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Limiting angle of Brownian motion on certain manifolds

Huiling Le

University of Nottingham, Department of Mathematics, University Park, Nottingham NG7 2RD, UK

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Summary. Suppose that $\mathbf{M}$ is a complete, simply connected Riemannian manifold of non-positive sectional curvature with dimension $m \geqq 3$. If, outside a fixed compact set, the sectional curvatures are bounded above by a negative constant multiple of the inverse of the square of the geodesic distance from a fixed point and below by another negative constant multiple of the square of the geodesic distance, then the angular part of Brownian motion on $\mathbf{M}$ tends to a limit as time tends to infinity, and the closure of the support of the distribution of this limit is the entire $\mathbf{S}^{m-1}$. This improves a result of Hsu and March.

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1 Introduction

The existence of non-constant bounded harmonic functions on general Riemannian manifolds has been investigated using both geometric and probabilistic methods. A detailed survey of recent progress on this area has been given in the introduction to [6]. The following is a summary of the results relevant to this paper.

Greene and Wu conjectured in [4] that a Cartan–Hadamard manifold, that is a complete, simply connected Riemannian manifold with non-positive sectional curvature, always possesses non-constant bounded harmonic functions if, outside a fixed compact set, the upper bound of its curvatures decays proportionally to the inverse of the square of the geodesic distance from a fixed point.

For a rotationally symmetric manifold of negative curvature, March proved in [10], by considering the condition for the invariant σ-field of Brownian motion to be non-trivial, that there exist non-constant bounded harmonic functions on the manifold if the radial curvatures at any point x are bounded above by

$-c/(r^2 \log r)$ for $c > c_m$, where r is the Riemannian distance from x to a fixed reference point of the manifold with respect to which it is rotationally symmetric, m denotes the dimension of the manifold and $c_2 = 1$, $c_m = 1/2$ for $m \geqq 3$. If, instead, the above bound is the lower bound of the radial curvatures, then there exist no non-constant bounded harmonic functions.

For a general simply connected manifold of negative curvature, one probabilistic method for constructing non-constant bounded harmonic functions is to consider the asymptotic behaviour of the angular component of Brownian motion on the manifold. For instance, Hsu and Kendall confirmed in [6] the Greene and Wu conjecture for the case of 2-dimensional manifolds using this method to extend ideas of [9, 5]. They proved that, under the appropriate Greene and Wu hypotheses, the angular component of Brownian motion converges to a limit as time tends to infinity and the closure of the support of the distribution of this limit is the entire circle of possible directions. For a manifold with dimension at least 3, Hsu and March proved in [5] a similar result if, off a given compact set, the sectional curvatures are bounded above by $-cr^{-2}$ for $c > 2$ and below by $-\tilde{c}r^{2\beta}$ for $\tilde{c} > 0$ and $\beta < 1 - 4/(1 + \sqrt{1 + 4c})$. Note that this requires β to approach zero as c approaches 2.

In this paper we obtain a similar result again but under more satisfyingly symmetric constraints than that of Hsu and March. Firstly we may take $\beta = 1$ irrespective of c and, except in dimension 3, c itself may be an arbitrary positive number. The strategy of our proof is a combination of that of [6] with ideas similar to those of Darling in [2]. In particular it is the introduction of a function analogous to Darling's persistence functions which enables us to weaken the hypotheses on the lower bound. Note that, if a manifold with dimension at least 3 has uncontrolled negative sectional curvatures then Brownian motion upon it may have a non-random limiting direction or no limiting direction at all (cf. [6]).

2 The main theorem

We assume throughout that $\mathbf{M}$ is a complete m-dimensional simply connected Riemannian manifold of non-positive sectional curvature, where $m > 2$. Then $\mathbf{M}$ is diffeomorphic to $\mathbb{R}^m$, with the diffeomorphism realised by the exponential map at any fixed reference point o in $\mathbf{M}$, so that $\mathbf{M}$ has global geodesic polar coordinates $(r, \theta) \in \mathbb{R}_+ \times \mathbf{S}^{m-1}$ with respect to o. In particular, $r(x)$ gives the distance between x and o. Suppose moreover that, for x outside a compact set, the sectional curvatures of $\mathbf{M}$ at x are bounded above by $-cr^{-2}(x)$ and below by $-\tilde{c}r^2(x)$, where c and $\tilde{c}$ are two arbitrary positive constants, except that we require $c > 3/4$ when $m = 3$. Without loss of generality we may take the compact set to be $\{x\colon\ r(x) \leqq r_1\}$, where r_1 is at least 1 and satisfies the further technical restrictions, which we shall require, that (1) $2^{3/2}\tilde{c}^{1/4} r_1 \sqrt{\log r_1} \geqq 1$ and (2) $\coth(2\sqrt{\tilde{c}}r_1) \leqq 2$.

Suppose that X is Brownian motion on $\mathbf{M}$ constructed on a probability space $(\Omega, \mathscr{F}, \mathscr{F}_t, \mathbb{P})$ and define $R_t = r(X_t)$, $\Theta_t = \theta(X_t)$, then our result is the following.

Theorem. *With the notation and hypotheses on* **M** *stated above*,

$$\mathbb{P}\left[\lim_{t\to\infty} \Theta_t \text{ exists}\right] = 1 .$$

The closure of the support of the probability law of the limit of Θ_t *is* $\mathbf{S}^{m-1}$.

