z+1)-\eta(z)]^2\{K_{N,\theta}(z,z)-2K_{N,\theta}(z,z+1)+K_{N,\theta}(z+1,z)\}\,.
\end{aligned} \tag{5.1}$$

Since, by Lemma 6.1, $\sup_x N|K_N(x+1)-K_N(x)| \leqq B_1$, since $\theta(\cdot)$ is a smooth function, and since $\eta(x)$ and $u_t(x)$ are smaller than or equal to 1, the last expression is bounded above by

$$C(\theta,B_1)e^{-\theta(z/N)}N^{-3}\sum_x e^{-\theta(x/N)}\,.$$

This proves the first inequality in Lemma 2.1. On the other hand, we have that

$$\begin{aligned}
\nabla_z F(\eta,x) &= \frac{2e^{-\sigma s}}{N^2}[1-2\eta(z)]\sum_x[\eta(x)-u_s(x)]K_{N,\theta}(x,z)\\
&\quad + \frac{e^{-\sigma s}}{N^2}e^{-2\theta(z/N)}K_N(0)\,.
\end{aligned} \tag{5.2}$$

To conclude the proof we just have to remember that the kernel $K_N(\cdot)$ is bounded in virtue of Lemma 6.1. □

Lemma 5.1 *Recall that u_t^N is the solution of the differential equation* (1.1). *Then,*

$$\begin{aligned}
&\partial_s\|\eta-u_s\|_{-1}^2 + L_N\|\eta-u_s\|_{-1}^2\\
&\quad = N^{-2}\sum_{x,y}[\eta(x)-u_s(x)]K_{N,\theta}(x,y)[N^2\Delta[\eta(\cdot)-u_s(\cdot)]](y)\\
&\qquad + 2N^{-2}\sum_{x,y}[\eta(x)-u_s(x)]K_{N,\theta}(x,y)[\tau_y r_0(\eta)-G(u_s(y))]\\
&\qquad + (1/2)N^{-1}\sum_x(\eta(x+1)-\eta(x))^2N[K_{N,\theta}(x+1,x+1)\\
&\qquad + K_{N,\theta}(x,x)-2K_{N,\theta}(x,x+1)] + N^{-2}\sum_x K_{N,\theta}(x,x)\tau_x r(\eta)\,.
\end{aligned} \tag{5.3}$$

Proof Since $u = u^N$ is the solution of equation 1.1, the time derivative of the H_{-1} norm of the difference $\eta - u_s$ is easy to compute. It is given by

$$\partial_s\|\eta-u_s\|_{-1}^2 = -2\sum_{x,y}\{\eta(x)-u_s(x)\}K_{N,\theta}(x,y)\{(N^2/2)(\Delta u_s)(y)+G(u_s(y))\}$$

To conclude the proof of the lemma we just need to recall from the proof of Lemma 2.1 the computation of $L_N\|\eta-u_s\|_{-1}^2$. □

We conclude this section by proving an upper bound for $\partial_s\|\eta-u_s\|_{-1}^2 + L_N\|\eta-u_s\|_{-1}^2$ that was needed in the proof of Theorem 1.1.

Lemma 5.2 *There exists positive finite constants $C_0(r,\theta)$, $\sigma = \sigma(r,\theta)$, $C_1(\theta)$ and $C_2(r)$, such that for every positive integer ℓ,*

$$\begin{aligned}\partial_s\|\eta - u_s\|_{-1}^2 + L_N\|\eta - u_s\|_{-1}^2 \leqq\ & C_0(r,\theta)\frac{\sqrt{\ell}}{\sqrt{N}} + \sigma\|\eta - u_s\|_{-1}^2 \\ & + C_1(\theta)\frac{\ell}{N}N^{-1}\sum_x |(N\nabla u)(x)|e^{-2\theta(x/N)} \\ & + C_2(r)N^{-1}\sum_x |\tau_x V_{r_0,\ell}(\eta)|e^{-2\theta(x/N)} \\ & + N^{-1}\sum_x |\tau_x V_{W,\ell}(\eta)|e^{-2\theta(x/N)}\ .\end{aligned}$$

Proof We obtained at Lemma 5.1 an explicit expression for $\partial_s\|\eta - u_s\|_{-1}^2 + L_N\|\eta - u_s\|_{-1}^2$. We will bound each term of this explicit expression separately.

