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Large deviations for two scaled diffusions

Robert Liptser

Department of Electrical Engineering-Systems, Tel Aviv University, Tel Aviv 69978, Israel, (e-mail: liptser@eng.tau.ac.il)

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Summary. We formulate large deviations principle (LDP) for diffusion pair $(X^\varepsilon, \xi^\varepsilon) = (X^\varepsilon_t, \xi^\varepsilon_t)$, where first component has a small diffusion parameter while the second is ergodic Markovian process with fast time. More exactly, the LDP is established for $(X^\varepsilon, v^\varepsilon)$ with $v^\varepsilon(dt, dz)$ being an occupation type measure corresponding to ξ^ε_t. In some sense we obtain a combination of Freidlin–Wentzell's and Donsker–Varadhan's results. Our approach relies on the concept of the exponential tightness and Puhalskii's theorem.

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1 Introduction

Let ε be a small positive parameter, $(X^\varepsilon, \xi^\varepsilon) = (X^\varepsilon_t, \xi^\varepsilon_t)_{t \geqq 0}$ be a diffusion pair defined on some stochastic basis $(\Omega, \mathscr{F}, \mathbf{F} = (\mathscr{F}_t)_{t \geqq 0}, \mathbf{P})$ by Itô's equations w.r.t. independent Wiener processes W_t and V_t:

$$\begin{aligned} dX^\varepsilon_t &= A(X^\varepsilon_t, \xi^\varepsilon_t)\, dt + \sqrt{\varepsilon} B(X^\varepsilon_t, \xi^\varepsilon_t)\, dW_t \, , \\ d\xi^\varepsilon_t &= \frac{1}{\varepsilon} b(\xi^\varepsilon_t)\, dt + \frac{1}{\sqrt{\varepsilon}} \sigma(\xi^\varepsilon_t)\, dV_t \end{aligned} \tag{1.1}$$

subject to fixed initial point (x_0, z_0).

Assume $(X^\varepsilon, \xi^\varepsilon)$ is an ergodic process in the following sense. Let $p(z)$ be the unique invariant density of ξ^ε,

$$v^{(p)}(dt, dz) = p(z)\, dt\, dz \, ,$$

and $\overline{X}_t$ is a solution of an ordinary differential equation $\dot{\overline{X}}_t = \overline{A}(\overline{X}_t)$ with $\overline{A}(x) = \int_R A(x,z)\, p(z)\, dz$ subject to the same initial point x_0. Then for any bounded continuous function $h(t,z)$ and $T > 0$

$$\begin{aligned} &\mathbf{P} - \lim_{\varepsilon\to 0} \int_0^T h(t,\xi_t^\varepsilon)dt = \int_0^T \int_R h(t,z)\nu^{(p)}(dt,dz)\,, \\ &\mathbf{P} - \lim_{\varepsilon\to 0} r_T(X^\varepsilon,\overline{X}) = 0\,, \end{aligned} \tag{1.2}$$

where r_T is the uniform metric on $[0,T]$. The above-mentioned ergodic property is a motivation to examine LDP for pair $(X^\varepsilon,\xi^\varepsilon)$, or more exactly for pair $(X^\varepsilon,\nu^\varepsilon)$, where $\nu^\varepsilon = \nu^\varepsilon(dt,dz)$ is an occupation measure on $(R_+ \times R, \mathscr{B}(R_+)\otimes\mathscr{B}(R))$ ($\mathscr{B}(R_+)$, $\mathscr{B}(R)$ are the Borel σ-algebras on R_+ and R respectively) corresponding to ξ^ε:

$$\nu^\varepsilon(\Delta\times\Gamma) = \int_0^\infty I(t\in\Delta,\ \xi_t^\varepsilon\in\Gamma)\,dt, \quad \Delta\in\mathscr{B}(R_+),\ \Gamma\in\mathscr{B}(R)\,. \tag{1.3}$$

A choice of ν^ε as the occupation measure is natural since the first ergodic property in (1.2) is nothing but

$$\mathbf{P} - \lim_{\varepsilon\to 0}\rho_T(\nu^\varepsilon,\nu^{(p)}) = 0\,,$$

where ρ_T is Levy–Prohorov's distance for restrictions of measures ν^ε and $\nu^{(p)}$ on $[0,T]\times R$. Also the first Itô's equation in (1.1) and the predictable quadratic variation $\langle M^\varepsilon\rangle_t$ of a martingale $M_t^\varepsilon = \int_0^t B(X_s^\varepsilon,\xi_s^\varepsilon)dW_s$ can be represented in the term of ν^ε:

$$X_t^\varepsilon = x_0 + \int_0^t\int_R A(X_s^\varepsilon,z)\nu^\varepsilon(ds,dz) + \sqrt{\varepsilon}M_t^\varepsilon\,,$$

$$\langle M^\varepsilon\rangle_t = \int_0^t\int_R B^2(X_s^\varepsilon,z)\nu^\varepsilon(ds,dz)\,.$$

The random measure ν^ε obeys the disintegration $\nu^\varepsilon(dt,dz) = dt\,K_{\nu^\varepsilon}(t,dz)$ with the transition kernel $K_{\nu^\varepsilon}(t,dz)$ being probabilistic Dirac's measure that is ν^ε values in space $\mathbb{M} = \mathbb{M}_{[0,\infty)}$ of σ-finite (locally in t) measures $\nu = \nu(dt,dz)$ on $(R_+\times R, \mathscr{B}(R_+)\otimes\mathscr{B}(R))$ obeying the disintegration $\nu(dt,dz) = K_\nu(t,dz)\,dt$ with the probabilistic transition kernel $K_\nu(t,dz)$ ($\int_R K_\nu(t,dz)\equiv 1$). X^ε values in the space $\mathbb{C} = \mathbb{C}_{[0,\infty)}$ of continuous function. Define metrics r and ρ in $\mathbb{C}$ and $\mathbb{M}$ respectively, letting

$$r(X',X'') = \sum_{k\geqq 1}\frac{r_k(X',X'')\wedge 1}{2^k} \quad\text{and}\quad \rho(\nu',\nu'') = \sum_{k\geqq 1}\frac{\rho_k(\nu',\nu'')\wedge 1}{2^k}\,.$$

Evidently ergodic properties (1.2) are equivalent to

$$\mathbf{P} - \lim_{\varepsilon\to 0}[r(X^\varepsilon,\overline{X}) + \rho(\nu^\varepsilon,\nu^{(p)})] = 0$$

and so for examination of the LDP for $(X^\varepsilon,\nu^\varepsilon)$ we choose the metric space $(\mathbb{C}\times\mathbb{M}, r\times\rho)$.

Recall the definition of LDP from Varadhan [1] adapted to our setting. The family $(X^\varepsilon, v^\varepsilon)$ obeys the LDP in the metric space $(\mathbb{C} \times \mathbb{M}, r \times \rho)$ if

(0) there exists a function $L(X, v)$, $X \in \mathbb{C}$, $v \in \mathbb{M}$, values in $[0, \infty]$, such that its level sets are compacts in $(\mathbb{C} \times \mathbb{M}, r \times \rho)$;

(1) for any open set G from $(\mathbb{C} \times \mathbb{M}, r \times \rho)$

$$\varliminf_{\varepsilon \to 0} \varepsilon \log \mathbf{P}((X^\varepsilon, v^\varepsilon) \in G) \geqq - \inf_{(X,v) \in G} L(X, v) \ ;$$

(2) for any closed set F from $(\mathbb{C} \times \mathbb{M}, r \times \rho)$

$$\varlimsup_{\varepsilon \to 0} \varepsilon \log \mathbf{P}((X^\varepsilon, v^\varepsilon) \in F) \leqq - \inf_{(X,v) \in F} L(X, v) \ .$$

The function $L(X, v)$, meeting in (0), (1), and (2), is named rate function (action functional in the terminology of Freidlin and Wentzell [2] or good rate function in the terminology of Stroock [3]).

Below we recall well known particular results in LDPs related to pair $(X^\varepsilon, \xi^\varepsilon)$ and give corresponding forms of rate functions which will be inherited by a rate function for our setting. Note at first LDP for family $\mu^\varepsilon(dz) = v^\varepsilon([0, 1], dz)$ (on the space of probability measures supplied by Levy–Prohorov's metric) proved by Donsker and Varadhan [4–7] for a wide class of Markov processes $\xi^\varepsilon_t = \xi_{t/\varepsilon}$. Corresponding rate function obeys an invariant form: for any probabilistic measure μ on R

$$I(\mu) = -\inf \int_R \frac{\mathscr{L} u(z)}{u(z)} \mu(dz) \ ,$$

where $\mathscr{L}$ is backward Kolmogorov's operator, respecting to ξ, and where "*inf*" is taken over all functions $u(z)$ from the domain of definition for the operator $\mathscr{L}$. For the diffusion case, Gärtner's type of $I(\mu)$ is well known [8]:

$$I(\mu) = \begin{cases} \frac{1}{8} \int_R \sigma^2(z) \left[\frac{m'(z)}{m(z)} - \frac{p'(z)}{p(z)} \right]^2 m(z)\, dz, & d\mu = m(z)\, dz, \ dm(z) = m'(z)\, dz \ , \\ \infty, & \text{otherwise} \ . \end{cases} \tag{1.4}$$

Freidlin–Wentzell's result [2] is devoted to LDP for diffusion X^ε with drift $A(x)$ and diffusion $B^2(x)$ (independent of z) in the space of continuous functions on every finite time interval, supplied by the uniform metric. A rate function, say, for $[0, T]$ time interval is given by

$$S(X) = \begin{cases} \frac{1}{2} \int_0^T \frac{[\dot{X}_t - A(X_t)]^2}{B^2(X_t)}\, dt, & dX_t = \dot{X}_t\, dt, \ X_0 = x_0 \ , \\ \infty, & \text{otherwise} \ . \end{cases} \tag{1.5}$$

Other type of LDP for a degenerate diffusion X^ε defined by the first equation in (1.1) with $B(x, z) \equiv 0$ and $\xi^\varepsilon_t = \xi_{t/\varepsilon}$, where ξ_t is Markov process values in a finite state space, also is well known from Freidlin [9]. In this case rate function has a form similar to (1.5) ($H(y,x)$ is some non negative function):

$$S(X) = \begin{cases} \int_0^T H(\dot{X}_t, X_t)\,dt, & dX_t = \dot{X}_t\,dt,\ X_0 = x_0\,, \\ \infty, & \text{otherwise}\,. \end{cases} \tag{1.6}$$

All above-mentioned LDPs are inspired by the examination of the LDP for $(X^\varepsilon, v^\varepsilon)$. In some sense, the LDP for $(X^\varepsilon, v^\varepsilon)$ is a combination of Donsker–Varadhan's and Freidlin–Wenzell's results. Namely LDP for v^ε is a generalization one for μ^ε while LDP for X^ε is implied by LDP for v^ε and for a diffusion martingale scaled by $\sqrt{\varepsilon}$. Hence, a rate function for $(X^\varepsilon, v^\varepsilon)$, is defined as a sum: $L(X,v) = L_1(X,v) + L_2(v)$, where $L_1(X,v)$ and $L_2(v)$ respect to X^ε and v^ε and what is more $L_1(X,v)$ has the same form as $S(x)$ in (1.5) with $A(X_t)$ and $B^2(X_t)$ replaced on $A_v(t,X_t) = \int_R A(X_t,z)K_v(t,dz)$ and $B_v^2(t,X_t) = \int_R B^2(X_t,z)K_v(t,dz)$, where $K_v(t,dz)$ is the transition kernel of measure v.

Note that $\xi_t^\varepsilon \in R$ and so the LDP for its occupation measure responds to a non compact diffusion case. Also note that diffusion parameter $B^2(x,z)$ is not assumed to be non singular and consequently $B^2(x,z) \equiv 0$ is admissible. The last allows to derive LDP for a singular diffusion parameter case from LDP for v^ε using the contraction principle of Varadhan [1] (continuous mapping method of Freidlin [10]). This result extends above-mentioned [9] for non compact case.

In contrast with Freidlin and Wentzell [2], Donsker and Varadhan [4–7], Gärtner [8], and Veretennikov [11,12], and many others (see e.g. Acosta [13], Dupuis and Elis [14]) our method of proof is based on Puhalskii's theorem [15,16] and relies concepts of exponential tightness and LD relative compactness.

The paper is organized as follows. In Sect. 2, we formulate the general assumptions and the main result. Section 3 contains the method of proving LDP which also has been used in [17]. In Sect. 4, we check the exponential tightness while in Sects. 5 and 6 the upper and lower bounds in local LDP are verified. The main results are proved in Sect. 7. All technical results are gathered in Appendix.

2 Assumptions. Main result

1. We fix the following conditions which are assumed to be fulfilled hereafter.

(A.1) $A(x,z)$ and $B(x,z)$ are continuous in (x,z), Lipschitz continuous in x uniformly in z, and $\sup_z (|A(0,z)| + |B(0,z)|) < \infty$;

(A.2) $\sigma^2(z)$ is bounded and uniformly positive function; it is continuously differentiable, having bounded and Lipschitz continuous derivative;

(A.3) $b(z)$ is Lipschitz continuous, satisfying

$$\lim_{|z|\to\infty} b(z)\operatorname{sign} z = -\infty\,.$$

It would be noted that (A.2) and (A.3) imply, so called, assumption (H*) from [6].

2. It is well known (see [18]) that under (A.2) and (A.3) ξ^ε is ergodic process obeying the unique invariant density

$$p(z) = \text{const.}\frac{\exp\left(2\int_0^z (b(y)\big/\sigma^2(y))\,dy\right)}{\sigma^2(z)}\ . \tag{2.1}$$

For any ν from $\mathbb{M}$ with the transition kernel $K_\nu(t,dz)$, define $K_\nu(t,dz)$-averaged drift $A_\nu(t,x) = \int_R A(x,z)K_\nu(t,dz)$ and diffusion parameter $B^2_\nu(t,x) = \int_R B^2(x,z)K_\nu(t,dz)$. If ν is absolutely continuous w.r.t. $\Lambda(dt,dz) = dtdz$, put

$$n(t,z) = \frac{d\nu}{d\Lambda}(t,z)\ . \tag{2.2}$$

If the density $n(t,z)$ is absolutely continuous w.r.t. dz: $d_z n(t,z) = n'_z(t,z)dz$, a function $n'_z(t,z)$ is chosen to be measurable in t,z.

Throughout the paper, we use conventions $0/0 = 0$ and $\min(\inf)(\emptyset) = \infty$.

For every $\nu \in \mathbb{M}$ and $X \in \mathbb{C}$ define two quantities (comp. (1.4) and (1.5)):

$$F(\nu) = \begin{cases} \int_0^\infty \int_R \sigma^2(z)\left[\frac{n'_z(t,z)}{n(t,z)} - \frac{p'(z)}{p(z)}\right]^2 n(t,z)\,dz\,dt, & d\nu = nd\Lambda,\ d_z n = n'_z\,dz\ , \\ \infty, & \text{otherwise}\ ; \end{cases}$$

$$S(X,\nu) = \begin{cases} \int_0^\infty \frac{[\dot X_t - A_\nu(t,X_t)]^2}{B^2_\nu(t,X_t)}\,dt, & dX = \dot X\,dt,\ X_0 = x_0\ , \\ \infty, & \text{otherwise}\ . \end{cases} \tag{2.3}$$

3. Now we are in the position to formulate the main result.

Theorem 2.1 *Under* (A.1), (A.2), *and* (A.3) *the family* $(X^\varepsilon,\nu^\varepsilon)$ *obeys the LDP in* $(\mathbb{C}\times\mathbb{M}, r\times\rho)$ *with rate function*

$$L(X,\nu) = \tfrac{1}{2}S(X,\nu) + \tfrac{1}{8}F(\nu)\ .$$

4. LDPs for families (X^ε) and (ξ^ε) run out from Theorem 2.1.

Corollary 2.1 (ν^ε) *obeys the LDP in* $(\mathbb{M},\rho)$ *with rate function* $\frac{1}{8}F(\nu)$.

