Probab. Theory Relat. Fields 106, 39–70 (1996)

Probability Theory and Related Fields

Wave front propagation and large deviations for diffusion–transmutation process

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Received: 31 October 1994 / In revised form: 13 November 1995

Summary. We study systems of reaction–diffusion equations of KPP-type with the coefficients and nonlinear terms slowly varying in the space variables. The long time behavior of the solution to such systems can be characterized by the motion of wave fronts. We describe the wave front motion, using the Feynman–Kac formula and the large deviation principle for the corresponding diffusion–transmutation process. We give a geometrical description of the motion in the examples and show some effects which appear in case of systems but not in the single RDE's.

Mathematics Subjects Classification (*1991*)*:* 60J60, 35K55

1 Introduction

An equation

$$\frac{\partial v(t,x)}{\partial t} = \frac{D}{2}\frac{\partial^2 v}{\partial x^2} + f(v), \quad t > 0, \ x \in R^1 , \tag{1.1}$$

is called KPP-equation (see [KPP]) if $D > 0$ is a constant and $f(v) = c(v)v$, where $c(v)$ is continuous, $c(v) > 0$ for $v < 1$, $c(v) < 0$ for $v > 1$ and $c = c(0) = \max_{0 \leqq v} c(v)$.

It was proved in [KPP] that the solution of (1.1) with initial condition

$$v(0,x) = \chi^-(x) = \begin{cases} 1, & x \leqq 0 , \\ 0, & x > 0 \end{cases}$$

for large t behaves as a running wave $\varphi(x - \alpha t)$. So the asymptotic behavior is characterized by the shape $\varphi(z)$ and the speed α of the wave. If we replace (1.1)

[1]Supported in part by ARO Grant DAAL03-92-G-0219
[2]Supported in part by NSF DMS 9207928

by an equation with space dependent coefficients then, in general, we can not expect such a regular behavior of the solution as $t \to \infty$. But if the dependence on x is slow, so that the diffusivity and the non-linear term actually depend on εx, $0 < \varepsilon \ll 1$, one can describe the asymptotic behavior of the solution for $\varepsilon \downarrow 0$, $t \sim \varepsilon^{-1}$. Let $v(t,x)$ be the solution of (1.1) with D and $f(v)$ replaced by $D(\varepsilon x)$ and $f(\varepsilon x, v)$ correspondingly, $v(0,x) = \chi^-(x)$. Then the equation for $u^\varepsilon(t,x) = v(\frac{t}{\varepsilon}, \frac{x}{\varepsilon})$ will have the form

$$\frac{\partial u^\varepsilon(t,x)}{\partial t} = \frac{\varepsilon D(x)}{2} \frac{\partial^2 u^\varepsilon}{\partial x^2} + \frac{1}{\varepsilon} f(x, u^\varepsilon), \quad u^\varepsilon(0,x) = \chi^-(x) .$$

A theory of such kind of equations in R^r as $\varepsilon \downarrow 0$ was developed in [F1, F2] and [F4] by probabilistic methods. Then an analytic proof and some generalizations of those results were given in [ES].

Generalizations of such results for PDE-systems of reaction-diffusion equation (RDE) type are of interest. A class of spatially homogeneous RDE systems which can be looked at as a generalization of KPP-equation was considered in [F1, F3] using the probabilistic approach. In [BES] a similar problem was considered for a wider class of spatially homogeneous systems.

Here we study the non-homogeneous in space case. As it is known [F1] even for one equation a number of new effects appear in the non-homogeneous case such as, for example, jumps of the wave fronts or a non-Markovian law of wave front propagation (see Example 1 in Sect. 4 of the present paper).

A Markov diffusion–transmutation process can be connected with PDE systems considered in this paper. If the system contains a small parameter, the corresponding process also depends on that parameter. The asymptotic behavior of the solutions of these systems is defined by a large deviation principle for the family of corresponding processes.

We consider the RDE systems of the following form:

$$(1.2) \qquad \frac{\partial u_k^\varepsilon(t,x)}{\partial t} = L_k^\varepsilon u_k^\varepsilon(t,x) + \frac{1}{\varepsilon} F_k(x, u^\varepsilon), \quad t > 0, \ x \in R^r, \ k = 1, 2, \dots, n .$$

Here $L_k^\varepsilon = \frac{\varepsilon}{2} \sum_{i,j=1}^r a_k^{ij}(x) \frac{\partial^2}{\partial x^i \partial x^j}$ are elliptic operators, $u^\varepsilon = (u_1^\varepsilon, \dots, u_n^\varepsilon)$, and $F_k(x,u)$, $1 \leqq k \leqq n$, are continuously differentiable in x and in u. The diffusion coefficients $a_k^{ij}(x)$ are assumed to be Lipschitz continuous and

$$\underline{A} \sum_1^r p_i^2 \leqq \sum_{i,j=1}^r a_k^{ij}(x) p_i p_j \leqq \bar{A} \sum_1^r p_i^2$$

for some positive $\underline{A}, \bar{A}$.

Three assumptions concerning the vector field $u \to F(x,u) = (F_1(x,u), \dots, F_n(x,u))$ indexed by $x \in R^r$ will be made.

(A1) There exists $B > 0$ such that for every $x \in R^r$ the vector field $F(x,u)$ points strictly inward from the boundary of the cube $[0,B]^n$, except at $u = 0$; $F(x,0) = 0$, and $\inf\{F_k(x,u)\colon \ 1 \leqq k \leqq n, \ u \in [0,B]^n, \ x \in R^r\} > -\infty$.

(A2) Let $c_{km}(x) = \partial F_k(x,0)/\partial u_m$; the $n \times n$ matrix $(c_{km}(x))$ is denoted by $c(x)$. Assume that $c_{km}(x)$ are Lipschitz continuous and

$$\sup\{c_{km}(x):\ 1 \leqq k,\ m \leqq n,\ x \in R^r\} = \overline{\beta} < \infty\ ,$$
$$\inf\{c_{km}(x):\ 1 \leqq k,\ m \leqq n,\ x \in R^r\} = \underline{\beta} > 0\ .$$

Note that $u = 0$ is an unstable equilibrium point for the field $F(x,u)$ for any $x \in R^r$.

(A3) $F_k(x,u) \leqq \sum_{m=1}^n c_{km}(x)u_m,\ 1 \leqq k \leqq n,\ x \in R^r,\ u \in [0,B]^n$.

For every $\gamma > 0$ there exists $B' = B'(\gamma) > 0$ (independent of $x \in R^r$) such that

$$F_k(x,u) \geqq \sum_{m=1}^n (c_{km}(x) - \gamma)u_m, \quad 1 \leqq k \leqq n,\ u \in [0,B']^n\ .$$

Remark 1 These assumptions are analogous to the KPP conditions in the single equation case. Assumption (A1) is the counterpart of the assumption that the nonlinear term in the KPP equation is negative for $u > 1$ and equal to zero at $u = 0$. But there is also a difference: In the KPP case, the point $u = 1$ is a stable attractor, and the solution converges to $u = 1$, a constant function, as $t \to \infty$. We do not assume that the vector field $F(x,u)$ in $[0,B]^n$ has a stable equilibrium point. Correspondingly, the statement of the main result (Theorem 1) will be weaker. The point is that in the case of systems an asymptotically stable equilibrium point of the vector field $F(x,u)$, as a rule, will not be an attractor for the solution of the RDE-systems. Actually, a similar effect does not allow to replace the cube $[0,B]^n$ by a general domain invariant for the dynamical system $du/dt = F(x,u),\ u \in R^n,\ x \in R^r$ is a parameter.

Assumption (A3) is the counterpart of the assumption that in the KPP case $F(u)/u$ has it's maximum at $u = 0$. This condition is essential and can unlikely be seriously weakened.

Assumption (A2) can be slightly weakened. At least, one can assume that $c_{ij}(x) > 0$ just for distinct i,j; $c_{i,i}(x)$ can be of any sign. One can also weaken the uniformity in x condition.

A Markov process $(X_t^\varepsilon, v_t^\varepsilon)$ in the state space $R^r \times \{1,\dots,n\}$ can be connected with the linear system (see [F1])

$$(1.3) \qquad \frac{\partial v_k^\varepsilon}{\partial t} = L_k^\varepsilon v_k^\varepsilon + \frac{1}{\varepsilon}\sum_{i=1}^n c_{ki}(x)(v_i^\varepsilon - v_k^\varepsilon), \quad 1 \leqq k \leqq n,\ t > 0,\ x \in R^r\ .$$

The process $(X_t^\varepsilon, v_t^\varepsilon)$ can be uniquely characterized as follows:

$$dX_t^\varepsilon = \varepsilon^{1/2}\sigma_{v_t^\varepsilon}(X_t^\varepsilon)dW_t, \quad X_0^\varepsilon = x \in R^r,\ t > 0\ ,$$

where W_t is a standard Wiener process in R^r, $\sigma_m(x),\ 1 \leqq m \leqq n$, are such that $\sigma_m(x)\sigma_m^*(x) = (a_m^{ij}(x))$; and v_t^ε is $\{1,\dots,n\}$-valued right-continuous process such that

$$P\{v_{t+\Delta}^\varepsilon = m \mid X_t^\varepsilon = y,\ v_t^\varepsilon = l\} = \frac{c_{lm}(y)}{\varepsilon}\Delta + o(\Delta),\ \Delta \downarrow 0\ ,$$
$$l,m \in \{1,2,\dots,n\},\ l \neq m,\ v_0^\varepsilon = k \in \{1,\dots,n\}\ .$$

The solutions of the Cauchy problem and of some initial-boundary value problems for system (1.3) can be written as expectations of proper functionals of the process $(X_t^\varepsilon, v_t^\varepsilon)$ (see [F1, Ch. 5; EF]).