Our proof, like that of [6], has three main ingredients. We first use a comparison of **M** with a rotationally symmetric manifold to obtain an inequality relating the 'angular' distance between two points to their metric distance. Secondly we compare the radial part R of X, starting from an arbitrary point $x_0 = (r, \theta)$ of **M**, with a suitable Bessel process to obtain a probabilistic lower bound, $p(r)$, on the rate of growth of R, where $p(r) \to 1$ as $r \to \infty$. Finally we obtain a sequence of stopping times T_n which, with probability at least $p(r)$, tend to infinity in such a way as to fit together with the other estimates to give the required result. That the T_n themselves tend to infinity follows from an estimate on the growth of ρ, the distance from the starting point x_0, obtained in the manner of Darling [2], from a bound on the Hessian of ρ.

3 The angular distance

We denote by $\delta(\theta_1, \theta_2)$ the distance, measured on the unit tangent sphere at the point o, between two of its points θ_1, θ_2. Let $\alpha = (1 + \sqrt{1+4c})/2$ so that $c = \alpha(\alpha - 1)$ and $\alpha > 1$. We fix ε such that $0 < \varepsilon < \min\{1, \alpha - 1\}$. For the following lemma we would only need $\varepsilon < 1$ but, when it is applied in Sect. 6, we shall also require $\alpha - \varepsilon > 1$.

Lemma 1 *There exists* $r_\varepsilon > 2r_1$ *such that if* $r(x_1) \geqq r_\varepsilon$ *and* $\operatorname{dist}(x_1, x_2) \leqq r^\varepsilon(x_1)$ *then*

$$\delta(\theta(x_1), \theta(x_2)) \leqq 2^\alpha \frac{\operatorname{dist}(x_1, x_2)}{r^\alpha(x_1)} .$$

Proof. Let $\tilde{\mathbf{M}}$ be an m-dimensional rotationally symmetric manifold with a pole $\tilde{o}$ and with the Riemannian metric given by $d\tilde{s}^2 = d\tilde{r}^2 + g^2(\tilde{r})\, d\tilde{\theta}^2$, where $(\tilde{r}, \tilde{\theta})$ is the geodesic polar coordinates around $\tilde{o}$. Then the radial curvature of $\tilde{\mathbf{M}}$ at $\tilde{x}$ is given by (cf. [4, p. 30])

$$-g''(\tilde{r}(\tilde{x}))/g(\tilde{r}(\tilde{x})) .$$

This will always be greater than or equal to the radial curvatures of **M** at x with $r(x) = \tilde{r}(\tilde{x})$ and equal to $-c\tilde{r}^{-2}(\tilde{x})$ if $\tilde{r} \geqq r_2$, for any given $r_2 > r_1$, provided we can find a smooth function $g \geqq 0$ defined on $\mathbb{R}_+$ satisfying the following conditions.
(1) $g(0) = 0$ and $g'(0) = 1$;
(2) for $t \leqq r_2$, $-g''(t)/g(t)$ is bounded below by the supremum of the sectional curvatures of **M** at x with $r(x) = t$;
(3) for $t \geqq r_2$, $g(t) = t^\alpha$.
However the existence of such smooth functions can be established as follows. Fix any smooth increasing function $h : \mathbb{R}_+ \to [0, 1]$ such that $h|_{[0, r_1]} = 0$ and

$h|_{[r_2,\infty)} = 1$. Write $K(t) = ch(t)t^{-2}$. Then the solution for the differential equation $g''(t)/g(t) = K(t)$ with the boundary conditions $g(0) = 0$ and $g'(0) = 1$ has the required properties.

Suppose first that $\tilde{x}_1$ and $\tilde{x}_2$ are two points in $\tilde{\mathbf{M}}\backslash\{\tilde{o}\}$ which are the images of v_1 and v_2 in the tangent space $\tau_{\tilde{o}}(\tilde{\mathbf{M}})$ at $\tilde{o}$, under the exponential map $\exp_{\tilde{o}}$ at $\tilde{o}$, and are such that $\text{dist}(\tilde{x}_1,\tilde{x}_2) \leqq \tilde{r}^{\varepsilon}(\tilde{x}_1)$. Write $\delta_0 = \delta(\tilde{\theta}(\tilde{x}_1),\tilde{\theta}(\tilde{x}_2))$. Without loss of generality, we may assume that $\delta_0 > 0$. Then the image, under $\exp_{\tilde{o}}$, of the linear subspace of $\tau_{\tilde{o}}(\tilde{\mathbf{M}})$ spanned by v_1 and v_2 is a 2-dimensional totally geodesic submanifold, $\tilde{\mathbf{M}}_0$, of $\tilde{\mathbf{M}}$ and the induced Riemannian metric structure on $\tilde{\mathbf{M}}_0$ still has the form

$$d\tilde{r}^2 + g^2(\tilde{r})\, d\tilde{\theta}^2 ,$$

where $(\tilde{r},\tilde{\theta})$ are the geodesic polar coordinates on $\tilde{\mathbf{M}}_0$ (cf. [4, pp. 25, 30]). If the geodesic segment $\tilde{x}(s)$ joining $\tilde{x}_1$ and $\tilde{x}_2$, which lies entirely in $\tilde{\mathbf{M}}_0$, has parameter $0 \leqq s \leqq \delta_0$ then, by the triangle inequality, we have

$$\tilde{r}(\tilde{x}(s)) \geqq \tilde{r}(\tilde{x}_1) - \text{dist}(\tilde{x}_1,\tilde{x}(s)) \geqq \tilde{r}(\tilde{x}_1) - \text{dist}(\tilde{x}_1,\tilde{x}_2), \quad 0 \leqq s \leqq \delta_0 .$$