We start with the fourth line of the right hand side of (5.3) which is the simplest one. Since the kernel $K_N(\cdot)$ is uniformly bounded, the fourth line is bounded by $C(r,\theta)N^{-1}$.

The third line is also simple to compute. Since, by Lemma 6.1, $2N[K_N(0) - K_N(1)] \leqq 1$ it is not difficult to see that this expression is bounded above by

$$\frac{1}{2N}\sum_x (\eta(x+1) - \eta(x))^2 e^{-2\theta(x/N)} + C(\theta)N^{-1}\ . \tag{5.4}$$

We now turn to the first line on the right hand side of (5.3). A simple summation by parts and the computation of the discrete laplacian applied to a product of two functions shows that this expression is equal to

$$\begin{aligned}& -N^{-2}\sum_{x,y}[\eta(x) - u_s(x)]e^{-\theta(x/N)}(-N^2\Delta K_N(x,\cdot))(y)[\eta(y) - u_s(y)]e^{-\theta(y/N)} \\ & + N^{-2}\sum_{x,y}[\eta(x) - u_s(x)]e^{-\theta(x/N)}K_N(x,y)(N^2\Delta e^{-\theta(\cdot/N)})(y)[\eta(y) - u_s(y)] \\ & + N^{-2}\sum_{x,y}[\eta(x) - u_s(x)]e^{-\theta(x/N)}(N\nabla K_N(x,\cdot))(y) \\ & \quad \times (N\nabla e^{-\theta(\cdot/N)})(y)[\eta(y) - u_s(y)] \\ & + N^{-2}\sum_{x,y}[\eta(x) - u_s(x)]e^{-\theta(x/N)}(N\nabla K_N(x,\cdot))(y-1) \\ & \quad \times (N\nabla e^{-\theta(\cdot/N)})(y-1)[\eta(y) - u_s(y)]\ .\end{aligned} \tag{5.5}$$

Since $K_N(\cdot)$ is the kernel $(I - N^2\Delta)^{-1}$, adding and subtracting the ℓ_2 norm of $\eta - u$, we obtain that the first line is equal to

$$-\|\eta - u\|_0^2 + \|\eta - u\|_{-1}^2\ . \tag{5.6}$$

Notice that the summation of this expression with (5.4) is equal to

$$-N^{-1}\sum_x [\eta(x)\eta(x+1) - 2\eta(x)u_s(x) + u_s(x)^2]e^{-2\theta(x/N)} + \|\eta - u\|_{-1}^2 + C(\theta)N^{-1}\ .$$

Since the expectation of $\eta(x)\eta(x+1)$ with respect to ν_ρ is equal to ρ^2, it is natural to use the one block estimate to replace $\eta(x)\eta(x+1)$ by $(\eta^\ell(x))^2$ and to perform a summation by parts to replace $\eta(x)$ by $\eta^\ell(x)$ in order to close the square. We rewrite thus the sum of the first line of (5.5) with (5.6) as

$$
\begin{aligned}
&- N^{-1}\sum_x [\eta^\ell(x) - u_s(x)]^2 e^{-2\theta(x/N)} + \|\eta - u_s\|_{-1}^2 \\
&\quad - N^{-1}\sum_x \left\{(2\ell'+1)^{-1}\sum_{|y-x|\leqq \ell'} \eta(y)\eta(y+1) - (\eta^\ell(x))^2\right\} e^{-2\theta(x/N)} \\
&\qquad + 2N^{-1}\sum_x \eta(x)[u_s(x) - u_s^{\ell'}(x)]e^{-2\theta(x/N)} + C(\theta)\ell N^{-1}\ .
\end{aligned}
$$

Here $\ell' = \ell - 1$ so that $\sum_{|y|\leqq \ell'}\eta(y)\eta(y+1)$ depends on η only through $\{\eta(-\ell),\dots,\eta(\ell)\}$.

Notice that the first term of this expression comes with a negative sign. We are thus allowed in the rest of the proof to bound any expression by a small constant times the ℓ_2 norm of $\eta^\ell - u_s$. In order to take advantage of this negative term, we will need to use the one block estimate to replace cylinder functions by functions that depend on $\eta^\ell(x)$. We shall do it systematically up to the end of the proof.