Let us define

$$Z_t^\varepsilon = \left(\int_0^t \chi_1(v_s^\varepsilon)\, ds, \ldots, \int_0^t \chi_n(v_s^\varepsilon)\, ds \right),$$

where χ_k is the indicator function of the point k. To study wave front propagation for system (1.2) we need the large deviation principle for the family $(X_t^\varepsilon, Z_t^\varepsilon)$, $0 \leqq t \leqq T$, as $\varepsilon \downarrow 0$. The action functional $\varepsilon^{-1} S_{0T}(\varphi, \mu)$ for this family, in the uniform topology, was found in [FL]:

$$S_{0T}(\varphi,\mu) = \begin{cases} \int_0^T \eta(\varphi_s, \dot\varphi_s, \dot\mu_s)\, ds \quad \text{if } \varphi \text{ and } \mu \text{ are} \\ \quad \text{absolutely continuous, } \sum_{k=1}^n \mu_s^k = s,\ 0 \leqq s \leqq T, \text{ and } \mu_s^k \\ \quad \text{are nondecreasing in } s,\ 1 \leqq k \leqq n\,, \\ \quad +\infty \ \textit{otherwise}. \end{cases}$$

Here $\varphi : [0,T] \to R^r$, $\mu = (\mu^1, \ldots, \mu^n) : [0,T] \to R^n$. The function $\eta(x,q,\beta)$ is the Legendre transformation in p and α of the principal eigenvalue $\lambda(x,p,\alpha)$ of the matrix

$$\hat c(x) + \Pi\,(x,p,\alpha)\,,$$

where $\hat c(x) = (\hat c_{ij}(x))$, $\hat c_{ij}(x) = c_{ij}(x)$ for $i \neq j$, $\hat c_{kk}(x) = -\sum_{j \neq k} c_{kj}(x)$ and $\Pi(x,p,\alpha)$ is the diagonal matrix with elements

$$\Pi_{kk} = \frac{1}{2} \sum_{i,j} a_k^{ij}(x) p_i p_j + \alpha_k; \quad k = 1, \ldots, n\,.$$

The function $\eta(x,q,\beta)$ is non-negative; $\eta(x,q,\beta) < \infty$ only for $\beta = (\beta_1, \beta_2, \ldots, \beta_n)$ such that $\beta_i \geqq 0$, $\sum_1^n \beta_i = 1$; $\eta(x;q,\beta) = 0$ only at $q = 0$ and $\beta = \beta(x)$ equal to the stationary distribution of the continuous time Markov chain with n states $\{1, 2, \ldots, n\}$ and transition intensity matrix $\hat c(x)$.

We add to (1.2) the initial conditions

$$(1.4) \qquad u_k^\varepsilon(0,x) = g_k(x) \in [0,B], \quad 1 \leqq k \leqq n,\ x \in R^r\,,$$

where g_k are continuous; $G_0 = \bigcup_{k=1}^n \mathit{supp}(g_k)$. Here $\mathit{supp}(g_k)$ is the closure of the set $\{x \in R^r :\ g_k(x) > 0\}$.

Denote by $\Lambda(x,p)$ the principle eigenvalue of the matrix

$$c(x) + \mathit{diag}\left(\frac{1}{2} p \cdot a_k(x) p \right), \quad p, x \in R^r\,,$$

which exists according to the Frobenius theorem. It is known that $\Lambda(x,p)$ is convex in $p \in R^r$. Let

$$(1.5) \qquad \zeta(x,q) = -\sup_{p \in R^r} [q \cdot p - \Lambda(x,p)], \quad x, q \in R^r\,,$$

which is concave in q.

The following simple relation follows from the fact that the Legendre transformation is an involution:

$$\zeta(x,q) = \sup_{\beta\in R^n} [b(x)\cdot\beta - \eta(x,q,\beta)], \quad x,q \in R^n , \tag{1.6}$$

where $b(x) = (\sum_{m=1}^n c_{1m}(x), \ldots, \sum_{m=1}^n c_{nm}(x))$; due to the properties of $\eta(x,q,\beta)$ mentioned above, the supremum in (1.6) can be taken just over the set $\{\beta = (\beta_1,\ldots,\beta_n):\ \beta_i \geqq 0, \sum_1^n \beta_i = 1\}$.
Let

$$V(t,x) = \sup\left\{ \min_{0\leqq a\leqq t} \int_0^a \zeta(\varphi_s,\dot\varphi_s)\,ds:\ \varphi \text{ is absolutely continuous}, \right.$$
$$\left. \varphi_0 = x,\ \varphi_t \in G_0 \right\},$$

$t > 0,\ x \in R^r$. Since one can take $a = 0$ we conclude that $-\infty < V(t,x) \leqq 0$.

The main result of this paper is the following.

Theorem 1 *Assume that* (A1)–(A3) *hold. Then the following relations for the solution* $u^\varepsilon(t,x) = (u_1^\varepsilon(t,x),\ldots,u_n^\varepsilon(t,x))$ *of problem* (1.2), (1.4) *hold*:
(i) $\lim_{\varepsilon\to 0} u_k^\varepsilon(t,x) = 0$ *for* $1 \leqq k \leqq n$, *uniformly in* (t,x) *from any compact subset of* $\{(t,x):\ t > 0,\ x \in R^r,\ V(t,x) < 0\}$,
(ii) $\underline{\lim}_{\varepsilon\to 0} u_k^\varepsilon(t,x) > 0$ *for* $1 \leqq k \leqq n$, *uniformly in* (t,x) *from any compact subset of* $(\{(t,x):\ t > 0,\ x \in R^r,\ V(t,x) = 0\})$, *where* (A) *means the interior of the set* A.

Moreover, the function $V(t,x)$ *is locally Lipschitz continuous in* $t > 0$, $x \in R^r$.

The remainder of this paper is organized into three sections. In Sect. 2 we present some properties of the function $V(t,x)$. In Sect. 3 we prove Theorem 1. In Sect. 4 we consider examples. We pay special attention to the geometric description of the motion of the wave fronts. In particular, under some conditions we describe the motion by a Huygens principle (see Sect. 4) in a proper Riemannian or Finsler metric and provide example showing that such a description is not always possible.

2 Properties of the function V

We prove some properties of the function ζ in Lemmas 2.1 and 2.2 and then use them to show that the function V is locally Lipschitz continuous in Lemma 2.3.

Lemma 2.1 *Let* $\underline{A}$, $\bar{A}$ *be as in the uniform ellipticity condition in the Introduction. Then*
(i) $-|q|^2/(2\bar{A}) + n\bar\beta \geqq \zeta(x,q) \geqq -|q|^2/(2\underline{A}) + n\underline{\beta}$, $x,q \in R^r$, *where* $\underline{\beta}$ *and* $\bar\beta$ *are defined in* (A2).
(ii) *There exists* $c > 0$, *independent of* $x,q \in R^r$ *such that*

$$\zeta(x,q) = -\max_{|p|\leqq c|q|} [q\cdot p - \Lambda(x,p)] .$$

(iii) *there exists* $c > 0$, *independent of* $x, q, \bar{q} \in R^r$ *such that*

$$|\zeta(x,\bar{q}) - \zeta(x,q)| \leqq c(|\bar{q}| + |q|)|\bar{q} - q| .$$

(iv) *there exists* $c > 0$, *independent of* $x, y, q \in R^r$ *such that*

$$|\zeta(x,q) - \zeta(y,q)| \leqq c(1 + |q|^2)|x - y| .$$

Proof. It is easy to see that for all $x, p \in R^r$

$$\Lambda(x,p) \geqq \frac{\underline{A}}{2}|p|^2 + \Lambda(x,0) \geqq \frac{\underline{A}}{2}|p|^2 + n\underline{\beta} ,$$

$$\Lambda(x,p) \leqq \frac{\bar{A}}{2}|p|^2 + \Lambda(x,0) \leqq \frac{\bar{A}}{2}|p|^2 + n\bar{\beta} ,$$

(i) then follows easily.

Since

$$\begin{aligned} q \cdot p - \Lambda(x,p) &\leqq q \cdot p - \frac{\underline{A}|p|^2}{2} - \Lambda(x,0) \\ &\leqq |p|\left(|q| - \frac{\underline{A}}{2}|p|\right) + [q \cdot 0 - \Lambda(x,0)] , \end{aligned}$$

which is less than $[q \cdot 0 - \Lambda(x,0)]$ when $|p| > \frac{2}{\underline{A}}|q|$, (ii) is true for $c = \frac{2}{\underline{A}}$.

It follows from (ii) that there exists $p^* = p^*(q)$ such that $|p^*| \leqq c|q|$ and

$$\zeta(x,q) = -[q \cdot p^* - \Lambda(x,p^*)] .$$

We then have

$$\begin{aligned} \zeta(x,\bar{q}) - \zeta(x,q) &= -\left\{\max_{|p| \leqq c|\bar{q}|}[\bar{q} \cdot p - \Lambda(x,p)]\right\} + [q \cdot p^* - \Lambda(x,p^*)] \\ &\leqq -[\bar{q} \cdot p^* - \Lambda(x,p^*)] + [q \cdot p^* - \Lambda(x,p^*)] = (q - \bar{q}) \cdot p^* . \end{aligned}$$

Exchange the roles of q and $\bar{q}$ to get

$$\zeta(x,q) - \zeta(x,\bar{q}) \leqq (\bar{q} - q) \cdot p^*(\bar{q}) .$$

These two inequalities imply (iii).