Corollary 2.2 (comp. [9]) (X^ε) *obeys the LDP in* $(\mathbb{C}, r)$ *with rate function* $S(X) = \inf_{\nu\in\mathbb{M}} L(X,\nu)$. *In particular, if* $B(x,z)\equiv 0$, *it is sufficient to take* "*inf*" *over all* ν *from* $\mathbb{M}$ *with the transition kernel* $K_\nu(t,dz)\equiv\mu(dz)$ *with* $d\mu = m(z)\,dz$ *such that the density* $m(z) = (d\mu/dz)(z)$ *is absolutely continuous w.r.t.* dz $(m'(z) = dm(z)/dz)$. *In this case, rate function*

$$S(X) = \begin{cases} \frac{1}{8}\int_0^\infty H(\dot X_t, X_t)\,dt, & dX = \dot X\,dt,\ X_0 = x_0\ , \\ \infty, & \textit{otherwise}\ , \end{cases} \tag{2.4}$$

where

$$H(y,x) = \inf \int_R \sigma^2(z) \left[\frac{m'(z)}{m(z)} - \frac{p'(z)}{p(z)} \right]^2 m(z)\, dz \,, \tag{2.5}$$

and where "inf" is taken over all above-mentioned measures μ *such that*

$$y = \int_R A(x,z) m(z)\, dz \,.$$

As an example, also the LDP for the family of the Donsker and Varadhan occupation measures $\mu^\varepsilon(dz) = \nu^\varepsilon([0,1] \times dz)$, corresponding to diffusion case, can be derived from Theorem 2.1. In fact, due to the contraction principle, (μ^ε) obeys the LDP with Gärtner's type rate function (see (1.4)) $I(\mu) = \inf \frac{1}{8} F(\nu)$, where "*inf*" is taken over all $\nu \in \mathbb{M}$ such that

$$\nu(dt,dz) = I(1 \geqq t)\, dt \mu(dz) + I(1 < t) \nu^{(p)}(dt,dz) \,.$$

3 Preliminaries

For proving LDP for the family $(X^\varepsilon, \nu^\varepsilon)$ in the metric space $(\mathbb{C} \times \mathbb{M}, r \times \rho)$ we apply Dawson–Gärtner's type theorem (see e.g. [19]). Following it the LDP in $(\mathbb{C} \times \mathbb{M}, r \times \rho)$ is implied by LDPs in the metric spaces $(\mathbb{C}_{[0,n]} \times \mathbb{M}_{[0,n]}, r_n \times \rho_n)$, $n \geqq 1$, where $\mathbb{C}_{[0,n]}$ is the space of continuous functions on the time interval $[0,n]$, $\mathbb{M}_{[0,n]}$ is the space of finite measures on $[0,n] \times R$, having probabilistic transition kernel w.r.t. dt, r_n is the uniform metric, and ρ_n is Levy–Prohorov's metric. The definition of the LDP in $(\mathbb{C}_{[0,n]} \times \mathbb{M}_{[0,n]}, r_n \times \rho_n)$ is given in terms of (0), (1), and (2) with obvious modifications. Moreover, if $L_n(X,\nu)$, $n \geqq 1$ are rate functions, corresponding to LDPs in $(\mathbb{C}_{[0,n]} \times \mathbb{M}_{[0,n]}, r_n \times \rho_n)$, $n \geqq 1$, then rate function in $(\mathbb{C} \times \mathbb{M}, r \times \rho)$ is defined as

$$L(X,\nu) = \sup_n L_n(X,\nu) \,. \tag{3.1}$$

Hence only the LDP in $(\mathbb{C}_{[0,T]} \times \mathbb{M}_{[0,T]}, r_T \times \rho_T)$ has to be checked for any $T > 0$. Our approach in proving the LDP in $(\mathbb{C}_{[0,T]} \times \mathbb{M}_{[0,T]}, r_T \times \rho_T)$, $T > 0$ relies on the concept of the exponential tightness and notions of LD relative compactness and local LDP. Below we give necessary definitions.

Definition 1 *The family* $(X^\varepsilon, \nu^\varepsilon)$ *is said to be exponentially tight in the metric space* $(\mathbb{C}_{[0,T]} \times \mathbb{M}_{[0,T]}, r_T \times \rho_T)$*, if there exists an increasing sequence of compacts* $(K_j)_{j \geqq 1}$ *such that*

$$\lim_j \overline{\lim_{\varepsilon \to 0}} \varepsilon \log \mathbf{P}((X^\varepsilon, \nu^\varepsilon) \in \{\mathbb{C}_{[0,T]} \times \mathbb{M}_{[0,T]}\} \backslash K_j) = -\infty \tag{3.2}$$

(*Deuschel and Stroock* [20], *Lynch and Sethuraman* [21]).

Definition 2 *The family* $(X^\varepsilon, \nu^\varepsilon)$ *is said to be LD relatively compact in* $(\mathbb{C}_{[0,T]} \times \mathbb{M}_{[0,T]}, r_T \times \rho_T)$*, if any decreasing to zero sequence* (ε_k) *contains* *further subsequence* $(\bar{\varepsilon}_k)$ $((\bar{\varepsilon}_k) \subseteq (\varepsilon_k))$ *such that the family* $(X^{\bar{\varepsilon}_k}, v^{\bar{\varepsilon}_k})$ *obeys the LDP in* $(\mathbb{C}_{[0,T]} \times \mathbb{M}_{[0,T]}, r_T \times \rho_T)$ (*with rate function* $L_T(X, v)$). (*Puhalskii* [15,16]).

Definition 3 *The family* $(X^\varepsilon, v^\varepsilon)$ *is said to obey the local LDP in* $(\mathbb{C}_{[0,T]} \times \mathbb{M}_{[0,T]}, r_T \times \rho_T)$ *with local rate function* $\widehat{L}_T(X, v)$, *if for any* (X, v) *from* $\mathbb{C}_{[0,T]} \times \mathbb{M}_{[0,T]}$

$$\begin{aligned} &\lim_{\delta\to 0}\overline{\lim_{\varepsilon\to 0}}\, \varepsilon \log \mathbf{P}((r_T(X^\varepsilon, X) + \rho_T(v^\varepsilon, v) \leqq \delta) \\ &\quad = \lim_{\delta\to 0}\underline{\lim_{\varepsilon\to 0}}\, \varepsilon \log \mathbf{P}((r_T(X^\varepsilon, X) + \rho_T(v^\varepsilon, v) \leqq \delta) \\ &\quad = -\widehat{L}_T(X, v) \end{aligned} \tag{3.3}$$

(*Freidlin and Wentzell* [2]).

The connecting component of these notions used in the proof of the next result is Puhalskii's theorem [15,16]. Below we formulate only the first part of it.

Theorem P *If* $(X^\varepsilon, v^\varepsilon)$ *is exponentially tight family in* $(\mathbb{C}_{[0,T]} \times \mathbb{M}_{[0,T]}, r_T \times \rho_T)$, *then it is LD relatively compact.*

The following result is a reformulation of Theorem 1.3 from [17].

Proposition 3.1 *The exponential tightness and the local LDP for the family* $(X^\varepsilon, v^\varepsilon)$ *in* $(\mathbb{C}_{[0,T]} \times \mathbb{M}_{[0,T]}, r_T \times \rho_T)$ *imply the LDP in* $(\mathbb{C}_{[0,T]} \times \mathbb{M}_{[0,T]}, r_T \times \rho_T)$ *for this family with* (*good*) *rate function* $L_T(X, v) \equiv \widehat{L}_T(X, v)$, *where* $\widehat{L}_T(X, v)$ *is the local rate function.*

4 Exponential tightness in $\mathbb{C}_{[0,T]} \times \mathbb{M}_{[0,T]}$

Theorem 4.1 *Under assumptions* (A.1), (A.2), *and* (A.3) *the family* $(X^\varepsilon, v^\varepsilon)$ *is exponentially tight in* $\mathbb{C}_{[0,T]} \times \mathbb{M}_{[0,T]}$.

Proof. Following Definition 1, (3.2) has to be checked. It is clear it takes place if

$$\begin{aligned} &\lim_{j}\overline{\lim_{\varepsilon\to 0}}\, \varepsilon \log \mathbf{P}(X^\varepsilon \in \mathbb{C}_{[0,T]} \setminus K'_j) = -\infty\,, \\ &\lim_{j}\overline{\lim_{\varepsilon\to 0}}\, \varepsilon \log \mathbf{P}(v^\varepsilon \in \mathbb{M}_{[0,T]} \setminus K''_j) = -\infty\,, \end{aligned} \tag{4.1}$$

where K'_j and K''_j are appropriate increasing sequences of compacts from $\mathbb{C}_{[0,T]}$ and $\mathbb{M}_{[0,T]}$ respectively. It is natural to use as compacts K'_j increasing sets of uniformly bounded and equicontinuous functions from $\mathbb{C}_{[0,T]}$ parametrized by j. Since the process $(X^\varepsilon_t, \xi^\varepsilon_t)_{t\geqq 0}$ is defined on a stochastic basis with the filtration $\mathbf{F}$ one can use Aldous–Puhalskii's type sufficient conditions (see [15], and also Theorem 3.1 in [17]) for C-exponential tightness:

$$
\begin{aligned}
&\lim_{j}\overline{\lim_{\varepsilon\to 0}}\ \varepsilon\log\mathbf{P}\left(\sup_{t\leqq T}|X^\varepsilon_t|>j\right)=-\infty\ ,\\
&\lim_{\delta\to 0}\overline{\lim_{\varepsilon\to 0}}\ \varepsilon\log\sup_{\tau\leqq T-\delta}\mathbf{P}\left(\sup_{t\leqq\delta}|X^\varepsilon_{\tau+t}-X^\varepsilon_\tau|>\eta\right)=-\infty,\quad\forall\eta>0\ ,
\end{aligned}
\tag{4.2}
$$

where τ is a stopping time w.r.t. the filtration $\mathbf{F}$. Following Theorem 3.1 in [17], (4.2) implies the validity of the first part in (4.1) with above-mentioned compacts K'_j of uniformly bounded and equicontinuous functions. Now, choose relevant compacts $K''_j,\ j\geqq 1$:

$$
K''_j=\bigcap_{m\geqq j}\left\{v\in\mathbb{M}_{[0,T]}:\int_0^T\int_{|z|>m}v(dt,dz)\ \leqq\ g(m)\right\},
\tag{4.3}
$$

where $g(y),\ y>0$ is positive continuous decreasing function with $\lim_{y\to\infty}g(y)=0$. In fact, if $v_k\in K''_j,\ k\geqq 1$ then we have for any $m\geqq j$ $\sup_k\int_0^T\int_{|z|>m}v_k\ (dt,dz)\leqq g(m)$ that is the set K''_j is tight and by Prohorov's theorem (see [22]) is relatively compact. On the other hand, since the set $\{z:|z|>m\}$ is open a limit of any converging sequence from K''_j also belongs to K''_j that is K''_j is compact in $(\mathbb{M}_{[0,T]},\rho_T)$. Evidently $K''_j\subseteq K''_{j+1}$. Below we choose a special function $g(y)$, suited to assumption (A.3), to satisfy the second part in (4.1).

We check the validity of (4.1) in the next two lemmas.

Lemma 4.1 *Under* (A.1) *the first relation in* (4.1) *holds.*

Lemma 4.2 *Under* (A.2) *and* (A.3) *the second relation in* (4.1) *holds.*

Proof of Lemma 4.1. Put

$$
Z^*_t=\sup_{t'\leqq t}|Z_{t'}|\ .
$$

By virtue of (A.1) we have $|A(x,z)|\leqq\ell(1+|x|)$. Therefore, with $t\leqq T$, we derive from (1.1)

$$
X^{\varepsilon *}_t\ \leqq\ |x_0|+\ell\int_0^t(1+X^{\varepsilon *}_s)\,ds+\sqrt{\varepsilon}M^{\varepsilon *}_T\ ,
\tag{4.4}
$$

where $M^\varepsilon_t=\int_0^t B(X^\varepsilon_s,\xi^\varepsilon_s)\,dW_s$. Due to Bellman–Gronwall's inequality, (4.4) implies $X^{\varepsilon *}_T\leqq\text{const.}(1+\sqrt{\varepsilon}M^{\varepsilon *}_T)$ with const., depending only on $|x_0|,\ell$, and T. Therefore, the first part of (4.2) holds if

$$
\lim_{j}\overline{\lim_{\varepsilon\to 0}}\,\varepsilon\log\mathbf{P}(M^{\varepsilon *}_T>j)=-\infty\ .
\tag{4.5}
$$

On the other hand, by Chebyshev's inequality $\mathbf{P}(M^{\varepsilon *}_T>j)\leqq j^{-1/\varepsilon}\mathbf{E}(M^{\varepsilon *}_T)^{1/\varepsilon}$ and so, $\varepsilon\log\mathbf{P}(M^{\varepsilon *}_T>j)\leqq-\log j+\varepsilon\log\mathbf{E}(M^{\varepsilon *}_T)^{1/\varepsilon}$. Thereby (4.5) holds if

$$
\overline{\lim_{\varepsilon\to 0}}\,\varepsilon\log\mathbf{E}(M^{\varepsilon *}_T)^{1/\varepsilon}<\infty\ .
\tag{4.6}
$$

Below we check the validity (4.6). Assuming $1/\varepsilon > 2$ and applying Itô's formula to $|M_t^\varepsilon|^{1/\varepsilon}$, we get

$$|M_t^\varepsilon|^{1/\varepsilon} = \frac{1}{\sqrt{\varepsilon}} \int_0^t |M_s^\varepsilon|^{1/\varepsilon - 1} \ (\operatorname{sign} M_s^\varepsilon) B(X_s^\varepsilon, \xi_s^\varepsilon)\, dW_s + \frac{1-\varepsilon}{2\varepsilon} \int_0^t |M_s^\varepsilon|^{1/\varepsilon - 2} B^2(X_s^\varepsilon, \xi_s^\varepsilon)\, ds \, ,$$

that is $|M_t^\varepsilon|^{1/\varepsilon}$ is a submartingale obeying a decomposition: $|M_t^\varepsilon|^{1/\varepsilon} = N_t^\varepsilon + U_t^\varepsilon$ with local martingale N_t^ε and predictable increasing process

$$U_t^\varepsilon = \frac{1-\varepsilon}{2\varepsilon} \int_0^t |M_s^\varepsilon|^{1/\varepsilon - 2} B^2(X_s^\varepsilon, \xi_s^\varepsilon)\, ds \, . \tag{4.7}$$

Then, due to a modification of Doob's inequality (see [23], Theorem 1.9.2)

$$\mathbf{E}\,(M_t^{\varepsilon *})^{1/\varepsilon} \leqq \left(\frac{1}{1-\varepsilon}\right)^{1/\varepsilon} \mathbf{E} U_t^\varepsilon \, . \tag{4.8}$$

Now evaluate from above $|M_s^\varepsilon|^{1/\varepsilon - 2} B^2(X_s^\varepsilon, \xi_s^\varepsilon)$. By virtue of (A.1) $|B(x,z)| \leqq \ell(1+|x|)$. Thereby, due to above-mentioned upper bound $X_T^{\varepsilon *} \leqq \text{const.}(1 + M_T^{\varepsilon *})$ which remains true with replacing T on s for any $s < T$, we arrive at

$$\begin{aligned} |M_s^\varepsilon|^{1/\varepsilon - 2} B^2(X_s^\varepsilon, \xi_s^\varepsilon) &\leqq \text{const.}(1 + |M_s^\varepsilon|^{1/\varepsilon - 2} + |M_s^\varepsilon|^{1/\varepsilon}) \\ &\leqq \text{const.}(1 + (M_s^{\varepsilon *})^{1/\varepsilon}) \, . \end{aligned}$$

