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Enlargement of the Wiener filtration by an absolutely continuous random variable via Malliavin's calculus

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Summary. The analytic treatment of problems related to the asymptotic behaviour of random dynamical systems generated by stochastic differential equations suffers from the presence of non-adapted random invariant measures. Semimartingale theory becomes accessible if the underlying Wiener filtration is enlarged by the information carried by the orthogonal projectors on the Oseledets spaces of the (linearized) system.

We study the corresponding problem of preservation of the semimartingale property and the validity of a priori inequalities between the norms of stochastic integrals in the enlarged filtration and norms of their quadratic variations in case the random element F enlarging the filtration is real valued and possesses an absolutely continuous law. Applying the tools of Malliavin's calculus, we give smoothness conditions on F under which the semimartingale property is preserved and a priori martingale inequalities are valid.

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1 Introduction

The ergodic theory of random dynamical systems provided the problems motivating this study of the relationship between Malliavin's calculus and the enlargement ("grossissement") of the Wiener filtration. They typically arise in the following context. Consider a linear Stratonovitch stochastic differential equation of the simple form

$$dx_t = A_0 x_t\, dt + \sum_{i=1}^{n} A_i x_t \circ dW_t^i$$

in $\mathbb{R}^d$, with $d \times d$-matrices A_i, $0 \leqq i \leqq n$, and an n-dimensional Wiener process $(W_t)_{t\in\mathbb{R}}$. Its fundamental (matrix) solution $(\phi(t, \cdot\,))_{t\in\mathbb{R}}$ gives an example of a random dynamical system (see [1]). The asymptotic properties of the solution trajectories are determined by its Lyapunov numbers and "Oseledets spaces" just as eigenvalues of $d \times d$-matrix and eigenspaces characterize the exponential growth of the solutions of a deterministic differential equation (see for example [1]). Indeed, in the spectral decompositions

$$\lim_{t \to \pm\infty} [\phi(t, \cdot)^* \phi(t, \cdot)]^{1/2|t|} = \sum_{i=1}^{r} e^{\lambda_i} Q_i^{\pm} ,$$

the Lyapunov numbers λ_i and random linear subspaces $U_i^{\pm}$ with corresponding orthogonal projectors $Q_i^{\pm}$, $1 \leqq i \leqq r$, appear. If a trajectory starts in $V_i^+ = U_i^+ \oplus \ldots \oplus U_r^+$, but not in V_{i+1}^+, its asymptotic exponential growth rate will be λ_i for t near $+\infty$, whereas for t near $-\infty$ the growth rate λ_i will be seen when starting in $V_i^- = U_{r+1-i}^- \oplus \ldots \oplus U_r^-$, and not in V_{i+1}^-. Hence the Oseledets spaces $E_i = V_i^+ \cap V_i^-$, $1 \leqq i \leqq r$, play the roles of deterministic eigenspaces. They are random and draw, due to the definition of V_i^+ and V_i^-, information from the whole history of W. The Oseledets spaces are invariant and, more importantly, the random invariant measures of the system take their support within them, and consequently are non-adapted with respect to the Wiener filtration. These random measures appear in many problems concerning the asymptotic behaviour of the system, for example in formulas of the type of Furstenberg–Khasminskii representing the Lyapunov exponents as spatial means, in a normal form theory for random dynamical systems generated by stochastic differential equations, the concept of "rotation numbers" which in analogy to the Lyapunov exponents characterize the asymptotic rotational behaviour, or the theory of linearization of random dynamical systems in the sense of Hartman–Grobman (see [21, 1, 2]).

The desire to use the powerful tools of semimartingale theory for the treatment of these problems conflicts with the non-adaptedness of the invariant measures. One way out of the conflict is the enlargement of filtrations. If R_i, $1 \leqq i \leqq r$, are the orthogonal projectors on the Oseledets spaces, one may enlarge the Wiener filtration $(\mathscr{F}_t)_{t \geqq 0}$ to get

$$\mathscr{G}_t = \mathscr{F}_t \vee \sigma(R_i\colon\ 1 \leqq i \leqq r),\ t \geqq 0 .$$

The obvious questions that arise at this point are among the classical questions of the "grossissement de filtrations":

1) Do $(\mathscr{F}_t)$-semimartingales remain semimartingales w.r.t. $(\mathscr{G}_t)$?
2) If yes, are stochastic integrals of integrands adapted w.r.t. the large filtration sufficiently well-behaved, i.e. are there a priori inequalities linking norms of these integrals with norms of their quadratic variations?

In a rather general framework, they have found answers in a series of deep theoretical works by Barlow [3], Jacod [8], Jeulin [9, 10], Chaleyat–Maurel and Jeulin [5], Meyer [12], Yoeurp [16] and Yor [17–20]. Our intention in this study is not to add to these far-reaching and powerful results, but to provide a satisfactory framework for a treatment of the above mentioned problems of ergodic theory by answering 1) and 2), thereby making as much use as possible of them.

Apart from trivial cases, our random variables of "grossissement initial" have no discrete laws. So our starting point had to be Jacod's criterion for a positive answer to 1). Let us model the random element by which we enlarge by F. The criterion states that if there exists a common measure μ on the space where F takes its values such that the conditional laws of F given $\mathscr{F}_t$ are absolutely continuous w.r.t. μ for all $t \geqq 0$, then any $(\mathscr{F}_t)$-semimartingale is a $(\mathscr{G}_t)$-semimartingale. If F takes its values in a finite dimensional Euclidean space, we may take Lebesgue measure for μ and try to formulate the absolute continuity criteria via the tools of Malliavin's calculus. For an application of Malliavin's calculus to an enlargement problem related to time reversal of diffusion see Pardoux [15], for "partial Malliavin's calculus" Nualart-Zakai [14] and Kusuoka and Stroock [11].

This way we find ourselves in a framework in which the beautiful results of Yor [18, 19] concerning a priori estimates in the sense of question 2) are not quite sufficient. Yor [18] treats the enlargement by a countable partition of the probability space, whereas Yor [19] takes care of the "grossissement progressif" which makes a random time into a stopping time. Our "grossissement initial" is with respect to an absolutely continuous random variable. Therefore having to deal with non-bounded positive martingales in the Girsanov formulation of the problem, we were led to extensions of the inequalities of Yor the conditions of which take a somewhat different form. We emphasize that the ideas and methods of both papers were of great importance hereby.

Now remember that the random vectors of enlargement in the situation we ultimately have in mind take their values in projective space or even on Grassmannian manifolds. Our original plan was to treat the real valued case first, and then pass on to the finite dimensional Euclidean and finally the case where F takes its values on a Riemannian manifold. But especially the fact that we had to extend results on some very basic questions of martingale inequalities made our manuscript grow fast. So we decided to just treat the real valued case here and defer the manifold valued case to a forthcoming paper.

Assuming therefore that F takes real values, we derive in Sect. 3 sufficient conditions on the Malliavin derivative DF under which Jacod's above mentioned criterion holds true. We show that the conditional law of F given $\mathscr{F}_t$ is absolutely continuous w.r. to Lebesgue measure provided $\int_t^\infty (D_u F)^2 \, du > 0$ P-a.s.

In Sect. 5 we give sufficient criteria to be verified by the Malliavin derivatives of F in order that the a priori inequalities between norms of stochastic integrals of $(\mathscr{G}_t)$-adapted processes and norms of their quadratic variations derived in Sect. 4 are valid. We prove that the existence of the second Malliavin derivative of F and integrability properties on $(\int_t^\infty (D_u F)^2 \, du)^{-1}$ are enough. Under some additional smoothness assumptions on F, in Sect. 6 an explicit formula for the compensating process of bounded variation appearing in the decomposition of local $(\mathscr{F}_t)$-martingales w.r.t. $(\mathscr{G}_t)$ is given. With the assistance of the formula of Clark–Ocone for representation of Wiener functionals the integrand of the process of bounded variation is seen to be a "logarithmic Malliavin derivative" of the conditional densities of F given $\mathscr{F}_t$, $t \geqq 0$. The applications of the results thus obtained to the problems of multiplicative ergodic theory sketched above will appear in a subsequent paper Imkeller [7].

2 Preliminaries and notation

Our basic probability space is the 1-dimensional canonical Wiener space $(\Omega, \mathscr{F}, P)$, equipped with the canonical Wiener process $W = (W_t)_{t \geqq 0}$. More precisely, $\Omega = C(\mathbb{R}_+; \mathbb{R})$ is the set of continuous functions on $\mathbb{R}_+$ starting at 0, $\mathscr{F}$ the σ-algebra of Borel sets with respect to uniform convergence on compacts of $\mathbb{R}_+$, P Wiener measure and W the coordinate process. The natural filtration $(\mathscr{F}_t)_{t \geqq 0}$ of W is assumed to be completed by the sets of P-measure 0. Let us briefly recall the basic concepts of Malliavin's calculus needed. We refer to Nualart [13] for a more detailed treatment.

Let $\mathscr{S}$ be the set of smooth random variables on $(\Omega, \mathscr{F}, P)$, i.e. of random variables of the form

$$F = f(W_{t_1}, \ldots, W_{t_n}), \quad f \in C_0^\infty(\mathbb{R}^n), \; t_1, \ldots, t_n \in \mathbb{R}_+ .$$

For $F \in \mathscr{S}$ we may define the Malliavin derivative

$$(DF)_s = D_s F = \sum_{i=1}^n \frac{\partial}{\partial x_i} f(W_{t_1}, \ldots, W_{t_n}) 1_{[0,t_i]}(s), \quad s \in \mathbb{R}_+ .$$

We may regard DF as a random element with values in $L^2(\mathbb{R}_+)$, and then define the Malliavin derivative of order k by k fold iteration of the above derivation. It will be denoted by $D^{\otimes k} F$, and is a random element with values in $L^2(\mathbb{R}_+^k)$. Its value at $(s_1, \ldots, s_k) \in \mathbb{R}_+^k$ is written $D^{\otimes k}_{s_1,\ldots,s_k}$.

If $S, T \geqq 0$, $S \leqq T$, $p \geqq 1$ and $k \in \mathbb{N}$, we denote by $\mathbb{D}_{p,k}([S,T])$ the Banach space given by the completion of $\mathscr{S}$ with respect to the norm

$$\|F\|_{p,k} = \|F\|_p + \sum_{1 \leqq j \leqq k} E\left(\left[\int_S^T (D^{\otimes j}_{s_1,\ldots,s_j} F)^2 ds_1 \ldots ds_j\right]^{p/2}\right)^{1/p}, \quad F \in \mathscr{S} .$$

More generally, if H is a Hilbert space and $\mathscr{S}_H$ the set of linear combinations of tensor products of elements of $\mathscr{S}$ with elements of H, $\mathbb{D}_{p,k}([S,T], H)$ will denote the closure of $\mathscr{S}_H$ w.r. to the norm

$$\|F\|_{p,k} = \| \, |F|_H \, \|_p + \sum_{1 \leqq j \leqq k} E\left(\left[\int_S^T |D^{\otimes j}_{s_1,\ldots,s_j} F|_H^2 ds_1 \ldots ds_j\right]^{p/2}\right)^{1/p}, \quad F \in \mathscr{S}_H ,$$

where the Malliavin derivatives of smooth functions are given in an obvious way, and $|\cdot|_H$ denotes the norm on H induced by the scalar product. These definitions are consistent. For example,

$$\|F\|_p + \|DF\|_{p,k-1} = \|F\|_{p,k}, \quad F \in \mathbb{D}_{p,k}([S,T]),$$

if $H = L^2([S,T])$ equipped with the canonical scalar product $\langle \cdot , \cdot \rangle_S^T$. It will usually be unambiguous from the environment of the formulas which interval $[S,T]$ we refer to. Not to overload the notation, we therefore do not index the norms with S, T.