We now turn to the second line of (5.5). We have just argued that the first step should be a one block estimate. We thus rewrite the second line of (5.5) as

$$
\begin{aligned}
&N^{-2}\sum_{x,y}[\eta(x) - u(x)]e^{-\theta(x/N)}K_N(x,y)(N^2\Delta e^{-\theta(\cdot/N)})(y)\big[\eta(y) - \eta^\ell(x)\big] \\
&\quad + N^{-2}\sum_{x,y}[\eta(x) - u(x)]e^{-\theta(x/N)}K_N(x,y)(N^2\Delta e^{-\theta(\cdot/N)})(y)\big[\eta^\ell(y) - u(y)\big]\ .
\end{aligned}
$$

Since $\theta(\cdot)$ is a smooth function and $K_N(\cdot)$ has a bounded discrete derivative, by summation by parts, the first line is bounded above by $C(\theta)\ell N^{-1}$. On the other hand, by Schwarz inequality, the second line is bounded above by

$$
(\varepsilon^{-1}/2)\|\eta - u_s\|_{-1}^2 + (\varepsilon/2)\|(N^2\Delta e^{-\theta(\cdot/N)})e^{\theta(\cdot/N)}[\eta^\ell - u_s]\|_{-1}^2
$$

for any positive ε. Since the H_{-1} norm is bounded above by the ℓ_2 norm and since the function $(N^2\Delta e^{-\theta(\cdot/N)})e^{\theta(\cdot/N)}$ is uniformly bounded, we have shown that the second line of (5.5) is bounded above by

$$
(\varepsilon^{-1}/2)\|\eta - u_s\|_{-1}^2 + (\varepsilon/2)\|\eta^\ell - u_s\|_0^2 + C(\theta)\ell N^{-1}
$$

for any positive ε.

We now turn to the third line of (5.5). Repeating the strategy adopted above, we first introduce by force $\eta^\ell(y)$ and rewrite the third line as

$$
\begin{aligned}
&N^{-2}\sum_{x,y}[\eta(x) - u_s(x)]e^{-\theta(x/N)}(N\nabla K_N(x,\cdot))(y)(N\nabla e^{-\theta(\cdot/N)})(y)[\eta(y) - \eta^\ell(y)] \\
&\quad + N^{-2}\sum_{x,y}[\eta(x) - u_s(x)]e^{-\theta(x/N)}(N\nabla K_N(x,\cdot))(y) \\
&\qquad \times (N\nabla e^{-\theta(\cdot/N)})(y)[\eta^\ell(y) - u_s(y)]\ . \qquad (5.7)
\end{aligned}
$$

The first line of this decomposition, by summation by parts, is equal to

$$N^{-2}\sum_{x,y} R_\theta(x)(2\ell+1)^{-1}\sum_{|z-y|\leqq\ell}(N\nabla K_N(x,\cdot))(z)\{(N\nabla e^{-\theta(\cdot/N)})(y)$$

$$-(N\nabla e^{-\theta(\cdot/N)})(z)\}\eta(y)$$

$$+N^{-2}\sum_{x,y} R_\theta(x)(2\ell+1)^{-1}\sum_{|z-y|\leqq\ell}\{(N\nabla K_N(x,\cdot))(y)-(N\nabla K_N(x,\cdot))(z)\}$$

$$\times(N\nabla e^{-\theta(\cdot/N)})(y)\eta(y)\,.$$

Here and below, to keep notation simple, we abbreviated $[\eta(x)-u(x)]e^{-\theta(x/N)}$ as $R_\theta(x)$. Since $\theta(\cdot)$ is a smooth function and $K_N(\cdot)$ has a bounded discrete derivative, the first line is bounded above by $C(\theta)\ell N^{-1}$.