Substituting the last upper bound in (4.7) and using (4.8) we obtain ($t \leqq T$) $\mathbf{E}(M_t^{\varepsilon *})^{1/\varepsilon} \leqq (\text{const.}/\varepsilon) \int_0^t [1 + \mathbf{E}(M_t^{\varepsilon *})^{1/\varepsilon}]\, ds$. Hence, by Bellman–Gronwall's inequality, an upper bound $\mathbf{E}(M_T^{\varepsilon *})^{1/\varepsilon} \leqq (\text{const.}T/\varepsilon) \exp\{(\text{const.}T/\varepsilon)\}$ holds and implies (4.6). Consequently the first part in (4.2) is valid. To check the second part in (4.2), first use obvious estimates:

$$\begin{aligned} &\mathbf{P}\left(\sup_{t \leqq \delta} |X_{\tau+t}^\varepsilon - X_\tau^\varepsilon| > \eta\right) \\ &\leqq \mathbf{P}\left(\sup_{t \leqq \delta} |X_{\tau+t}^\varepsilon - X_\tau^\varepsilon| > \eta,\ X_T^{\varepsilon *} \leqq j\right) + \mathbf{P}(X_T^{\varepsilon *} > j) \\ &\leqq 2 \max\left[\mathbf{P}\left(\sup_{t \leqq \delta} |X_{\tau+t}^\varepsilon - X_\tau^\varepsilon| > \eta,\ X_T^{\varepsilon *} \leqq j\right), \mathbf{P}(X_T^{\varepsilon *} > j)\right] . \end{aligned}$$

Thence, due to proved above the first part of (4.2), the validity of the second part follows if

$$\lim_{\delta \to 0} \overline{\lim_{\varepsilon \to 0}}\, \varepsilon \log \sup_{\tau \leqq T - \delta} \mathbf{P}\left(\sup_{t \leqq \delta} |X_{\tau+t}^\varepsilon - X_\tau^\varepsilon| > \eta,\ X_T^{\varepsilon *} \leqq j\right) = -\infty, \qquad j \geqq 1,\ \eta > 0 \, . \tag{4.9}$$

The simplest way for verifying (4.9) consists in checking the validity of both

$$\lim_{\delta\to 0}\overline{\lim_{\varepsilon\to 0}}\,\varepsilon\log\sup_{\tau\leqq T-\delta}\mathbf{P}\left(\sup_{t\leqq\delta}\left|\int_\tau^{\tau+t}A(X_s^\varepsilon,\xi_s^\varepsilon)\,ds\right|>\eta,\ X_T^{\varepsilon *}\leqq j\right)=-\infty\,,$$

$$\lim_{\delta\to 0}\overline{\lim_{\varepsilon\to 0}}\,\varepsilon\log\sup_{\tau\leqq T-\delta}\mathbf{P}\left(\sup_{t\leqq\delta}\left|\sqrt{\varepsilon}\int_\tau^{\tau+t}B(X_s^\varepsilon,\xi_s^\varepsilon)\,dW_s\right|>\eta,\ X_T^{\varepsilon *}\leqq j\right)=-\infty\,. \tag{4.10}$$

Obviously, the first part in (4.10) holds. To verify the second, note that the process $Y_t^\varepsilon=\sqrt{\varepsilon}\int_\tau^{\tau+t}B(X_s^\varepsilon,\xi_s^\varepsilon)\,dW_s$ is continuous martingale w.r.t. the new filtration $\mathbf{F}^\tau=(\mathscr{F}_{\tau+t})_{t\geqq 0}$ (see Chap. 4, Sect. 7 in [23]). It has the predictable quadratic variation $\langle Y^\varepsilon\rangle_t=\varepsilon\int_\tau^{\tau+t}B^2(X_s^\varepsilon,\xi_s^\varepsilon)\,ds$. Also define a positive continuous local martingale (w.r.t. the same filtration $\mathbf{F}^\tau$)

$$Z_t^\varepsilon=\exp(\lambda Y_t^\varepsilon-\tfrac{1}{2}\lambda^2\langle Y^\varepsilon\rangle_t),\quad \lambda\in R\,, \tag{4.11}$$

which is simultaneously a supermartingale (see [23], Problem 1.4.4) and so for any Markov time σ (w.r.t. $\mathbf{F}^\tau$) $\mathbf{E}Z_\sigma^\varepsilon\leqq 1$. Take $\sigma=\inf\{t\leqq\delta:\ |Y_t^\varepsilon|\geqq\eta\}$. Evidently the second part of (4.10) holds if

$$\lim_{\delta\to 0}\overline{\lim_{\varepsilon\to 0}}\,\varepsilon\log\sup_{\tau\leqq T-\delta}\mathbf{P}(Y_\sigma^\varepsilon\geqq\eta\quad(\text{or}\leqq-\eta),\ \sigma\leqq\delta,\ X_T^{\varepsilon *}\leqq j)=-\infty\,. \tag{4.12}$$

By virtue of an obvious inequality $\mathbf{E}Z_\sigma^\varepsilon I(Y_\sigma^\varepsilon\geqq\eta,\ X_T^{\varepsilon *}\leqq j)\leqq 1$ we find that

$$\varepsilon\log\mathbf{P}(Y_\sigma^\varepsilon\geqq\eta,\ \sigma\leqq\delta,\ X_T^{\varepsilon *}\leqq j)\leqq-\sup_{\lambda>0}\left[\lambda\eta-\text{const.}\frac{\lambda^2}{2}\delta\varepsilon\right] \tag{4.13}$$

and since

$$\sup_{\lambda>0}\left[\lambda\eta-\text{const.}\frac{\lambda^2}{2}\delta\varepsilon\right]=\frac{\eta^2}{2\,\text{const.}\,\delta\varepsilon}$$

(4.12) with "$\geqq\eta$" is implied by (4.13). The validity (4.12) with "$\leqq-\eta$" is proved in the same way.

Proof of Lemma 4.2. It is clear that $\{\nu^\varepsilon\in\mathbb{M}_{[0,T]}\backslash K_j''\}=\{\ell(j,\nu^\varepsilon)<\infty\}$, where

$$\ell(j,\nu)=\min\left\{m\geqq j:\int_0^T\int_{|z|>m}\nu(dt,dz)>g(m)\right\}\,. \tag{4.14}$$

Therefore, the second part of (4.1) is equivalent to

$$\lim_j\overline{\lim_{\varepsilon\to 0}}\,\varepsilon\log\mathbf{P}(\ell(j,\nu^\varepsilon)<\infty)=-\infty\,. \tag{4.15}$$

To verify (4.15), choose a special function $g(y)$ satisfying the above-mentioned properties. To this end introduce non linear operator

$$\mathscr{D}=b(z)\frac{\partial}{\partial z}+\frac{\sigma^2(z)}{2}\left[\frac{\partial^2}{\partial z^2}+\left(\frac{\partial}{\partial z}\right)^2\right] \tag{4.16}$$

and choose a non negative twice continuously differentiable function $u(z)$ such that

$$
\begin{aligned}
&-\sup_{v\in R}\mathscr{D}u(v) = -d > -\infty\ ,\\
&\lim_{j\to\infty}\inf_{|z|>j}\left[-\mathscr{D}u(z) + \sup_{v\in R}\mathscr{D}u(v)\right] = \infty\ .
\end{aligned}
\tag{4.17}
$$

Under assumptions (A.2) and (A.3) one can take any of function $u(z)$ with properties: $u(0) = 0$, $u'(z) = \operatorname{sign} z$, $|z| > 1$, and $0 \leqq u''(z) \leqq 1$. With chosen $u(z)$ put

$$
g(y) = \inf_{|z|>y}\left[-\mathscr{D}u(z) + \sup_{v}\mathscr{D}u(v)\right]^{-1/2}\ . \tag{4.18}
$$

Introduce a positive continuous local martingale (the martingale property is checked by Itô's formula)

$$
Z_t^\varepsilon = \exp\left(u(\xi_t^\varepsilon) - u(\xi_0) - \int_0^t \mathscr{D}u(\xi_s^\varepsilon)\,ds\right)\ . \tag{4.19}
$$

It is simultaneously a supermartingale (see Problem 1.4.4. in [23]) and so $\mathbf{E}Z_T^\varepsilon \leqq 1$. The last implies

$$
\mathbf{E}I(\ell(j,v^\varepsilon) < \infty)Z_T^\varepsilon \leqq 1\ . \tag{4.20}
$$

Inequality (4.20) can be sharpened by changing of Z_T^ε on its lower bound on the set $\{\ell(j,v^\varepsilon) < \infty\}$ which can be chosen non random. Taking into account that $\int_0^T \mathscr{D}u(\xi_s^\varepsilon)\,ds = \int_0^T\int_R \mathscr{D}u(z)v^\varepsilon(ds,dz)$ and $\ell(j,v^\varepsilon) \geqq j$ we arrive at

$$
\begin{aligned}
\log Z_T^\varepsilon &\geqq -u(\xi_0) - \frac{dT}{\varepsilon} + \frac{1}{\varepsilon}\int_0^T \int_{|z|>\ell(j,v^\varepsilon)} [-\mathscr{D}u(z) + d]\, v^\varepsilon(ds,dz)\\
&\geqq -u(\xi_0) - \frac{dT}{\varepsilon} + \frac{1}{\varepsilon}\inf_{|z|>\ell(j,v^\varepsilon)}[-\mathscr{D}u(z) + d]\int_0^T \int_{|z|>\ell(j,v^\varepsilon)} v^\varepsilon(ds,dz)\\
&\geqq -u(\xi_0) - \frac{dT}{\varepsilon} + \frac{1}{\varepsilon}\inf_{|z|>\ell(j,v^\varepsilon)}[-\mathscr{D}u(z) + d]^{1/2}\\
&\geqq -u(\xi_0) - \frac{dT}{\varepsilon} + \frac{1}{\varepsilon}\inf_{|z|>j}[-\mathscr{D}u(z) + d]^{1/2}\ \ (= \log Z_*)\ .
\end{aligned}
$$

Thereby, from (4.20), with Z_T^ε repalced on Z_*, we derive

$$
\varepsilon \log \mathbf{P}\big(\ell(j,v^\varepsilon) < \infty\big) \leqq \varepsilon u(\xi_0) + dT - \inf_{|z|>j}[-\mathscr{D}u(z) + d]^{1/2}\ ,
$$

i.e. (4.15) is implied by (4.17).

5 Upper bound for local LDP in $\mathbb{C}_{[0,T]} \times \mathbb{M}_{[0,T]}$

In this section, we consider family $(X^\varepsilon, v^\varepsilon)$ from $\mathbb{C}_{[0,T]} \times \mathbb{M}_{[0,T]}$. Parallel to $F(v)$ and $S(X,v)$, given in (2.3), let us define $F_T(v)$ and $S_T(X,v)$ by changing integrals "$\int_0^\infty$" in (2.3) on "$\int_0^T$". Put

$$L_T(X,v) = \tfrac{1}{2} S_T(X,v) + \tfrac{1}{8} F_T(v) \,. \tag{5.1}$$

Theorem 5.1 *Under* (A.1), (A.2), *and* (A.3) *for every* (X,v) *from* $\mathbb{C}_{[0,T]} \times \mathbb{M}_{[0,T]}$

$$\overline{\lim_{\delta\to 0}}\, \overline{\lim_{\varepsilon\to 0}}\, \varepsilon \log \mathbf{P}(r_T(X^\varepsilon, X) + \rho_T(v^\varepsilon, v) \leqq \delta) \leqq -L_T(X,v) \,.$$

Proof of this theorem is based on

Lemma 5.1 *Assume* (A.1), (A.2), *and* (A.3). *Then for every piece wise constant function* $\lambda(t) = \sum_i \lambda(t_i) I(t_i \leqq t < t_{i+1})$ (*with not overlapping intervals* $[t_i, t_{i+1})$), *and for every compactly supported in z and continuously differentiable* (*once in t and twice in z*) *function* $u(t,z)$, *and* $X \in \mathbb{C}_{[0,T]}$, $v \in \mathbb{M}_{[0,T]}$

$$\overline{\lim_{\delta\to 0}}\, \overline{\lim_{\varepsilon\to 0}}\, \varepsilon \log \mathbf{P}(r_T(X^\varepsilon, X) + \rho_T(v^\varepsilon, v) \leqq \delta)$$

$$\leqq -\Bigg\{ \sum_i \lambda(t_i)[X_{T\wedge t_{i+1}} - X_{T\wedge t_i}] - \int_0^T \int_R \lambda(t) A(X_t, z) v(dt, dz)$$

$$- \frac{1}{2} \int_0^T \int_R \lambda^2(t) B^2(X_t, z) v(dt, dz) \Bigg\} + \int_0^T \int_R \mathscr{D} u(t,z) v(dt, dz) \,,$$

where $\mathscr{D}$ *is the non linear operator defined in* (4.16).

Proof. The following well known fact will be used hereafter. If N_t ($N_0 = 0$) is continuous local martingale and $\langle N \rangle_t$ is its predictable quadratic variation, then the exponential process $Z_t = \exp(N_t - (1/2)\langle N \rangle_t)$ is a continuous local martingale too, and what is more if N'_t, N''_t are continuous local martingales ($N'_0 = N''_0 = 0$) with the mutual predictable quadratic variation $\langle N', N'' \rangle_t \equiv 0$ and Z'_t, Z''_t are corresponding exponential processes, then the process $Z'_t Z''_t$ is also local martingale which, being positive, is a supermartingale too (Problem 1.4.4 in [23]) and so $\mathbf{E} Z'_t Z''_t \leqq 1$, $t \geqq 0$.

Let $\lambda(t)$ and $u(t,z)$ be functions involved in Lemma 5.1. Put

$$N'_t = \frac{1}{\sqrt{\varepsilon}} \int_0^t \lambda(s) B(X^\varepsilon_s, \xi^\varepsilon_s)\, dW_s \,,$$

$$N''_t = \frac{1}{\sqrt{\varepsilon}} \int_0^t u'_z(s, \xi^\varepsilon_s) \sigma(\xi^\varepsilon_s)\, dV_s \,.$$

Evidently

$$\langle N'\rangle_t = \frac{1}{\varepsilon}\int_0^t \lambda^2(s)B^2(X_s^\varepsilon, \xi_s^\varepsilon)\,ds\,,$$
$$\langle N''\rangle_t = \frac{1}{\varepsilon}\int_0^t u_z'^2(s,\xi_s^\varepsilon)\sigma^2(\xi_s^\varepsilon)\,ds\,. \tag{5.2}$$

Since Wiener processes W_t and V_t are independent and so $\langle N', N''\rangle_t \equiv 0$ a process

$$Z_t = \exp(N_t' + N_t'' - \tfrac{1}{2}[\langle N'\rangle_t + \langle N''\rangle_t]) \tag{5.3}$$

is local martingale and also a supermartingale with

$$\mathbf{E}Z_t \leqq 1, \quad t \geqq 0\,. \tag{5.4}$$

Note that

$$N_t' = \frac{1}{\varepsilon}\int_0^t \lambda(s)[dX_s^\varepsilon - A(X_s^\varepsilon, \xi_s^\varepsilon)\,ds] \tag{5.5}$$

and also find similar representation for N_t''. Due to Itô's formula we obtain

$$\frac{1}{\sqrt{\varepsilon}}\int_0^t u_z'(s,\xi_s^\varepsilon)\sigma(\xi_s^\varepsilon)\,dV_s = u(t,\xi_t^\varepsilon) - u(0,\xi_0) - \int_0^t u_t'(s,\xi_s^\varepsilon)\,ds - \frac{1}{\varepsilon}\int_0^t \mathscr{L}u(s,\xi_s^\varepsilon)\,ds\,,$$

where $\mathscr{L} = b(z)(\partial d/\partial z) + (\sigma^2(z)/2)(\partial^2/z^2)$ and consequently

$$N_t'' = u(t,\xi_t^\varepsilon) - u(0,\xi_0) + \int_0^t u_t'(s,\xi_s^\varepsilon)\,ds - \frac{1}{\varepsilon}\int_0^t \mathscr{L}u(s,\xi_s^\varepsilon)\,ds\,. \tag{5.6}$$