If D is considered as a linear operator with values in $L^2(\Omega \times [S,T])$, its adjoint, the "Shorokhod integral" from S to T, will be denoted by δ_S^T. We shall have the opportunity to work on products $\Omega \times \Omega$ of our canonical Wiener space with itself. In this case, to identify the number of the coordinate with respect to which the Malliavin derivative resp. the Shorokhod integral is taken, we write $D^1, D^2, \delta^1, \delta^2$ etc. If on $\mathscr{F} \otimes \mathscr{F}$ we consider the measure $P \otimes P$ and take expectations w.r. to one component while fixing the other, an index E_i with the expectation will indicate the number of the component of integration, if there is ambiguity, $i = 1, 2$. The domains of the respective Shorokhod integrals are denoted by $\mathrm{dom}(\delta)$, $\mathrm{dom}(\delta^1)$ etc.

3 The absolute continuity of conditional laws

Let $(\Omega, \mathscr{F}, P)$ be the canonical 1-dimensional Wiener space of continuous functions on $\mathbb{R}_+$ starting at 0. Assume that $F \in L^2(\Omega, \mathscr{F}, P)$ is a random variable, and let

$$\mathscr{G}_t = \mathscr{F}_t \vee \sigma(F), \quad t \geqq 0 ,$$

be the canonical filtration enlarged by the information present in F. We emphasize that $(\mathscr{F}_t)_{t \geqq 0}$ is supposed to satisfy the usual conditions, hence so does $(\mathscr{G}_t)_{t \geqq 0}$. We shall answer the question: under which conditions is a semimartingale w.r.t. $(\mathscr{F}_t)_{t \geqq 0}$ still a semimartingale w.r.t. $(\mathscr{G}_t)_{t \geqq 0}$? Jacod [8] showed that this is the case provided the conditional laws of F given $\mathscr{F}_t$ possess densities with respect to a common reference measure. We shall assume that Lebesgue measure is this common reference measure, and use Malliavin's criterion for absolute continuity to provide densities for the conditional laws. To represent conditional laws we shall use the following transformations on Wiener space. For $t \geqq 0$, let

$$\begin{aligned} S_t : \ \Omega \times \Omega \quad &\to \quad \Omega \\ (\omega_1, \omega_2) \quad &\to \quad \left(u \quad \to \quad \begin{matrix} \omega_1(u), \quad u \leqq t \\ \\ \omega_1(t) + \omega_2(u - t), \quad u > t \end{matrix} \right) . \end{aligned}$$

Then it is obvious that S_t is $\mathscr{F}_t \otimes \mathscr{F} - \mathscr{F}$-measurable, and the Markov property for Brownian motion simply states that

$$(P \otimes P) \circ S_t^{-1} = P, \quad t \geqq 0 . \tag{1}$$

In terms of these transformations, the conditional laws of F have a simple representation.

Lemma 1 *Let $t \geqq 0$. Then*

$$(\omega, C) \to P(\{F \circ S_t(\omega, \ \cdot \ ) \in C\})$$

is a regular conditional probability of F given $\mathscr{F}_t$.

Proof. First of all, we have

$\omega \to F \circ S_t(\omega, \ \cdot \ )$ is $\mathscr{F}_t$-measurable, hence
$\omega \to P(\{F \circ S_t(\omega, \ \cdot \ ) \in C\})\, \mathscr{F}_t$-measurable for $C \in \mathscr{B}(\mathbb{R})$.

Moreover, for $A \in \mathscr{F}_t$ we have

$$S_t^{-1}[A] = A \times \Omega .$$

Hence the transformation theorem for measures implies for $C \in \mathscr{B}(\mathbb{R})$

$$\begin{aligned} &P(A \cap \{F \in C\}) \\ &= P \otimes P(S_t^{-1}[A] \cap \{F \circ S_t \in C\}) \quad ((1)) \\ &= P \otimes P(A \times \Omega \cap \{F \circ S_t \in C\}) \\ &= \int_A P(\{F \circ S_t(\omega, \cdot) \in C\}) dP(\omega) \quad \text{(Fubini)} \end{aligned}$$

This is what had to be shown. □

Remark. As was pointed out by a referee, Lemma 1 reproves a version of what is known as "Dawson's formula".

Let us next see how Malliavin derivatives and Shorokhod integrals behave when passing from Ω to the product $\Omega \times \Omega$ via S_t.

For an L^2-function on $\Omega \times \Omega$ we denote by D^1 resp. D^2 the Malliavin derivatives with respect to the first resp. second variable, and by $\mathbb{D}^1_{p,1}$ resp. $\mathbb{D}^2_{p,1}$ etc. the respective Sobolev spaces, $p \geqq 1$.

Lemma 2 *Let* $0 \leqq t < T, F \in \mathbb{D}_{2,1}([0,T])$. *Then*

$$F \circ S_t \in \mathbb{D}^1_{2,1}([0,t]) \cap \mathbb{D}^2_{2,1}([0, T-t])$$

and we have

$$\left.\begin{aligned} D^1_\cdot [F \circ S_t] &= D_\cdot F \circ S_t \\ D^2_\cdot [F \circ S_t] &= D_{t+\cdot} F \circ S_t \end{aligned}\right\} \quad \lambda \otimes P \otimes P\text{-}a.s.$$

Proof. A usual completion argument boils the assertion down to a statement about $F \in \mathscr{S}$, the space of smooth cylinder functions.
Assume F is of the form

$$F = f(W_{t_1}, \ldots, W_{t_n}) ,$$

where $t_1, \ldots, t_k \leqq t, \; t_{k+1}, \ldots, t_n > t, \; f \in C_0^\infty(\mathbb{R}^n)$.

Let us denote by W^1, W^2 the first resp. second coordinate canonical processes on $\Omega \times \Omega$. Then we have

$$F \circ S_t = f(W^1_{t_1}, \ldots, W^1_{t_n}, W^1_t + W^2_{t_{k+1}-t}, \ldots, W^1_t + W^2_{t_n - t}) .$$

Hence for $u \leqq t$

$$D^1_u[F \circ S_t] = \sum_{i=1}^n \frac{\partial}{\partial x_i} f(W^1_{t_1}, \ldots, W^1_{t_k}, W^1_t + W^2_{t_{k+1}-t}, \ldots, W^1_t + W^2_{t_n - t}) 1_{[0, t_i \wedge t]}(u)$$

and

$$D_u F \circ S_t = \sum_{i=1}^{n} \frac{\partial}{\partial x_i} f(W^1_{t_1}, \ldots, W^1_{t_k}, W^1_t + W^2_{t_{k+1}-t}, \ldots, W^1_t + W^2_{t_n - t}) 1_{[0,t_i]}(u) \,.$$

This implies

$$D_u F \circ S_t = D^1_u [F \circ S_t] \,,$$

as asserted. Moreover, for $0 \leqq u \leqq T - t$

$$D^2_u [F \circ S_t] = \sum_{i=k+1}^{n} \frac{\partial}{\partial x_i} f(W^1_{t_1}, \ldots, W^1_{t_k}, W^1_t + W^2_{t_{k+1}-t}, \ldots, W^1_t + W^2_{t_n - t}) 1_{[0,t_i - t]}(u)$$

and

$$D_{t+u} F \circ S_t = \sum_{i=1}^{n} \frac{\partial}{\partial x_i} f(W^1_{t_1}, \ldots, W^1_{t_k}, W^1_t + W^2_{t_{k+1}-t}, \ldots, W^1_t + W^2_{t_n - t}) 1_{[0,t_i]}(t+u)$$

$$= \sum_{i=k+1}^{n} \frac{\partial}{\partial x_i} f(W^1_{t_1}, \ldots, W^1_{t_k}, W^1_t + W^2_{t_{k+1}-t}, \ldots, W^1_t + W^2_{t+t_n - t}) 1_{[0,t_i - t]}(u) \,,$$

hence also

$$D_{t+u} F \circ S_t = D^2_u [F \circ S_t] \,.$$

This completes the proof. □

We are ready to give a criterion for the absolute continuity of conditional laws of F.

Theorem 1 *Assume that* $0 \leqq t < T, F \in \mathbb{D}_{2,1}([0,T])$. *Then the conditional law of* F *given* $\mathscr{F}_t$ *is* P*-a.s. absolutely continuous w.r. to Lebesgue measure, if*

$$\int_t^T (D_u F)^2 \, du > 0 \;\; P\text{-}a.s.$$

Proof. According to Lemma 1, a version of the regular conditional law of F given $\mathscr{F}_t$ is given by

$$(\omega, C) \to P(\{F \circ S_t(\omega, \cdot) \in C\}), \quad \omega \in \Omega, C \in \mathscr{B}(\mathbb{R}) \,.$$

Now according to the hypothesis, we have

$$\int_t^T (D_u F)^2 \, du \circ S_t = \int_0^{T-t} (D^2_u [F \circ S_t])^2 \, du > 0, \quad P \otimes P\text{-a.s. (Lemma 2)} \,,$$

hence by Fubini's theorem

$$\int_0^{T-t} (D^2_u [F \circ S_t])^2 (\omega, \cdot) \, du > 0 \quad P\text{-a.s. for } P\text{-a.e. } \omega \in \Omega \,.$$

This implies by Nualart [13, p. 89] that for P-a.e. $\omega \in \Omega$ the P-law of $F \circ S_t(\omega, \cdot)$ is absolutely continuous with respect to Lebesgue measure. This is what had to be shown. □

Remark. Theorem 1 is a special case of the more general Theorem 4.2 of Nualart and Zakai [14], an elaborated version of which can also be found as Theorem 5.2.7 of Bouleau and Hirsch [4]. In both references the notation is comparable. To explain the relationship with our result, we stick to the notation of [4]. Given $0 < t < T$, we there have to choose $\Omega = C_0([0,T]; \mathbb{R})$, the Wiener measure m and $X = (W_s)_{0 \leqq s \leqq t}$. For the Dirichlet space $\mathbb{D}$ we take $\mathbb{D}_{2,1}([0,T])$, the classical Dirichlet space over Ω. Then for $F \in \mathbb{D}$ we get

$$D_s^X F = 1_{[t,T]}(s) D_s F \ ,$$

and the Dirichlet form $\mathscr{E}^X$ associated with D^X, given by the formula

$$\mathscr{E}^X(F) = \tfrac{1}{2} E(\langle D^X F, D^X F \rangle) \ ,$$

is closed due to Proposition 5.2.5b of [4]. Hence in this setting the hypotheses of Theorem 5.2.7 are fulfilled and it implies that if $F \in \mathbb{D}$ and

$$\Gamma^X(F) = \langle D^X F, D^X F \rangle = \int_t^T (D_s F)^2 \, ds > 0$$

m-a.s., then F possesses a conditional density given $\sigma(X) = \mathscr{F}_t$. Theorem 5.2.9 of [4] gives a multidimensional version of this result.