Since K_N has a singularity at the origin, to bound the second line, we shall fix a positive integer $A\geqq 2\ell$ and decompose the sum according to the value of $|x-y|$. For the values of x and y such that $|x-y|\leqq A$, since everything is bounded, the sum is bounded by $C(\theta)AN^{-1}$. On the other hand, for values of x and y such that $|x-y|\geqq A$, since, by Lemma 6.1, the discrete Laplacian of $K_N(\cdot)$ at z is bounded by $N/|z|$, the sum is bounded above by

$$C(\theta)N^{-2}\sum_{|x-y|\geqq A} e^{-\theta(x/N)}e^{-\theta(y/N)}$$

$$\times\left|(2\ell+1)^{-1}\sum_{|z-y|\leqq\ell}\{(N\nabla K_N(x,\cdot))(y)-(N\nabla K_N(x,\cdot))(z)\}\right|$$

$$\leqq C'(\theta)N^{-2}\sum_{|x-y|\geqq A} e^{-\theta(x/N)}e^{-\theta(y/N)}\frac{\ell}{|x-y|-\ell}\,.$$

Since we chose $A\geqq 2\ell$, this sum is bounded above by $C(\theta)\ell A^{-1}$. Minimizing over $A\geqq 2\ell$ and recollecting all terms, we obtain that the first line of (5.7) is bounded above by $C(\theta)\ell^{1/2}N^{-1/2}$.

We now turn to the second line of (5.7). By Schwarz inequality it is bounded above by

$$(\varepsilon/2)C(\theta)N^{-1}\sum_y[\eta^\ell(y)-u_s(y)]^2e^{-2\theta(y/N)}$$

$$+(\varepsilon^{-1}/2)N^{-1}\sum_y\left\{N^{-1}\sum_x R_\theta(x)\,[N\nabla K_N(x,\cdot)](y)\right\}^2$$

for any positive ε. The first term is just the ℓ_2 norm of $\eta^\ell-u_s$, while the second, by summation of parts, may be rewritten as

$$(\varepsilon^{-1}/2)N^{-1}\sum_y[K_N*R_\theta](y)[(-N^2\Delta K_N)*R_\theta](y)\,.$$

Here $f*g$ denotes the convolution of two functions f and g in $\ell_2(\mathbb{Z})$. A simple argument involving Fourier transforms shows that this expression is bounded above by $\varepsilon^{-1}/2$ times the H_{-1} norm of $\eta - u_s$. In conclusion, we proved that the third line of (5.5) is bounded above by

$$C(\theta)\varepsilon\,\|\eta^\ell - u_s\|_0^2 + (\varepsilon^{-1}/2)\|\eta - u_s\|_{-1}^2 + C(\theta)\ell^{1/2}N^{-1/2}\ .$$

By similar reasons the fourth line of (5.5) is bounded by the same expression. It is now time to summarize what we did up to this point. Choosing an appropriate ε, we proved that the sum of the first, third and fourth line of (5.3) are bounded above by

$$-(1/2)\|\eta^\ell - u_s\|_0^2 + C(\theta)\|\eta - u_s\|_{-1}^2 + C_0(r,\theta)\frac{\sqrt{\ell}}{\sqrt{N}}$$
$$+ C_1(\theta)\frac{\ell}{N}N^{-1}\sum_x |(N\nabla u)(x)|e^{-2\theta(x/N)} + N^{-1}\sum_x |\tau_x V_{W,\ell}(\eta)|e^{-2\theta(x/N)}$$

for all $\ell \geqq 1$ and some finite constant $C(\theta)$.

We now turn to the second line on the right hand side of (5.3). Again, the first step consists in applying the one block estimate. We therefore rewrite this expression as

$$2N^{-2}\sum_{x,y} R_\theta(x)K_N(x,y)e^{-\theta(y/N)}\left\{\tau_y r_0(\eta) - (2\ell'+1)^{-1}\sum_{|z-y|\leqq\ell'}\tau_z r_0(\eta)\right\}$$
$$+ 2N^{-2}\sum_{x,y} R_\theta(x)K_N(x,y)e^{-\theta(y/N)}\left\{(2\ell'+1)^{-1}\sum_{|z-y|\leqq\ell'}\tau_z r_0(\eta) - G(\eta^\ell(y))\right\}$$
$$+ 2N^{-2}\sum_{x,y} R_\theta(x)K_N(x,y)e^{-\theta(y/N)}\{G(\eta^\ell(y)) - G(u_s(y))\}\ . \tag{5.8}$$

In this formula $\ell' = \ell(r_0)$ stands for $\ell - s(r_0)$, where $s(r_0)$ denotes the length of the support of r_0 and is defined in Sect. 2 just after (2.4).