(5.4) implies an obvious inequality

$$\mathbf{E}I(r_T(X^\varepsilon, X) + \rho_T(v^\varepsilon, v) \leqq \delta)Z_T \leqq 1\,, \tag{5.7}$$

which can be sharpened by changing of Z_T by its lower bound. To this end evaluate from below $\log Z_T$ on the set $\{r_T(X^\varepsilon, X) + \rho_T(v^\varepsilon, v) \leqq \delta\}$. For both $N_T' - \frac{1}{2}\langle N'\rangle_T$ and $N_T'' - \frac{1}{2}\langle N''\rangle_T$ we get

$$\begin{aligned}
N_T' - \tfrac{1}{2}\langle N'\rangle_T
&\geqq \frac{1}{\varepsilon}\sum_i \lambda(t_i)[X_{T\wedge t_{i+1}} - X_{T\wedge t_i}] \\
&\quad - \frac{1}{\varepsilon}\int_0^T\int_R \lambda(t)A(X_t,z)v(dt,dz) - \frac{1}{2}\int_0^T\int_R \lambda^2(t)B^2(X_t,z)v(dt,dz) \\
&\quad - \frac{1}{\varepsilon}\Bigg\{\sum_i |\lambda(t_i)|[|X_{T\wedge t_{i+1}}^\varepsilon - X_{T\wedge t_{i+1}}| + |X_{T\wedge t_i}^\varepsilon - X_{T\wedge t_i}|] \\
&\quad + \int_0^T |\lambda(t)||A(X_t^\varepsilon,\xi_t^\varepsilon) - A(X_t,\xi_t^\varepsilon)|\,ds + \frac{1}{2}\int_0^T \lambda^2(t)|B^2(X_t^\varepsilon,\xi_t^\varepsilon) - B^2(X_t,\xi_t^\varepsilon)|\,ds \\
&\quad + \left|\int_0^T\int_R [\lambda(t)A(X_t,z)| + \lambda^2(t)B^2(X_t,z)][v^\varepsilon - v](dz,dt)\right|\Bigg\}
\end{aligned} \tag{5.8}$$

and

$$
\begin{aligned}
N_T'' - \tfrac{1}{2}\langle N''\rangle_T
&= -\frac{1}{\varepsilon}\int_0^T [\mathscr{L}u(s,\xi_s^\varepsilon) + \tfrac{1}{2}u_z^2(s,\xi_s^\varepsilon)]\,ds + u(T,\xi_T^\varepsilon) - u(0,\xi_0) - \int_0^T u_t(s,\xi_s^\varepsilon) \\
&= -\frac{1}{\varepsilon}\int_0^T\int_R \mathscr{D}u(s,z)v^\varepsilon(dz,ds) + u(T,\xi_T^\varepsilon) - u(0,\xi_0) - \int_0^T u_t(s,\xi_s^\varepsilon)\,ds \\
&\geqq -\frac{1}{\varepsilon}\int_0^T\int_R \mathscr{D}u(s,z)v(dz,ds) \\
&\quad - \frac{1}{\varepsilon}\Bigg\{\left|\int_0^T\int_R \mathscr{D}u(s,z)[v^\varepsilon - v](dz,ds)\right| \\
&\quad + \varepsilon|u(T,\xi_T^\varepsilon)| + \varepsilon|u(0,\xi_0)| + \varepsilon\int_0^T |u_t(s,\xi_s^\varepsilon)|\,ds\Bigg\}\,. \qquad (5.9)
\end{aligned}
$$

The terms in the curly brackets in the right hand sides of (5.8) and (5.9) are random variables. Nevertheless, they can be evaluated from above on the set $\{r_T(X^\varepsilon,X) + \rho_T(v^\varepsilon.v) \leqq \delta\}$ by non random quantities. Evidently $\int_0^T\int_R |\lambda(t)|$ $|A(X_t^\varepsilon,\xi_t^\varepsilon) - A(X_t,\xi_t^\varepsilon)|\,ds \leqq \text{const.}T\delta$ and $\frac{1}{2}\int_0^T \lambda^2(t)|B^2(X_t^\varepsilon,\xi_t^\varepsilon) - B^2(X_t,\xi_t^\varepsilon)|\,ds \leqq$ const.$\delta\int_0^T[1+|X_t|]\,ds$. Denote by $H(s,z) = \lambda(s)A(X_s,z) + \lambda^2(s)B^2(X_s,z)/2$. Since $\lambda(s)$ is piece wise constant function without loss of a generality one can assume that it is simply constant. Then function $H(s,z)$ is bounded continuous function and so, by Lemma A.1 (see Appendix) for any $\gamma > 0$ and $k \geqq 1$ there exist increasing continuous function $h_k^\gamma(y), y \geqq 0$ with $h_k^\gamma(0) = 0$ and decreasing sequence φ_k, $k \geqq 1$ with $\lim_k \varphi_k = 0$ both dependent on $H(s,z)$ and v only such that

$$
\left|\int_0^T\int_R H(s,z)[v^\varepsilon - v](ds,dz)\right| \leqq \gamma + h_k^\gamma(\delta) + \varphi_k\,.
$$

Further, by the remark to Lemma A.1

$$
\left|\int_0^T\int_R \mathscr{D}u(t,z)[v^\varepsilon - v](dt,dz)\right| \leqq \gamma + h^\gamma(\delta)\,,
$$

where $h^\gamma(y)$ is an increasing continuous function with $h^\gamma(0) = 0$ depending on $\mathscr{D}u(s,z)$ and v only.

Hence, we arrive to the lower bounds (with positive const.'s):

$$
\begin{aligned}
N_T' - \tfrac{1}{2}\langle N'\rangle_T \geqq \frac{1}{\varepsilon}\Bigg[&\sum_i \lambda(t_i)[X_{T\wedge t_{i+1}} - X_{T\wedge t_i}] - \int_0^T\int_R \lambda(t)A(X_t,z)v(dt,dz) \\
&- \int_0^T\int_R \lambda^2(t)B^2(X_t,z)v(dt,dz)\Bigg] - \frac{\text{const.}}{\varepsilon}(\gamma + h_k^\gamma(\delta) + \varphi_k)\quad (5.10)
\end{aligned}
$$

and

$$N''_T - \tfrac{1}{2}\langle N''\rangle_T \geqq -\frac{1}{\varepsilon}\int_0^T\int_R \mathscr{D}u(t,z)v(dt,dz) - \frac{\text{const.}}{\varepsilon}(\varepsilon+\gamma+h^\gamma(\delta)+\varphi_k))\,. \tag{5.11}$$

By virtue of (5.10) and (5.11) one can choose a non random lower bound:

$$\begin{aligned}\log Z_T \geqq \frac{1}{\varepsilon}\Bigg[&\sum_i \lambda(t_i)[X_{T\wedge t_{i+1}} - X_{T\wedge t_i}] - \int_0^T\int_R \lambda(t)A(X_t,z)v(dt,dz)\\ &- \int_0^T\int_R \lambda^2(t)B^2(X_t,z)v(dt,dz) - \frac{1}{\varepsilon}\int_0^T\int_R \mathscr{D}u(t,z)v(dt,dz)\Bigg]\\ &- \frac{\text{const.}}{\varepsilon}(\varepsilon+\gamma+h^\gamma(\delta)+h_k^\gamma(\delta)+\varphi_k).\\ = \log Z_*\,.\end{aligned}$$

Hence and from (5.7), with replacing of Z_T on Z_*, it follows

$$\begin{aligned}&\varepsilon\log \mathbf{P}(r_T(X^\varepsilon,X)+\rho(v^\varepsilon,v)\leqq\delta)\\ &\leqq -\Bigg[\sum_i \lambda(t_i)[X_{T\wedge t_{i+1}} - X_{T\wedge t_i}] - \int_0^T\int_R \lambda(t)A(X_t,z)v(dt,dz)\\ &\quad - \frac{1}{2}\int_0^T\int_R \lambda^2(t)B^2(X_t,z)v(dt,dz)\Bigg] + \int_0^T\int_R \mathscr{D}u(t,z)v(dt,dz)\\ &\quad + \text{const.}\{(\varepsilon+\gamma+h^\gamma(\delta)+h_k^\gamma(\delta)+\varphi_k)\}\,.\end{aligned} \tag{5.12}$$

The desired result holds since the term in the curly brackets of the right hand side in (5.12) goes to zero if limit "$lim_{k\to\infty}\, lim_{\gamma\to 0}\, lim_{\delta\to 0}\, \lim_{\varepsilon\to 0}$" is taken.

Proof of Theorem 5.1. Follows from Lemmas 5.1, A.2, and A.3 (see Appendix) since

$$\begin{aligned}&-\sup_\lambda\Bigg\{\sum_i \lambda(t_i)[X_{T\wedge t_{i+1}} - X_{T\wedge t_i}] - \int_0^T\int_R \lambda(t)A(X_t,z)v(dt,dz)\\ &- \frac{1}{2}\int_0^T\int_R \lambda^2(t)B^2(X_t,z)v(dt,dz)\Bigg\} + \inf_u \int_0^T\int_R \mathscr{D}u(t,z)v(dt,dz)\\ &= -[\tfrac{1}{2}S_T(X,v)+\tfrac{1}{8}F_T(v)] = -L_T(X,v)\,.\end{aligned}$$

6 Lower bound for local LDP in $\mathbb{C}_{[0,T]} \times \mathbb{M}_{[0,T]}$

Theorem 6.1 *Under* (A.1), (A.2), *and* (A.3), *for every* (X, v) *from* $\mathbb{C}_{[0,T]} \times \mathbb{M}_{[0,T]}$

$$\varliminf_{\delta\to 0}\varliminf_{\varepsilon\to 0} \varepsilon \log \mathbf{P}(r_T(X^\varepsilon, X) + \rho_T(v^\varepsilon, v) \leqq \delta) \geqq -L_T(X, v)\,.$$

Evidently for X, v such that $L_T(X, v) = \infty$ it is nothing to prove. Therefore we consider below only the case $L_T(X, v) < \infty$ which distinguishes subsets from $\mathbb{C}_{[0,T]} \times \mathbb{M}_{[0,T]}$:
(i) $dX_t \ll dt$ and $\frac{1}{2}S_T(X, v) = \frac{1}{2}\int_0^T ([\dot{X}_t - A_v(t, X_t)]^2/B_v^2(t, X_t))\,dt < \infty$;
(ii) $dv = n\,d\lambda$, $d_z n = n'_z\,dz$ and $\frac{1}{8}F_T(v) = \frac{1}{2}\int_0^T\int_R (v_v^2(t,z)/\sigma^2(z))n(t,z)\,dz\,dt < \infty$,
where

$$v_v(t,z) = \frac{\sigma^2(z)}{2}\left[\frac{n'_z(t,z)}{n(t,z)} - \frac{p'(z)}{p(z)}\right]. \tag{6.1}$$

It is convenient to consider further subset (ii′) of (ii):
(ii′) The function $v_v(t,z)$ is compactly supported in z and continuously differentiable in (t,z), having bounded partial derivatives.
The central role in proving Theorem 6.1 plays

Lemma 6.1 *Assume* (i), (ii′), *and* $\inf_{x,z}\ B^2(x,z) > 0$. *Then for any* $\delta > 0$ *and* $\gamma > 0$ *there exists an increasing continuous function* $h_\gamma(y)$ *with* $h_\gamma(0) = 0$, *depending on* $v_v^2(s,z)/\sigma^2(z)$ *and* v *only*, *such that*

$$\varliminf_{\varepsilon\to 0} \varepsilon \log \mathbf{P}(r_T(X^\varepsilon, X) + \rho_T(v^\varepsilon, v) \leqq \delta) \geqq -L_T(X, v) - \gamma - h_\gamma(\delta)\,.$$

Proof. Put

$$\begin{aligned} b_v(t,z) &= b(z) + v_v(t,z)\,,\\ G_v(t,x,z) &= \frac{\dot{X}_t - A_v(t,X_t)}{B_v(t,X_t)}B(x,z) + A(x,z) \end{aligned} \tag{6.2}$$

and parallel to $(X_t^\varepsilon, \xi_t^\varepsilon)$ introduce, on the same stochastic basis, new diffusion pair $(\widetilde{X}_t^\varepsilon, \widetilde{\xi}_t^\varepsilon)$:

$$\begin{aligned} d\widetilde{X}_t^\varepsilon &= G_v(t, \widetilde{X}_t^\varepsilon, \widetilde{\xi}_t^\varepsilon)\,dt + \sqrt{\varepsilon}B(\widetilde{X}_t^\varepsilon, \widetilde{\xi}_t^\varepsilon)\,dW_t\,,\\ d\widetilde{\xi}_t^\varepsilon &= \frac{1}{\varepsilon}b_v(t, \widetilde{\xi}_t^\varepsilon)\,dt + \frac{1}{\sqrt{\varepsilon}}\sigma(\widetilde{\xi}_t^\varepsilon)\,dV_t \end{aligned} \tag{6.3}$$

subject to the same initial point $(x_0,\ \xi_0)$. Also denote by $\widetilde{v}^\varepsilon(dt, dz)$ the occupation measure corresponding to $\widetilde{\xi}^\varepsilon$: $\widetilde{v}^\varepsilon(\Delta \times \Gamma) = \int_0^\infty I(t \in \Delta, \widetilde{\xi}_t^\varepsilon \in \Gamma)\,dt$.

By virtue of the formula $b(z) = \frac{1}{2}(p'(z)/p(z)) + \sigma'(z)\sigma(z)$ (see (2.1)) we get

$$\frac{2b_v(t,z)}{\sigma^2(z)} = \frac{n'_z(t,z)}{n(t,z)} + 2\frac{\sigma'(z)}{\sigma(z)}$$

and so

$$p_v(t,z) = c(t)\frac{\exp(2\int_0^z \frac{b_v(t,y)}{\sigma^2(y)}\,dy)}{\sigma^2(z)}\,,$$

with norming constant $c(t)$ such that $\int_R p_v(t,z) = 1$, coincides with $n(t,z)$. Then by Lemma A.5 (see Appendix)

$$\mathbf{P} - \lim_{\varepsilon\to 0} \rho_T(\widetilde{v}^\varepsilon, v) = 0 \quad \text{and} \quad \mathbf{P} - \lim_{\varepsilon\to 0} r_T(\widetilde{X}^\varepsilon, X) = 0 \, . \tag{6.4}$$

Denote by Q^ε and $\widetilde{Q}^\varepsilon$ distributions of $(X_t^\varepsilon, \xi_t^\varepsilon)_{t\leqq T}$, $(\widetilde{X}_t^\varepsilon, \widetilde{\xi}_t^\varepsilon)_{t\leqq T}$ respectively. By Theorem 7.18 (Chap. 7 in [24]) Q^ε is absolutely continuous w.r.t. $\widetilde{Q}^\varepsilon$ and

$$\frac{dQ^\varepsilon}{d\widetilde{Q}^\varepsilon}(\widetilde{X}^\varepsilon, \widetilde{\xi}^\varepsilon) = \exp\left(\frac{1}{\sqrt{\varepsilon}} M_T^\varepsilon - \frac{1}{2\varepsilon}\langle M^\varepsilon\rangle_T + \frac{1}{\sqrt{\varepsilon}} M_T - \frac{1}{2\varepsilon}\langle M\rangle_T\right) , \tag{6.5}$$

where

$$M_t^\varepsilon = -\int_0^t \frac{v_v(s, \widetilde{\xi}_s^\varepsilon)}{\sigma(\widetilde{\xi}_s^\varepsilon)}\, dV_s \quad \text{and} \quad M_t = -\int_0^t \frac{\dot{X}_s - A_v(s, X_s)}{B_v(s, X_s)} dW_s \, ,$$

$$\langle M^\varepsilon\rangle_t = \int_0^t \frac{v_v^2(s, \widetilde{\xi}_s^\varepsilon)}{\sigma^2(\widetilde{\xi}_s^\varepsilon)}\, ds \quad \text{and} \quad \langle M\rangle_t = \int_0^t \frac{[\dot{X}_s - A_v(s, X_s)]^2}{B_v^2(s, X_s)}\, ds \, .$$

By virtue of (6.5) and the rule of changing for probability measure we obtain

$$\mathbf{P}(r_T(X^\varepsilon, X) + \rho_T(v^\varepsilon, v) \leqq \delta) = \mathbf{E}\left[\frac{dQ^\varepsilon}{d\widetilde{Q}^\varepsilon}(\widetilde{X}^\varepsilon, \widetilde{\xi}^\varepsilon) I(r_T(\widetilde{X}^\varepsilon, X) + \rho_T(\widetilde{v}^\varepsilon, v) \leqq \delta)\right] . \tag{6.6}$$