Despite these facts we chose to keep our original proof of Theorem 1 for two reasons. Firstly, it is more elementary and direct than the one given in the general setting by Nualart and Zakai [14] or Bouleau and Hirsch [4]. Secondly, it fits better in our framework since it puts to work the technique of factorization of the Wiener space which will be explicitly employed in Sects. 5 and 6.

Corollary 1 *Assume that* $F \in \mathbb{D}_{2,1}([0,T])$ *for any* $T > 0$, *and that for* $0 \leqq t$ *there exists* $T > t$ *such that*

$$\int_t^T (D_u F)^2 \, du > 0 \quad P\text{-a.s.} \, .$$

Then any $(\mathscr{F}_t)$*-semimartingale is a* $(\mathscr{G}_t)$*-semimartingale*

Proof. According to Theorem 1 for any $t \geqq 0$ the regular conditional law of F given $\mathscr{F}_t$ possesses a density w.r.t. Lebesgue measure. According to Jacod [8, p. 15] this implies that the semimartingale property is preserved. □

4 The integrability of the compensator

To give estimates of the moments of the compensator of local martingales in the larger filtration, in this section we shall always assume that Jacod's [8] criterion is fulfilled with respect to Lebesgue measure as common reference measure. According to Corollary 1 this is the case if $F \in \mathbb{D}_{2,1}([0,T])$ for $T > 0$ and for any $t > 0$ there exists $T > t$ such that

$$\int_t^T (D_u F)^2 \, du > 0, \quad P\text{-a.s.}$$

We first recall some results of Jacod [8, pp. 18–22], which will be essential to the following. First of all, there exists a version of the conditional densities measurable in all these variables. More precisely, there exists a function

$$(\omega, t, x) \to p(\omega, t, x)$$

measurable with respect to $\mathscr{F} \otimes \mathscr{B}(\mathbb{R}_+) \otimes \mathscr{B}(\mathbb{R})$ such that
(2) $(p(\cdot, t, x))_{t \geqq 0}$ is a cadlag, P-a.s. continuous $(\mathscr{F}_t)$-martingale for any $x \in \mathbb{R}$,
(3) $p(\cdot, t, x)\lambda(dx)$ is a version of the regular conditional law of F given $\mathscr{F}_t$, for any $t \geqq 0$ (see [8, pp. 18, 19]).
The P-a.s. continuity in (2) stems from the fact that $(\mathscr{F}_t)_{t \geqq 0}$ is the Wiener filtration. If we define for $x \in \mathbb{R}, a \geqq 0$

$$T_a^x = \inf\{t \geqq 0 : \ p(\cdot, t-, x) \leqq a\},$$

then T_a^x is an $(\mathscr{F}_t)$-stopping time, and T_a^F a $(\mathscr{G}_t)$-stopping time, such that
(4) $p(\cdot, \cdot, x) > 0$ and $p(\cdot, \cdot -, x) > 0$ on $[0, T_0^x[$, $p(\cdot, \cdot, x) = 0$ on $[T_0^x, \infty[$,
(5) $T_0^F = \infty$ P-a.s., and $T_{1/n}^F \uparrow \infty$ P-a.s. (see [8, pp. 19, 20]).
Moreover, the proof of Theorem 2.1 of Jacod [8, p. 20] contains the statement that there exists a process

$$(\omega, t, x) \to \alpha(\omega, t, x),$$

which is product measurable and satisfies
(6) $\alpha(\cdot, \cdot, x)$ is $(\mathscr{F}_t)$-adapted, and

$$p(\cdot, t, x) = \int_0^t \alpha(\cdot, s, x) \, dW_s + p(x), \quad t \geqq 0$$

for any $x \in \mathbb{R}$,
where $p(x) = p(\cdot, 0, x)$ is the density of F with respect to λ.
Finally, if we define

$$k(\omega, t, x) = \begin{cases} \frac{\alpha(\omega, t, x)}{p(\omega, t, x)} & \text{if } p(\omega, t, x) > 0, \\ 0 & \text{else,} \end{cases}$$

we obtain a product measurable process which satisfies
(7) $k(\cdot, \cdot, x)$ is $(\mathscr{F}_t)$-adapted, for any $x \in \mathbb{R}$,
(8) $k(\cdot, \cdot, F) = \frac{\alpha(\cdot, \cdot, F)}{p(\cdot, \cdot, F)}$,
due to (5), and, most importantly, for any local $(\mathscr{F}_t)$-martingale $M = \int_0^\cdot \beta_s dW_s$ we have
(9) $\widetilde{M}_t = M_t - \int_0^t \beta_s k(\cdot, s, F) \, ds$
is a local $(\mathscr{G}_t)$-martingale (see [8, Théorème 2.1]).

Our aim will be to derive a priori estimates for the moments of the compensator in (9), and this way to obtain imbedding results for martingale spaces w.r.t. $(\mathscr{F}_t)_{t \geqq 0}$ in martingale spaces w.r.t. $(\mathscr{G}_t)_{t \geqq 0}$. Hereby we shall be guided by Yor [18, 19], where the cases of "grossissement initial" with respect to a countable family of sets in Ω resp. "grossissement progressif" by a random variable that has to become a stopping time in the larger filtration, are treated. The key step will consist in an estimate for the potential of the process

$$A_t = \int_0^t k^2(\cdot, s, F)\, ds = \int_0^t \frac{\alpha(\cdot, s, F)^2}{p(\cdot, s, F)} ds\,, \quad t \geqq 0 \qquad \text{(see (8))}.$$

To obtain this estimate, let us localize along the sequences of $(\mathscr{F}_t)$-stopping times

$$S_n^x = \inf\left\{ t \geqq 0 : \int_0^t k(\cdot, s, x)^2\, ds \geqq n \right\}, \quad n \in \mathbb{N}, \quad x \in \mathbb{R}\,.$$

Note that for $n \in \mathbb{N}$

S_n^F is a $(\mathscr{G}_t)$-stopping time,

which according to (9) has the property

(10) $S_n^F \uparrow \infty \ (n \to \infty)$.

Let us consider the $(\mathscr{F}_t)$-stopping times

$$U_n^x = S_n^x \wedge T_{1/n}^x, \quad x \in \mathbb{R}, \quad n \in \mathbb{N}\,,$$

and the increasing processes

$$A_t^n = A_{t \wedge U_n^F}\,, \quad n \in \mathbb{N},\ t \geqq 0\,,$$

as well as the martingales

$$M_t^n(x) = \int_0^t 1_{[0, U_n^x[}(s) k(\cdot, s, x)\, dW_s\,.$$

By definition of the stopping times we have

$$M_t^n(x) = \int_0^t 1_{[0, U_n^x[}(s) \frac{\alpha(\cdot, s, x)}{p(\cdot, s, x)}\, dW_s, \quad t \geqq 0,\ n \in \mathbb{N},\ x \in \mathbb{R}\,.$$

Abbreviate

$$N_t^n(x) = p(\cdot, t \wedge U_n^x, x), \quad t \geqq 0,\ x \in \mathbb{R},\ n \in \mathbb{N}\,.$$

Then (6) and Itô's formula give for $x \in \mathbb{R},\ n \in \mathbb{N},\ t \geqq 0$

(11) $\quad 1_{\{U_n^x > 0\}}[\ln N_t^n(x) - \ln p(x)]$

$$= \int_0^t 1_{[0, U_n^x[}(s)\, \frac{1}{N_s^n(x)}\, dN_s^n(x) - \frac{1}{2} \int_0^t 1_{[0, U_n^x[}(s)\, \frac{1}{N_s^n(x)^2}\, d\langle N_s^n(x) \rangle_s$$

$$= M_t^n(x) - \frac{1}{2} \int_0^t 1_{[0, U_n^x]}(s)\, \frac{\alpha^2(\cdot, s, x)}{p^2(\cdot, s, x)}\, ds\,.$$

Equation (11) will give the following estimate of the potential of A^n.

Lemma 3 *Let $s,t \geqq 0$, $s \leqq t$, $n \in \mathbb{N}$. Then*

$$E(A_t^n - A_s^n | \mathscr{G}_s) = 2\, 1_{\{U_n^F > 0\}} [\ln N_s^n(F) - E(\ln N_t^n(F) | \mathscr{G}_s)] \, .$$

Proof. Let $X : \mathbb{R} \times \Omega \to \mathbb{R}$ be a bounded product measurable random variable. Then, as is seen by a monotone class argument starting with indicators of $\{F \in C\} \times A$, $C \in \mathscr{B}(\mathbb{R})$, $A \in \mathscr{F}$, we have

$$E(X(F, \cdot)|\mathscr{G}_t) = E(X(x, \cdot)|\mathscr{F}_t)|_{x=F} \, . \tag{12}$$

Now we start with (11), applying (12) twice, once to

$$X(x, \cdot) = \int_s^t 1_{[0, U_n^x[}(u) \frac{\alpha^2(\cdot, u, x)}{p^2(\cdot, u, x)} \, du \, ,$$

once to

$$X(x, \cdot) = \ln N_t^n(x), \quad x \in \mathbb{R} \, .$$

The resulting equation is

$$\begin{aligned} &E(A_t^n - A_s^n | \mathscr{G}_s) \\ &= E\left(\int_s^t 1_{[0, U_n^F[}(u) \frac{\alpha(\cdot, u, F)^2}{p(\cdot, u, F)^2} \, du | \mathscr{G}_s \right) \\ &= E\left(\int_s^t 1_{[0, U_n^x[}(u) \frac{\alpha(\cdot, u, x)^2}{p(\cdot, u, x)^2} \, du \middle| \mathscr{F}_s \right) \Bigg|_{x=F} \\ &= 2\, 1_{\{U_n^x > 0\}} (E(\ln N_s^n(x) - \ln N_t^n(x) | \mathscr{F}_s))|_{x=F} \\ &= 2\, 1_{\{U_n^F > 0\}} [\ln N_s^n(F) - E(\ln N_t^n(F) | \mathscr{G}_s)] \, . \end{aligned} \tag{13}$$

Here we have used the martingale property of $M^n(x)$, $n \in \mathbb{N}$, $x \in \mathbb{R}$. This completes the proof. □

The following inequality combines the observation of Lemma 3 with the inequality of Burkholder–Davis–Gundy.

Lemma 4 *For $T > 0$, $p > 1$, $n \in \mathbb{N}$, we have*

$$E((A_T^n)^p)^{1/p} \leqq p \frac{2p-1}{p-1} E(1_{\{U_n^F > 0\}} \sup_{0 \leqq t \leqq T} |\ln N_t^n(F)|^p)^{1/p} \, .$$

Proof. By Lemma 3 and the inequality of Burkholder–Davis–Gundy for rough increasing processes (see [12, p. 138]) we have

$$\begin{aligned} &E((A_T^n)^p)^{1/p} \\ &\leqq p\, E(1_{\{U_n^F > 0\}} \sup_{0 \leqq t \leqq T} |\ln N_t^n(F) - E(\ln N_T^n(F) | \mathscr{G}_t)|^p)^{1/p} \\ &\leqq p \left[E\left(1_{\{U_n^F > 0\}} \sup_{0 \leqq t \leqq T} |\ln N_t^n(F)|^p \right)^{1/p} \right. \\ &\quad \left. + E\left(1_{\{U_n^F > 0\}} \sup_{0 \leqq t \leqq T} |E(\ln N_T^n(F) | \mathscr{G}_t)|^p \right)^{1/p} \right] . \end{aligned} \tag{14}$$

Now apply Doob's inequality to the second term in (14) and compare with the first. This gives the desired result. □

Our next task will be to estimate $\sup_{0\leqq t\leqq T} |\ln N_t^n(F)|^p$.