Recall that we assume the functions $c_{km}(x)$, $a_k^{ij}(x)$ to be Lipschitz continuous. This implies the existence of $c_1 > 0$ such that

$$|\Lambda(x,p) - \Lambda(y,p)| \leqq c_1(1 + |p|^2)|x - y| \quad \text{for all } x, y, p \in R^r .$$

Thus, by (ii) (renotate the constant c by c_2)

$$\begin{aligned} \zeta(x,q) &= \min_{|p| \leqq c_2|q|}[-q \cdot p + \Lambda(x,p)] \\ &\leqq \min_{|p| \leqq c_2|q|}[-q \cdot p + \Lambda(y,p)] + \max_{|p| \leqq c_2|q|}[\Lambda(x,p) - \Lambda(y,p)] \\ &\leqq \zeta(y,q) + c_1(1 + c_2^2|q|^2)|x - y| \\ &\leqq \zeta(y,q) + c_1(1 + c_2^2)(1 + |q|^2)|x - y| . \end{aligned}$$

Because the roles of x and y can be exchanged, (iv) is proved with $c = c_1(1 + c_2^2)$. $\square$

Lemma 2.2 *Let* φ *be such that* $\int_0^t \zeta(\varphi_s, \dot{\varphi}_s)\, ds \neq -\infty$. *The following two properties hold.*
(i) *Let* $\varphi_s^b = \varphi_{bs}$, $0 \leqq s \leqq b^{-1}t$, $b > 0$. *Then*

$$\lim_{b \to 1} \left| \int_0^{t/b} \zeta(\varphi_s^b, \dot{\varphi}_s^b)\, ds - \int_0^t \zeta(\varphi_s, \dot{\varphi}_s)\, ds \right| = 0\,.$$

(ii) *Let* $\tilde{\zeta}(x,q)$ *be defined as* $\zeta(x,q)$ *for functions* $c_{ij}(x)$ *replaced by* $\tilde{c}_{ij}(x)$. *Then*

$$\int_0^t \tilde{\zeta}(\varphi_s, \dot{\varphi}_s)\, ds \to \int_0^t \zeta(\varphi_s, \dot{\varphi}_s)\, ds \text{ as } \gamma \equiv \sup_{x \in R^r,\, 1 \leqq i,j \leqq n} |\tilde{c}_{ij}(x) - c_{ij}(x)| \to 0\,.$$

Proof. A change of variable $s = bs'$ shows that

$$\begin{aligned}\left| \int_0^{t/b} \zeta(\varphi_{s'}^b, \dot{\varphi}_{s'}^b)\, ds' - \int_0^t \zeta(\varphi_s, \dot{\varphi}_s)\, ds \right| &= \left| \int_0^t b^{-1} \zeta(\varphi_s, b\dot{\varphi}_s)\, ds - \int_0^t \zeta(\varphi_s, \dot{\varphi}_s)\, ds \right| \\ &\leqq b^{-1} \int_0^t |\zeta(\varphi_s, b\dot{\varphi}_s)|\, ds - \zeta(\varphi_s, \dot{\varphi}_s)|\, ds \\ &\quad + |b^{-1} - 1| \left| \int_0^t \zeta(\varphi_s, \dot{\varphi}_s)\, ds \right|.\end{aligned}$$

It is clear that the second term in the righthand side tends to 0 as $b \to 1$. The first term in the righthand side is bounded, via Lemma 2.1(iii) and (i), by

$$b^{-1} c(b+1)\, |b-1| \int_0^t |\dot{\varphi}_s|^2\, ds \leqq b^{-1} c(b+1) |b-1| 2\bar{A} \int_0^t (n\bar{\beta} - \zeta(\varphi_s, \dot{\varphi}_s))\, ds\,,$$

which again tends to 0 as $b \to 1$. Statement (i) is proved.

Let $\tilde{\Lambda}(x,p)$ be defined as $\Lambda(x,p)$ for function $c_{ij}(x)$ replaced by $\tilde{c}_{ij}(x)$. Then,

$$|\tilde{\Lambda}(x,p) - \Lambda(x,p)| \leqq n\gamma\,.$$

Hence,

$$|\tilde{\zeta}(x,q) - \zeta(x,q)| \leqq n\gamma$$

from which (ii) follows. □

Lemma 2.3 *The function* $V(t,x)$ *is locally Lipschitz continuous in* $t > 0$, $x \in R^r$: *For any compact subset* F *of* $\{(t,x)\colon t > 0,\ x \in R^r\}$ *there exists* $K = K_F$ *such that* $|V(s,x) - V(t,y)| \leqq K(|t-s| + |x-y|)$ *for* $(s,x),(t,y) \in F$.

Proof. Throughout the proof c means a constant which may vary from place to place. The proof consists of two steps.

Step 1 For given $t > 0$, $x \in R^r$, there exist constants $A, \delta > 0$ such that

$$|V(T,z) - V(T,y)| \leqq A|z-y| \quad \text{for } |y-x| < \delta,\ |z-x| < \delta \quad \text{and } |T-t| < \delta\,.$$

It follows from Lemma 2.1(i) that a constant $M = M_{t,x,\delta}$ exists such that

$$V(T,y) = \sup_{\varphi\in C_M,\varphi_0=y,\varphi_T\in G_0}\ \min_{0\leqq a\leqq T}\int_0^a \zeta(\varphi_s,\dot{\varphi}_s)\,ds$$

for $|x-y| \leqq \delta$, $|T-t| \leqq \delta$, where

$$C_M = \left\{\varphi\in C_{0,t+\delta}\colon \int_0^{t+\delta}|\dot{\varphi}_s|^2\,ds \leqq M\right\}.$$

Let $\phi_s = \phi_s[\varphi] = \varphi_s + (T-s)q$, $q = \frac{1}{T}(z-y)$. Then $\phi_0 = z$, $\phi_T = \varphi_T \in G_0$ if $\varphi_0 = y$, $\varphi_T \in G_0$, and

$$\begin{aligned}
&|V(T,z) - V(T,y)| \\
&\quad\leqq \sup_{\varphi\in C_M,\varphi_0=y,\varphi_T\in G_0}\int_0^T|\zeta(\phi_s,\dot{\phi}_s) - \zeta(\varphi_s,\dot{\varphi}_s)|\,ds \\
&\quad\leqq \sup_{\varphi\in C_M,\varphi_0=y,\varphi_T\in G_0}\int_0^T|(\zeta(\phi_s\dot{\phi}_s| - \zeta(\varphi_s,\dot{\phi}_s) + (\zeta(\varphi_s,\dot{\phi}_s) - \zeta(\varphi_s\dot{\varphi}_s))|\,ds \\
&\quad\leqq \sup_{\varphi\in C_M,\varphi_0=y,\varphi_T\in G_0}\int_0^T\{c|x-y| + c|q|\int_0^T(|\dot{\varphi}_s| + |\dot{\phi}_s|)\,ds\}\,,
\end{aligned}$$

where the last inequality uses Lemma 2.1(iii) and (iv). By the Schwartz inequality

$$\int_0^T|\dot{\varphi}_s|\,ds \leqq \left(T\int_0^T|\dot{\varphi}_s|^2\,ds\right)^{1/2} \leqq (TM)^{1/2}\,.$$

Using the same bound for $\int_0^T|\dot{\phi}_s[\varphi]|\,ds$ and the fact that $q = \frac{1}{T}(z-y)$ we derive step 1.

Step 2 There exist $K,\delta > 0$ such that

$$|V(T_1,x) - V(T,x)| \leqq K(T_1 - T) \quad \text{for } t+\delta > T_1 > T > t-\delta\,.$$

Let $\theta = T_1 - T$. Then

$$\begin{aligned}
V(T_1,x) &\leqq \sup\left\{\min_{\theta\leqq a\leqq T+\theta}\int_0^a\zeta(\varphi_s,\dot{\varphi}_s)\,ds\colon\ \varphi_0 = x,\ \varphi_{T+\theta}\in G_0,\ \varphi\in C_M\right\} \\
&\leqq \sup\left\{n\bar{\beta}\theta - \int_0^\theta|\dot{\varphi}_s|^2\,ds/(2\bar{A}) + V(T,\varphi_\theta)\colon\right. \\
&\qquad\qquad \left.\varphi_0 = x,\ \varphi_{T+\theta}\in G_0,\ \varphi\in C_M\right\},
\end{aligned}$$

where the last inequality uses Lemma 2.1(i). The result of step 1 guarantees

$$|V(T,\varphi_\theta) - V(T,x)| \leqq A|\varphi_\theta - x|\,.$$

Combining these two estimates we derive

$$\begin{aligned} &V(T_1,x) - V(T,x) \\ &\quad \leqq n\bar{\beta}\theta + \sup\left\{ -\int_0^\theta |\dot{\varphi}_s|^2\, ds/(2\bar{A}) + A|\varphi_\theta - x| :\ \varphi_0 = x \text{ and } \varphi \in C_M \right\} . \end{aligned}$$

Set $u = |\varphi_\theta - x| = |\int_0^\theta \dot{\varphi}_s\, ds|$ and apply the Schwartz inequality

$$\int_0^\theta |\dot{\varphi}_s|^2\, ds \ \geqq\ \frac{u^2}{\theta} .$$

Then,

$$V(T_1,x) - V(T,x) \ \leqq\ n\bar{\beta}\theta + \sup_{u \geqq 0}(Au - u^2/(2\bar{A}\theta)) = (n\bar{\beta} + A^2\bar{A}/2)\theta .$$