The desired lower bound can be derived from (6.6) provided that a relevant lower bound for the right hand side of (6.6) can be found. Use an obvious inequality:

$$\begin{aligned} &I(r_T(\widetilde{X}^\varepsilon, X) + \rho_T(\widetilde{v}^\varepsilon, v) \leqq \delta) \\ &\quad \geqq I(r_T(\widetilde{X}^\varepsilon, X) + \rho_T(\widetilde{v}^\varepsilon, v) \leqq \delta, |M_T^\varepsilon| \leqq k,\ |M_T| \leqq k) \end{aligned}$$

and estimate from below $\log(dQ^\varepsilon/d\widetilde{Q}^\varepsilon)(\widetilde{X}^\varepsilon, \widetilde{\xi}^\varepsilon)$ on the set $\{r_T(\widetilde{X}^\varepsilon, X) + \rho_T(\widetilde{v}^\varepsilon, v) \leqq \delta\} \cap \{|M_T^\varepsilon| \leqq k\} \cap \{|M_T| \leqq k\}$. Noticing that $\frac{1}{2}\langle M\rangle_T = \frac{1}{2} S(X, v)_T$ and $\frac{1}{2}\int_0^T \int_R (v_v^2(s,z)/\sigma^2(z)) n(t,z)\, dz\, dt = \frac{1}{8} F_T(v)$ we obtain

$$\begin{aligned} \log \frac{dQ^\varepsilon}{d\widetilde{Q}^\varepsilon}(\widetilde{X}^\varepsilon, \widetilde{\xi}^\varepsilon) &\geqq -\frac{2k}{\sqrt{\varepsilon}} - \frac{1}{\varepsilon} L_T(X, v) \\ &\quad - \frac{1}{2\varepsilon}\left|\int_0^T \int_R \frac{v_v^2(s,z)}{\sigma^2(z)} m[\widetilde{v}^\varepsilon(dt, dz) - n(t,z)\, dz\, dt\right| \, . \end{aligned}$$

By the remark to Lemma A.1 (see Appendix), for any $\gamma > 0$ there exists increasing continuous function $h_\gamma(y)$ with $h\gamma(0) = 0$, depending on $v_v^2(s,z)/\sigma^2(z)$ and v only, such that

$$\frac{1}{2}\left|\int_0^T \int_R \frac{v_v^2(s,z)}{\sigma^2(z)} [\widetilde{v}^\varepsilon(dt, dz) - n(t,z)\, dz\, dt\right| \leqq \gamma + h_\gamma(\delta) \, .$$

Then the lower bound for the right hand side of (6.6) is the following: $-(2k/\sqrt{\varepsilon}) - (1/\varepsilon)[L_T(X,v) + \gamma + h_\gamma(\delta)]$. It implies

$$\varepsilon \log \mathbf{P}(r_T(X^\varepsilon, X) + \rho_T(v^\varepsilon, v) \leqq \delta)$$
$$\geqq -L_T(X,v) - 2k\sqrt{\varepsilon} - \gamma - h_\gamma(\delta) + \varepsilon \log \mathbf{P}(r_T(\widetilde{X}^\varepsilon, X) + \rho_T(\widetilde{v}^\varepsilon, v) \leqq \delta,$$
$$|M_T^\varepsilon| \leqq k,\ |M_T| \leqq k)\ .$$

Thus the statement of the lemma holds since

$$\lim_k \lim_{\varepsilon\to 0} \mathbf{P}(r_T(\widetilde{X}^\varepsilon, X) + \rho_T(\widetilde{v}^\varepsilon, v) \leqq \delta,\ |M_T^\varepsilon| \leqq k,\ |M_T| \leqq k) = 1$$

what follows by virtue of (6.4), obvious $\lim_k \mathbf{P}(|M_T| > k) = 0$ and

$$\mathbf{P}(|M_T^\varepsilon| > k) \leqq \frac{\mathbf{E}|M_T^\varepsilon|^2}{k^2} = \frac{\mathbf{E}\langle M^\varepsilon\rangle_T}{k^2} \leqq \frac{\text{const.}}{k^2} \to 0, \quad k \to \infty\ .$$

Proof of Theorem 6.1. Assume (i), (ii), and $\inf_{x,z} B^2(x,z) > 0$. Due to Lemma A.4 (see Appendix), one can choose a sequence $v^{(k)},\ k \geqq 1$ of measures such that for every k the function $v_{v^{(k)}}(t,z)$ satisfies (ii′) and what is more $\rho(v, v^{(k)}) \to 0,\ L_T(X, v^{(k)}) \to L_T(X,v)$. On the other hand, by Lemma 6.1 for any $\delta > 0$ and $\gamma > 0$ there exist increasing continuous function $h_{\gamma,k}(y)$ with $h_{\gamma,k}(0) = 0$, depending on $v^{(k)}$, such that

$$\varliminf_{\varepsilon\to 0} \varepsilon \log \mathbf{P}(r_T(X^\varepsilon, X) + \rho_T(v^\varepsilon, v^{(k)}) \leqq \delta) \geqq -L_T(X, v^{(k)}) - \gamma - h_{\gamma,k}(\delta)\ .$$

Choose $k_0(\delta)$ such that for any $k \geqq k_0(\delta)$ we have $0 < \delta - \rho_T(v, v^{(k)}) \leqq \delta/2$. Then, taking into account the triangular inequality: $\rho_T(v^\varepsilon, v) \leqq \rho_T(v^\varepsilon, v^{(k)}) + \rho_T(v, v^{(k)})$, we arrive to a lower bound:

$$\varliminf_{\varepsilon\to 0} \varepsilon \log \mathbf{P}(r_T(X^\varepsilon, X) + \rho_T(v^\varepsilon, v) \leqq \delta)$$
$$\geqq \varliminf_{\varepsilon\to 0} \varepsilon \log \mathbf{P}(r_T(X^\varepsilon, X) + \rho_T(v^\varepsilon, v^{(k)}) \leqq \delta/2)$$
$$\geqq -L_T(X, v^{(k)}) - \gamma - h_{\gamma,k}(\delta/2)\ .$$

The right hand side of the last inequality converges to $-L_T(X,v)$ if limit "$lim_k\ lim_{\gamma\to 0}\ lim_{\delta\to 0}$" is taken.

Assume only (i) and (ii). Parallel to the process X_t^ε introduce new diffusion $X_t^{\varepsilon,\beta},\ \beta \neq 0$:

$$dX_t^{\varepsilon,\beta} = A(X_t^{\varepsilon,\beta}, \xi_t^\varepsilon)\,dt + \sqrt{\varepsilon}[B(X_t^{\varepsilon,\beta}, \xi_t^\varepsilon)\,dW_t + \beta\,dW_t']$$

subject to the same initial point x_0, where W_t' is a Wiener process independent of (W_t, ξ_t^ε). The diffusion parameter here is $B^2(x,z) + \beta^2$ and so, due to be proved above,

$$\varliminf_{\delta\to 0} \varliminf_{\varepsilon\to 0} \varepsilon \log \mathbf{P}(r_T(X^{\varepsilon,\beta}, X) + \rho_T(v^\varepsilon, v) \leqq \delta) \geqq -L_T^\beta(X,v)\ ,$$

where $L_T^\beta(X,v) = \frac{1}{2}S_T^\beta(X,v) + \frac{1}{8}F_T(v)$, and where

$$S_T^\beta(X,v) = \int_0^\infty \frac{[\dot{X}_t - A_v(y,X_t)]^2}{B_v^2(t,X_t) + \beta^2}\, dt .$$

Evidently $\lim_{\beta\to 0} S_T^\beta(X,v) = S_T(X,v)$. On the other hand, by Lemma A.6 (see Appendix)

$$\lim_{\beta\to 0}\overline{\lim_{\varepsilon\to 0}}\, \varepsilon \log \mathbf{P}(r_T(X^{\varepsilon,\beta}, X^\varepsilon) > \eta) = -\infty .$$

To get the desired result, we combine both these facts. Namely, using the triangular inequality: $r_T(X^\varepsilon, X) \leqq r_T(X^{\varepsilon,\beta}, X^\varepsilon) + r_T(X^{\varepsilon,\beta}, X)$ and taking $\eta = \delta/2$ we arrive to an upper bound

$$\begin{aligned}
&\mathbf{P}(r_T(X^{\varepsilon,\beta}, X) + \rho_T(v^\varepsilon, v) \leqq \delta)\\
&\quad\leqq \mathbf{P}(r_T(X^\varepsilon, X) + \rho_T(v^\varepsilon, v) \leqq \delta/2) + \mathbf{P}(r_T(X^{\varepsilon,\beta}, X^\varepsilon) > \delta/2)\\
&\quad\leqq 2\max[\mathbf{P}(r_T(X^\varepsilon, X) + \rho_T(v^\varepsilon, v) \leqq \delta/2), \mathbf{P}(r_T(X^{\varepsilon,\beta}, X^\varepsilon) > \delta/2)] ,
\end{aligned}$$

which implies

$$\underline{\lim}_{\delta\to 0}\,\underline{\lim}_{\varepsilon\to 0}\, \varepsilon \log \mathbf{P}(r_T(X^\varepsilon, X) + \rho_T(v^\varepsilon, v) \leqq \delta) \geqq -\lim_{\beta\to 0} L_T^\beta(X,v) = -L_T(X,v) .$$

Other approach for establishing lower bound with singular diffusion parameter can be found in Puhalskii [25].

7 Proof of main result

Proof of Theorem 3.1. Due to Theorems 4.1, 5.1, and Proposition 3.1 the family $(X^\varepsilon, v^\varepsilon)$ obeys the LDP in $(\mathbb{C}_{[0,k]} \times \mathbb{M}_{[0,k]}, r_k \times \rho_k)$ with rate function $L_k(X,v)$. Then it obeys the LDP in the metric space $(\mathbb{C} \times \mathbb{M}, r \times \rho)$ with rate function $\sup_k L_k(X,v) = L(X,v)$.

Proof of Corollary 2.1. The result holds since $\inf_{X\in\mathbb{C}} S(X,v)$ is attained at X_t^0, being a solution of a differential equation: $\dot{X}_t = A_v(t, X_t^0)$ subject to $X_0^0 = x_0$, and so $S(X^0, v) = 0$.

Proof of Corollary 2.2. The first statement is obvious.

Assume $B^2(x,z) \equiv 0$. In this case $S(X,v) = 0$ for any X_t being a solution of a differential equation $\dot{X}_t = \int_R A(X_t,z)n(t,z)\,dz$ subject to $X_0 = x_0$; otherwise $S(X,v) = \infty$. Therefore

$$S(X) = \begin{cases} \frac{1}{8}\inf_{v:\dot{X}_t = \int_R A(X_t,z)n(t,z)\,dz,\ X_0 = x_0} F(v) & \\ \infty, & \text{otherwise} . \end{cases}$$

On the other hand, since $F(v)<\infty$ implies $dv = n\,d\lambda$, $d_z n = n'_z\,dz$, assuming measurability in t of function

$$H(t,\dot{X}_t,X_t) = \inf_{v:\dot{X}_t=\int_R A(X_t,z)n(t,z)\,dz,\,X_0=x_0} \int_R \sigma^2(z)\left[\frac{n'_z(t,z)}{n(t,z)} - \frac{p'(z)}{p(z)}\right]^2 n(t,z)\,dz \tag{7.1}$$

we arrive at independent of t function $H(t,y,x)\equiv H(y,x)$, or by other words, "*inf*" in (7.1) can be taken over all measures v with densities $n(t,z)\equiv m(z)$. The last means the desired result holds if the function

$$H(y,x) = \inf_{m:\begin{cases} dm=m'\,dz \\ y=\int_R A(x,z)m(z)\,dz\end{cases}} \int_R \sigma^2(z)\left[\frac{m'(z)}{m(z)} - \frac{p'(z)}{p(z)}\right]^2 m(z)\,dz \tag{7.2}$$

is measurable. We check this by showing that level sets of $H(y,x)$ are closed.

Let $c\geqq 0$ be fixed and (y_n,x_n), $n\geqq 1$ be a sequence from $\{(y,x): H(y,x)\leqq c\}$ converging to a limit point (y_0,x_0). Show that $H(y_0,x_0)\leqq c$. By virtue of assumption (A.1) the set $\mathscr{A}(y,x)=\{m: y=\int_R A(x,z)m(z)\,dz\}$ is closed in the Levy–Prohorov metric that is for every fixed (y,x) there exists a density $m^{(y,x)}$ from $\mathscr{A}(y,x)$ such that

$$H(y,x) = \begin{cases} \int_R \sigma^2(z)\left[\frac{(m^{(y,x)}(z))'}{m^{(y,x)}(z)} - \frac{p'(z)}{p(z)}\right]^2 m^{(y,x)}(z)\,dz, & dm^{(y,x)} = (m^{(y,x)})'\,dz\,, \\ \infty, & \text{otherwise}\,. \end{cases} \tag{7.3}$$

Note that the function $H(y,x)$, defined in (7.3), obeys a following property: there exists a measure $v^{(y,x)}$ from $\mathbb{M}_{[0,1]}$, having density $m^{(y,x)}(z)$ w.r.t. $dt\,dz$, such that $H(y,x)=F_1(v^{(y,x)})$. Since $\frac{1}{8}F_1(v)$ is good rate function level sets $\{y,x\colon H(y,x)\leqq c\}$ are compacts. Therefore $H(y_0,x_0)\leqq c$.

Appendix

1. Evaluation via Levy–Prohorov's metric

Lemma A.1 *Let $T>0$, $v',v''\in\mathbb{M}_{[0,T]}$, $\rho_T(v',v'')=q$, and $f(t,z)$ be bounded continuous function. Then for any $\gamma>0$ and $k\geqq 1$ one can choose increasing continuous function $h^\gamma_k(y)$, $y\geqq 0$ with $h^\gamma_k(0)=0$ and decreasing sequence φ_k, $k\geqq 1$ with $\lim_k\varphi_k=0$ both depending on $f(t,z)$ and only from one of v' or v'' such that*

$$\left|\int_0^T\int_R f(t,z)[v'-v''](dt,\,dz)\right| \leqq \gamma + h^\gamma_k(q) + \varphi_k\,.$$

Remark. If $f(t,z)$ is bounded compactly supported continuous function, then the statement of the lemma remains true with $h^\gamma_k(y)\equiv h^\gamma(y)$ and $\varphi_k\equiv 0$.

Proof. Assume $f(t,z)$ is continuously differentiable (one in z and twice in (t,z)) and compactly supported in z. Denote by $F'(t,z) = \nu'([0,t] \times (-\infty, z])$ that is $F'(t,z)$ is the distribution function correspoding to ν'. Integrating by parts we get

$$\int_0^T \int_R f(t,z)\nu'(dt,\, dz) = -\int_0^T \int_R \left[\frac{\partial f(t,z)}{\partial z} + \frac{\partial^2 f(t,z)}{\partial t \partial z}\right] F'(t,z)\, dz\, dt$$

and consequently (F'' is the distribution functions corresponding to ν'')

$$\left|\int_0^T \int_R f(t,z)[\nu' - \nu''](dt,\, dz)\right| \leqq \int_0^T \int_R |F'(t,z) - F''(t,z)| m(t,z)\, dz\, dt\ ,$$

where $m(t,z) = |(\partial/\partial z) f(t,z)| + |(\partial^2/\partial t \partial z) f(t,z)|$.

Assume $f(t,z)$ is compactly supported in z and continuous only. Then, approximating it by compactly supported and continuously differentiable in z function $f^\gamma(t,z)$ in a sense $\sup_{t,z} |f(t,z) - f^\gamma(t,z)| \leqq \gamma/2T$, due to the foregoing proof, we get

$$\left|\int_0^T \int_R f(t,z)[\nu' - \nu''](dt,\, dz)\right| \leqq \gamma + \int_0^T \int_R |F'(t,z) - F''(t,z)| m^\gamma(t,z)\, dz\, dt$$

with $m^\gamma(t,z) = |(\partial/\partial z) f^\gamma(t,z)| + |(\partial^2/\partial t \partial z) f^\gamma(t,z)|$.