By summation of parts since $r_0(\eta)$ and the discrete derivatives of the kernel K_N are bounded, the first line is bounded above by $C(\theta)\ell N^{-1}$.

By Schwarz inequality and since the H_{-1} norm is bounded above by the ℓ_2 norm, the second line is bounded by

$$\varepsilon\left\|(2\ell'+1)^{-1}\sum_{|z|\leqq\ell'}\tau_z r_0 - G(\eta^\ell(0))\right\|_{-1}^2 + \varepsilon^{-1}\,\|\eta - u_s\|_0^2$$

for every positive ε. Since r_0 is a bounded function the first term is bounded by

$$C(r)\varepsilon N^{-1}\sum_x e^{-\theta(x/N)}|\tau_x V_{\ell,r_0}|\ .$$

Here we used again notation introduced in (2.4).

At last, by similar reasons, the third line of (5.8) is bounded above by

$$\varepsilon^{-1}\|\eta - u_s\|_{-1}^2 + \varepsilon\|G(\eta^\ell(0)) - G(u_s)\|_0^2 \leqq \varepsilon^{-1}\|\eta - u_s\|_{-1}^2 + \varepsilon\|G'\|_\infty^2\|\eta^\ell(0) - u_s\|_0^2$$

Thus the second line of (5.3) is bounded above by

$$2\varepsilon^{-1}\|\eta - u_s\|_{-1}^2 + \varepsilon\|G'\|_\infty^2 \|\eta\ell(0) - u_s\|_0^2$$
$$+ C(r)\varepsilon N^{-1} \sum_x e^{-\theta(x/N)} |\tau_x V_{\ell, r_0}| + C(\theta)\ell N^{-1} .$$

To conclude the proof of the lemma, we just have to collect all terms. □

6 Appendix

In this last section we present some results concerning the operator $(I - N^2\Delta)^{-1}$ used throughout the article and on solutions of the differential equation (1.1).

For a fixed positive integer N endow the space $\ell^2(\mathbb{Z}) = \{f : \mathbb{Z} \to \mathbb{R}; \sum_x f(x)^2 < \infty\}$ with the inner product

$$\langle f, g \rangle = \frac{1}{N} \sum_x f(x) g(x) .$$

Moreover, for two functions f and g in $\ell_2(\mathbb{Z})$, we represent by $f * g$ the convolution of f with $g : (f * g)(x) = N^{-1} \sum_y f(y) g(x - y)$. Recall also from Sect. 2 that for f in $\ell_2(\mathbb{Z})$ we denote, respectively, by ∇f and Δf the discrete derivative and discrete laplacian of f:

$$(\nabla f)(z) = f(z+1) - f(z)$$
$$(\Delta f)(z) = f(z+1) + f(z-1) - 2f(z) .$$

For f in $\ell^2(\mathbb{Z})$ define the Fourier transform $\hat{f} : [-\pi, \pi] \to \mathbb{C}$ of f by

$$\hat{f}(\theta) = \frac{1}{N} \sum_x f(x) e^{i\theta x} .$$

It is straightforward to see that the inverse Fourier transform is then given by

$$\check{f}(x) = \frac{N}{2\pi} \int_{-\pi}^{\pi} f(\theta) e^{-i\theta x} \, d\theta .$$

Define $K_N : \mathbb{Z} \to \mathbb{R}$ by

$$K_N(x) = \frac{N}{2\pi} \int_{-\pi}^{\pi} \frac{\cos(\theta x)}{1 + 2N^2(1 - \cos\theta)} \, d\theta$$

Then an easy computation involving Fourier transforms shows that

$$(I - N^2\Delta)^{-1} f(x) = (K_N * f)(x) .$$

$K_N(\,\cdot\,)$ is a regular function with a singularity at the origin. More precisely, fix $0 < a < 1$. The Fourier transform of the sequence $a^{|x|}$, $x \in \mathbb{Z}$, is equal to

$$\frac{1}{N} \frac{1 - a^2}{(1-a)^2 + 2a(1 - \cos\theta)} = \frac{1+a}{\sqrt{a}} \frac{1}{1 + 2N^2(1 - \cos\theta)}$$

if $N = \sqrt{a}/(1-a)$. In particular,

$$K_N(x,y) \;=\; \frac{\sqrt{a}}{1+a} a^{|x-y|} \tag{6.1}$$

with a the solution of $N = \sqrt{a}/(1-a)$. We now summarize the properties of the kernel K_N needed in the proof of the exponential estimates.