Step 2 is completed once we show that $V(T_1,x) - V(T,x) \geqq 0$ for $T_1 \geqq T$:

$$\begin{aligned} V(T_1,x) \ &\geqq\ \sup\left\{ \min_{0 \leqq a \leqq T+\theta} \int_0^a \zeta(\varphi_s, \dot{\varphi}_s)\, ds :\ \varphi_s = x \right. \\ &\qquad\qquad \left. \text{for } 0 \leqq s \leqq \theta \text{ and } \varphi_{T+\theta} \in G_0 \right\} \\ &\geqq\ V(T,x) , \end{aligned}$$

where the last inequality uses Lemma 2.1(i). □

We introduce two other functions, V_0 and V_1 which are more familiar in the literature of wave front propagation cf. [ES, F3]. We shall prove that $V = V_0 = V_1$ in Lemma 2.4. A functional $\tau:\ C([0,t], R^r) \to [0,t]$ is called a stopping time if τ depends only on $\varphi_s,\ 0 \leqq s \leqq u$, when restricted to $\{\tau \leqq u\}$. Let Σ_t be the collection of all stopping times not greater than t. If F is a closed subset of $[0,t] \times R^r$ and $\{0\} \times R^r \subset F$, then

$$\tau_F \equiv \min\{s:\ s \geqq 0 \text{ and } (t - s, \varphi_s) \in F\}$$

is clearly a stopping time not greater than t. Let Θ_t be the collection of τ_F. Let

$$\begin{aligned} V_0(t,x) &= \inf_{\tau \in \Sigma_t} \left\{ \sup \int_0^\tau \zeta(\varphi_s, \dot{\varphi}_s)\, ds :\ \varphi \text{ is absolutely continuous,} \right. \\ &\qquad\qquad \left. \varphi_0 = x \text{ and } \varphi_t \in G_0 \right\} , \\ V_1(t,x) &= \inf_{\tau \in \Theta_t} \left\{ \sup \int_0^\tau \zeta(\varphi_s, \dot{\varphi}_s)\, ds :\ \varphi \text{ is absolutely continuous,} \right. \\ &\qquad\qquad \left. \varphi_0 = x \text{ and } \varphi_t \in G_0 \right\}, \quad t > 0,\ x \in R^r . \end{aligned}$$

Due to (1.6), the functions $V(t,x)$, $V_0(t,x)$, $V_1(t,x)$ can be expressed through $\eta(x,q,\beta)$ instead of $\zeta(x,q)$. For example,

$$V_1(t,x) = \inf_{\tau\in\Theta_t} \sup\left\{ \int_0^{\tau(\varphi)} \left[\sum_{k,j=1}^{n} c_{kj}(\varphi_s)\dot\mu_s^k - \eta(\varphi_s,\dot\varphi_s,\dot\mu_s) \right] ds : \varphi \in C_{0t},\ \varphi_0 = x, \right.$$
$$\varphi_t \in G_0;\ \mu_s = (\mu_s^1,\ldots,\mu_s^n),\ \mu_s^i \text{ are non-decreasing}$$
$$\left. \text{absolutely continuous functions on } [0,t],\ \sum_{i=1}^{n} \mu_s^i = s \right\} .$$

This representation shows the connection between $V_1(t,x)$ and the action functional for the family $(X_t^\varepsilon, Z_t^\varepsilon)$.

The definition of V_0 and V_1 differs only in the admissible set of stopping times. It is easy to see that $V \leqq V_0 \leqq V_1$. We prove

Lemma 2.4 *The three functions V, V_0 and V_1 are equal.*

Proof. It suffices to prove that $V_1 \leqq V$. Suppose that $V_1(T,X) > V(T,X)$ for some $T > 0$, $x \in R^r$. We shall produce a contradiction. Let $F = \{(s,y)\colon V(s,y) = 0 \text{ or } s = 0\} \cap \{s \leqq T\}$. Since V is (Lipschitz) continuous, the set F is closed and $\tau = \tau_F \in \Theta_T$. The definition of V_1 guarantees the existence of $\varphi\colon [0,T] \to R^r$, $\varphi_0 = X$, $\varphi_T \in G_0$ such that

$$\int_0^{\tau[\varphi]} \zeta(\varphi_s,\dot\varphi_s)\,ds > V(T,X)\,. \tag{2.1}$$

If $\tau[\varphi] < T$, the definition of τ yields

$$V(T-\tau[\varphi], \varphi_{\tau[\varphi]}) = 0\,, \tag{2.2}$$

and

$$V(T-b,\varphi_b) < 0 \quad \text{for } 0 \leqq b \leqq \tau[\varphi]\,. \tag{2.3}$$

Equality (2.2) ensures that for any $\delta > 0$ there exists a reconstruction of φ in the time interval $(\tau[\varphi], T]$ such that for the new φ

$$\int_0^a \zeta(\varphi_s,\dot\varphi_s)\,ds \geqq \int_0^{\tau[\varphi]} \zeta(\varphi_s,\dot\varphi_s)\,ds - \delta \quad \text{for } \tau[\varphi] \leqq a \leqq T\,. \tag{2.4}$$

One then concludes from (2.3) that

$$\int_0^b \zeta(\varphi_s,\dot\varphi_s)\,ds > \int_0^{\tau[\varphi]} \zeta(\varphi_s,\dot\varphi_s)\,ds \quad \text{for } 0 \leqq b \leqq \tau[\varphi]\,. \tag{2.5}$$

Then (2.4) and (2.5) imply that $V(T,X) \geqq \int_0^{\tau[\varphi]} \zeta(\varphi_s,\dot\varphi_s)\,ds - \delta$, for any $\delta > 0$, which contradicts (2.1). If $\tau[\varphi] = T$, (2.3) still holds and implies that

$$V(T,X) \geqq \min_{0\leqq a \leqq T} \int_0^a \zeta(\varphi_s,\dot\varphi_s)\,ds = \int_0^T \zeta(\varphi_s,\dot\varphi_s)\,ds\,.$$

This contradicts (2.1) and completes the proof. □

Let

$$V^*(t,x) = \sup_{\varphi_0 = x,\, \varphi_t \in G_0} \int_0^t \zeta(\varphi, \dot{\varphi}_s)\, ds, \quad t > 0,\ x \in R^r .$$

This function takes a form simpler than V.

We say that condition (N) is fulfilled, if

$$\text{(N)} \qquad V^*(t,x) = \sup\left\{ \int_0^t \zeta(\varphi_s, \dot{\varphi}_s)\, ds :\ \varphi_0 = x,\ \varphi_t \in G_0 \right.$$

and

$$\left. V^*(t-s, \varphi_s) < 0 \text{ for } 0 < s < t \right\}, \text{ whenever } V^*(t,x) \leqq 0 .$$

Condition (N) is fulfilled when, for example, $a_k^{ij}(x)$ and $c_{km}(x)$ are constants.

Lemma 2.5 *If* (N) *is fulfilled, then* $V(t,x) = \min(V^*(t,x), 0)$.

Proof. In view of Lemma 2.4 it suffices to prove that (i) and (ii) hold:
(i) $V_1(t,x) \geqq V^*(t,x) \geqq V(t,x)$ when $V^*(t,x) \leqq 0$.
(ii) $V(t,x) = 0$ when $V^*(t,x) > 0$.

Statement (ii) follows from statement (i) and the monotonicity of V and V^* in t. The second inequality of (i) is obvious. Next, we prove the first inequality. From the condition (N) and the definition of $V^*(t,x)$ one can conclude that for any $\delta > 0$ there exists $\phi_s = \phi_s^\delta,\ 0 \leqq s \leqq t,\ \phi_0 = x,\ \phi_t \in G_0$, such that

$$V^*(t,x) \leqq \int_0^t \zeta(\phi_s, \dot{\phi}_s)\, ds + \delta ,$$

$$V^*(t-s, \phi_s) < 0 \quad \text{for } 0 < s < t .$$

Then we obtain

$$\begin{aligned} V_1(t,x) &\geqq \inf_{\tau \in \Theta_t} \int_0^{\tau[\phi]} \zeta(\phi_s, \dot{\phi}_s)\, ds \geqq \inf_{0 \leqq a \leqq t} \int_0^a \zeta(\phi_s, \dot{\phi}_s)\, ds \\ &= \int_0^t \zeta(\phi_s, \dot{\phi}_s)\, ds \geqq V^*(t,x) - \delta , \end{aligned}$$

where the equality follows from the fact that $\int_0^t \zeta(\phi_s, \dot{\phi}_s)\, ds < 0$ for $0 \leqq a < t$ since $V^*(t-a, \phi_a) < 0$. Since δ is an arbitrary positive number we conclude that $V_1(t,x) \geqq V^*(t,x)$. □

Remark 1 It follows from Lemma 2.5 that $\{V(t,x) < 0\} = \{V^*(t,x) < 0\}$ and $(\{V(t,x) = 0\}) = (\{V^*(t,x) \geqq 0\})$, if (N) is fulfilled.

Remark 2 One should compare our condition (N) to that given in [F2], in which $n = 1$ (one reaction–diffusion equation) is considered and

$$\text{(2.6)} \qquad \begin{aligned} V^*(t,x) &= \sup_{\varphi = x,\, \varphi_t \in G_0} \left[\int_0^t c(\varphi_s)\, ds - S_{0t}(\varphi) \right], \\ S_{0t}(\varphi) &= \frac{1}{2} \int_0^t \sum_{i,j=1}^r a_{ij}(\varphi_s) \dot{\varphi}_s^i \dot{\varphi}_s^j\, ds, (a_{ij}(x)) = (a^{ij}(x))^{-1} . \end{aligned}$$

Recall from (1.6) that

$$\zeta(x,q) = \sup_{\beta_k \geqq 0,\, \sum_{k=1}^n \beta_k=1} \left[\left(\sum_{\substack{m=1 \\ k=1}}^{n} c_{km}(x)\beta_k \right) - \eta(x,q,\beta) \right].$$

It is clear that our V^* function reduces to (2.6) when $n = 1$.