In the general case, one can choose a decomposition $f(t,z) = f_k(t,z) + g_k(t,z)$, where $f_k(t,z)$ is continuous compactly supported in z on the interval $[-k,k]$ function while $g_k(t,z) \equiv 0$ on the interval $[-(k-1/2),(k-1/2)]$ and is bounded: $|g_k(t,z)| \leqq L$. Then by foregoing result we get

$$\left|\int_0^T \int_R f(t,z)[\nu' - \nu''](dt,\, dz)\right| \leqq \gamma + \int_0^T \int_R |F'(t,z) - F''(t,z)| m_k^\gamma(t,z)\, dz\, dt$$
$$+ L \int_0^T \int_{|z| > k-1/2} [\nu' + \nu''](dt,\, dz)\ ,$$

where $m_k^\gamma(t,z) = |(\partial/\partial z) f_k^\gamma(t,z)| + |(\partial^2/\partial t \partial z) f_k^\gamma(t,z)|$. Evaluate from above the last integral from the right hand side. To this end, choose an increasing sequences $z_k \nearrow \infty$, $k \to \infty$ such that $z_k \leqq k - 1/2$ and for every k z_k and $-z_k$ are points of continuity for the distribution function $F'(T,z)$. Then

$$\int_0^T \int_{|z|>k-1/2} [\nu' + \nu''](dt,\, dz) \leqq 2\int_0^T \int_{|z|>z_k} \nu'(dt,\, dz) + \left|\int_0^T \int_{|z|>z_k} [\nu' - \nu''](dt,\, dz)\right|$$
$$\leqq 2\int_0^T \int_{|z|>z_k} \nu'(dt,\, dz) + |F'(T,z_k) - F''(T,z_k)|$$
$$+ |F'(T,-z_k) - F''(T,-z_k)|\ .$$

Now evaluate from above $|F'(t,z) - F''(t,z)|$ via q and $F'(t,z)$. From the definition of the Levy–Prohorov metric (see e.g. [22, 26]) it follows: $q + F'(t-q, z-q) - F'(t,z) \leqq F'(t,z) - F''(t,z) \leqq q + F'(t+q, z+q) - F'(t,z)$ and so

$$|F'(t,z) - F''(t,z)| \leqq q + [F'(t+q, z+q) - F'(t-q, z-q)] \,.$$

Hence, combining all obtained upper estimates, we arrive at the desired result with

$$\begin{aligned} h_k^\gamma(y) = y\left(2L + \int_0^T \int_R m_k^\gamma(t,z)\,dt\,dz\right) \\ + \int_0^T \int_R [F'(t+y, z+y) - F'(t-y, z-y)] m_k^\gamma(t,z)\,dz\,dt \\ + L|F'(T+y, z_k+y) - F'(T-y, z_k-y)| \\ + L|F'(T+y, -z_k+y) - F'(T-y, -z_k-y)| \end{aligned}$$

and

$$\varphi_k = 2L \int_0^T \int_{|z|>z_k} \nu'(dt,\, dz) \,.$$

The same proof takes place with F'' instead of F'.

2. The Fenchel–Legendre transform

Let $\lambda(t) = \sum_i \lambda(t_i) I(t_i \leqq t < t_{i+1})$ with non overlapping intervals $[t_i, t_{i+1})$. For any $X \in \mathbb{C}_{[0,T]}$ and $\nu \in \mathbb{M}_{[0,T]}$ put $\int_0^T \lambda(t)\,dX_t = \sum_i \lambda(t_i)[X_{T\wedge t_{i+1}} - X_{T\wedge t_i}]$, $A_\nu(t, X_t) = \int_R A(X_t, z) K_\nu(t,\, dz)$, and $B_\nu^2(t, X_t) = \int_R B^2(X_t, z) K_\nu(t,\, dz)$. Let $\mathscr{D}$ be non linear operator defined in (4.16).

Lemma A.2 *For any $X \in \mathbb{C}_{[0,T]}$ and $\nu \in \mathbb{M}_{[0,T]}$*

$$\sup \int_0^T \left[\lambda(t)\,dX_t - (A_\nu(t,X_t) - \tfrac{1}{2}\lambda^2(t) B_\nu^2(t,X_t))\,dt\right.$$

$$= \begin{cases} \frac{1}{2}\int_0^T \dfrac{[\dot{X}_t - A_\nu(t,X_t)]^2}{B_\nu^2(t,X_t)}\,dt, & dX_t = \dot{X}_t\,dt\,, \\ \infty, & \textit{otherwise}\,, \end{cases}$$

where "sup" is taken over all piece wise constant functions $\lambda(t)$.

Lemma A.3 *For any $\nu \in \mathbb{M}_{[0,T]}$*

$$\inf \int_0^T \int_R \mathscr{D}u(t,z)\nu(dt,\, dz)$$

$$= \begin{cases} -\frac{1}{8}\int_0^T \int_R \sigma^2(z) \left[\frac{n_z'(t,z)}{n(t,z)} - \frac{p'(z)}{p(z)}\right]^2 n(t,z)\,dz\,dt, & d\nu = n\,d\lambda,\ d_z n = n_z'\,dz\,, \\ -\infty, & \textit{otherwise}\,, \end{cases}$$

where "inf" is taken over all continuously differentiable (once in t and twice in z) compactly supported in z functions u(t, z).

Proof of Lemma A.2. For $dX_t \ll dt$ the result follows from Lemma 6.1 in [17] (see also Lemma 2.1 in [27]). For $dX_t = \dot{X}_t\,dt$ by Lemma 6.1 [17] "$sup \int_0^T$" is equal $\int_0^T \sup_{\lambda \in R}\{\lambda(\dot{X}_t - A_v(t, X_t)) - \frac{1}{2}\lambda^2 B_v^2(t, X_t))\}\,dt = \frac{1}{2}\int_0^T([\dot{X}_t - A_v(t, X_t)]/B_v^2(t, X_t))\,dt$.

Proof of Lemma A.3. Assume $dv = n\,d\lambda$, $d_z n = n'_z\,dz$. Due to (2.1), $p'(z)/p(z) = (2b(z) - 2\sigma(z)\sigma'(z))/\sigma^2(z)$ and so $b(z) = \frac{1}{2}[\sigma^2(z)(p'(z)/p(z)) + 2\sigma(z)\sigma'(z)]$. Putting $v(t, z) = u'_z(t, z)$ and taking into account the formula for $b(z)$ we arrive at

$$\int_0^T\int_R \mathscr{D}u(t,z)n(t,z)\,dz\,dt = \frac{1}{2}\int_0^T\int_R \left\{\left[\sigma^2(z)\frac{p'(z)}{p(z)} + 2\sigma(z)\sigma'(z)\right]v(t,z) + \sigma^2(z)(v'_z(t,z) + v^2(t,z))\right\} n(t,z)\,dz\,dt\,. \tag{8.1}$$

Then, integrating by parts,

$$\int_R \sigma^2(z)(v'_z(t,z)n(t,z)\,dz = -\int_R v(t,z)[2\sigma(z)\sigma'(z)n(t,z) + \sigma^2(z)n'_z(t,z)]\,dz\,,$$

we obtain

$$\int_0^T\int_R \mathscr{D}u(t,z)n(t,z)\,dz\,dt = \frac{1}{2}\int_0^T\int_R \sigma^2(z)\left(v^2(t,z)n(t,z) + v(t,z)\left[\frac{p'(z)}{p(z)}n(t,z) - n_z(t,z)\right]\right)dz\,dt\,. \tag{8.2}$$

(8.2) and the method of proving for lemma 6.1 in [17] imply

$$\begin{aligned}\inf\int_0^T\int_R \mathscr{D}u(t,z)n(t,z)\,dz\,dt &= \frac{1}{2}\int_0^T\int_R \sigma^2(z)\inf_{v\in R}\left(v^2 n(t,z) + v\left[\frac{p'(z)}{p(z)}n(t,z) - n_z(t,z)\right]\right)dz\,dt \\ &= -\frac{1}{8}\int_0^T\int_R \sigma^2(z)\left[\frac{n_z(t,z)}{n(t,z)} - \frac{p'(z)}{p(z)}\right]^2 n(t,z)\,dz\,dt\,.\end{aligned}$$

Thus for "$dv = n\,d\lambda, d_z n = n'_z\,dz$", the result holds.

Assume $dv = n\lambda, d_z n \ll dz$. Show that $\inf\int_0^T\int_R \mathscr{D}u(t,z)n(t,z)\,dz\,dt = -\infty$. To this end, take $u(t,z) \equiv u(z)$ and put $v(z) = u'(z)$. The function $v(z)$ is compactly supported and continuously differentiable and, in particular, has the finite total variation. Put $n(z) = \int_0^T n(t,z)\,dt$ and $w(z) = \frac{1}{2}\sigma^2(z)n(z)$. It is clear that there exists a positive constant, say, ℓ such that $I(v) = \ell \int_R [v^2(z) + |v(z)|]n(z)\,dz + \int_R w(z)\,dv(z)$ is an upper bound for the right hand side of (8.1). Show that $I(v)$ can be chosen less than any negative quantity. Use the fact that $I(v)$ is well defined not only for compactly supported and continuously differentiable function $v(z)$ but also for any compactly supported function $v^\alpha(z)$ obeying finite total variation. Assume that there exists a family of $v^\alpha(z)$, $\alpha \in (0, 1]$ such that

$$\lim_{\alpha\to 0} I(v^\alpha) = -\infty \tag{8.3}$$

and every function $v^\alpha(z)$ obeys an approximation by $v^\alpha_m(z)$, $m \geqq 1$ of continuously differentiable compactly supported functions in a sense

$$\lim_m I(v^\alpha_m) = I(v^\alpha)\ . \tag{8.4}$$

We show that under (8.3) and (8.4) the desired result holds. In fact, for fixed α one can choose a number m_α such that $|I(v^\alpha) - I(v^\alpha_{m_\alpha})| \leqq 1$. Hence we obtain

$$\inf \int_0^T \int_R \mathscr{D}u(t,z)n(t,z)\,dz\,dt \leqq I(v^\alpha_{m_\alpha}) \leqq 1 + I(v^\alpha) \to -\infty, \quad \alpha \to 0\ .$$

Therefore, only (8.3) and (8.4) have to be checked. Since $d_z n \not\ll dz$ the function $n(z)$ is not absolutely continuous and $w(z)$ is inherited the same property. Therefore by the definition of the negation for absolute continuity [28] a constant k can be chosen such that for any $\alpha > 0$ there exists a positive constant c and non overlapping intervals $(z'_i, z''_i) \in [-c, c]$, such that $\sum_i |w(z''_i) - w(z'_i)| \geqq k$ and $\sum_i \int_{z'_i}^{z''_i} n(z)\,dz \leqq \alpha$. Put

$$v^\alpha(z) = \begin{cases} -\dfrac{1}{\sqrt{\alpha}}\,\mathrm{sign}\,[w(z''_i) - w(z'_i)], & z'_i < z \leqq z''_i\ , \\ 0, & \text{otherwise}\ . \end{cases}$$

Show that (8.3) holds. Evaluate from above $I(v^\alpha)$:

$$\begin{aligned} I(v^\alpha) &= \ell \int_R [(v^\alpha(z))^2 + |v^\alpha(z)|]n(z)\,dz + \int_R w(z)\,dv^\alpha(z) \\ &\leqq \ell\left(\frac{1}{\alpha} + \frac{1}{\sqrt{\alpha}}\right)\sum_i \int_{z'_i}^{z''_i} n(z)\,dz + \sum_i w(z'_i)[v^\alpha(z''_i) - v^\alpha(z'_i)] \\ &\leqq \ell(1 + \sqrt{\alpha}) + \sum_i w(z'_i)[v^\alpha(z''_i) - v^\alpha(z'_i)]\ . \end{aligned}$$

Now, summizing by parts, we find $\sum_i w(z'_i)[v^\alpha(z''_i) - v^\alpha(z'_i)] = -\sum_i v^\alpha(z''_i)[w(z''_i) - w(z'_i)]$. On the other hand, from the definition of $v^\alpha(z)$ it follows $\sum_i v^\alpha(z''_i)[w(z''_i) - w(z'_i)] = (1/\sqrt{\alpha})\sum_i |w(z''_i) - w(z'_i)| \geqq k/\sqrt{\alpha}$. Thereby

$$I(v^\alpha) \leqq \ell(1 + \sqrt{\alpha}) - \frac{k}{\sqrt{\alpha}} \to \infty, \quad \alpha \to 0\ .$$

Evidently to satisfy (8.4), it is sufficient to choose approximating functions $v_m^\alpha(z)$, $m \geqq 1$ which are compactly supported and continuously differentiable and such that $\lim_m v_m^\alpha(z) = v^\alpha(z)$ in every point of continuity of $v^\alpha(z)$.

Assume $v \not\ll \lambda$. Put $K^v(dz) = \int_0^T K_v(t, dz)\, dt$ and note that $K^v(dz) \not\ll dz$. Use Lebesgue's decomposition: $K^v(dz) = q(z)\, dz + K^\perp(dz)$, where $q(z)$ is a density of absolutely continuous part of $K^v(dz)$ and $K^\perp(dz)$ is its singular part. Taking $u(t,z) \equiv u(z)$ which is compactly supported, say, on $[-c, c]$ we find

$$\int_0^T \mathscr{D}u(z) v(dt,\, dz) = \int_{-c}^{c} \mathscr{D}u(z) q(z)\, dz + \int_{-c}^{c} \mathscr{D}u(z) K^\perp(dz)\, .$$

Since $|u'(z)| \leqq |u'(0)| + \int_{-c}^{c} |u''(y)|\, dy$ there exists constant, say, ℓ, such that $\int_{-c}^{c} \mathscr{D}u(z) q(z)\, dz \leqq \ell(1 + \int_{-c}^{c} |u''(y)|\, dy)$ and so we arrive to an upper estimate

$$\int_0^T \mathscr{D}u(z) v(dt,\, dz) \leqq \ell \left(1 + \int_{-c}^{c} |u''(z)|\, dz\right) + \frac{1}{2} \int_{-c}^{c} \sigma^2(z) u''(z) K^\perp(dz)\, .$$

Then, using the singularity of $K^\perp(dz)$ and dz, one can choose $u''(z)$ such that the second integral is less than any negative quantity while the first remains bounded.

3. Approximation of rate function

For "$dX = \dot{X}_t\, dt,\ dv = n\, d\lambda,\ d_z n = n_z'\, dz$" denote by

$$S_T(X, v) = \int_0^T \frac{[\dot{X}_t - A_v(t, X_t)]^2}{B_v^2(t, X_t)}\, dt\ ,$$

$$F_T(v) = \int_0^T \int_R \sigma^2(z) \left[\frac{n_z'(t,z)}{n(t,z)} - \frac{p'(z)}{p(z)}\right]^2 n(t,z)\, dz\, dt\ .$$

Also note one to one correspondence between density $n(t,z)$ and function $v_v(t,z)$ defined in (6.1):

$$n(t,z) = n(t,0) \frac{p(z)}{p(0)} \exp\left(2 \int_0^z \frac{v_v(t,y)}{\sigma^2(y)}\, dy\right). \tag{8.5}$$

Put

$$\phi(t) = \int_R |n_z'(t,y)|\, dy\ . \tag{8.6}$$

Lemma A.4 *Let $B^2(x,z) \geqq \beta^2 > 0$. If $S_T(X, v) < \infty$, $F_T(v) < \infty$, then v can be approximated by a sequence of measures $v^{(k)}$, $k \geqq 1$, satisfying the property: $dv^{(k)} = n^{(k)} d\lambda$, $d_z n^{(k)} = n_z^{(k)}\, dz$, such that the function $v_{v^{(k)}}(t,z)$, corresponding to $n^{(k)}(t,z)$, is compactly supported in z and continuously differentiable in (t,z)*

and what is more

$$\lim_k \rho_T(\nu, \nu^{(k)}) = 0 ,$$

$$\lim_k S_T(X, \nu^{(k)}) = S_T(X, \nu) , \tag{8.7}$$

$$\lim_k F_T(\nu^{(k)}) = F_T(\nu) .$$

Proof. Introduce a chain of expanding subclasses of measures ν characterized in terms of $n(t,z)$ and $v_\nu(t,z)$:

0) $v_\nu(t,z)$ is compactly supported in z and continuously differentiable in (t,z);

1) $v_\nu(t,z)$ is compactly supported in z and bounded;

2) $v_\nu(t,z)$ is compactly supported, $\inf_{t \leqq T, z\in R} (n(t,z)/p(z)) > 0$ and $\sup_{t\leqq T} [n(t,0) + \phi(t)] < \infty$;

3) $v_\nu(t,z)$ is compactly supported, $\inf_{t \leqq T, z\in R} (n(t,z)/p(z)) > 0$;

4) $v_\nu(t,z)$ is compactly supported;

5) $v_\nu(t,z)$ satisfies the assumptions of the lemma.