For this purpose it is necessary to give a general estimate for such an expression. Let $f\colon\ [0,T]\to \mathbb{R}$ be a nonnegative cadlag function such that $f(0)=1$ and set

$$i_T = \inf_{0\leqq t\leqq T} f(t), \quad s_T = \sup_{0\leqq t\leqq T} f(t)\,.$$

Then obviously

$$i_T \leqq 1 \leqq s_T\,,$$

and so

$$\text{(15)}\qquad \sup_{0\leqq t\leqq T} |\ln f(t)| = \ln\frac{1}{i_T}\,1_{\{s_T\leqq 1/i_T\}} + \ln s_T\,1_{\{s_T>1/i_T\}} \leqq \ln\frac{1}{i_T} + \ln s_T\,.$$

Let us now consider the following martingales. Take

$$K_t^n(x) = \begin{cases} \dfrac{N_t^n(x)}{p(x)}, & p(x)>0\,,\\ 0, & p(x)=0\,,\end{cases}$$

$t \geqq 0$, $n\in\mathbb{N}$, $x\in\mathbb{R}$. Then obviously on $\{U_n^F>0\}$,

$$\text{(16)}\qquad K_0^n(F)=1\,.$$

Hence we may estimate for $p>1$, $n\in\mathbb{N}$, $T>0$, with a universal constant c_p,

(17)

$$\begin{aligned} &E\left(\sup_{0\leqq t\leqq T} |\ln N_t^n(F)|^p\,1_{\{U_n^x>0\}}\right)\\ &\quad = \int_{\mathbb{R}} E\left(N_T^n(x)\sup_{0\leqq t\leqq T}|\ln N_t^n(x)|^p\,1_{\{U_n^x>0\}}\right)dx \quad (N^n(x)\ \mathscr{F}_{U_n^x}\text{-measurable})\\ &\quad \leqq c_p\left[\int_{\mathbb{R}} p(x)|\ln p(x)|^p\,dx + \int_{\mathbb{R}} p(x)E\left(K_T^n(x)\sup_{0\leqq t\leqq T}|\ln K_t^n(x)|^p\right)dx\right]\\ &\quad \leqq c_p\left[\int_{\mathbb{R}} p(x)|\ln p(x)|^p\,dx + \int_{\mathbb{R}} p(x)E\left(K_T^n(x)\left(\ln\sup_{0\leqq t\leqq T}K_t^n(x)\right)^p\right)dx\right.\\ &\qquad \left. + \int_{\mathbb{R}} p(x)E\left(K_T^n(x)\left(\ln\frac{1}{\inf_{0\leqq t\leqq T}K_t^n(x)}\right)^p\right)dx\right]\,. \end{aligned}$$

Let us first consider the last term in (17). It may be treated in a similar fashion as in Yor [19]. Indeed, as we shall see, the finiteness of the first term on the rhs of (17) is sufficient.

Lemma 5 *Let* $T > 0, p > 1$. *Then*

$$\sup_{n\in\mathbb{N}} \int_{\mathbb{R}} p(x) E\left(K_T^n(x)\left(\ln \frac{1}{\inf_{0\leqq t\leqq T} K_t^n(x)}\right)^p\right) dx < \infty \,.$$

Proof. For abbreviation, put

$$I_T = \inf_{0\leqq t\leqq T} K_t^n(x)\,,$$

fixing $n \in \mathbb{N}$, $x \in \mathbb{R}$. Moreover, let

$$\sigma_b = \inf\{t \geqq 0:\ K_t^n(x) \leqq b\} \wedge T,\ \ b > 0\,.$$

It is clear that σ_b is an $(\mathscr{F}_t)$-stopping time. The law of I_T can then be estimated as follows. We have for $b \in]0,1]$ by Doob's optional stopping theorem

$$\begin{aligned}
E(1_{\{I_T<b\}} K_T^n(x)) &= E(1_{\{\sigma_b<T\}} K_T^n(x)) \\
&= E(1_{\{\sigma_b<T\}} K_{\sigma_b}^n(x)) \\
&= bP(I_T < b)\,.
\end{aligned} \tag{18}$$

Equation (18) yields for $p > 1,\ T > 0$

$$\begin{aligned}
&\int_{\mathbb{R}} E\left(\left(\ln\frac{1}{I_T}\right)^p K_T^n(x)\right) p(x)\,dx \\
&= E\left(\left(\ln\frac{1}{\inf_{0\leqq t\leqq T} K_t^n(F)}\right)^p\right) \\
&= p\int_1^\infty \lambda^{p-1} P\left(\inf_{0\leqq t\leqq T} K_t^n(F) < e^{-\lambda}\right) d\lambda \\
&= p\int_1^\infty \lambda^{p-1}\int_{\mathbb{R}} E(1_{\{I_T<e^{-\lambda}\}} K_T^n(x))\,p(x)\,dx\,d\lambda \\
&\leqq p\int_1^\infty \lambda^{p-1} e^{-\lambda}\,d\lambda < \infty \quad ((18))\,.
\end{aligned} \tag{19}$$

This completes the proof. □

The estimate of the second term in (17) is harder. We shall use the following lemma.

Lemma 6 *Let* $p \geqq 0, (X_t)_{t\geqq 0}$ *a positive, cadlag, P-a.s. continuous martingale such that* $X_0 = 1$. *Let* $T > 0$, *and set*

$$X_T^* = \sup_{0\leqq t\leqq T} X_t\,.$$

Then, there is a universal c_p *such that*

$$E(X_T^*(\ln X_T^*)^p) \leqq c_p(1 + E(X_T(\ln^+ X_T)^{p+1}))\,.$$

Proof. First of all, Dellacherie and Meyer [6, p.19] yields

$$E(X_T^*(\ln X_T^*)^p) = \frac{1}{p+1} E(X_T(\ln X_T^*)^{p+1}) + E(X_T(\ln X_T^*)^p) \,. \tag{20}$$

Moreover, for $x, y > 0, x < y, y \geqq 1$, we have

$$x \ln y - x \ln^+ x \leqq \frac{y}{e} \quad \text{(see [6, p. 20])} \,,$$

hence for $p > 1$

$$\begin{aligned} x(\ln y)^p - x(\ln^+ x)^p &\leqq x p \int_{x \vee 1}^{y} \frac{(\ln t)^{p-1}}{t} \, dt \\ &\leqq p(\ln y)^{p-1} x(\ln y - \ln^+ x) \\ &\leqq p(\ln y)^{p-1} \frac{y}{e} \,. \end{aligned} \tag{21}$$

As a consequence of (21), we may write

$$\begin{aligned} E(X_T^*(\ln X_T^*)^p) \leqq \; & \frac{1}{p+1} E(X_T(\ln^+ X_T)^{p+1}) \\ & + \frac{1}{e} E(X_T^*(\ln X_T^*)^p) + E(X_T(\ln X_T^*)^p) \,, \end{aligned} \tag{22}$$

that is, using again (20)

$$\begin{aligned} & E(X_T^*(\ln X_T^*)^p) \\ & \leqq \frac{e}{e-1} \left[\frac{1}{p+1} E(X_T(\ln^+ X_T)^{p+1}) + E(X_T(\ln X_T^*)^p) \right] \\ & \leqq \frac{e}{e-1} \left[\frac{1}{p+1} E(X_T(\ln^+ X_T)^{p+1}) \right. \\ & \qquad \left. + \frac{p}{e} E(X_T^*(\ln X_T^*)^{p-1}) + E(X_T(\ln^+ X_T)^p) \right] \,. \end{aligned} \tag{23}$$

From this formula it is clear how to obtain the desired inequality by induction, for it may be proved by simpler arguments for the interval $p \in]0, 1]$. □

We are ready to estimate the second term in (17).

Lemma 7 *Let* $p > 1$, $T > 0$. *Assume that*

$$E(\ln^+ N_T(F)^p) < \infty \,, \tag{24}$$

and

$$\int_{\mathbb{R}} p(x) \left(\ln^+ \frac{1}{p(x)} \right)^p dx < \infty \,. \tag{25}$$

Then we have

$$\sup_{n\in\mathbb{N}} \int_{\mathbb{R}} p(x)\, E\left(K_T^n(x)\left(\ln \sup_{0\leqq t\leqq T} K_t^n(x)\right)^p\right) dx < \infty\,,$$

and in particular

$$\int_{\mathbb{R}} p(x)|\ln p(x)|^p\, dx < \infty\,.$$

Proof. Recall the definition of $N(x)$ which was given in the remarks before Lemma 3. By convexity and Doob's optional stopping theorem we have first of all

$$\begin{aligned}(26)\qquad \sup_{n\in\mathbb{N}} E(\ln^+ N_T^n(F)^p) &= \sup_{n\in\mathbb{N}} \int_{\mathbb{R}} E(N_T^n(x)(\ln^+ N_T^n(x))^p)\, dx\\ &\leqq \int_{\mathbb{R}} E(N_T(x)(\ln^+ N_T(x))^p)\, dx\\ &= E(\ln^+ N_T(F)^p) < \infty\,.\end{aligned}$$

Note that this also implies

$$\int_{\mathbb{R}} p(x)(\ln^+ p(x))^p\, dx < \infty\,,$$

so that we have already proved

$$\int_{\mathbb{R}} p(x)\,|\ln p(x)\,|^p\, dx < \infty\,.$$

To prove the first inequality, fix $n \in \mathbb{N}$, and let for $x \in \mathbb{R}, b \geqq 1$

$$\tau_b = \inf\{t \geqq 0:\ K_t^n(x) \geqq b\} \wedge T\,.$$

Then τ_b is an $(\mathscr{F}_t)$-stopping time, and, denoting

$$S_T(x) = \sup_{0\leqq t\leqq T} K_t^n(x)\,,$$

we have the analogue of (18)

$$(27)\qquad E(1_{\{S_T(x)>b\}}K_T^n(x)) = b\,P(S_T(x) > b)\,.$$

Hence, we obtain an analogue of (19):

$$\begin{aligned}(28)\qquad &\int_{\mathbb{R}} p(x)\ E((\ln S_T(x))^p\, K_T^n(x))\, dx\\ &= p\int_1^\infty \lambda^{p-1} \int_{\mathbb{R}} e^\lambda P(S_T(x) > e^\lambda)\, p(x)\, dx\, d\lambda\\ &= p\int_e^\infty (\ln t)^{p-1} \int_{\mathbb{R}} P(S_T(x) > t)\ p(x)\, dx\, dt\\ &\leqq p\int_{\mathbb{R}} E((\ln S_T(x))^{p-1} S_T(x))\ p(x)\, dx \quad \text{(Fubini)}\,.\end{aligned}$$

We next apply Lemma 6 to the last expression in (28). An appeal to (26) finishes the proof. □

We are ready to state the main boundedness result.

Lemma 8 *Let* $p > 1, T > 0$, *and assume that* (24) *and* (25) *are satisfied. Then we have*

$$E(A_T^p) < \infty\,.$$

Proof. By Lemma 4, we have to show that (17) is bounded in $n \in \mathbb{N}$. But this is a consequence of Lemmas 5 and 7 together with (24) and (25). □

Here is the main result of this section.