3 Proof of Theorem 1

The proof goes according to the following plan: First, as shown in Lemma 2.4, one can replace $V(t,x)$ by $V_1(t,x)$. Then we prove in Lemma 3.1 a comparison result, which allows to reduce the proof of both statements of Theorem 1 to equations with simpler nonlinearities. In Lemma 3.2 we prove the first statement of the Theorem by replacing in (1.2) the function $F_k(x,u)$ by $\tilde{F}_k(x,u) = \sum_{j=1}^n c_{k,j}(x)u_j + \gamma u_k(1 - u_k/M')$ with proper $\gamma, M' > 0$. The solution of problem (1.2), (1.4) with the nonlinear terms $\tilde{F}_k(x,u)$ can be bounded from above using the Feynman–Kac formula and the upper large deviation bound for the Markov process corresponding to system (1.3).

The second statement of Theorem 1 follows from Lemmas 3.4 and 3.6. Lemma 3.4 is proved using a comparison with a single KPP-type equation given in Lemma 3.3. The proof of Lemma 3.6 is based on the lower large deviation bound and the fact that $F_k(x,u)$, for small $|u|$, not only can be bounded from above but also can be approximated by $\sum_{j=1}^n c_{kj}(x)u_j$. Lemma 3.5 contains a simple auxiliary statement.

Lemma 3.1 *Assume that conditions* (A1)–(A3) *are fulfilled. Then the following statements hold*;

(i) *The cube* $[0,B]^n$ *is an invariant region for problem* (1.2), *that is, if the initial conditions* $(g_1(x),\ldots,g_n(x)) \in [0,B]^n$ *for* $x \in R^r$ *then* $u^\varepsilon(t,x) \in [0,B]^n$ *for any* $t \geqq 0,\ x \in R^r$.

(ii) *Let* $f(v),\ v \in R^1$, *be continuously differentiable and* $f(v) \geqq 0$ *for* $0 \leqq v \leqq B$. *Define* $\tilde{F}_k(x;u) = \sum_{j=1}^n c_{kj}(x)u_j + f(u_k)$, *and let* $\tilde{u}^\varepsilon(t,x) = (\tilde{u}_1^\varepsilon(t,x),\ldots,\tilde{u}_n^\varepsilon(t,x))$ *be the solution of problem* (1.2)–(1.4) *with* $F_k(x,u)$ *replaced by* $\tilde{F}_k(x;u)$, $k = 1,\ldots,n$,

Then $\tilde{u}_k^\varepsilon(t,x) \geqq u_k^\varepsilon(t,x)$ *for any* $t \geqq 0,\ x \in R^r,\ k = 1,\ldots,n$.

(iii) *Let* $f(z),\ z \in R^1$, *be a continuously differentiable function such that* $F_k(x;u_1,\ldots,u_n) \geqq f(u_k)$. *Denote* $\bar{u}_k^\varepsilon(t,x)$ *the solution of the Cauchy problem*

$$\frac{\partial \bar{u}_k^\varepsilon}{\partial t} = L_k^\varepsilon \bar{u}_k^\varepsilon + \frac{1}{\varepsilon} f(\bar{u}_k^\varepsilon), \quad t > 0,\ x \in R^r,$$

$$\bar{u}_k^\varepsilon(0,x) = g_k(x),$$

where L_k^ε *is the same as in* (1.2) *and* $g_k(x)$ *is the same as in* (1.4). *Then* $u_k^\varepsilon(t,x) \geqq \bar{u}_k^\varepsilon(t,x)$ *for* $t \geqq 0,\ x \in R^r$.

Proof. Statement (i) follows from Theorem 14.7 in [S].

To prove (ii) denote by $v_k(t,x)$ the difference $\tilde{u}_k^\varepsilon(t,x) - u_k^\varepsilon(t,x)$. From the mean value theorem we derive that for given $\tilde{u}_k^\varepsilon(t,x)$ and $u_k^\varepsilon(t,x)$ continuous functions $\tilde{c}_k(t,x)$, the dependence on ε being omitted, exist such that

$$f(\tilde{u}_k^\varepsilon(t,x)) - f(u_k^\varepsilon(t,x))$$
$$= \tilde{c}_k(t,x)(\tilde{u}_k^\varepsilon(t,x) - u_k^\varepsilon(t,x)\,, \quad k = 1,\dots,n,\ t \geqq 0, \quad x \in R^r\,.$$

Let $c_k(x) = \sum_{j=1}^n c_{kj}(x)$, $\hat{c}_k(t,x) = c_k(x) + \tilde{c}_k(t,x)$, $g_k(t,x) = \tilde{F}_k(x,u^\varepsilon) - F_k(x,u^\varepsilon)$. The functions $v_k(t,x)$, $k = 1,\dots,n$, satisfy the equations

$$\frac{\partial v_k}{\partial t} = L^\varepsilon v_k + \frac{1}{\varepsilon}[\tilde{F}_k(x,\tilde{u}^\varepsilon) - F_k(x,u^\varepsilon)] = L_k^\varepsilon v_k$$
$$+ \frac{1}{\varepsilon}\left[\sum_{j=1}^n c_{kj}(x)(v_j - v_k) + \hat{c}_k(t,x)v_k + g_k(t,x)\right],$$
$$v_k(0,x) = 0,\ t > 0,\ x \in R^r,\ k = 1,\dots,n\,.$$

Let $(X_t^\varepsilon, v_t^\varepsilon)$ be the Markov process in $R^r \times \{1,\dots,n\}$ corresponding to the system (1.3). Then the functions $v_k(t,x)$ can be represented by the Feynman–Kac formula

$$v_k(t,x) = \frac{1}{\varepsilon} E_{x,k} \int_0^t g_{v_t^\varepsilon}(t-s, X_s^\varepsilon) \exp\left\{\frac{1}{\varepsilon}\int_0^s \hat{c}_{v_s^\varepsilon}(t-s_1, X_{s_1}^\varepsilon)\,ds_1\right\} ds.$$

Since $g_k(t,x) = \tilde{F}_k(x,u^\varepsilon) - F_k(x,u^\varepsilon) \geqq f(u_k^\varepsilon) \geqq 0$ by (A3), statement (ii) follows from this representation and statement (i).

To prove (iii) put $w_k(t,x) = u^\varepsilon(t,x) - \bar{u}_k^\varepsilon(t,x)$. The function $w_k(t,x)$ satisfies the equation

$$\frac{\partial w_k}{\partial t} = L_k^\varepsilon w_k + \frac{1}{\varepsilon}[F_k(x,u^\varepsilon) - f(u_k^\varepsilon)] + \frac{1}{\varepsilon}[f(u_k^\varepsilon) - f(\bar{u}_k^\varepsilon))$$
$$= L_k^\varepsilon w_k + \frac{1}{\varepsilon} f'(\hat{u}_k^\varepsilon) w_k + \frac{1}{\varepsilon}[F_k(x,u^\varepsilon) - f(u_k^\varepsilon)]\,,$$
$$w_k(0,x) = 0\,.$$

Here $\hat{u}_k^\varepsilon = \hat{u}_k^\varepsilon(t,x)$ is an intermediate point between $\bar{u}_k^\varepsilon$ and u_k^ε. Since $F_k(x,u) - f(u_k) \geqq 0$, we conclude from the maximum principle for linear parabolic equations that $w_k^\varepsilon(t,x) = u_k^\varepsilon(t,x) - \bar{u}_k^\varepsilon(t,x) \geqq 0$. □

Remark. Statements (ii) and (iii) are special cases of a more general comparison theorem for system (1.2). The general result also can be proved using the probabilistic representation. One can derive it using an analytic approach as well (see Theorem 4.4.1 in [LLV]).

Lemma 3.2 *Let $u^\varepsilon(t,x)$ be the solution of problem* (1.2), (1.4), *and ϕ be a compact subset of the set $\{(t,x)\colon t > 0,\ x \in R^r,\ V_1(t,x) < 0\}$. Then*

$$\lim_{\varepsilon\downarrow 0} u_k^\varepsilon(t,x) = 0 \text{ uniformly in } (t,x) \in \phi, \quad k = 1,\dots,n\,.$$

Proof. Let

$$c_k(x) = \sum_{j=1}^{n} c_{kj}(x) ,$$

$$|c| = \max_{x \in R^r,\, 1 \leqq k \leqq n} c_k(x) ,$$

$$\tilde{F}_k(x, v) = \tilde{F}_{k,\gamma,M}(x, v) = \sum_{j=1}^{n} c_{kj}(x)(v_j + v_k) + \left(\gamma v_k \left(1 - \frac{v_k}{M'}\right)\right) ,$$

where, $\gamma, M' > 0$. Consider the system

$$\frac{\partial v_k^\varepsilon(t,x)}{\partial t} = L_k^\varepsilon v_k^\varepsilon + \frac{1}{\varepsilon} \tilde{F}_k(x, v^\varepsilon), \quad t > 0,\ x \in R^r ,$$
$$v_k^\varepsilon(0,x) = g_k(x), \quad 1 \leqq k \leqq n . \tag{3.1}$$

Let $M' > B$; γ will be chosen later. One can check that then

$$\tilde{F}_k(x, v) \geqq \sum_j c_{kj}(x) v_j, \quad v \in [0, B]^n .$$

It follows from the statement (ii) of Lemma 3.1

$$0 \leqq u_k^\varepsilon(t,x) \leqq v_k^\varepsilon(t,x), \quad t \geqq 0,\ x \in R^r,\ 1 \leqq k \leqq n . \tag{3.2}$$

Now to prove Lemma 3.2 it is enough to check that $v_k^\varepsilon(t,x) \to 0$ as $\varepsilon \downarrow 0$ uniformly in $(t,x) \in \phi$.