The proof is based on the following fact. If measure ν from class "i" $(i = 1, \dots, 5)$ can be approximated by $\nu^{(k)}$, $k \geqq 1$ from class "$i-1$" in a sense (8.7), then the statement of the lemma holds.

Assume $\nu^{(k)}$, $k \geqq 1$ is such that

$$\Lambda_T - \lim_k n^{(k)}(t,z) = n(t,z) \quad (\Lambda_T(dt,\, dz) = I_{[0,T]}\, dt\, dz) ,$$

$$\lim_k F_T(\nu^{(k)}) = F_T(\nu) . \tag{8.8}$$

Then by Scheffe's theorem [29, 22] we have $\lim_k \int_0^T \int_R |n(t,z) - n^{(k)}(t,z)|\, dt\, dz = 0$ that is $\nu^{(k)}$ converges to ν in the total variation norm which implies convergence in Levy–Prohorov's metric too: $\rho_T(\nu, \nu^{(k)}) \to 0$. Since $A_{\nu^{(k)}}(t,X_t) = \int_R A(X_t,z) n^{(k)}(t,z)\, dz$ and $B^2_{\nu^{(k)}}(t,X_t) = \int_R B^2(X_t,z) n^{(k)}(t,z)\, dz$ by Lebesgue dominated theorem $S_T(X, \nu^{(k)}) \to S_T(X, \nu)$. Therefore, for all steps of approximations only (8.8) has to be checked.

Assume ν is from class "1". Approximate $v_\nu(t,z)$ by $v_\nu^{(k)}(t,z)$:

$$\lim_k \int_0^T \int_R [v_\nu(t,z) - v_\nu^{(k)}(t,z)]^2 (1 + n(t,z))\, dt\, dz = 0 ,$$

where for all k the functions $v_\nu^{(k)}(t,z)$, $k \geqq 1$ are compactly supported continuously differentiable in (t,z). Without loss of a generality one can assume that all function are bounded by the same constant. Similarly to (8.5) define a density of $\nu^{(k)}$:

$$n^{(k)}(t,z) = n^{(k)}(t,0) \frac{p(z)}{p(0)} \exp\left(\int_0^z \frac{v_\nu^{(k)}(t,y)}{\sigma^2(z)}\, dy \right) , \tag{8.9}$$

with

$$n^{(k)}(t,0) = \left(\int_R \frac{p(z)}{p(0)} \exp\left(\int_0^z \frac{v_v^{(k)}(t,y)}{\sigma^2(z)}\, dy \right) dz \right)^{-1} .$$

Put

$$v^{(k)}(dt,\, dz) = n^{(k)}(t,z)\, dt\, dz\, .$$

Evidently $v^{(k)}$ belongs to class “0”. It is easy to check that

$$v_v^{(k)}(t,z) \equiv v_{v^{(k)}}(t,z) \tag{8.10}$$

and the validity of the first part in (8.8). To verify the second part in (8.8), note that

$$F_T(v^{(k)}) = 4\int_0^T \int_R \frac{(v_{v^{(k)}}(t,z))^2}{\sigma^2(z)} n^{(k)}(t,z)\, dz\, dt$$

and consequently

$$\begin{aligned} |F_T(v) - F_T(v^{(k)})| &\leqq \text{const.} \int_0^T \int_R |n(t,z) - n^{(k)}(t,z)| dt dz \\ &\quad + \text{const.} \int_0^T \int_R |v_v^2(t,z) - (v_{v^{(k)}}(t,z))^2| dt\, dz \\ &\quad \to 0, \quad k \to \infty. \end{aligned}$$

Assume v is from class “2”. For the definiteness assume that there exists positive constant z_0 such that $v_v(t,z) \equiv 0$ out of $[-z_0, z_0]$. Put $v_v^{(k)}(t,z) = v_v(t,z) I(|n_z'(t,z)| \leqq k)$, define $n^{(k)}(t,z)$ by (8.9) and take $v^{(k)}$ with this density. It belongs to class “1”. Herewith, $v_v^{(k)}(t,z)$ is defined by (8.10). It is clear that the first part in (8.8) holds and below we check the validity of the second part. We have

$$F_T(v^{(k)}) = 4\int_0^T \int_{|z|\leqq z_0} \frac{v_v^2(t,z)}{\sigma^2(z)} I(|n_z(t,z)| \leqq k) n^{(k)}(t,z)\, dz\, dt\, ,$$

$$F_T(v) = 4\int_0^T \int_{|z|\leqq z_0} \frac{v_v^2(t,z)}{\sigma^2(z)} n(t,z)\, dz\, dt\, .$$

The required convergence $F_T(v^{(k)}) \to F_T(v)$ holds by Lebesgue dominated theorem since $n^{(k)}(t,z) \leqq p(z) \exp(2\phi(t)) \leqq \text{const.} n(t,z)$.

Assume v is from class “3”. Putting $v_v^{(k)}(t,z) = v_v(t,z) I(n(t,0) + \phi(t) \leqq k)$ we arrive at

$$n^{(k)}(t,z) = \begin{cases} n(t,z), & n(t,0) + \phi(t) \leqq k\, , \\ p(z), & n(t,0) + \phi(t) > k\, , \end{cases}$$

and since $n^{(k)}(t,0) \leqq k + p(0)$ and $\phi^{(k)}(t) \leqq k + \int_R |p'(z)|\, dz$ measure $v^{(k)}$ with density $n^{(k)}(t,z)$ belongs to class “2”. It is clear that the first part in (8.8) holds

and

$$|F_T(v)-F_T(v^{(k)})| = 4\int\limits_0^T \int\limits_{|z|\leqq z_0} \frac{v_v^2(t,z)}{\sigma^2(z)} I(n(t,0)+\phi(t)>k)\,p(z)\,dz\,dt$$

$$\leqq \text{const.}\int\limits_0^T \int\limits_{|z|\leqq z_0} \frac{v_v^2(t,z)}{\sigma^2(z)} I(n(t,0)+\phi(t)>k)n(t,z)\,dz\,dt$$

$$\to 0, \quad k\to\infty.$$

Assume v is from class "4". Put $n^{(k)}(t,z)=c^{(k)}(t)(n(t,z)\vee p(z))$, where $c^{(k)}(t)=(\int_R(n(t,z)\vee p(z))\,dz)^{-1}$ is norming constant. $v^{(k)}$ with this density belongs to class "3". The first part in (8.8) holds and what is more $\lim_k c^{(k)}(t)=1$. On the other hand, since $v_{v^{(k)}}(t,z)=v_v(t,z)I(n(t,z)\geqq p(z)/k)$ we obtain

$$F_T(v^{(k)}) = 4\int\limits_0^T \int\limits_R \frac{v_v^2(t,z)}{\sigma^2(z)} I(n(t,z)\geqq p(z)/k)c^{(k)}(t)n(t,z)\,dz\,dt$$

$$\to 4\int\limits_0^T \int\limits_R \frac{v_v^2(t,z)}{\sigma^2(z)} n(t,z)dzdt = F_T(v)\ .$$

Assume v is from class "5". Put $v_v^{(k)}(t,z)=v_v(t,z)T(|z|\leqq k)$ and define $n^{(k)}(t,z)$ by (8.9). Then

$$n^{(k)}(t,z) = n^{(k)}(t,0)\begin{cases} p(z)\dfrac{n(t,k)}{p(k)}, & z>k\ ,\\ n(t,z), & |z|\leqq k\ ,\\ p(z)\dfrac{n(t,-k)}{p(-k)}, & z<-k\ . \end{cases}$$

Taking $v^{(k)}$ with this density and noticing that $\lim_k n^{(k)}(t,0)=1$ we find

$$F_T(v^{(k)}) = 4\int\limits_0^T \int\limits_{|z|\leqq k} \frac{(v_v(t,z))^2}{\sigma^2(z)} c^{(k)}(t)n(t,z)\,dz\,dt \to F_T(v)\ ,$$

i.e. both parts in (8.8) hold.

4. Ergodic property

Consider diffusion pair $(\widetilde{X}_t^\varepsilon,\widetilde{\xi}_t^\varepsilon)$ defined by Itô's differential equations w.r.t. independent Wiener processes W_t and V_t:

$$\begin{aligned} d\widetilde{X}_t^\varepsilon &= G(t,\widetilde{X}_t^\varepsilon,\widetilde{\xi}_t^\varepsilon)\,dt + \sqrt{\varepsilon}B(\widetilde{X}_t^\varepsilon,\widetilde{\xi}_t^\varepsilon)\,dW_t\ ,\\ d\widetilde{\xi}_t^\varepsilon &= \frac{1}{\varepsilon}b(t,\widetilde{\xi}_t^\varepsilon)\,dt + \frac{1}{\sqrt{\varepsilon}}\sigma(\widetilde{\xi}_t^\varepsilon)\,dV_t \end{aligned} \tag{8.11}$$

subject to (x_0,z_0), where $B(x,z)$ and $\sigma(z)$ are functions involving in (1.1). Assume $b(t,z)$ is continuous it (t,z), continuously differentiable in t, Lipschitz continuous in z uniformly in t, and $zb(t,z)$ is negative for large $|z|$ uniformly in t. Also assume that

$$G(t,x,z) = \frac{\dot{X}_t - A_p(t,x)}{B_p(t,x)} B(x,z) + A(x,z) , \tag{8.12}$$

where $A(x,z)$ involves in (1.1), $\dot{X}_t$ is the Radon–Nykodim derivative of absolute continuous function X_t from $\mathbb{C}$ with $X_0 = x_0$, $A_p(t,x) = \int_R A(x,z)\,p(t,z)\,dz$, $B_p(t,x) = \sqrt{\int_R B^2(x,z)\,p(t,z)\,dz}$, and where (comp. (2.1))

$$p(t,z) = c(t) \frac{\exp(2\int_0^z \frac{b(t,y)}{\sigma^2(y)}\,dy)}{\sigma^2(z)}$$

with norming function $c(t)$ such that $\int_R p(t,z)\,dz = 1$. Introduce an occupation measure $\widetilde{v}^\varepsilon(dt,dz)$: $\widetilde{v}^\varepsilon(\Delta \times \Gamma) = \int_0^\infty I(t \in \Delta, \widetilde{\xi}_t^\varepsilon \in \Gamma)\,dt$ and put $v(dt,dz) = p(t,z)\,dz\,dt$.

Lemma A.5

$$\mathbf{P} - \lim_{\varepsilon \to 0} \rho_T(\widetilde{v}^\varepsilon, v) = 0 \quad \textit{and} \quad \mathbf{P} - \lim_{\varepsilon \to 0} r_T(\widetilde{X}^\varepsilon, X) = 0 \,.$$

Proof. It is clear, the first statement of the lemma is equivalent to: for any bounded and continuous function $h(t,z)$ $\int_0^T \int_R h(t,\widetilde{\xi}_t^\varepsilon)\,dt \to \int_0^T \int_R h(t,z) p(t,z)\,dz\,dt$ in probability or, for $h^0(t,z) = h(t,z) - \int_0^T \int_R h(t,y)\,p(t,y)\,dy\,dt$,

$$\mathbf{P} - \lim_{\varepsilon \to 0} \int_0^T \int_R h^0(t,\widetilde{\xi}_t^\varepsilon)\,dt = 0 \,.$$

First we check it for continuously differentiable in t,z function $h(t,z)$, having bounded partial derivatives. Straightforward calculation brings Kolmogorov's type differential equation (t is fixed):

$$\frac{1}{2} \frac{\partial}{\partial z} (\sigma^2(x)\,p(t,x)) = b(t,z)\,p(t,z) \,.$$

A conjugate differential equation

$$\frac{1}{2} \sigma^2(z) \frac{\partial w(t,z)}{\partial z} + b(t,z) = h^0(t,z) \tag{8.13}$$

obeys a solution

$$w(t,z) = \frac{2}{\sigma^2(z)\,p(t,z)} \int_{-\infty}^{z} h^0(t,y)\,p(t,y)\,dy \,.$$

It is clear that properties of $h(t,z)$ are inherited by $w(t,z)$ and so function $u(t,z) = \int_0^z w(t,y)\,dy$ is continuously differentiable once in t and twice in z and what is more, due to the boundness of $w(t,z)$, there exists a positive constant, say ℓ, such that $|u(t,z)| \leqq \ell|z|$ and $|u_t(t,z)| \leqq \ell|z|$. Applying Itô's formula to $u(t,\widetilde{\xi}^\varepsilon_t)$ and taking into account that $w(t,z)$ is solution of differential equation (8.13) we find $u(T,\widetilde{\xi}^\varepsilon_T) = u(0,\xi_0) + \int_0^T u'_t(t,\widetilde{\xi}^\varepsilon_t)\,dt$ $+(1/\sqrt{\varepsilon})\int_0^T w(t,\widetilde{\xi}^\varepsilon_t)\sigma(\widetilde{\xi}^\varepsilon_t)\,dV_t + (1/\varepsilon)\int_0^T h^0(t,\widetilde{\xi}^\varepsilon_t)\,dt$ that is

$$\int_0^T h^0(t,\widetilde{\xi}^\varepsilon_t)\,dt = \varepsilon u(T,\widetilde{\xi}^\varepsilon_T) - \varepsilon u(0,\xi_0) - \varepsilon\int_0^T u'_t(t,\widetilde{\xi}^\varepsilon_t)\,dt - \sqrt{\varepsilon}\int_0^T w(t,\widetilde{\xi}^\varepsilon_t)\sigma(\widetilde{\xi}^\varepsilon_t)\,dV_t\,. \tag{8.14}$$

The second term in the right hand of (8.14) converges to zero; the last term converges to zero in probability since by Problem 1.9.2 in [23] the mentioned convergence follows from $\varepsilon\int_0^T w^2(t,\widetilde{\xi}^\varepsilon_t)\sigma^2(\widetilde{\xi}^\varepsilon_t)\,dt \to 0$; other two terms converge to zero in probability if $\lim_{\varepsilon\to 0}\varepsilon^2\mathbf{E}\sup_{t\leqq T}(\widetilde{\xi}^\varepsilon_t)^2 = 0$. To check the last, apply Itô's formula to $(\varepsilon\widetilde{\xi}^\varepsilon_t)^2$:

$$(\varepsilon\widetilde{\xi}^\varepsilon_t)^2 = (\varepsilon\xi_0)^2 + 2\varepsilon\int_0^t b(s,\widetilde{\xi}^\varepsilon_s)\widetilde{\xi}^\varepsilon_s\,ds + \varepsilon\int_0^t \sigma^2(\widetilde{\xi}^\varepsilon_s)\,ds + 2\varepsilon^{3/2}\int_0^t \widetilde{\xi}^\varepsilon_s\sigma(\widetilde{\xi}^\varepsilon_s)\,dV_s\,.$$

The function $b(s,z)$ is such that $zb(t,z)$ is negative for large $|z|$ what implies $(\varepsilon\widetilde{\xi}^\varepsilon_t)^2 \leqq (\varepsilon\xi_0)^2 + T\varepsilon\,\text{const.} + 2\varepsilon^{3/2}\int_0^t \widetilde{\xi}^\varepsilon_s\sigma(\widetilde{\xi}^\varepsilon_s)\,dV_s$. Thereby $\mathbf{E}(\varepsilon\widetilde{\xi}^\varepsilon_t)^2 \leqq (\varepsilon\xi_0)^2 +$ $T\varepsilon$const. In turn, using Doob's inequality (see e.g. Theorem 1.9.1 in [23]), we arrive to $\mathbf{E}\sup_{t\leqq T}(\varepsilon\widetilde{\xi}^\varepsilon_t)^2 \leqq (\varepsilon\xi_0)^2 + T\varepsilon\,\text{const.} + \text{const.}\varepsilon^3\int_0^T \mathbf{E}(\varepsilon\widetilde{\xi}^\varepsilon_t)^2\,dt$ and, due to the above obtained upper bound for $\mathbf{E}(\varepsilon\widetilde{\xi}^\varepsilon_t)^2$, the result holds.