Theorem 2 *Let* $r, p, q > 1$ *such that* $1/r = 1/p + 1/q, T > 0$. *Assume that* u *is a* $(\mathscr{G}_t)$*-adapted process which is locally square integrable* P*-a.s. Then there is a universal constant* $c_{r,q}$ *such that*

$$\left\| \sup_{0 \leqq t \leqq T} \left| \int_0^t u_s \, dW_s \right| \right\|_r \leqq c_{r,q} \left\| \left[\int_0^T u_s^2 \, ds \right]^{1/2} \right\|_q$$

if

$$E((\ln^+ N_T(F))^p < \infty, \int_{\mathbb{R}} p(x) \left(\ln^+ \frac{1}{p(x)} \right)^p dx < \infty\,.$$

Proof. By the usual stopping and completion argument, we may assume that u is bounded. (9) tells us that

$$\begin{aligned} \widetilde{W}_t &= W_t - \int_0^t k(\,\cdot\,, s, F)\, ds \\ &= W_t - \int_0^t \frac{\alpha(\,\cdot\,, s, F)}{N_s(F)}\, ds \end{aligned}$$

is a $(\mathscr{G}_t)$-Wiener process. Hence

$$\int_0^\cdot u_s \, dW_s = \int_0 u_s \, d\widetilde{W}_s + \int_0^\cdot u_s \, k(\,\cdot\,, s, F)\, ds$$

and therefore

$$\begin{aligned} &\left\| \sup_{0 \leqq t \leqq T} \left| \int_0^t u_s \, dW_s \right| \right\|_r \\ &\quad \leqq \left\| \sup_{0 \leqq t \leqq T} \left| \int_0^t u_s \, d\widetilde{W}_s \right| \right\|_r + \left\| \sup_{0 \leqq t \leqq T} \left| \int_0^t u_s \, k(\,\cdot\,, s, F) ds \right| \right\|_r \\ &\quad \leqq c_1 \left\| \left[\int_0^T u_s^2 \, ds \right]^{1/2} \right\|_r + \left\| \left[\int_0^T u_s^2 \, ds \right]^{1/2} \left[\int_0^T k(\,\cdot\,, s, F)^2 ds \right]^{1/2} \right\|_r \end{aligned}$$

$$\leqq c_1 \left\| \left[\int_0^T u_s^2\, ds \right]^{1/2} \right\|_q + \left\| \left[\int_0^T u_s^2\, ds \right]^{1/2} \right\|_q \quad \left\| A_T^{1/2} \right\|_p$$

$$= \left[c_1 + \left\| A_T^{1/2} \right\|_p \right] \left\| \left[\int_0^T u_s^2\, ds \right]^{1/2} \right\|_q$$

According to Lemma 8,

$$c_{r,q} = c_1 + \| A_T^{1/2} \|_p < \infty$$

due to the hypotheses. This completes the proof. □

Theorem 2 is a purely martingale theoretic result. We now have to return to the framework of Malliavin's calculus to look for conditions on F under which the hypotheses of Theorem 2 are valid.

5 The representation of conditional densities

We now return to the methods of Malliavin's calculus. It provides the necessary tools to describe the conditional densities of F explicitly under sufficient regularity conditions. These representations play an important role in our analysis, since they will be the starting point for the study of the hypotheses of Theorem 2. Indeed, regularity assumptions in terms of Malliavin's calculus concerning F will guarantee that Theorem 2 is applicable. This way we gain control over the compensator in the canonical decomposition of $(\mathscr{F}_t)$-martingales with respect to the enlarged filtration.

Using the representation of conditional laws found in Sect. 1, let us now derive representations of their densities. Hereby, Shorokhod's integral will enter the scene. In Sect. 1, we made use of a switch between the space Ω and the space $\Omega \times \Omega$ by means of the measure preserving maps S_t. We made the transport of Malliavin derivatives explicit. Let us now exhibit how Shorokhod integrals and conditional expectations are transported.

Remark. For $t \geqq 0$ and an integrable random variable H on Wiener space the statement

$$E(H|\mathscr{F}_t) = \int_\Omega H \circ S_t(\,\cdot\,, \omega_2) P(d\omega_2)$$

is a special case of Lemma 1.

Recall that the Shorokhod integral on Ω will be denoted by δ, and the Shorokhod integrals on the respective components of $\Omega \times \Omega$ by δ^1, δ^2. For integrals from s to t we write δ_s^t etc, and $\langle u, v\rangle_s^t$ for $\int_s^t u_r v_r dr$, u, v square integrable. The following lemma deals with the transfer of Shorokhod integrals.

Lemma 9 *Let* $0 \leqq r \leqq s \leqq t$, $u \in \mathrm{dom}(\delta_s^t)$. *Then*

$$u_{r+\cdot} \circ S_r \in \mathrm{dom}((\delta^2)_{s-r}^{t-r}) \quad P\text{-}a.s., \ and,$$

if defined trivially on the exceptional set, we have

$$(\delta^2)_{s-r}^{t-r}(u_{r+\cdot} \circ S_r) = \delta_s^t(u) \circ S_r \quad (P \otimes P\text{-}a.s.).$$

Proof. Let $H \in \mathbb{D}_{2,1}([s,t]), G \in \mathbb{D}_{2,1}([0,r])$ $\mathscr{F}_r$-measurable, bounded. Then by Lemma 2

$$H \circ S_r \in \mathbb{D}_{2,1}^2([s-r, t-r])$$

and

$$D^2[H \circ S_r] = D_{r+\cdot} H \circ S_r \ .$$

Hence

$$\begin{aligned} &E(G\,E(\langle u_{r+\cdot} \circ S_r, D^2(H \circ S_r)\rangle_{s-r}^{t-r})) \\ &\quad = E \otimes E(G \circ S_r \langle u_{r+\cdot} \circ S_r, D^2(H \circ S_r)\rangle_{s-r}^{t-r}) \\ &\quad = E \otimes E(\langle u_{r+\cdot} \circ S_r, D^2((G \cdot H) \circ S_r)\rangle_{s-r}^{t-r}) \\ &\quad = E \otimes E(\langle u, D(G \cdot H)\rangle_s^t \circ S_r) \\ &\quad = E(\langle u, D(G \cdot H)\rangle_s^t) \\ &\quad = E(\delta_s^t(u) G \cdot H) \quad (u \in \mathrm{dom}(\delta_s^t)) \\ &\quad = E \otimes E(\delta_s^t(u) \circ S_r\,(G \cdot H) \circ S_r) \\ &\quad = E(G\,E(\delta_s^t(u) \circ S_r\, H \circ S_r)) \ . \end{aligned}$$

This equation generalizes immediately to general $\mathscr{F}_r$-measurable bounded G. Hence we obtain

$$E(\langle u_{r+\cdot} \circ S_r,\ D^2(H \circ S_r)\rangle_{s-r}^{t-r}) = E(\delta_s^t(u) \circ S_r\, H \circ S_r) \quad P\text{-a.s.}$$

Hence P-a.s. we have

$$u_{r+\cdot} \circ S_r \in \mathrm{dom}((\delta^2)_{s-r}^{t-r})$$

and (defining the integral trivially on the set of measure 0)

$$\delta_{s-r}^{t-r}(u_{r+\cdot} \circ S_r) = \delta_s^t(u) \circ S_r \ .$$

This is the asserted equation. □

We are ready for the representation formula of conditional densities.

Theorem 3 *Let* $0 \leqq t \leqq T$. *Assume that* $F \in \mathbb{D}_{2,1}([0,T])$ *and*

$$\frac{DF}{\langle DF, DF\rangle_t^T} \in \mathrm{dom}(\delta_t^T) \ .$$

Then the P*-a.s. bounded and continuous function*

$$p^+(\omega, t, x) = E\left(1_{\{F>x\}}\delta_t^T\left(\frac{DF}{\langle DF, DF\rangle_t^T}\right) \circ S_t(\omega, \cdot)\right) ,$$

$\omega \in \Omega, x \in \mathbb{R}$, *is a version of the density of the regular conditional law of* F *given* $\mathscr{F}_t$.

Proof. According to Lemma 2, for P-a.e. $\omega \in \Omega$, we have

$$F \circ S_t(\omega, \cdot) \in \mathbb{D}^2_{2,1}([0, T-t])$$

and

$$D^2[F \circ S_t](\omega, \cdot) = D_{t+\cdot} F \circ S_t(\omega, \cdot) . \tag{29}$$

To abbreviate, let $G = F \circ S_t$. Then (29) implies also that for P-a.e. $\omega \in \Omega$

$$\frac{D_{t+\cdot} F}{\langle DF, DF \rangle_t^T} \circ S_t(\omega, \cdot) = \frac{D^2 G(\omega, \cdot)}{\langle D^2 G, D^2 G \rangle_0^{T-t}(\omega, \cdot)} . \tag{30}$$

Moreover, Lemma 9 allows us to affirm that for P-a.e. $\omega \in \Omega$

$$\frac{D_{t+\cdot} F}{\langle DF, DF \rangle_t^T} \circ S_t(\omega, \cdot) \in \mathrm{dom}((\delta^2)_0^{T-t}) \tag{31}$$

and

$$(\delta^2)_0^{T-t}\left(\frac{D^2 G}{\langle D^2 G, D^2 G \rangle_0^{T-t}}\right)(\omega, \cdot) = \delta_s^t\left(\frac{DF}{\langle DF, DF \rangle_t^T}\right) \circ S_t(\omega, \cdot) . \tag{32}$$

Now we take up the arguments of Nualart [13, p. 80]. Let $[a, b] \subset \mathbb{R}$ an interval and

$$\psi = 1_{[a,b]}, \quad \varphi^+(y) = \int_{-\infty}^{y} \psi(z)\, dz .$$

Then for P-a.e. $\omega \in \Omega$ his arguments give

$$E(\psi(G)(\omega, \cdot)) = E\left(\varphi^+(G)(\delta^2)_0^{T-t}\left(\frac{D^2 G}{\langle D^2 G, D^2 G \rangle_0^{T-t}}\right)(\omega, \cdot)\right) . \tag{33}$$

Now use the preceding statements to translate this result back into the language of conditional laws. We have for P-a.e. $\omega \in \Omega$

$$\begin{aligned} P(\omega, t, [a, b]) &= P(F \circ S_t(\omega, \cdot) \in [a, b]) \\ &= P(G(\omega, \cdot) \in [a, b]) \\ &= E(\psi(G)(\omega, \cdot)) \\ &= E\left(\varphi^+(F)\delta_t^T\left(\frac{DF}{\langle DF, DF \rangle_t^T}\right) \circ S_t(\omega, \cdot)\right) \\ &= E\left(\int_{-\infty}^{F} \psi(x)\, dx\, \delta_t^T\left(\frac{DF}{\langle DF, DF \rangle_t^T}\right) \circ S_t(\omega, \cdot)\right) \\ &= \int_a^b E\left(1_{\{F > x\}}\delta_t^T\left(\frac{DF}{\langle DF, DF \rangle_t^T}\right) \circ S_t(\omega, \cdot)\right) dx . \end{aligned}$$

This gives the desired formula. It remains to remark that the integrand clearly is a bounded continuous function due to dominated convergence. □

Remark. There is another canonical version of the conditional density given by Theorem 3. As an immediate consequence of the zero mean property of the Skorokhod integral it is seen to be given by the formula

$$p^-(\omega,t,x) = -E\left(1_{\{F<x\}}\delta_t^T\left(\frac{DF}{\langle DF,DF\rangle_t^T}\right)\circ S_t(\omega,\cdot)\right),$$

$\omega\in\Omega$, $x\in\mathbb{R}, 0\leqq t<T$. It will be used together with p^+ in the proof of Theorem 4.