System (3.1) is of the same type as system (1.2), (1.4). One can introduce the function $\tilde{V}_1(t,x)$ for the system (3.1) in the same way as the function $V_1(t,x)$ was introduced for (1.2). It is easy to check that the vector field $\tilde{F}(x, v) = (\tilde{F}_1(x, v), \ldots, \tilde{F}_n(x, v))$ satisfies condition (A1) with B replaced by $M = \frac{M'(|c|+\gamma)}{\gamma} + B$, and then

$$0 \leqq v_k(t,x) \leqq M \quad \text{for all } t \geqq 0,\ x \in R^r,\ 1 \leqq k \leqq n .$$

Since

$$\tilde{c}_{ki}(x) = \left. \frac{\partial \tilde{F}_k(x, v)}{\partial v_i} \right|_{v=0} = c_{ki}(x) \quad \text{for } k \neq i \quad \text{and} \quad \tilde{c}_{kk}(x) = c_{kk}(x) + \gamma ,$$

it follows from (1.6), the definition of $V_1(t,x)$ and Lemma 2.4, that

$$\tilde{V}_1(t,x) \leqq V_1(t,x) + \gamma t . \tag{3.3}$$

Since $V_1(t,x)$ is continuous (see Lemmas 2.3 and 2.4),

$$\max\{V_1(t,x) \colon (t,x) \in \phi\} = -\beta < 0 .$$

Let

$$T = \max\{t \colon (t,x) \in \phi \text{ for some } x \in R^r\} ,$$

$T < \infty$ because of compactness of ϕ. It follows from (3.3) that $\gamma > 0$ can be chosen so small that $\tilde{V}_1(t,x) < -\frac{\beta}{2}$ for $(t,x) \in \phi$.

Let $(X_t^\varepsilon, v_t^\varepsilon)$ be the Markov process in the state space $R^r \times \{1,\dots,n\}$ such that its generator A acts on a smooth function $h(x,k)$ as follows:

$$Ah(x,k) = L_k^\varepsilon h(x,k) + \frac{1}{\varepsilon}\sum_{j=1}^n c_{kj}(x)(h(x,j) - h(x,k)) \, .$$

Such a process exists and is unique (see, for example, [F1, Sk]). Using the Feynman–Kac formula one can write the following relation for the solution of problem (3.1)

$$v_k^\varepsilon(t,x) = E_{x,k} v^\varepsilon_{v^\varepsilon_\tau}(t-\tau, X^\varepsilon_\tau) \exp\left\{\frac{1}{\varepsilon}\int_0^\tau \tilde{c}^\gamma_{v^\varepsilon_s}(X^\varepsilon_s, v^\varepsilon(t-s, X^\varepsilon_s))\,ds\right\}, \tag{3.4}$$

where $\tilde{c}_k^\gamma(x,v) = (\gamma + c_k(x))(1 - \frac{v_k}{M})$, and τ is any stopping time for the process $(X_t^\varepsilon, v_t^\varepsilon)$ such that $P_{x,k}\{\tau \leqq t\} = 1$. We conclude from (3.4) that

$$0 \leqq v_k^\varepsilon(t,x) \leqq ME_{x,k}\chi_{t,G_0} \exp\left\{\frac{1}{\varepsilon}\int_0^\tau (\tilde{c}^\gamma_{v^\varepsilon_s}(X^\varepsilon_s, v^\varepsilon(t-s, X^\varepsilon_s))\,ds\right\} = D_k^\varepsilon(t,x) \, , \tag{3.5}$$

where χ_{t,G_0} is the indicator function of the complement of the set $\{\tau = t, X_t^\varepsilon \notin G_0\}$, $G_0 = \{x \in R^r\colon\ \sum_1^n g_i(x) > 0\}$.

Now, since $\tilde{V}_1(t,x) < -\frac{\beta}{2}$ for $(t,x) \in \phi$, the definition of V_1 in Sect. 2 implies that there exist a closed set F and the corresponding stopping time $\tau^* = \tau_F \in \Theta_t$ such that

$$\sup\left\{\int_0^{\tau*}\left[\gamma + \left(\sum_k c_k(\varphi_s)\dot{\mu}_s^k) - \eta(\varphi_s, \dot{\varphi}_s, \dot{\mu}_s\right)\right]ds\colon \varphi_0 = x,\ \varphi_t \in G_0\right\} < -\frac{\beta}{2} \, .$$

Denote χ_l the indicator of the set

$$S_l = \left\{\frac{t(l-1)}{m} \leqq \tau^* < \frac{tl}{m}\right\}, \quad l = 1,\dots,m \, .$$

Then taking into account that $c_k(x) \geqq 0$, we can write:

$$\begin{aligned} D_k^\varepsilon(t,x) \leqq{} & M\sum_{l=1}^m E_{x,k}\chi_l\chi_{t,G_0} \exp\left\{\frac{1}{\varepsilon}\int_0^{tl/m}(\gamma + c_{v^\varepsilon_s}(X^\varepsilon_s))\,ds\right\} \\ & + ME_{x,k}\chi_{\tau^* = t}\chi_{t,G_0} \exp\left\{\frac{1}{\varepsilon}\int_0^t (\gamma + c_{v^\varepsilon_s}(X^\varepsilon_s)\,ds\right\} . \end{aligned} \tag{3.6}$$

Notice that the set $\{\phi\colon \phi_0 = x,\ \phi_t \in G_0,\ \phi \in S_l\}$ is contained in

$T_l = \{\phi\colon \phi_0 = x,\ \phi_t \in G_0,\ (t-s, \phi(s)) \in F$ for some $(l-1)t/m \leqq s \leqq lt/m\}$, which is closed with respect to the supremum norm. Using the large deviation principle, we get

$$\lim_{\varepsilon\downarrow 0} \varepsilon \ln E_{x,k}\chi_l \chi_{t,G_0} \exp\left\{\frac{1}{\varepsilon}\int_0^{tl/m} (c_{v_s^\varepsilon}(X_s^\varepsilon)+\gamma)\,ds\right\}$$

$$\leqq \sup\left\{\int_0^{tl/m}\left(\gamma+\sum_k c_k(\varphi_s)\dot{\mu}_s^k\right)ds - S_{0,tl/m}(\varphi,\mu)\colon \phi\in T_l\right\}. \tag{3.7}$$

Note that the supremum in the righthand side of (3.7) is bounded from above by

$$\sup\left\{\int_0^{\tau*}\left[\left(\gamma+\sum_i c_i(\varphi_s)\dot{\mu}_s^i\right)-\eta(\varphi_s,\dot{\varphi}_s,\dot{\mu}_s)\right]ds\colon \phi\in T_l\right\}$$

$$+\frac{t}{m}(\gamma+|c|)\leqq -\frac{\beta}{2}+\frac{t(\gamma+|c|)}{m}, \tag{3.8}$$

for $l = 1,\ldots,m$ and μ as in (3.7). Now choosing $m > \frac{4t}{\beta}(\gamma+|c|)$ we derive from (3.7) and (3.8) that

$$\overline{\lim_{\varepsilon\downarrow 0}}\, \varepsilon \ln E_{x,k}\chi_l \chi_{t,G_0} \exp\left\{\frac{1}{\varepsilon}\int_0^{\tau*} c_{v_s^\varepsilon}(X_s^\varepsilon)\,ds\right\} \leqq -\frac{\beta}{4}. \tag{3.9}$$

A similar bound holds for the last term in the righthand side of (3.6):

$$\overline{\lim_{\varepsilon\downarrow 0}}\, \varepsilon \ln E_{x,k}\chi_{\tau^*} = {}_t\chi_{t,G_0} \exp\left\{\frac{1}{\varepsilon}\int_0^{t} c_{v_s^\varepsilon}(X_s^\varepsilon)\,ds\right\} \leqq -\frac{\beta}{4}. \tag{3.10}$$

The statement of Lemma 3.2 follows from (3.2), (3.5)–(3.10). □

Consider an auxiliary problem

$$\frac{\partial u^\varepsilon(t,x)}{\partial t} = \frac{\varepsilon}{2}\sum_{i,j=1}^{r} a^{ij}(x)\frac{\partial^2 u^\varepsilon}{\partial x^i \partial x^j} + \frac{1}{\varepsilon}f(u^\varepsilon)$$

$$= \varepsilon L u^\varepsilon + \frac{1}{\varepsilon}f(u^\varepsilon),\quad t>0,\ x\in G\subset R^r,\ u^\varepsilon(t,x)\Big|_{x\in\partial G} = 0,$$

$$u^\varepsilon(0,x) = g^\varepsilon(x) \geqq 0,\ \varepsilon > 0. \tag{3.11}$$

We assume that L is an elliptic operator with bounded smooth enough coefficients, $g^\varepsilon(x)$ is bounded and continuous except on maybe a finite number of smooth manifolds of dimension $r-1$, where it has simple discontinuities: $\lim g^\varepsilon(x)$ exists when x approaches a point on the discontinuity manifold from one side and equal to a continuous function on the manifold. The nonlinear term in (3.11) is assumed to be of KPP type: there exists $\mu > 0$ such that $f(u) = c(u)u$, where $c(u)$ is Lipschitz continuous, $c(u) > 0$ for $0 < u < \mu$, $c(\mu) = 0$, $c = c(0) = \max_{u\geqq 0} c(u)$, $c(u) < 0$ for $u > \mu$. One can check that $0 \leqq u^\varepsilon(t,x) \leqq \max(\mu, \sup_x g(x))$ (see [F3]). Denote $\rho(\cdot,\cdot)$ the Riemannian metric in R^r corresponding to the form $\sum_{i,j=1}^n a_{ij}(x)\,dx^i\,dx^j$, $(a_{ij}(x)) = (a^{ij}(x))^{-1}$. Let $G = G_h = \{x\in R^r,\ \rho(x,0) < h\}$, $h > 0$, and let ∂G be a smooth $(r-1)$-dimensional manifold.