Thus, there exists an $r_\varepsilon \geqq 2r_2$ such that, if $\tilde{r}(\tilde{x}_1) \geqq r_\varepsilon$, then $2\tilde{r}(\tilde{x}(s)) \geqq \tilde{r}(\tilde{x}_1)$, $\forall\, 0 \leqq s \leqq \delta_0$. This implies that, if $\tilde{r}(\tilde{x}_1) \geqq r_\varepsilon$, then

$$\begin{aligned}\text{dist}(\tilde{x}_1,\tilde{x}_2) &= \int_0^{\delta_0} \sqrt{(\tilde{r}'(\tilde{x}(s)))^2 + g^2(\tilde{r}(\tilde{x}(s)))}\, ds \\ &\geqq \int_0^{\delta_0} g(\tilde{r}(\tilde{x}(s)))\, ds = \int_0^{\delta_0} \tilde{r}^{\alpha}(\tilde{x}(s))\, ds \geqq 2^{-\alpha}\tilde{r}^{\alpha}(\tilde{x}_1)\delta_0 .\end{aligned}$$

Now, suppose that x_1 and x_2 are two points in $\mathbf{M}\backslash\{o\}$ such that $\text{dist}(x_1,x_2) \leqq r^{\varepsilon}(x_1)$. Choose two points $\tilde{x}_i$ in $\tilde{\mathbf{M}}$ such that $\tilde{r}(\tilde{x}_i) = r(x_i)$, $i = 1, 2$, and $\delta(\tilde{\theta}(\tilde{x}_1),\tilde{\theta}(\tilde{x}_2)) = \delta(\theta(x_1),\theta(x_2))$. Then there is a linear isomorphism I between the tangent spaces $\tau_0(\mathbf{M})$ and $\tau_{\tilde{o}}(\tilde{\mathbf{M}})$ such that the corresponding polar map, $\exp_{\tilde{o}} \circ I \circ \exp_o^{-1}$, takes x_i to $\tilde{x}_i$. Thus, by the Rauch Comparison Theorem (cf. [1, p. 30]),

$$\text{dist}(\tilde{x}_1,\tilde{x}_2) \leqq \text{dist}(x_1,x_2) \leqq \tilde{r}^{\varepsilon}(\tilde{x}_1) ,$$

and the result follows from the above. □

4. The rate of growth of the radial part of X

Write $\partial_{\mathbf{M}}$ for the radial tangent vector field $\partial/\partial r$ on $\mathbf{M}$. Since the Ricci curvature of a radial tangent vector can be expressed as the sum of the sectional curvatures of the planes spanned by the radial tangent vector together with each of a set of tangent vectors which are orthonormal to it and orthogonal each other (cf. [11, p. 88]), the Ricci curvature of $\partial_{\mathbf{M}}$ at x is less than or equal to the Ricci curvature of $\partial_{\mathbb{R}^m}$ at any $\hat{x}$ such that the radial component of $\hat{x}$ is equal to $r(x)$. By the Laplacian Comparison Theorem (cf. [4, p. 26]), we have

$$\Delta r \geqq \frac{m-1}{r} \quad \forall r > 0$$

as $(m-1)/r$ is the Laplacian of the radial function on $\mathbb{R}^m$ (cf. [4, p. 30]). Thus the Itô stochastic differential equation for the radial part R_t of X satisfies

$$dR_t = dB_t + \frac{1}{2}\Delta R_t\, dt \geqq dB_t + \frac{m-1}{2R_t}\, dt\ ,$$

where B is Brownian motion on $\mathbb{R}$. Since Brownian motion in $\mathbb{R}^m$ is transient for $m \geqq 3$, it follows that R will tend to infinity as time tends to infinity. Outside our compact set, $\{x\colon\ r(x) \leqq r_1\}$, we can obtain a lower bound for the rate of growth of the radial component of R of X.

Let $\hat{\alpha} = (m-1)\alpha$, BES_r^β be the Bessel process of index β starting at r and denote by $\mathbb{P}^{r,\theta}$ the conditional probability measure obtained from $\mathbb{P}$ by conditioning on $X_0 = (r, \theta)$.

Lemma 2 *For $r > r_1^2$ and for all sufficiently small η*

$$\mathbb{P}^{r,\theta}[R_t \geqq \sqrt{r} \vee t^{1/2-\eta},\ \forall t] \geqq p(r)\ ,$$

where

$$p(r) = \{1 - (r_1/r)^{\hat{\alpha}-1}\}\{\mathbb{P}[\mathrm{BES}_1^{\hat{\alpha}}(t) \geqq t^{1/2-\eta},\ \forall t \geqq r^{1/(1-2\eta)}] - r^{-(\hat{\alpha}-1)/2}\}\ .$$