The second criterion of Theorem 3 is hard to verify. Let us give a sufficient criterion for its validity.

Corollary 2 *Let* $0\leqq t<T$, $p,q>1$ *such that* $1/p+1/q=\frac{1}{2}$. *Assume that* $F\in\mathbb{D}_{p,2}([0,T])$ *and*

$$\frac{1}{\langle DF,DF\rangle_t^T}\in L^q(\Omega,\mathscr{F},P).$$

Then the P-*a.s. bounded and continuous function*

$$p^+(\omega,t,x) = E\left(1_{\{F>x\}}\delta_t^T\left(\frac{DF}{\langle DF,DF\rangle_t^T}\right)\circ S_t(\omega,\cdot)\right),$$

$\omega\in\Omega$, $x\in\mathbb{R}$, *is a version of the density of the regular conditional law of* F *given* $\mathscr{F}_t$.

Proof. According to Nualart [13, p. 72], we have to show that

$$\left\|\frac{DF}{\langle DF,DF\rangle_t^T}\right\|_{1,2}<\infty, \tag{34}$$

where the norm is taken with respect to $[t,T]$.

For this sake, let us consider more closely the Malliavin derivative of the integrand. For $t\leqq u\leqq T$ we have

$$D_u\left[\frac{DF}{\langle DF,DF\rangle_t^T}\right] = \frac{D_uDF}{\langle DF,DF\rangle_t^T} - \frac{2DF}{(\langle DF,DF\rangle_t^T)^2}\langle D_uDF,DF\rangle_t^T.$$

Hence

$$\begin{aligned}
&\left(\left\langle D\left[\frac{DF}{\langle DF,DF\rangle_t^T}\right],D\left[\frac{DF}{\langle DF,DF\rangle_t^T}\right]\right\rangle_t^T\right)^{1/2} \\
&\leqq \frac{1}{\langle DF,DF\rangle_t^T}(\langle D^{\otimes 2}F,D^{\otimes 2}F\rangle_t^T)^{1/2} \\
&\quad + 2\frac{(\langle DF,DF\rangle_t^T)^{1/2}}{(\langle DF,DF\rangle_t^T)^2}\left(\int_t^T(\langle D_uDF,DF\rangle_t^T)^2\,du\right)^{1/2} \\
&\leqq \frac{1}{\langle DF,DF\rangle_t^T}(\langle D^{\otimes 2}F,D^{\otimes 2}F\rangle_t^T)^{1/2}
\end{aligned} \tag{35}$$

$$+ \frac{2}{(\langle DF, DF\rangle_t^T)^2} \langle DF, DF\rangle_t^T (\langle D^{\otimes 2}F, D^{\otimes 2}F\rangle_t^T)^{1/2} \quad \text{(Cauchy–Schwarz)}$$

$$\leqq \frac{3}{\langle DF, DF\rangle_t^T} \langle D^{\otimes 2}F, D^{\otimes 2}F\rangle^{1/2} .$$

From (35) it is clear that (34) reduces to the hypotheses $F \in \mathbb{D}_{p,2}([0,T])$, $1/\langle DF, DF\rangle_t^T \in L^q(\Omega, \mathscr{F}, P)$, by Hölder's inequality. □

We are now ready to combine Theorem 2 with Theorem 3, to obtain a regularity result for the compensator in the enlarged filtration $(\mathscr{G}_t)_{t \geqq 0}$.

Theorem 4 *Let $S, T > 0$, $S < T$. Assume $r, p, q > 1$ are such that $1/r = 1/p + 1/q$. Suppose furthermore that $F \in \mathbb{D}_{2,1}([0,T])$, that for $0 \leqq t \leqq S$ we have $\langle DF, DF\rangle_t^T > 0$ P-a.s., and that for $s = 0, S$ we have*

$$\frac{DF}{\langle DF, DF\rangle_s^T} \in \mathrm{dom}(\delta_s^T) .$$

Finally, suppose that putting

$$X_s = \delta_s^T \left(\frac{DF}{\langle DF, DF\rangle_s^T} \right), \quad s = 0, S ,$$

we have

$$E\left(|F|\,|X_0| \left(\ln^+ \frac{1}{|X_0|} \right)^p \right) < \infty , \tag{36}$$

$$E\left(|F|\,|X_S| \left(\ln^+ \frac{1}{|X_S|} \right)^p \right) < \infty . \tag{37}$$

Then for any $(\mathscr{G}_t)$-adapted P-a.s. locally square integrable process u we have with a universal constant $c_{r,q}$

$$\left\| \sup_{0 \leqq t \leqq S} \left| \int_0^t u_s \, dW_s \right| \right\|_r \leqq c_{r,q} \left\| \left[\int_0^S u_s^2 \, ds \right]^{1/2} \right\|_q .$$

Proof. For $s = 0, S$ let $p^+(\cdot, s, \cdot)$ be the conditional densities provided by Theorem 3. They are well defined by Theorem 1 and represented by the formulas of Theorem 3. All we have to show is that the hypotheses of Theorem 2 are consequences of (1), (2). Let us do this for $s = S$. Note first that for any $x \in \mathbb{R}, \omega \in \Omega$ we have, due to the convexity of the function

$$x \to x(\ln^+ x)^p ,$$

and Jensen's inequality

$$p^+(\omega, S, x)(\ln p^+(\omega, S, x))^p \leqq E([1_{\{F>x\}}|X_S|(\ln^+ 1_{\{F>x\}}|X_S|)^p] \circ S_S(\omega, \cdot)) ,$$

and a similar inequality for p^-. Hence,

$$\begin{aligned} E((\ln^+ p^+(\cdot, S, F))^p) &= \int_{\mathbb{R}} E(p^+(\cdot, S, x)(\ln^+ p^+(\cdot, S, x)))^p \, dx \\ &= \int_0^\infty E(p^+(\cdot, S, x)(\ln^+ p^+(\cdot, S, x)))^p \, dx \\ &\quad + \int_{-\infty}^0 E(p^-(\cdot, S, x)(\ln^+ p^-(\cdot, S, x)))^p \, dx \\ &\leqq \int_0^\infty E(1_{\{F>x\}} |X_S| (\ln^+ |X_S|)^p) \, dx \\ &\quad + \int_{-\infty}^0 E(1_{\{F<x\}} |X_S| (\ln^+ |X_S|)^p) \, dx \\ &= E(|F| \, |X_S| \, (\ln^+ |X_S|)^p) \, . \end{aligned}$$

This boils the first condition of Theorem 2 down to (37).

In the same way, the second one is related to (36). This completes the proof. □

Let us now answer the question, under which conditions (36) and (37) are satisfied. They easily follow from conditions of the type of Corollary 1.

Corollary 3 *Suppose that* $\alpha, \beta, \gamma > 1$ *are such that*

$$\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} < 1, \ r, q > 1 \text{ such that } r < q \, .$$

Let $0 \leqq S < T$.
Assume that $F \in L^\alpha(\Omega, \mathscr{F}, P) \cap \mathbb{D}_{\beta,2}([0, T])$, *and*

$$\frac{1}{\langle DF, DF \rangle_t^T} \in L^\gamma(\Omega, \mathscr{F}, P), \quad s = 0, S \, .$$

Then there exists a constant $c_{r,q}$ *such that for any* $(\mathscr{G}_t)$*-adapted* P*-a.s. locally square integrable process* u *we have*

$$\left\| \sup_{0 \leqq t \leqq S} \left| \int_0^t u_s \, dW_s \right| \right\|_r \leqq c_{r,q} \left\| \left[\int_0^S u_s^2 \, ds \right]^{1/2} \right\|_q \, .$$

Proof. Let p be such that

$$\frac{1}{r} = \frac{1}{p} + \frac{1}{q} \, .$$

Choose $\varepsilon > 0$ such that

$$1 + \varepsilon = \frac{1 - 1/\alpha}{1/\beta + 1/\gamma} \, ,$$

which is possible due to the hypotheses. We have to verify (36) and (37).

Let us concentrate on (36). We first apply Hölder's inequality to obtain in the notation of the theorem

$$E(|F|\,|X_0|(\ln^+|X_0|)^p) \leqq \|F\|_\alpha \|X_0(\ln^+|X_0|)^p\|_{\alpha/(\alpha-1)} .$$

Now, choose a constant c_1, such that

$$(\ln^+|x|)^p \leqq c_1|x|^\varepsilon, \quad x \in \mathbb{R} .$$

Then by Nualart [13], p. 72, we have

$$\begin{aligned}\|X_0(\ln^+|X_0|)^p\|_{\alpha/(\alpha-1)} &\leqq c_1\|\,|X_0|^{1+\varepsilon}\|_{\alpha/(\alpha-1)} \\ &= c_1\|X_0\|^{1/(1+\varepsilon)}_{(\alpha/(\alpha-1))(1+\varepsilon)} \\ &\leqq c_2\left\|\frac{DF}{\langle DF,DF\rangle_0^T}\right\|^{1/(1+\varepsilon)}_{1/(1/\beta+1/\gamma)^2} .\end{aligned}$$

Now we can proceed just as in the proof of Corollary 1, in which the role of 2 is taken by $(1/\beta + 1/\gamma)^{-1}$, those of p, q by β and γ. This completes the proof. □

6 The compensator in terms of a logarithmic Malliavin derivative

In Jacod [8], the conditional densities of F given $\mathscr{F}_t$ were shown to be martingales in t. Since we are working with the Wiener filtration, they can be represented as Ito integrals of adapted processes α, as in (6). The purpose of the following investigations will be to establish more precisely the link between α and the conditional density. We shall show that the compensator of W w.r.t. $(\mathscr{G}_t)_{t\geqq 0}$ is indeed given in terms of a logarithmic Malliavin derivative of p. For this comparison, however, we shall need additional smoothness hypotheses on F. They shall be investigated in the following.

Lemma 10 *Let* $0 \leqq t < T$, $X \in \mathbb{D}_{2,1}([0,T])$.
Then

$$\omega \to E(X \circ S_t(\omega, \cdot)) \in \mathbb{D}_{2,1}([0,t]) \quad \textit{and for } 0 \leqq u \leqq t$$
$$D_u E_2(X \circ S_t(\cdot,\cdot)) = E_2(D_u X \circ S_t(\cdot,\cdot)) .$$

Proof. This is a special case of a more general result stating that the Malliavin derivative commutes with the conditional expectation provided the conditioning σ-field is generated by a Gaussian subspace of Wiener space. □

Modifying the formula of Theorem 1 a bit, we obtain the following representation of the conditional densities. We remark that we now work with strict regularity assumptions on F, since we want to keep the statements relatively simple. We recall that the Sobolev norms with which we have to work satisfy an inequality of the Hölder type.

Remark. Assume that $r, p, q > 1$ such that $1/r = 1/p + 1/q$. Let $k \in \mathbb{N}, T > 0$ and $X \in \mathbb{D}_{p,k}([0,T]), Y \in \mathbb{D}_{q,k}([0,T])$. Then $X \cdot Y \in \mathbb{D}_{r,k}([0,T])$ and with a universal constant $c_{p,q}$ we have

$$\|X \cdot Y\|_{r,k} \leqq c_{p,q} \|X\|_{p,k} \|Y\|_{q,k} \,. \tag{39}$$

This is a simple consequence of the inequality of Cauchy–Schwarz (see also [22, p. 50, Proposition 1.10]).