Lemma 6.1 *There exists an universal constant B_1 so that*

$$\sup_{x\in\mathbb{Z}} |K_N(x)| \leqq K_N(0) \leqq B_1, \qquad \sup_{x\in\mathbb{Z}} |N\nabla K_N(x)| \leqq B_1\,,$$

$$\sup_{x\in\mathbb{Z}} \left\{ |N^2 \Delta K_N(x)| - B_1 \frac{N}{1+|x|} \right\} \leqq 0 \quad \text{and} \quad N|K_N(1) - K_N(0)| \leqq 1/2 \tag{6.2}$$

for every $N \geqq 1$.

The proof is standard and thus omitted.

We now turn to the proof of some regularity of the solution of the differential equation (1.1).

Lemma 6.2 *For each N and each initial condition $u^N : \mathbb{Z} \to [0,1]$ there exists a unique solution u^N of the equation*

$$\begin{cases} \partial_t\, u_t^N(x) = \dfrac{N^2}{2}(\Delta u_t^N)(x) + G(u_t^N(x))\,, \\ u_0^N(\,\cdot\,) = u^N(\,\cdot\,)\,. \end{cases}$$

Moreover the solution is positive, bounded by 1 *and there exists a constant B_2 depending only on G and θ such that*

$$\int_0^t ds\, \frac{1}{N} \sum_x |(N\nabla u_s)(x)|^2 e^{-2\theta(x/N)} \;\leqq\; 1 + B_2 t\,.$$

Proof Existence and uniqueness in finite volume are proven by usual contraction methods (cf. [Sm, Chapter 11] for instance). That the solution is positive and bounded by 1 follows from the maximum principle since $G(0) \geqq 0$ and $G(1) \leqq 0$. A standard argument proves the existence of a solution for the infinite volume problem from existence in finite volume. Uniqueness in infinite volume follows, for instance, by H_{-1} methods.

We now prove a bound on the ℓ_2 norm of the derivative of solutions. By a summation by parts and since the solution is bounded by 1, we have that

$$\partial_t \frac{1}{N} \sum_{|x|\leqq A} (u_t(x))^2 e^{-2\theta(x/N)} \;\leqq\; -\frac{1}{N} \sum_{|x|\leqq A} (N\nabla u_t^N(x))^2 e^{-2\theta(x/N)}$$

$$+ \frac{2}{N} \sum_{|x|\leqq A} u_t^N(x)\, G(u_t^N(x)) e^{-2\theta(x/N)} + CNe^{-2\theta(A/N)}$$

$$- \frac{1}{N} \sum_{|x|\leqq A} u_t^N(x+1)\,(N\nabla u_t^N(x)) N\{e^{-2\nabla\theta(x/N)} - 1\} e^{-2\theta(x/N)}$$

for some universal constant C and every positive integer A. Therefore, integrating over time and applying the elementary inequality $2ab \leqq a^2 + b^2$, we get that

$$\int_0^t ds\, \frac{1}{2N} \sum_{|x| \leqq A} (N\nabla u_s^N(x))^2 e^{-2\theta(x/N)} \leqq \frac{1}{N} \sum_{|x| \leqq A} (u_0^N(x))^2 e^{-2\theta(x/N)}$$

$$- \frac{1}{N} \sum_x (u_t^N(x))^2 e^{-2\theta(x/N)} + \frac{2}{N} \int_0^t \sum_{|x| \leqq A} u_s^N(x)\, G(u_s^N(x)) e^{-2\theta(x/N)}\, ds$$

$$+ CtNe^{-2\theta(A/N)} + \frac{1}{2N} \int_0^t \sum_{|x| \leqq A} (u_s^N(x+1))^2 \,[N\{e^{-2\nabla\theta(x/N)} - 1\}]^2 e^{-2\theta(x/N)}\, ds\,.$$

We conclude the proof of the lemma letting A increase to ∞ and recalling that the solution is bounded by 1 in absolute value. □

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