Proof. The Laplacian of the radial function $\tilde{r}$ on $\tilde{\mathbf{M}}$, where $\tilde{\mathbf{M}}$ is constructed as in the proof of Lemma 1, is given by

$$\Delta\tilde{r} = (m-1)\frac{g'(\tilde{r})}{g(\tilde{r})}$$

and so, for $\tilde{r} \geqq r_2$, $\Delta\tilde{r} = \hat{\alpha}/\tilde{r}$. Since the Ricci curvature of $\mathbf{M}$ for $\partial_{\mathbf{M}}$ at x is less than or equal to the Ricci curvature of $\tilde{\mathbf{M}}$ for $\partial_{\mathbf{M}}$ at any $\tilde{x}$ such that $\tilde{r}(\tilde{x}) = r(x)$, the Laplacian Comparison Theorem implies that

$$\Delta r \geqq \frac{\hat{\alpha}}{r} \quad \text{for } r \geqq r_2\ .$$

Therefore, we have that, when $R_t \geqq r_2$,

$$dR_t \geqq dB_t + \frac{\hat{\alpha}}{2R_t}\, dt\ .$$

Now, the solution of the stochastic differential equation

$$dY_t = dB_t + \frac{\hat{\alpha}}{2Y_t}\, dt\ ,$$

where the Brownian motion B is the martingale part of R, is a Bessel process $\mathrm{BES}^{\hat{\alpha}}$. Then, by comparison of R with this $\mathrm{BES}^{\hat{\alpha}}$, we have, for all $r \geqq r_2^2$

and all sufficiently small η,

$$\begin{aligned}
&\mathbb{P}^{r,\theta}[R_t \geqq \sqrt{r} \vee t^{1/2-\eta},\ \forall t] \\
&\quad \geqq \mathbb{P}^{r,\theta}[R_t \geqq \sqrt{r} \vee t^{1/2-\eta},\ \forall t \,|\, \mathrm{BES}_r^{\hat\alpha} \text{ never hits the level } r_2] \\
&\qquad \times \mathbb{P}^{r,\theta}[\mathrm{BES}_r^{\hat\alpha} \text{ never hits the level } r_2] \\
&\quad \geqq \mathbb{P}[\mathrm{BES}_r^{\hat\alpha}(t) \geqq \sqrt{r} \vee t^{1/2-\eta},\ \forall t] \times \mathbb{P}[\mathrm{BES}_r^{\hat\alpha} \text{ never hits the level } r_2]\,.
\end{aligned}$$

Now,

$$\mathbb{P}[\mathrm{BES}_r^{\hat\alpha} \text{ never hits the level } r_2] = 1 - (r_2/r)^{\hat\alpha - 1}$$

(cf. [8, pp. 195, 238]) and

$$\begin{aligned}
&\mathbb{P}[\mathrm{BES}_r^{\hat\alpha}(t) \geqq \sqrt{r} \vee t^{1/2-\eta},\ \forall t] \\
&\quad = \mathbb{P}[\mathrm{BES}_r^{\hat\alpha}(t) \geqq \sqrt{r},\ \forall\, 0 \leqq t < r^{1/(1-2\eta)},\ \text{and } \mathrm{BES}_r^{\hat\alpha}(t) \geqq t^{1/2-\eta}, \\
&\qquad \forall t \geqq r^{1/(1-2\eta)}] \\
&\quad = \mathbb{P}[\mathrm{BES}_r^{\hat\alpha}(t) \geqq \sqrt{r},\ \forall t,\ \text{and } \mathrm{BES}_r^{\hat\alpha}(t) \geqq t^{1/2-\eta},\ \forall t \geqq r^{1/(1-2\eta)}] \\
&\quad = \mathbb{P}[\mathrm{BES}_r^{\hat\alpha}(t) \geqq t^{1/2-\eta},\ \forall t \geqq r^{1/(1-2\eta)}] \\
&\qquad - \mathbb{P}[\mathrm{BES}_r^{\hat\alpha}(t) \geqq t^{1/2-\eta},\ \forall t \geqq r^{1/(1-2\eta)},\ \text{and } \mathrm{BES}_r^{\hat\alpha}(t) < \sqrt{r} \text{ for some } t \geqq 0] \\
&\quad \geqq \mathbb{P}[\mathrm{BES}_r^{\hat\alpha}(t) \geqq t^{1/2-\eta},\ \forall t \geqq r^{1/(1-2\eta)}] - \mathbb{P}[\mathrm{BES}_r^{\hat\alpha} \text{ hits the level } \sqrt{r}] \\
&\quad \geqq \mathbb{P}[\mathrm{BES}_r^{\hat\alpha}(t) \geqq t^{1/2-\eta},\ \forall t \geqq r^{1/(1-2\eta)}] - r^{-(\hat\alpha-1)/2}\,.
\end{aligned}$$

Thus, the required result follows by letting $r_2 \downarrow r_1$. □

Since almost surely $\mathrm{BES}_1^{\hat\alpha}(t) \geqq t^{1/2-\eta}$ for sufficiently large t (cf. [12]), $p(r)$ will tend to 1 as r tends to infinity, and so $\mathbb{P}^{r,\theta}[R_t \geqq \sqrt{r} \vee t^{1/2-\eta},\ \forall t]$ tends to 1, uniformly with respect to θ, as r tends to infinity.

5 The sequence of stopping times

In this section we study the sequence of stopping times T_n, required for the proof of the theorem. We fix $x_0 \in \mathbf{M}\backslash\{o\}$ such that $r_0 = r(x_0) > r_1$ and write $\rho(x) = \mathrm{dist}(x, x_0)$. Then the sectional curvatures at x, for all $x \in \mathbf{M}$, are bounded below by

$$-\tilde c\{r_0 + \rho(x)\}^2\,.$$

Define the function ϕ on $\mathbb{R}_+$ by

$$\phi(t) = a\sqrt{\log(r_0 + t)}, \quad a = 2^{-1/2}\tilde c^{-1/4}\,.$$

Then

$$\phi(0) = a\sqrt{\log(r_0)} \geqq 0 \quad \text{and} \quad \phi'(t) = \frac{a}{2}\,\frac{1}{(r_0 + t)\sqrt{\log(r_0 + t)}}\,.$$

Thus ϕ' is decreasing and $\phi'(0) = a/(2r_0\sqrt{\log(r_0)}) < 1$, by the hypothesis (1) made on r_1 in Sect. 2, so that in particular $\phi''(t) \leqq 0$.