Lemma 11 *Let* $0 \leqq t \leqq T, k \in \mathbb{N}_0, F \in \mathbb{D}_{p,k+1}([0,T])$ *such that* $(\langle DF, DF\rangle_t^T)^{-1} \in L^p(\Omega, \mathscr{F}, P),\ p \geqq 1$.
Then the $L^2([0,T])$*-valued random variable*

$$X_u = \frac{DF}{\langle DF, DF\rangle_u^T}, \quad 0 \leqq u \leqq T\,,$$

satisfies

$$\sup_{0 \leqq u \leqq t} \|X_u\|_{p,k} < \infty \quad \textit{for any } \ p \geqq 1\,.$$

Proof. According to the rules of differentiation for D, there is an $L^2([0,T]^{l+1})$-valued random variable Z_l such that

$$\begin{aligned}\int_0^t \cdots \int_0^t (D^{\otimes l}_{s_1 \dots s_l} X_u)^2\, ds_1 \dots ds_l &\leqq \frac{Z_l}{(\langle DF, DF\rangle_u^T)^{l+1}} \\ &\leqq \frac{Z_l}{(\langle DF, DF\rangle_t^T)^{l+1}}, \quad 0 \leqq l \leqq k, 0 \leqq u \leqq t\,.\end{aligned}$$

Now apply the inequality of the preceding remark and take the sup over $0 \leqq u \leqq t$ on the left hand side. □

Corollary 4 *Let* $T > 0, 0 \leqq t < T$. *Under the assumptions of Lemma* 11 *for* $k = 2$ *we have*

$$\sup_{0 \leqq u \leqq t} \|\delta_u^T(X_u)\|_{p,1} < \infty\,,$$

$$\sup_{0 \leqq u \leqq t} \|X_u \delta_u^T(X_u)\|_{p,1} < \infty\,.$$

Proof. First of all, Nualart [13, Proposition 1.5.4] yields that there exists a constant c_1 such that

$$\|\delta_u^T(X_u)\|_{p,1} \leqq c_1 \|X_u\|_{p,2}$$

for $u \in [0,t]$. Hence the first inequality comes directly from the lemma. For the second, we have in addition to invoke the inequality (39). □

The following proposition gives representations of the conditional densities under additional regularity assumptions.

Proposition 1 *Let* $0 \leqq t < T$. *Assume* $F \in \mathbb{D}_{p,3}([0,T])$ *and* $(\langle DF, DF\rangle_t^T)^{-1} \in L^p(\Omega, \mathscr{F}, P),\ p \geqq 1$. *Then the function*

$$p^+(\omega, t, x) = E((F - x)^+ \delta_t^T(X_t \delta_t^T(X_t)) \circ S_t(\omega, \cdot))\,,$$

$\omega \in \Omega, x \in \mathbb{R}$, *is a version of the density of the regular conditional law of F given* $\mathscr{F}_t$, *where* $X_t = DF/\langle DF, DF\rangle_t^T$.

Proof. According to Corollary 4, we have

$$X_t \delta_t^T(X_t) \in \mathrm{dom}\,(\delta_t^T)\,.$$

Hence we may proceed exactly as in the proof of Theorem 3, replacing X_t with $X_t\delta_t^T(X_t)$ and $\psi = 1_{[a,b]}$ with $\psi = 1_{]x,\infty[}$. □

Proposition 1 allows us to obtain a formula for the Malliavin derivative of the regular conditional density.

Proposition 2 *Let* $0 \leqq t < T$. *Assume that* $F \in \mathbb{D}_{p,3}([0,T])$ *and* $(\langle DF, DF\rangle_t^T)^{-1} \in L^p(\Omega, \mathscr{F}, P)$, $p \geqq 1$. *Then in the notation of Proposition* 1, $p^+(\,\cdot\,, t, x) \in \mathbb{D}_{p,1}([0,t])$ *for* $p \geqq 1, x \in \mathbb{R}$, *and*

$$D_r\, p^+(\omega, t, x) = E(1_{\{F>x\}}[D_rF\;\delta_t^T(X_t\delta_t^T(X_t)) \\ + \langle DF, D_r[X_t\delta_t^T(X_t)]\rangle_t^T] \circ S_t(\omega,\,\cdot\,))\,,$$

$0 \leqq r \leqq t, \omega \in \Omega, x \in \mathbb{R}$, *is a version of the Malliavin derivative of the density of the regular conditional law of F given* $\mathscr{F}_t$.

Proof. Suppose first that $u \in \mathbb{D}_{p,2}([0,T], L^2([0,T]))$, and $X \in \mathbb{D}_{p,1}([0,T])$ for $p \geqq 1$. Then Lemma 10 and the duality of D and δ_t^T yield the following chain of equations for $0 \leqq r \leqq t$:

$$\begin{aligned} (40)\qquad & D_rE_2(X\,\cdot\,\delta_t^T(u)\circ S_t(\,\cdot\,,\,\cdot\,)) \\ &= E_2(D_r[X\,\cdot\,\delta_t^T(u)]\circ S_t(\,\cdot\,,\,\cdot\,)) \\ &= E_2([D_rX\,\cdot\,\delta_t^T(u) + X\,\cdot\,\delta_t^T(D_ru)]\circ S_t(\,\cdot\,,\,\cdot\,)) \quad (r \leqq t) \\ &= E_2([D_rX\,\cdot\,\delta_t^T(u) + \langle DX, D_ru\rangle_t^T]\circ S_t(\,\cdot\,,\,\cdot\,))\,. \end{aligned}$$

Now the right hand side of (40) converges if we approximate u by a sequence of functions $(u_n)_{n\in\mathbb{N}}$ in $\mathbb{D}_{p,1}([0,T], L^2([0,T]))$. Hence the Malliavin differentiability extends to expressions containing these functions. Now replace X by $(F-x)^+, x \in \mathbb{R}$, and u by $X_t\delta_t^T(X_t)$ to obtain the desired formula. Hereby keep in mind that $X_t\delta_t^T(X_t) \in \mathbb{D}_{p,1}([0,T], L^2([0,T]))$ due to Corollary 4. □

By the remark made at the beginning of Sect. 3 and the law of iterated conditional expectations Proposition 2 immediately gives us versions of conditional derivatives.

Corollary 5 *Let* $T > t \geqq 0$. *Assume that* $F \in \mathbb{D}_{p,3}([0,T])$, *and* $(\langle DF, DF\rangle_t^T)^{-1} \in L^p(\Omega, \mathscr{F}, P)$ *for* $p \geqq 1$. *Then for any* $x \in \mathbb{R}$ *the Malliavin derivative* $D_r\,p^+(\,\cdot\,, t, x)$ *given by Theorem* 3 *satisfies*

$$\begin{aligned} & E(D_r\,p^+(\omega, t, x)|\mathscr{F}_r) \\ &\quad = E(1_{\{F>x\}}[D_rF\;\delta_t^T(X_t\delta_t^T(X_t)) + \langle DF, D_r[X_t\delta_t^T(X_t)]\rangle_t^T]\circ S_r(\omega,\,\cdot\,))\,, \\ &\qquad \textit{for } P\otimes\lambda\textit{-a.e.}\quad (\omega, r) \in \Omega\times[0,t]\,. \end{aligned}$$

We shall now make use of the formula of Clark–Ocone (see [13, p. 45]) to identify the process α in the representation of the conditional densities of F. Under the conditions of Proposition 2 it yields the relationship

$$p^+(\cdot, t, x) = E(p^+(\cdot, t, x)) + \int_0^t E(D_r p^+(\cdot, t, x) \mid \mathscr{F}_r)\, dW_r \, .$$

Let us consider the stochastic integrand more closely. Lemma 10 essentially tells us that Malliavin derivative and conditional expectation can be interchanged. Taking into account that $p^+(\cdot, t, x)$ is a martingale we therefore obtain formally

$$E(D_r p^+(\cdot, t, x) \mid \mathscr{F}_r) = D_r p^+(\cdot, r, x) \tag{41}$$

for a fixed $r \in [0,1]$. But the Malliavin derivative DX of a random variable X is an element of $L^2(\mathbb{R}_+ \times \Omega)$, hence its value $D_r X$ for r fixed is not well defined unless the function $r \to D_r X$ possesses some additional regularity. We shall give a sufficient criterion under which the Malliavin derivative of $p^+(\cdot, r, x)$ is left continuous at r. This will enable us to write (41) for all r.

Lemma 12 *Let $T > t \geqq 0$. Assume that $F \in \mathbb{D}_{p,3}([0,T])$ and $(\langle DF, DF \rangle_t^T)^{-1} \in L^p(\Omega, \mathscr{F}, P)$ for $p \geqq 1$. Assume moreover that*

$$r \to D_r F \, , \tag{42}$$

$$r \to D_r DF \, , \tag{43}$$

$$r \to D_r D^{\otimes 2} F \tag{44}$$

are continuous respectively as mappings from $[0,t]$ to $L^2(\Omega)$ resp. $L^2([t,T] \times \Omega)$ resp. $L^2([t,T]^2 \times \Omega)$.
Then the mapping $r \to D_r p^+(\cdot, s, x)$ is left continuous in $L^1(\Omega, \mathscr{F}, P)$ at $s \in [0,t]$, $x \in \mathbb{R}$.

Proof. We write p instead of p^+. Fix $s \in [0,t]$. Then according to Corollary 5 a version of $Dp(\cdot, s, x)$ in the notation of Lemma 11 is given by

$$D_r p(\omega, s, x) = E(1_{\{F>x\}}[D_r F\, \delta_s^T(X_s \delta_s^T(X_s)) + \langle DF, D_r[X_s \delta_s^T(X_s)] \rangle_s^T \circ S_s(\omega, \cdot))$$

$\omega \in \Omega, r \in [0,s]$.
Now for $r \leqq s$, writing $D_r^s = D_s - D_r$, we have by Jensen's inequality

$$E(|D_r^s p(\cdot, s, x)|) \leqq E(|D_r^s F|\, |\delta_s^T(X_s \delta_s^T(X_s))|) + E(|\langle DF, D_r[X_s \delta_s^T(X_s)] \rangle_s^T|) \, . \tag{45}$$

It is immediately clear how the first term on the rhs of (45) may be estimated. One has to use Hölder's inequality, Nualart [13, p. 72] and Corollary 3. (42) forces the expression to 0 as r approaches s from below. Let us discuss more precisely the more difficult second term. First note that by $r \leqq s$

$$D_r^s[X_s\, \delta_s^T(X_s)] = D_r^s X_s \cdot \delta_s^T(X_s) + X_s \cdot \delta_s^T(D_r^s X_s) \, . \tag{46}$$