If $h(t,z)$ is bounded and continuous only, it can be approximated by smooth functions $h_m(t,z)$, $m \geqq 1$ in the following sense: for any $k \geqq 1$ $\lim_m \sup_{t\leqq T,\,|z|\leqq k}$ $|h(t,z) - h_m(t,z)| = 0$. Since for every $h_m(t,z)$ the statement of the lemma is proved, it holds for $h(t,z)$ if

$$\lim_m \int_0^T\int_R |h(t,z) - h_m(t,z)|\,p(t,z)\,dz\,dt = 0\,,$$

$$\mathbf{P} - \lim_m \overline{\lim_{\varepsilon\to 0}} \int_0^T |h(t,\widetilde{\xi}^\varepsilon_t) - h_m(t,\widetilde{\xi}^\varepsilon_t)|n(t,z)\,dt = 0\,.$$

The first takes place since $\lim_k \int_0^T\int_{|z|>k} n(t,z)\,dz\,dt = 0$ while the second from $\mathbf{P} - \lim_m \overline{\lim}_{\varepsilon\to 0}\int_0^T I(|\widetilde{\xi}^\varepsilon_t)| > k)\,dt = 0$ and the fact that one can choose smooth bounded functions $g_k(z)$, $k \geqq 1$ such that $I(|z| > k) \leqq g_k(z)$, $\lim_k g_k(z) = 0$, $z \in R$ and by proved above $\int_0^T g_k(\widetilde{\xi}^\varepsilon_t)\,dt \to \int_0^T\int_R g_k(z)p(t,z)\,dz\,dt \to 0$, $k \to \infty$.

To check the second statement, put $\Delta_t = \widetilde{X}^\varepsilon_t t - X_t$. From the first equation in (8.11) we find

$$\Delta_t = \int_0^t \left[\frac{\dot{X}_t - A_p(s,X_s)}{B_p(s,X_s)} B(\widetilde{X}^\varepsilon_s, \widetilde{\xi}^\varepsilon_s) + A(\widetilde{X}^\varepsilon_s, \widetilde{\xi}^\varepsilon_s) - \dot{X}_s\right] ds + \sqrt{\varepsilon}\int_0^t B(\widetilde{X}^\varepsilon_s, \widetilde{\xi}^\varepsilon_s)\, dW_s$$

$$= \int_0^t \left[\frac{\dot{X}_t - A_p(s,X_s)}{B_p(s,X_s)} (B(\widetilde{X}^\varepsilon_s, \widetilde{\xi}^\varepsilon_s) - B(X_s, \widetilde{\xi}^\varepsilon_s))\right] ds$$

$$+ \int_0^t \left[\frac{\dot{X}_s - A_p(s,X_s)}{B_p(s,X_s)} (B(X_s, \widetilde{\xi}^\varepsilon_s) - B_p(s,X_s))\right] ds$$

$$+ \int_0^t (A(\widetilde{X}^\varepsilon_s, \widetilde{\xi}^\varepsilon_s) - A(X_s, \widetilde{\xi}^\varepsilon_s))\, ds + \int_0^t (A(X_s, \widetilde{\xi}^\varepsilon_s) - A_p(s,X_s))\, ds$$

$$+ \sqrt{\varepsilon}\int_0^t B(\widetilde{X}^\varepsilon_s, \widetilde{\xi}^\varepsilon_s)\, dW_s\,.$$

For brevity put $\varphi_s = (\dot{X}_s - A_p(s,X_s))/B_p(s,X_s)$. Then by the Lipschitz continuity of $A(x,z), B(x,z)$ in x uniformly in z, say, with constant ℓ, we obtain

$$|\Delta_t| \leqq \ell \int_0^t (1 + |\varphi_s|)\Delta_s\, ds + \sup_{t\leqq T}\left|\int_0^t \varphi_s(B(X_s, \widetilde{\xi}^\varepsilon_s) - B_p(s,X_s))\, ds\right|$$

$$+ \sup_{t\leqq T}\left|\int_0^t (A(X_s, \widetilde{\xi}^\varepsilon_s) - A_p(s,X_s))\, ds\right| + \sqrt{\varepsilon}\sup_{t\leqq T}\left|\int_0^t B(\widetilde{X}^\varepsilon_s, \widetilde{\xi}^\varepsilon_s)\, dW_s\right|\,.$$

Therefore, by Gronwall–Bellman's inequality

$$\sup_{t\leqq T}|\Delta_t| \leqq \exp\left(\ell\int_0^T (1 + |\varphi_s|)\, ds\right)$$

$$\times \left[\sup_{t\leqq T}\left|\int_0^t \varphi_s(B(X_s, \widetilde{\xi}^\varepsilon_s) - B_p(s,X_s))\, ds\right|\right.$$

$$\left. + \sup_{t\leqq T}\left|\int_0^t (A(X_s, \widetilde{\xi}^\varepsilon_s) - A_p(s,X_s))\, ds\right| + \sqrt{\varepsilon}\sup_{t\leqq T}\left|\int_0^t B(\widetilde{X}^\varepsilon_s, \widetilde{\xi}^\varepsilon_s)\, dW_s\right|\right]\,.$$

Hence, the second statement holds if

$$\mathbf{P} - \lim_{\varepsilon\to 0}\sup_{t\leqq T}\sqrt{\varepsilon}\sup_{t\leqq T}\left|\int_0^t B(\widetilde{X}^\varepsilon_s, \widetilde{\xi}^\varepsilon_s)\, dW_s\right| = 0 \tag{8.15}$$

and for any measurable function ψ_s such that $\int_0^T \psi_s^2 ds < \infty$ and any continuous function $C(x,z)$, being Lipschitz's continuous in x uniformly in z,

$$\mathbf{P} - \lim_{\varepsilon\to 0}\sup_{t\leqq T}\left|\int_0^t \psi_s(C(X_s, \widetilde{\xi}^\varepsilon_s) - C_p(s,X_s))\, ds\right| = 0\,, \tag{8.16}$$

where $C_p(s,X_s) = \int_R C(X_s,z)\,p(s,z)\,dz$. It can be shown (see e.g. the method of proving the statement (2) of Theorem 4.6, Chap. 4 in [24]) that $\sup_{t \leqq T} \mathbf{E}(\widetilde{X}^\varepsilon_t)^2 \leqq$ const. and so $\mathbf{E}\int_0^T B^2(\widetilde{X}^\varepsilon_s, \widetilde{\xi}^\varepsilon_s)\,ds \leqq$ const. Consequently, by Doob's inequality (see e.g. Theorem 1.9.1 [23]) we get $\mathbf{E}\sup_{t\leqq T} |\sqrt{\varepsilon}\int_0^t B(\widetilde{X}^\varepsilon_s, \widetilde{\xi}^\varepsilon_s)\,dW_s|^2 \leqq \varepsilon$const. that is (8.15) holds. To check the validity of (8.16) with $\psi_s C(x,z) \geqq 0$, note that, due to Problem 5.3.2 in [23], it is sufficient to prove

$$\mathbf{P} - \lim_{\varepsilon\to 0} \int_0^t \psi_s(C(X_s,\widetilde{\xi}^\varepsilon_s) - C_p(s,X_s))\,ds = 0, \quad \forall t \leqq T\,, \tag{8.17}$$

and what is more, due to an arbitrariness of ψ_s and $C(x,z)$, (8.17) implies (8.16) in the general case since one can use separately (8.17) for positive $(\psi_s C(x,z))^+$ and negative $(\psi_s C(x,z))^-$ parts. Therefore, (8.17) remains to be verified. If ψ_s is bounded and continuous, (8.17) takes place by virtue of the first statement of the lemma. If only $\int_0^T \psi_s^2\,ds < \infty$, approximate ψ_s by bounded and continuous functions $\psi_s^{(k)}$, $k \geqq 1$ such that $\lim_k \int_0^T (\psi_s - \psi_s^{(k)})^2\,ds = 0$ and, due to the boundness in z of $C(x,z)$ and Cauchy–Schwartz's inequality, we find that

$$\left|\int_0^t (\psi_s - \psi_s^{(k)})(C(X_s,\widetilde{\xi}^\varepsilon_s - C_p(s,X_s))\,ds\right| \leqq \text{const.}\sqrt{\int_0^T (\psi_s - \psi_s^{(k)})^2\,ds} \to 0, \quad k\to\infty$$

that is (8.17) takes place since it holds for every $\psi_s^{(k)}$.

5. LD-regularization

Parallel to X^ε_t, defined in (1.1), determine new diffusion $X^{\varepsilon,\beta}_t$ with uniformly non singular diffusion parameter $B^2(x,z) + \beta^2$, $\beta^2 > 0$, letting

$$dX^{\varepsilon,\beta}_t = A(X^{\varepsilon,\beta}_t, \xi^\varepsilon_t)\,dt + \sqrt{\varepsilon}[B(X^{\varepsilon,\beta}_t,\xi^\varepsilon_t)\,dW_t + \beta\,dW'_t] \tag{8.18}$$

subject to the same initial point x_0, where W'_t is a Wiener process independent of (W_t, ξ^ε_t).

Lemma A.6 *Under assumption* (A.1) *for every* $T > 0$ *and* $\eta > 0$

$$\lim_{\beta\to 0}\overline{\lim_{\varepsilon\to 0}}\,\varepsilon\log\mathbf{P}(r_T(X^{\varepsilon,\beta}, X^\varepsilon) > \eta) = -\infty\,.$$

Proof. Put $\Delta_t = X^{\varepsilon,\beta}_t - X^\varepsilon_t$, and

$$a_1(x',x'',z) = \frac{A(x'',z) - A(x',z)}{x'' - x'}, \qquad a_2(x',x'',z) = \frac{B(x'',z) - B(x',z)}{x''-x'}\,,$$

where for $x' = x''$ $a_i(x',x',z)$, $i = 1,2$ are Radon–Nikodym's derivatives. Note that for $x' \neq x''$ $a_i(x',x'',z)$, are bounded, say, by constant ℓ, and so $a_i(x',x',z)$ can be taken bounded by the same constant. For brevity, denote by $\alpha_i(t) = a_i(X^{\varepsilon,\beta}_t, X^\varepsilon_t, \xi^\varepsilon_t)$, $i = 1,2$. (8.18) and (1.1) imply: $\Delta_t = \int_0^t \alpha_1(s)\Delta_s\,ds + \sqrt{\varepsilon}\int_0^t$

$\alpha_2(s)\Delta_s\,dW_s + \sqrt{\varepsilon}\beta W'_t$. Letting $\mathscr{E}_t = \exp(\int_0^t[\alpha_1(s) - (\varepsilon/2)\alpha_2^2(s)]\,ds + \sqrt{\varepsilon}\int_0^t \alpha_2 (s)dW_s)$ and using Itô's formula, we find that $\Delta_t = \sqrt{\varepsilon}\beta\mathscr{E}_t\int_0^t \mathscr{E}_s^{-1}dW'_s$ and thereby

$$\sup_{t\leqq T}|\Delta_t| \leqq \sqrt{\varepsilon}\beta\sup_{t\leqq T}\mathscr{E}_t\sup_{t\leqq T}\left|\int_0^t \mathscr{E}_s^{-1}\,dW'_s\right| .$$

Put $\Gamma_N = \{1/N \leqq \inf_{t\leqq T}\mathscr{E}_t \leqq \sup_{t\leqq T}\mathscr{E}_t \leqq N\}$ and use an upper estimate

$$\begin{aligned}\mathbf{P}\left(\sup_{t\leqq T}|\Delta_t| > \eta\right) &\leqq \mathbf{P}\left(\sup_{t\leqq T}|\Delta_t| > \eta, \Gamma_N\right) + \mathbf{P}(\Omega\setminus\Gamma_N)\\ &\leqq 2\max\left[\mathbf{P}\left(\sup_{t\leqq T}|\Delta_t| > \eta, \Gamma_N\right), \mathbf{P}(\Omega\setminus\Gamma_N)\right]\end{aligned}$$

which implies, due to the boundness of $\alpha_i(s),\ i = 1,2$, the desired statement if

$$\begin{aligned}&\lim_N \overline{\lim_{\varepsilon\to 0}}\,\varepsilon\log\mathbf{P}\left(\sqrt{\varepsilon}\sup_{t\leqq T}\left|\int_0^t \alpha_2(s)\,dW_s\right| > N\right) = -\infty,\\ &\lim_{\beta\to 0}\overline{\lim_{\varepsilon\to 0}}\,\varepsilon\log\mathbf{P}\left(\sqrt{\varepsilon}\beta\sup_{t\leqq T}\left|\int_0^t \mathscr{E}_s^{-1}dW'_s\right| > \eta, \Gamma_N\right) = -\infty, \quad \forall N \geqq 1 . \quad (8.19)\end{aligned}$$

Let $\tau = \{t \leqq T : |\int_0^t \alpha_2(s)\,dW_s| > (N/\sqrt{\varepsilon})\}$ and $\sigma = \{t \leqq t : |\int_0^t \mathscr{E}_s^{-1}\,dW'_s| > (\eta/\sqrt{\varepsilon}\beta)\}$. It is clear that (8.19) is equivalent to:

$$\begin{aligned}&\lim_N \overline{\lim_{\varepsilon\to 0}}\,\varepsilon\log\mathbf{P}\left(\sqrt{\varepsilon}\int_0^\tau \alpha_2(s)\,dW_s \geqq N \ (\text{or} \leqq -N)\right) = -\infty ,\\ &\lim_{\beta\to 0}\overline{\lim_{\varepsilon\to 0}}\,\varepsilon\log\mathbf{P}\left(\sqrt{\varepsilon}\beta\int_0^\sigma \mathscr{E}_s^{-1}dW'_s \geqq \eta \ (\text{or} \leqq -\eta), \Gamma_N\right) = -\infty, \quad \forall N \geqq 1 .\end{aligned} \quad (8.20)$$

Below we check (8.20). To this end with $\lambda \in R$, introduce continuous local martingales: $Z_t^1 = \exp(\lambda\int_0^t \alpha_2(s)\,dW_s - (\lambda^2/2)\int_0^t \alpha_2^2(s)\,ds)$ and $Z_t^2 = \exp(\lambda\int_0^t \mathscr{E}_s^{-1}dW'_s - (\lambda^2/2)\int_0^t \mathscr{E}_s^{-2}\,ds)$, where each of them is a supermartingale too (see Problem 1.4.4. in [23]) that is $\mathbf{E}Z_\tau^1 \leqq 1$ and $\mathbf{E}Z_\sigma^2 \leqq 1$. Then we use obvious inequalities: $\mathbf{E}I(\sqrt{\varepsilon}\int_0^\tau \alpha_2(s)\,dW_s \geqq N)Z_\tau^1 \leqq 1$ and $\mathbf{E}I(\sqrt{\varepsilon}\beta\int_0^\sigma \mathscr{E}_s dW'_s \geqq \eta,\ \Gamma_N)Z_\sigma^2 \leqq 1$. Since for $\lambda > 0$, $\log Z_\tau^1 \geqq (\lambda N/\sqrt{\varepsilon}) - (\lambda^2\ell^2 T/2)$ and $\log Z_\sigma^1 \geqq (\lambda\eta/\sqrt{\varepsilon}\beta) - (\lambda^2 N^2 T/2)$ on sets $\{\sqrt{\varepsilon}\int_0^\tau \alpha_2(s)\,dW_s \geqq N\}$ and $\{\sqrt{\varepsilon}\beta\int_0^\sigma \mathscr{E}_s dW'_s \geqq \eta, \Gamma_N\}$ respectively, we arrive at (8.20) for "$\geqq N$" and "$\geqq \eta$", taking $\lambda^1 = (N/\sqrt{\varepsilon}\ell^2 T)$ and $\lambda^2 = (\eta/\sqrt{\varepsilon}\beta N^2 T)$. For "$\leqq -N$" and "$\leqq -\eta$" the validity of (8.20) is proved in the same way.

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