Lemma 3 *For $\rho \geqq 1$, the Hessian of ρ acting on any unit tangent vector which is orthogonal to a radial tangent vector is bounded by*

$$\frac{1}{\phi(\rho)\phi'(\rho)} = 2\frac{r_0+\rho}{a^2} .$$

Proof. By the Hessian Comparison Theorem (cf. [4, p. 19]), we only need to show that this is true for an m-dimensional rotationally symmetric manifold with radial curvature function $-\tilde{c}(r_0+\hat{r})^2$. Write the Riemannian metric of such a manifold as $d\hat{r}^2 + f^2(\hat{r})\,d\hat{\theta}^2$, where f satisfies the Jacobi equation $f''(t) = \tilde{c}(r_0+t)^2 f(t)$ with $f(0)=0$ and $f'(0)=1$. Then the Hessian of $\hat{r}$ acting on any tangent vector orthonormal to a radial tangent vector is equal to $f'(\hat{r})/f(\hat{r})$ (cf. [4, p. 30]). On $[0,1]$, $f''(t) \leqq 4\tilde{c}r_0^2 f(t)$ and so $(f'/f)(t) \leqq 2r_0\sqrt{\tilde{c}}\coth(2r_0\sqrt{\tilde{c}}t)$, since the function on the right hand side is the Hessian of the distance function on a Riemannian manifold with constant sectional curvature $-4\tilde{c}r_0^2$ acting on any tangent vector orthonormal to a radial tangent vector. Thus, in particular, $(f'/f)(1) \leqq 2r_0\sqrt{\tilde{c}}\coth(2\,r_0\sqrt{\tilde{c}})$. Now consider $\tilde{f} : [1,\infty) \to \mathbb{R}_+$ such that $\tilde{f}(t) = \exp(\tilde{a}(r_0+t)^2)$, where $\tilde{a} = \sqrt{\tilde{c}}\coth(2r_0\sqrt{\tilde{c}})$. Then

$$(\tilde{f}''/\tilde{f})(t) \geqq 4\tilde{c}\coth^2(2r_0\sqrt{\tilde{c}})(r_0+t)^2 > \tilde{c}(r_0+t)^2 = (f''/f)(t)$$

and

$$(\tilde{f}'/\tilde{f})(t) = 2\sqrt{\tilde{c}}\coth(2r_0\sqrt{\tilde{c}})(r_0+t) ,$$

so that $(\tilde{f}'/\tilde{f})(1) \geqq 2r_0\sqrt{\tilde{c}}\coth(2r_0\sqrt{\tilde{c}}) \geqq (f'/f)(1)$. Since

$$\{\tilde{f}f' - f\tilde{f}'\}' = f\tilde{f}\{f''/f - \tilde{f}''/\tilde{f}\} \leqq 0 ,$$

it follows that, on $[1,\infty)$, $(f'/f)(t) \leqq (\tilde{f}'/\tilde{f})(t)$. Thus the required bound follows from the second hypothesis on r_1, made in Sect. 2. □

We next use this bound on the Hessian of ρ to obtain a bound on the growth of $\rho_t = \rho(X_t)$ in a manner similar to Darling's use of 'persistence functions' in [2]. This type of argument and the estimates that result from it originated in Kallenberg–Sztencel [7]. Since we follow the proof of Darling's Proposition 5.2 and Theorem 2.1 quite closely, we just sketch the argument.

Lemma 4 *If ρ is sufficiently large, then*

$$\mathbb{P}[\sup\{\rho_s :\ 0 \leqq s \leqq t\} \geqq \rho] \leqq \{\mathbb{P}[B_1 \geqq 1]\}^{-1}\exp\left\{-\frac{\log(r_0+\rho)}{32\sqrt{\tilde{c}}t}\right\},$$

where B is standard Brownian motion on $\mathbb{R}^1$ starting from 0.

Proof. If we denote by $(\rho,\tilde{\theta})$ the global geodesic polar coordinates with respect to x_0, then

$$t = [X]_t = [\rho]_t + [\tilde{\theta}(X)]_t$$

where $[X]_t = \int g_{\mathbf{M}}(X_s)(dX_s, dX_s)$ and $[\rho]$ and $[\tilde{\theta}(X)]$ are similarly defined. Thus we have $0 \leqq [\rho]_t \leqq t$ and, in fact, $[\rho]_t < t$ a.s. Define

$$H_t = \phi(\rho_t) - \phi(1) - \sqrt{t - [\rho]_t} \, .$$

Then $H_t > 0$ implies that $\rho_t > 1$. Write H_t^+ for the positive part of H_t and decompose dH_t^+ as $dH_t^+ = dM_t + dL_t - dV_t$, such that M is a local martingale, L is the local time of H at zero, V is a process of locally bounded variation and $M_0 = V_0 = L_0 = 0$. Since for any vector field v

$$\mathrm{Hess}^\rho(\partial/\partial\rho, v) = 0$$

(cf. [2, Lemma 1.4]), we have

$$\mathrm{Hess}^\rho(dX_t, dX_t) = \mathrm{Hess}^\rho(d\tilde{\Theta}_t, d\tilde{\Theta}_t) \, ,$$

where $\tilde{\Theta}_t = \tilde{\theta}(X_t)$, and thus

$$dV_t = \frac{1}{2} 1_{\{H_t > 0\}} \{2d\sqrt{t - [\rho]_t} - \phi'(\rho_t)\mathrm{Hess}^\rho(d\tilde{\Theta}_t, d\tilde{\Theta}_t) - \phi''(\rho_t)d[\rho]_t\} \, .$$