Moreover,

$$D_r^s X_s = D_r^s \frac{DF}{\langle DF, DF\rangle_s^T} = \frac{D_r^s DF}{\langle DF, DF\rangle_s^T} - \frac{2DF}{(\langle DF, DF\rangle_s^T)^2}\langle D_r^s DF, DF\rangle \ , \tag{47}$$

and hence

$$\begin{aligned} D_r^s DX_s &= D_r^s\left[\frac{D^{\otimes 2}F}{\langle DF, DF\rangle_s^T} - \frac{2DF}{(\langle DF, DF\rangle_s^T)^2}\langle D^{\otimes 2}F, DF\rangle\right] \\ &= \frac{D_r^s D^{\otimes 2}F}{\langle DF, DF\rangle_s^T} - \frac{2D^{\otimes 2}F}{(\langle DF, DF\rangle_s^T)^2}\langle D_r^s DF, DF\rangle \\ &\quad - \frac{2D_r^s DF}{(\langle DF, DF\rangle_s^T)^2}\langle D^{\otimes 2}F, DF\rangle \\ &\quad + \frac{8DF}{(\langle DF, DF\rangle_s^T)^3}\langle D_r^s DF, DF\rangle\langle D^{\otimes 2}F, DF\rangle \\ &\quad - \frac{2DF}{(\langle DF, DF\rangle_s^T)^2}[\langle D_r^s D^{\otimes 2}F, DF\rangle + \langle D^{\otimes 2}F, D_r^s DF\rangle] \ . \end{aligned} \tag{48}$$

Using just Cauchy–Schwarz's inequality several times we get

$$\langle D_r^s X_s, D_r^s X_s\rangle^{1/2} \leqq 3\frac{(\langle D_r^s DF, D_r^s DF\rangle_s^T)^{1/2}}{\langle DF, DF\rangle_s^T} \ , \tag{49}$$

and

$$\begin{aligned} \langle D_r^s DX_s, D_r^s DX_s\rangle^{1/2} &\leqq 3\frac{(\langle D_r^s D^{\otimes 2}F, D_r^s D^{\otimes 2}F\rangle_s^T)^{1/2}}{\langle DF, DF\rangle_s^T} \\ &\quad + 14\frac{(\langle D^{\otimes 2}F, D^{\otimes 2}F\rangle_s^T)^{1/2}(\langle D_r^s DF, D_r^s DF\rangle_s^T)^{1/2}}{(\langle DF, DF\rangle_s^T)^{3/2}} \end{aligned} \tag{50}$$

(46) and (49) give

$$\begin{aligned} &|\langle DF, D_r^s(X_s\delta_s^T(X_s))\rangle_s^T| \\ &= |\langle DF, D_r^s X_s\rangle_s^T \delta_s^T(X_s) + \delta_s^T(D_r^s X_s)| \\ &\leqq (\delta_s^T(X_s))(\langle DF, DF\rangle_s^T)^{1/2}(\langle D_r^s X_s, D_r^s X_s\rangle_s^T)^{1/2} + |\delta_s^T(D_r^s X_s)| \\ &\leqq 3(\delta_s^T(X_s))\frac{(\langle D_r^s DF, D_r^s DF\rangle_s^T)^{1/2}}{(\langle DF, DF\rangle_s^T)^{1/2}} + |\delta_s^T(D_r^s X_s)| \ . \end{aligned}$$

Hence by Hölder's inequality and Nualart [13, p. 72] and (50)

$$(51)\quad E(|\langle DF, D_r^s(X_s\,\delta_s^T(X_s))\rangle_s^T|)$$

$$\leqq c_1\|(\langle D_r^s DF, D_r^s DF\rangle_s^T)^{1/2}\|_2 + \|D_r^s X_s\|_{2,1}$$

$$\leqq c_2[\|(\langle D_r^s DF, D_r^s DF\rangle_s^T)^{1/2}\|_2 + \|(\langle D_r^s D^{\otimes 2}F, D_r^s D^{\otimes 2}F\rangle_s^T)^{1/2}\|_2]$$

with suitable constants c_1, c_2. Due to (43) and (44), the rhs of (51) converges to 0 as $r \uparrow s$. This completes the proof. □

Lemma 12 has prepared the proof of the main result of this section.

Theorem 5 *Let* $0 \leqq t < T$. *Assume that* $F \in \mathbb{D}_{p,3}([0,T])$ *and* $(\langle DF, DF\rangle_t^T)^{-1} \in L^p(\Omega, \mathscr{F}, P)$ *for* $p \geqq 1$. *Assume that*

$$r \to D_r F\ ,$$

$$r \to D_r DF\ ,$$

$$r \to D_r D^{\otimes 2} F$$

are continuous as mappings from $[0,t]$ *to* $L^2(\Omega)$ *resp.* $L^2([t,T]\times\Omega)$ *resp.* $L^2([t,T]^2\times\Omega)$. *Then for* $x \in \mathbb{R}$ *we have*

$$(52)\qquad p(\,\cdot\,,t,x) = \int_0^t D_u\, p(\,\cdot\,,u,x)\, dW_u + p(x)\ .$$

Moreover, for any local $(\mathscr{F}_t)$*-martingale* $M = \int_0^{\cdot} \beta_s\, dW_s$ *we have*

$$\widetilde{M}_t = M_t - \int_0^t \beta_s \left.\frac{D_s\, p(\,\cdot\,,s,x)}{p(\,\cdot\,,s,x)}\right|_{x=F} ds, \quad 0 \leqq t < T\ ,$$

is a $(\mathscr{G}_t)$*-local martingale. In particular,*

$$\widetilde{W}_t = W_t - \int_0^t \left.\frac{D_s\, p(\,\cdot\,,s,x)}{p(\,\cdot\,,s,x)}\right|_{x=F} ds$$

is a $(\mathscr{G}_t)$*-Wiener process.*

Proof. The second and third assertion follow obviously from (52). See (6)–(9). Fix $x \in \mathbb{R}$. We may use p^+ and write p again. The formula of Clark–Ocone (see [13, p. 45]) is applicable due to Corollary 4 and gives

$$(53)\qquad p(\,\cdot\,,t,x) = p(x) + \int_0^t E(D_s\, p(\,\cdot\,,t,x)|\mathscr{F}_s)\, dW_s\ .$$

Now fix $0 \leqq s \leqq t$. Lemma 10 and the remark made at the beginning of Sect. 3 allow us to write

$$E(D_r\, p(\,\cdot\,,t,x)|\mathscr{F}_s) = D_r E(p(\,\cdot\,,t,x)|\mathscr{F}_s) = D_r\, p(\,\cdot\,,s,x)$$

for λ-a.e. $r \in [0,s]$.

But by Lemma 12, this process is left continuous in $L^1(\Omega, \mathscr{F}, P)$ at s. Hence

$$E(D_s p(\,\cdot\,, t, x)|\mathscr{F}_s) = \lim_{r \uparrow s} D_r p(\,\cdot\,, s, x) = D_s p(\,\cdot\,, s, x)\,.$$

This yields the desired formula, apart from a canonical measurability argument. □

Proposition 2 and Corollary 5 indicate that even if $r \mapsto D_r p^+(\,\cdot\,, s, x)$ is not left continuous at s, and thus $D_r p^+(\,\cdot\,, r, x)$ has no canonical meaning, we can still obtain a version of the result of Theorem 5. Indeed, the right hand side of the formula given in Proposition 2 still makes sense if we set $t = r$. We shall therefore, abusing the notation a bit, continue to write in the sequel

$$\begin{aligned} &D_r p^+(\omega, r, x) \\ &\quad = E(1_{\{F>x\}}[D_r F\, \delta_r^T(X_r \delta_r^T(X_r)) + \langle DF, D_r[X_r \delta_r^T(X_r)]\rangle_r^T] \circ S_r(\omega,\,\cdot\,))\,, \end{aligned}$$

$0 \leqq r < T$. We shall use Corollary 5 to show that the process $D_r p^+(\,\cdot\,, r, x)$ defined this way still yields the formulas obtained in Theorem 5. For this purpose we just have to extend the estimates given in Lemma 11 and Corollary 4 respectively to prove continuity of the mappings

$$t \mapsto \delta_t^T(X_t \delta_t^T(X_t)) \quad \text{and } t \mapsto X_t \delta_t^T(X_t))$$

in appropriate Sobolev norms.

Lemma 13 *Let* $0 \leqq t < T$. *Assume* $F \in \mathbb{D}_{p,3}([0,T])$ *and* $(\langle DF, DF\rangle_t^T)^{-1} \in L^p(\Omega, \mathscr{F}, P)$, $p \geqq 1$. *Then the mapping* $s \mapsto \delta_s^T(X_s \delta_s^T(X_s))$ *is continuous with respect to* $\|\cdot\|_p$, *the mapping* $s \mapsto X_s \delta_s^T(X_s)$ *continuous with respect to* $\|\cdot\|_{p,1}$ *on* $[0,t]$ *for any* $p \geqq 1$.

Proof. Fix $p \geqq 1$. Using the remark preceding Lemma 11 and Corollary 4, we find constants c_1, c_2 and $q \geqq 1$ such that for $0 \leqq u \leqq v \leqq t$

$$\begin{aligned} &\|\delta_v^T(X_v \delta_v^T(X_v)) - \delta_u^T(X_u \delta_u^T(X_u))\|_p \\ &\quad \leqq \|\delta_u^v(X_v \delta_v^T(X_v))\|_p + \|\delta_u^T((X_v - X_u)\delta_v^T(X_v))\|_p \\ &\qquad + \|\delta_u^T(X_u \delta_u^v(X_v))\|_p + \|\delta_u^T(X_u \delta_u^T(X_v - X_u))\|_p \\ &\quad \leqq c_1 \|1_{[u,v]}\|_q + c_2 \|\langle DF, DF\rangle_u^v\|_{q,2}\,. \end{aligned}$$

Due to our hypotheses, dominated convergence applies and yields the convergence to 0 of the rhs of the above inequality as $|v - u| \to 0$. This implies the first one of the continuity properties stated. The argument for the second one is evidently simpler. □

We obtain the following generalization of Theorem 5.

Theorem 6 *Let* $0 \leqq t < T$. *Assume* $F \in \mathbb{D}_{p,3}([0,T])$ *and* $(\langle DF, DF\rangle_t^T)^{-1} \in L^p(\Omega, \mathscr{F}, P)$, $p \geqq 1$. *Then the process*

$$D_r p^+(\omega, r, x)$$
$$= E(1_{\{F>x\}}[D_r F\, \delta_r^T(X_r \delta_r^T(X_r)) + \langle DF, D_r[X_r \delta_r^T(X_r)]\rangle_r^T] \circ S_r(\omega\ \cdot\ ))\,,$$

$\omega \in \Omega$, $x \in \mathbb{R}$, $0 \leqq r \leqq t$, *is well defined and fulfills the assertions of Theorem* 5.

Proof. We continue to write p instead of p^+. Let $(\mathbf{J}_n)_{n\in N}$ be a sequence of partitions of $[0,t]$ by nontrivial intervals $J = [s_J, t_J]$ the mesh of which converges to 0 as $n \to \infty$. Let

$$X_n(s) = \sum_{J \in \mathbf{J}_n} E(D_s p(\ \cdot\ , t_J, x)|\mathscr{F}_s) 1_J(s)\,,$$

$n \in \mathbb{N}, 0 \leqq s \leqq t$. Then the theorem of Clark–Ocone gives

$$p(\ \cdot\ , t, x) = p(x) + \int_0^t X_n(s)\, dW_s$$

for any $n \in \mathbb{N}$. Moreover, as a consequence of Lemma 13 we have $X_n \to D.p(\ \cdot\ ,\ \cdot\ , x)$ in $L^2(\Omega \times [0,t])$. Since evidently the limit process is adapted, we obtain the desired formula

$$p(\ \cdot\ , t, x) = p(x) + \int_0^t D_s p(\ \cdot\ , s, x)\, dW_s\,,$$

from which the formula for the compensator of W in the enlarged filtration follow readily. □

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