Lemma 3.3 *Assume that for any* $\delta > 0$ *there exists* $\varepsilon_0 > 0$ *such that* $g^\varepsilon(x) > e^{-\delta/\varepsilon}$ *for* $|x| < e^{-2\delta/\varepsilon}$ *and* $\varepsilon \in (0, \varepsilon_0)$.

Then

$$\lim_{\varepsilon \downarrow 0} u^\varepsilon(t,x) = \mu$$

for the points (t,x) *such that* $\rho(x,0) < \min(t\sqrt{2c}, h)$. *The convergence is uniform in* (t,x) *from any compact subset of* $\{(t,x):\ t > 0,\ x \in R^r$ *and* $\rho(x,0) < \min(t\sqrt{2c}, h)\}$.

Proof. Let

$$g^\varepsilon_\delta(x) = \begin{cases} e^{-\delta/\varepsilon}, & \text{for } |x| \leqq e^{-2\delta/\varepsilon}\,, \\ 0, & \text{for } |x| > e^{-2\delta/\varepsilon}\,, \end{cases}$$

The solution of problem (3.11) with $g^\varepsilon(x) = g^\varepsilon_\delta(x)$ is denoted $u^\varepsilon_\delta(t,x)$. It follows from Theorem 6.2.2 of [F1] that

$$\lim_{\varepsilon \to 0} u^\varepsilon_\delta(t,x) = 0 \quad \text{if } \rho(x,0) > t\sqrt{2c}\,, \tag{3.12}$$

uniformly in any compact subset of $\{(t,x):\ t > 0,\ h > \rho(x,0) > t\sqrt{2c}\}$.

Now, let us prove that

$$\varliminf_{\varepsilon \downarrow 0} \varepsilon \ln u^\varepsilon_\delta(t,x) > -6\delta \tag{3.13}$$

if $\rho = (x,0) = t\sqrt{2c} < h$. Using the Feynman–Kac formula we get

$$u^\varepsilon_\delta(t,x) = E_x\, g^\varepsilon_\delta(X^\varepsilon_t) \exp\left\{\frac{1}{\varepsilon}\int_0^t c(u^\varepsilon_\delta(t-s, X^\varepsilon_s))\, ds\right\} \chi_{\tau > t}\,, \tag{3.14}$$

where X^ε_t is the diffusion process in R^r governed by the operator εL, $\tau = \min\{t : X^\varepsilon_t \notin G\}$ and $\chi_{\tau>t}$ is the indicator function of the set $\{\tau > t\}$. Let $\hat{\varphi}_s, 0 \leqq s \leqq t$, be the minimal geodesic in the metric ρ connecting points x and 0; $\hat{\varphi}_0 = x$, $\hat{\varphi}_t = 0$, $\rho(x,0) < h$, with the parametrization proportional to the Riemannian length. Let $\kappa > 0$ be so small that

$$3c\kappa < \delta, \qquad \frac{\rho^2(0,x)}{t - 3\kappa} - \frac{\rho^2(0,x)}{t} < \delta, \qquad \frac{\rho^2(0, \hat{\varphi}_{t-3\kappa})}{\kappa} < \delta\,, \tag{3.15}$$

and $\mu_1 > 0$ be such that the transition density $p^\varepsilon(t,x,y)$ of the process X^ε_t satisfies the inequality

$$p^\varepsilon(\kappa, x, y) > \exp\left\{-\frac{\delta}{\varepsilon}\right\} \quad \text{for } |x| < \mu_1,\ |y| < \mu_1\,. \tag{3.16}$$

The existence of such μ_1 follows, for example, from the Varadhan's result [V]:

$$\lim_{\varepsilon \downarrow 0} \varepsilon \ln p^\varepsilon(t,x,y) = -\frac{\rho^2(x,y)}{2t}\,.$$

Consider the function $\bar{\varphi}_s$, $0 \leqq s \leqq t-\kappa$, defined as follows:

$$\bar{\varphi}_s = \begin{cases} x & \text{if } 0 \leqq s \leqq \kappa\,, \\ \hat{\varphi}_{s-\kappa} & \text{if } \kappa \leqq s \leqq t-2\kappa\,, \\ \tilde{\varphi}_{s-(t-2\kappa)} & \text{if } t-2\kappa \leqq s \leqq t-\kappa\,, \end{cases}$$

where $\tilde{\varphi}_s$ is the minimal geodesic connecting points $\hat{\varphi}_{t-3\kappa}$ and 0 in time $[0,\kappa]$ with the parameter proportional to the Riemannian length. The function $\bar{\varphi}_s$, $0 \leqq s \leqq t-\kappa$, starts at x and ends at 0. Denote

$$\mu_2 = \max_{\kappa \leqq s \leqq t-2\kappa} |\hat{\varphi}_s - \bar{\varphi}_s|, \qquad \mu = \tfrac{1}{2}\min(\mu_1, \mu_2)\,.$$

Note that the closure of the Euclidean μ-neighborhood Γ_μ of the curve $(t-s, \bar{\varphi}_s)$, $\kappa \leqq s \leqq t-2\kappa$, belongs to the set $\{(t,x)\colon \rho(x,0) > t\sqrt{2c}\}$. Thus, due to (3.12), there exists $\varepsilon_0 > 0$ such that $c(u^\varepsilon(t-s,x)) > c - \frac{\delta}{t}$ for $(t-s, \bar{\varphi}_s) \in \Gamma_\mu$, $\varepsilon < \varepsilon_0$. Denote $\chi^\mu_{\bar{\varphi}}$ the indicator of Γ_μ. Then we have from (3.14) and the large deviation principle for $\varepsilon > 0$ small enough

$$\begin{aligned} u^\varepsilon_\delta(t,x) &\geqq E_x u^\varepsilon_\delta(\kappa, X^\varepsilon_{t-\kappa})\chi^\mu_{\bar{\varphi}} \exp\left\{\frac{1}{\varepsilon}\int_0^{t-\kappa} c(u^\varepsilon_\delta(t-s, X^\varepsilon_s))\,ds\right\} \\ &\geqq E_x \min_{|y| \leqq \mu} u^\varepsilon_\delta(\kappa, y) \exp\left\{\frac{1}{\varepsilon}\left[c(t-\kappa) - S^*_{0,t-\kappa}(\bar{\varphi}) - \delta\right]\right\}\,, \end{aligned} \tag{3.17}$$

where $\frac{1}{\varepsilon}S^*_{0T}(\bar{\varphi})$ is the action functional for the process X^ε_s as $\varepsilon \downarrow 0$. As shown in [F1, Sect. 6.2]

$$S^*_{0,t-\kappa}(\bar{\varphi}) = \frac{\rho^2(x, \hat{\varphi}_{t-3\kappa})}{2(t-3\kappa)} + \frac{\rho^2(\hat{\varphi}_{t-3\kappa}, 0)}{2\kappa}\,.$$

We conclude from (3.15) that

$$S^*_{0,t-\kappa}(\bar{\varphi}) < \frac{\rho^2(x,0)}{2t} + \delta\,. \tag{3.18}$$

One can derive from (3.14) and (3.16) that

$$u^\varepsilon_\delta(\kappa, x) > e^{-3\delta/\varepsilon} \quad \text{for } |x| \leqq \mu\,. \tag{3.19}$$

Gathering bounds (3.17)–(3.19), we derive (3.13).

Since $g^\varepsilon(x) \geqq g^\varepsilon_\delta(x)$ we conclude that

$$u^\varepsilon(t,x) \geqq e^{-10\delta/\varepsilon} \quad \text{if } \rho(x,0) = t\sqrt{2c} < h$$

and ε is small enough. Taking into account that $\delta > 0$ is arbitrary we get

$$\lim_{\varepsilon \downarrow 0} \varepsilon \ln u^\varepsilon(t,x) = 0\,, \tag{3.20}$$

if $\rho(x,0) = t\sqrt{2c} < h$. The statement of Lemma 3.3 follows from (3.20), (3.14) and the strong Markov property by standard arguments (see Sect. 6.2 in [F1] or [F2]), and we omit them. □

Let

$$K^A_{t_0,x_0} = \{(t,x):\ t > t_0,\ |x - x_0| < A(t - t_0)\}\ ,$$

$$D^A_{t_0,x_0} = \{(t,x):\ 0 < t < t_0,\ |x_0 - x| < A(t_0 - t)\}\ .$$

Lemma 3.4 *Assume that for some* $k_0 \in \{1,\dots,n\}$, $t_0 \geqq 0$, $x_0 \in R^r$

$$\lim_{\varepsilon\downarrow 0} \varepsilon \ln u^{\varepsilon}_{k_0}(t_0,x_0) = 0\ .$$

Then there exists $A > 0$ *such that*

$$\underline{\lim}_{\varepsilon\downarrow 0}\, u^{\varepsilon}_k(t,x) > 0\ ,$$

uniformly in (t,x) *from any compact subset of* $K^A_{t_0,x_0}$ *and any* $k \in \{1,\dots,n\}$.