It then follows from Lemma 3 and the definition of H^+ that

$$dV_t \geqq -\frac{1}{2}\phi''(\rho_t)d[\rho]_t \geqq 0$$

and so $L_t = -\inf\{M_s - V_s : \ 0 \leqq s \leqq t\}$ by Skorohod's lemma. Hence

$$H_t^+ \leqq M_t - \inf\{M_s : \ 0 \leqq s \leqq t\} \, .$$

Expressing the local martingale M as random time changed Brownian motion, that is, $M_t = \tilde{B} \circ [M]_t$ and noting that $\tilde{B}_t - \inf\{\tilde{B}_s : \ 0 \leqq s \leqq t\}$ is the modulus of another Brownian motion, we have

$$H_t^+ \leqq |B \circ [M]_t| \, ,$$

so that $\phi(\rho_t) \leqq |B \circ [M]_t| + \phi(1) + \sqrt{t - [\rho]_t}$. Since $[M]_t \leqq [\rho]_t \leqq t$, it follows that

$$\phi(\sup\{\rho_s : \ 0 \leqq s \leqq t\}) \leqq \phi(1) + \sup\{|B_s| + \sqrt{t-s} : \ 0 \leqq s \leqq t\} \, .$$

Thus, if $\phi^{-1}(\rho) > 1$,

$$\begin{aligned} &\mathbb{P}[\sup\{\rho_s : \ 0 \leqq s \leqq t\} \geqq \phi^{-1}(\rho)] \\ &\quad \leqq 2\mathbb{P}[\sup\{B_s + \sqrt{t-s} : \ 0 \leqq s \leqq t\} \geqq \rho - \phi(1)] \end{aligned}$$

as $-B$ is also a Brownian motion. By Brownian motion scaling and the strong Markov property of B at the stopping time $\inf\{s \in [0,t] : \ B_t + \sqrt{t-s} \geqq \rho - \phi(1)\}$, we then have

$$\mathbb{P}[\sup\{\rho_s : \ 0 \leqq s \leqq t\} \geqq \phi^{-1}(\rho)] \leqq \frac{\mathbb{P}[\sup\{B_s : \ 0 \leqq s \leqq t\} \geqq \rho - \phi(1)]}{\mathbb{P}[B_1 \geqq 1]} \, ,$$

and so the standard properties of Brownian motion show that, if $\rho > 1$,

$$\begin{aligned}\mathbb{P}[\sup\{\rho_s:\ 0 \leqq s \leqq t\} \geqq \rho] &\leqq \frac{\mathbb{P}[\sup\{B_s:\ 0 \leqq s \leqq t\} \geqq \phi(\rho) - \phi(1)]}{\mathbb{P}[B_1 \geqq 1]} \\ &= 2\frac{\mathbb{P}[B_t \geqq \phi(\rho) - \phi(1)]}{\mathbb{P}[B_1 \geqq 1]} \\ &\leqq \frac{\exp\{-(\phi(\rho) - \phi(1))^2/2t\}}{\mathbb{P}[B_1 \geqq 1]}\ .\end{aligned}$$

When ρ is sufficiently large, we have

$$\begin{aligned}\sqrt{\log(r_0+\rho)} - \sqrt{\log(r_0+1)} &= \frac{\log(r_0+\rho) - \log(r_0+1)}{\sqrt{\log(r_0+\rho)} + \sqrt{\log(r_0+1)}} \\ &\geqq \frac{\log(r_0+\rho) - \log(r_0+1)}{2\sqrt{\log(r_0+\rho)}} \\ &\geqq \frac{1}{4}\sqrt{\log(r_0+\rho)}\ ,\end{aligned}$$

which gives the stated result. □

We now define the sequence of stopping times

$$\begin{aligned}T_0 &= 0 \\ T_{n+1} &= \inf\{t > T_n:\ \text{dist}\left(X_t, X_{T_n}\right) = \left(R_{T_n}\right)^{\varepsilon}\}\end{aligned}$$

with $\inf\{\emptyset\} = \infty$.

Lemma 5 *If $R_0 = r > r_1^2$, then there is a positive constant $\tilde{\beta}$ such that with probability at least $p(r)$ $T_n \geqq \tilde{\beta} n$ for all sufficiently large n.*

Proof. When T_n is finite, define

$$L_{n+1} = T_{n+1} - T_n\ .$$

Then, for $\beta > 0$ such that $1 - 2\exp\{-1/(32\beta\sqrt{\tilde{c}})\} = l > 0$ and for $R_{T_n} > r_1$, we have by Lemma 4 that

$$\begin{aligned}p_n &\equiv \mathbb{P}[L_{n+1} \geqq \beta\ \log(R_{T_n} + (R_{T_n})^{\varepsilon})\ |\ \mathscr{F}_{T_n}] \\ &\geqq 1 - 2\exp\left\{-\frac{1}{32\beta\sqrt{\tilde{c}}}\right\} = l > 0\ .\end{aligned}$$

For a sequence of i.i.d. random variables $V_1, V_2, \ldots$ with uniform distribution on [0, 1] and independent of $\mathscr{F}_\infty = \bigvee_n \mathscr{F}_{T_n}$, define, for $n \geqq 0$,

$$U_{n+1} = \begin{cases} 1 & \text{if } L_{n+1} > \beta\ \log(R_{T_n} + (R_{T_n})^{\varepsilon}) \text{ and } p_n V_{n+1} < 1\ , \\ 0 & \text{otherwise}\ . \end{cases}$$

Then U_{n+1} takes values 0 and 1 and is measurable with respect to the σ-field generated by $\mathscr{F}_{T_{n+1}}$ and $V_1, \ldots, V_n$. Lemma 2 and the fact that $R_0 = r >$

r_1^2 imply that, with probability at least $p(r)$, $R_t > r_1$ for all t. Hence, the $\{U_n\colon n \geqq 1\}$ constructed in this way form a sequence of i.i.d. non-degenerate $\{0,1\}$-valued random variables and, with probability at least $p(r)$,

$$\mathbb{P}[U_{n+1} = 1 \mid \mathscr{F}_{T_n}, V_1, \ldots, V_n] = l\ .$$

Since

$$L_{n+1} = T_{n+1} - T_n \geqq \beta \log(R_{T_n} + (R_{T_n})^{\varepsilon})U_{n+1}$$

then, with probability at least $p(r)$,

$$L_{n+1} \geqq \beta U_{n+1}$$

and so

$$T_{n+1} \geqq \beta S_{n+1}\ ,$$

where $S_n = \sum_{1 \leqq k \leqq n} U_k$. Since S_n/n tends to l almost surely when n tends to infinity, we have the required result. □

6 Proof of the Theorem

If $r \geqq r_\varepsilon^2$ then, with $\mathbb{P}^{r,\theta}$-probability at least $p(r)$, $R_t \geqq \sqrt{r} \vee t^{1/2-\eta}$ for all t, so that, with $\mathbb{P}^{r,\theta}$-probability at least $p(r)$, $R_{T_n} \geqq r_\varepsilon$ for all n. Choosing η small enough such that $(1-2\eta)(\alpha-\varepsilon) > 1$ then, it follows from the proof of Lemma 5 that the intersection of the events $\{T_n \geqq \tilde{\beta} n\}$ and $\{R_t \geqq \sqrt{r} \vee t^{1/2-\eta},\ \forall t\}$ occurs with probability at least $p(r)$ and so, with $\mathbb{P}^{r,\theta}$-probability at least $p(r)$,

$$\begin{aligned}
4^{-\alpha} \sup_{t \geqq 0} \delta^2(\theta, \Theta_t) &\leqq 4^{-\alpha} \sum_{n=0}^{\infty} \sup_{T_n \leqq t \leqq T_{n+1}} \delta^2(\Theta_{T_n}, \Theta_{T_n + t}) \\
&\leqq \sum_{n=0}^{\infty} (R_{T_n})^{2(\varepsilon-\alpha)} \quad \text{by Lemma 1} \\
&\leqq \sum_{n=0}^{\infty} \{\sqrt{r} \vee T_n^{1/2-\eta}\}^{2(\varepsilon-\alpha)} \quad \text{by Lemma 2} \\
&\leqq \sum_{0 \leqq n \leqq N} r^{\varepsilon-\alpha} + \sum_{n=0}^{\infty} \{r \vee (\tilde{\beta} n)^{1-2\eta}\}^{\varepsilon-\alpha} \quad \text{by Lemma 5} \\
&= N r^{\varepsilon-\alpha} + \sum_{0 \leqq \tilde{\beta} n \leqq r^{1/(1-2\eta)}} r^{\varepsilon-\alpha} + \sum_{\tilde{\beta} n > r^{1/(1-2\eta)}} \{\tilde{\beta} n\}^{(1-2\eta)(\varepsilon-\alpha)} \\
&\leqq \{N + \tilde{\beta}^{-1}\} r^{2\lambda} + \tilde{\beta}^{-1} \int_{r^{1/(1-2\eta)}}^{\infty} (x-1)^{(1-2\eta)(\varepsilon-\alpha)}\, dx \\
&\leqq \{N + \tilde{\beta}^{-1}\} r^{2\lambda} - \frac{\tilde{\beta}^{-1}}{(1-2\eta)(\varepsilon-\alpha)+1} (r-1)^{\frac{(1-2\eta)(\varepsilon-\alpha)+1}{1-2\eta}} \\
&\leqq C_0^2 (r-1)^{2\lambda}\ ,
\end{aligned}$$

where $N = N(\omega)$ is a positive integer such that $T_n \geqq \tilde{\beta} n$ for all $n \geqq N$,

$$\lambda = \frac{\varepsilon - \alpha}{2} + \frac{1}{2(1-2\eta)} < 0 \quad \text{and}$$

$$C_0(\omega) = \sqrt{N + \left\{1 - \frac{1}{(1-2\eta)(\varepsilon-\alpha)+1}\right\} \tilde{\beta}^{-1}} \, .$$

Write τ_r for $\inf\{t \geqq 0\colon R_t = r\}$. Then, for $r > r_\varepsilon^2$, we have

$$\mathbb{P}\left[\sup_{t \geqq 0} \delta(\Theta_{\tau_r}, \Theta_{\tau_r + t}) \leqq 2^\alpha C_0(\omega)(r-1)^\lambda\right] \geqq p(r)\,. \qquad (*)$$

Let $k_1 = 1$ and $k_n = n \vee \min\{k\colon k$ satisfies the inequality $(I)\}$ for $n \geqq 2$, where the inequality (I) is

$$(I) \qquad 1 - \mathbb{P}[\mathrm{BES}_1^{\hat{\alpha}} \geqq t^{1/2-\eta},\ \forall t \geqq k^{1/(1-2\eta)}] \leqq n^{-(\hat{\alpha}-1)/2}\,.$$

By the remark at the end of Sect. 4, such $\{k_n\colon n \geqq 1\}$ exist. Define

$$A_n = \{\omega\colon \text{there exist } s,\, t > \tau_{k_n}(\omega) \text{ such that } \delta(\Theta_s, \Theta_t)(\omega) > 2^{1+\alpha} C_0(\omega)(k_n - 1)^\lambda\}\,.$$

Then, since, by the triangle inequality, $\delta(\theta_1, \theta_2) \leqq \delta(\theta_1, \theta_0) + \delta(\theta_2, \theta_0)$ for any three points θ_i, $i = 0, 1, 2$, on the unit tangent sphere at the pole,

$$\begin{aligned}
\mathbb{P}[A_n] &\leqq \mathbb{P}[\omega\colon \text{ there exist } s,\ t > \tau_{k_n}(\omega) \text{ such that } \delta(\Theta_s, \Theta_{\tau_{k_n}})(\omega) \\
&\qquad + \delta(\Theta_t, \Theta_{\tau_{k_n}})(\omega) > 2^{1+\alpha} C_0 (k_n - 1)^\lambda] \\
&= 1 - \mathbb{P}[\omega\colon \forall s, t \geqq \tau_{k_n}(\omega),\ \delta(\Theta_s, \Theta_{\tau_{k_n}})(\omega) \\
&\qquad + \delta(\Theta_t, \Theta_{\tau_{k_n}})(\omega) \leqq 2^{1+\alpha} C_0 (k_n - 1)^\lambda] \\
&= 1 - \mathbb{P}\Big[\omega\colon \sup_{s,t > \tau_{k_n}(\omega)} \{\delta(\Theta_s, \Theta_{\tau_{k_n}}) \\
&\qquad + \delta(\Theta_t, \Theta_{\tau_{k_n}})\}(\omega) \leqq 2^{1+\alpha} C_0 (k_n - 1)^\lambda\Big] \\
&= 1 - \mathbb{P}\Big[\omega\colon \sup_{t > \tau_{k_n}(\omega)} \delta(\Theta_t, \Theta_{\tau_{k_n}})(\omega) \leqq 2^\alpha C_0 (k_n - 1)^\lambda\Big] \\
&\leqq 1 - p(k_n) \quad \text{when } n > r_\varepsilon \\
&\leqq (r_1/k_n)^{\hat{\alpha}-1} + k_n^{-(\hat{\alpha}-1)/2} + n^{-(\hat{\alpha}-1)/2} \quad \text{when } n > r_\varepsilon^2 \\
&\leqq 3n^{-(\hat{\alpha}-1)/2} \quad \text{when } n > r_\varepsilon^2\,.
\end{aligned}$$

It follows from $m \geqq 3$, and $c > 3/4$ if $m = 3$, that $\hat{\alpha} = (m-1)\alpha = (m-1)(1 + \sqrt{1+4c})/2 > 3$ so that

$$\sum_{n \geqq 1} \mathbb{P}[A_n] < \infty$$

and, by the Borel–Cantelli lemma, we have

$$\mathbb{P}[A_n \text{ happens infinitely often}] = 0 \,.$$

By Lemma 5, $C_0(\omega) < \infty$ almost surely and so it follows that $C_0(\omega)(k_n - 1)^{\lambda}$ tends to zero as n tends to infinity. Hence

$$\begin{aligned}\mathbb{P}\left[\lim_{t\to\infty} \Theta_t \text{ exists}\right] &\geqq \mathbb{P}[\omega\colon \exists N \geqq 1 \text{ s.t. } \forall n \geqq N \text{ and } \forall s,t > \tau_{k_n}(\omega) \\ &\qquad \delta(\Theta_s,\Theta_t)(\omega) \leqq 2^{1+\alpha} C_0(\omega)(k_n - 1)^{\lambda}] \\ &= \mathbb{P}[\omega\colon \exists N \geqq 1 \text{ s.t. } \forall n \geqq N \ \omega \in A_n^c] \\ &= \mathbb{P}\left[\bigcup_{n\geqq 1}\bigcap_{k\geqq n} A_k^c\right] = 1 \,,\end{aligned}$$

where A_k^c denotes the complementary set of A_k. It is clear that $\lim_{t\to\infty}\Theta_t$ is invariant with respect to time t.

Finally, since the Laplace–Beltrami operator is uniformly elliptic in any ball of $\mathbf{M}$, and so the support of $\mathbb{P}^{r,\theta}$ is the class of continuous mappings from $\mathbb{R}_+$ to $\mathbf{M}$ starting at (r,θ) (cf. [13, p. 169]), we have, for any non-empty open set U in $\mathbf{M}$,

$$\mathbb{P}^{r,\theta}[\Theta_{\tau_{r'}} \in U \text{ for sufficiently large } r'] > 0 \,.$$

This, together with $(*)$, shows that the closure of the support of the probability law of the limit of Θ_t is the entire $\mathbf{S}^{m-1}$ (cf. [5]).

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