Proof. Let $\underline{\beta}$ be as in condition (A2), $B' = B'(\frac{\beta}{2}) < B$ as in condition (A3). Let $-M$ be the infimum in (A1). Let $f(u)$ be a continuously differentiable function on $[0,\infty)$ such that

$$f(u) = \begin{cases} \frac{\beta}{2}\cdot u(1 - \frac{2u}{B'}) & \text{if } 0 \leqq u \leqq \frac{B'}{2}\ , \\ -M & \text{if } u > B' \end{cases}$$

and let $f(u)$ decreases on $(\frac{B'}{2}, B')$.

Consider the following system of uncoupled equations:

$$\frac{\partial v^{\varepsilon}_k}{\partial t} = L^{\varepsilon}_k v^{\varepsilon}_k + \frac{1}{\varepsilon} f(v^{\varepsilon}_k), \quad t > 0,\ x \in R^r\ ,$$

$$v^{\varepsilon}_k(0,x) = u^{\varepsilon}_k(t_0,x), \quad k = 1,\dots,n\ .$$

Because of conditions (A1)–(A3)

$$F_k(x,u) \geqq f(u_k), \quad x \in R^r,\ u \in [0,B]^n,\ k = 1,\dots,n\ .$$

It is readily checked that the vector field f satisfies (A1)–(A3). Thus, according to Lemma 3.1(i) and (iii),

$$u^{\varepsilon}_k(t_0 + t,x) \geqq v^{\varepsilon}_k(t,x), \quad t \geqq 0,\ x \in R^r,\ 1 \leqq k \leqq n\ . \tag{3.21}$$

Note that if functions $u^{\varepsilon}_1(t,x),\dots,u^{\varepsilon}_n(t,x)$ satisfy (1.2), (1.4), then $\bar{u}^{\varepsilon}_k(t,x) = u^{\varepsilon}_k(\varepsilon t, \varepsilon x)$, $k = 1,\dots,n$, satisfy equations

$$\frac{\partial \bar{u}^{\varepsilon}_k}{\partial t} = \frac{1}{2}\sum_{i,j=1}^{n} a^{ij}_k(\varepsilon x)\frac{\partial^2 \bar{u}^{\varepsilon}_k}{\partial x^i \partial x^j} + F_k(\varepsilon x;\ \bar{u}^{\varepsilon}_1,\dots,\bar{u}^{\varepsilon}_n)\ ,$$

$$t > 0, \quad x \in R^r,\ k = 1,\dots,n\ ,$$

and according to Lemma 3.1(i), $0 \leqq \bar{u}^{\varepsilon}_k(t,x) \leqq B$. Taking into account that the equations for $\bar{u}^{\varepsilon}_k$ are parabolic uniformly in $\varepsilon \in (0,1]$ and have bounded coefficients and their first derivatives, we conclude from the standard apriori bounds for the solutions of (linear) parabolic equations (see, for example, [Fri, Theorem 4 in Ch. 7]) that $|\nabla_x \bar{u}_k^\varepsilon(t,x)| < \text{const.} < \infty$ uniformly in $\varepsilon \in (0,1]$. Thus $|\nabla_x u_k^\varepsilon(t,x)| \leqq \text{const.}\, \varepsilon^{-1}$. The last bound implies, that if $\lim_{\varepsilon\downarrow 0} \varepsilon \ln u_{k_0}^\varepsilon(t_0,x_0) = 0$ then for any $\delta > 0$ there exists ε_0 such that

$$u_{k_0}^\varepsilon(t_0,x) > e^{-\delta/\varepsilon} \quad \text{for } |x - x_0| < e^{-2\delta/\varepsilon},\ 0 < \varepsilon < \varepsilon_0 .$$

Then we derive from Lemma 3.3, that there exists $A > 0$ such that for any compact subset K of the cone K_{0,x_0}^A

$$\lim_{\varepsilon\downarrow 0} v_{k_0}^\varepsilon(t,x) = \frac{B'}{2} \quad \text{uniformly in } (t,x) \in K .$$

Thus due to (3.21)

$$\varliminf_{\varepsilon\to 0} u_{k_0}^\varepsilon(t,x) \geqq \frac{B'}{2} ,$$

uniformly in (t,x) from any compact subset of the cone K_{t_0,x_0}^A.

To finish the proof of Lemma 3.4 we need to show that for all $k = 1,2,\ldots,n$ and some $\kappa > 0$

$$\varliminf_{\varepsilon\to 0} u_k^\varepsilon(t,x) \geqq \kappa , \tag{3.22}$$

uniformly in any compact subset of K_{t_0,x_0}^A.

It follows from (A1)–(A3) and the mean value theorem that

$$F_k(x,u) = \sum_{j=1}^n c_{kj}(x,u)u_j, \quad 1 \leqq k \leqq n,\ x \in R^r,\ u \in R^n ,$$

where $c_{kj}(x,u)$ are continuous and bounded and $c_{kj}(x,u) \geqq 0$ for $x \in R^r$, $|u| < \kappa_1$ for some $\kappa_1 > 0$. Let $\Xi_s^\varepsilon = (t_s, X_s^\varepsilon, v_s^\varepsilon)$ be the Markov process in the state space $R^1 \times R^r \times \{1,\ldots,n\}$ governed by the generator A:

$$Ah(s,x,k) = -\frac{\partial h}{\partial s} + L^\varepsilon h + \frac{1}{\varepsilon}\sum_{j=1}^n c_{kj}(x,u^\varepsilon(s,x))\,(h(s,x,j) - h(s,x,k)) ,$$

where $u^\varepsilon(s,x)$ is the solution of problem (1.2), (1.4). The process Ξ_s is defined at least in the domain $\{(s,x,k)\colon |u^\varepsilon(s,x)| < \kappa_1\}$; $t_s = t_0 - s$.

Suppose (3.22) is not true: there exist k_1, a compact $K \in K_{t_0,x_0}^A$, $\kappa_2, 0 < \kappa_2 < \frac{1}{2}\min(\kappa_1, B')$, and a sequence $(t^{\varepsilon'}, x^{\varepsilon'}) \in K$, such that

$$\lim_{\varepsilon'\downarrow 0} u_{k_1}^{\varepsilon'}\left(t^{\varepsilon'}, x^{\varepsilon'}\right) < \kappa_2 < \tfrac{1}{2}\min(\kappa_1, B') .$$

Denote

$$\Lambda^{\varepsilon'} = \{(s,y,k)\colon u_k^{\varepsilon'}(s,y) < 2\kappa_2\} .$$

$$\tau^{\varepsilon'} = \tfrac{1}{2}\min\left(\left(t^{\varepsilon'} - t_0\right),\ \min\left\{s\colon (t_s, X_s^{\varepsilon'}, v_s^{\varepsilon'}) \notin \Lambda^{\varepsilon'}\right\}\right) .$$

Taking into account that $c_k^{\varepsilon'}(s,x) = \sum_{j=1}^n c_{kj}(\kappa, u^{\varepsilon'}(s,x)) > 0$ in $\Lambda^{\varepsilon'}$ we conclude from the Feynman–Kac formula that

$$u_{k_1}^{\varepsilon'}\left(t^{\varepsilon'}, x^{\varepsilon'}\right) \geqq E_{t^{\varepsilon'},x_{\varepsilon'},k_1} u^{\varepsilon'}_{v_{\tau^{\varepsilon'}}^{\varepsilon'}}\left(t^{\varepsilon'} - \tau^{\varepsilon'}, X^{\varepsilon'}_{\tau^{\varepsilon'}}\right) . \tag{3.23}$$

The difference $t^{\varepsilon'} - t_0$ is bounded from below by a positive constant since all $(t^{\varepsilon'}, x^{\varepsilon'}) \in K \subset K^A_{t_0,x_0}$. Taking into account the positivity of the jumping intensities $c_{k,j}(x,u)$ and the result for $u^{\varepsilon}_{k_0}(t,x)$ we can conclude that

$$P_{t^{\varepsilon'},x^{\varepsilon'},k_1}\left\{\tau^{\varepsilon'} < \tfrac{1}{2}\left(t^{\varepsilon'} - t_0\right)\right\} \to 1 \quad \text{as } \varepsilon' \downarrow 0 ,$$

and thus

$$P_{t^{\varepsilon'},x^{\varepsilon'},k_1}\left\{u^{\varepsilon'}_{v_{\tau^{\varepsilon'}}^{\varepsilon'}}\left(t_{\varepsilon'} - \tau^{\varepsilon'}, X^{\varepsilon'}_{\tau^{\varepsilon'}}\right) \geqq 2\kappa_2\right\} \to 1 \quad \text{as } \varepsilon \downarrow 0 . \tag{3.24}$$

From (3.23) and (3.24) we conclude that

$$\varliminf_{\varepsilon' \to 0} u_{k_1}^{\varepsilon'}\left(t^{\varepsilon'}, x^{\varepsilon'}\right) \geqq \kappa_2 .$$

This contradiction proves (3.22). □

Denote

$$\mathscr{E}^{(\varepsilon')} = \left\{(t,x) \colon t > 0,\ x \in R^r,\ \varliminf_{\varepsilon' \to 0} u_k^{\varepsilon'}(t,x) = 0 \text{ for some } k = 1, \ldots, n\right\} .$$

Here (ε') is a sequence such that $\varepsilon' > 0$, $\varepsilon' \to 0$.

Lemma 3.5 (i) *If* $(t_0, x_0) \in \mathscr{E}^{(\varepsilon')}$ *then there exists* $A > 0$ *such that*

$$\varlimsup_{\varepsilon' \downarrow 0} \varepsilon' \ln u_k^{\varepsilon'}(t,x) < 0$$

for any point $(t,x) \in D^A_{t_0,x_0}$ *and any* $k = 1, \ldots, n$.
(ii) *For any compact* K *contained in the interior* $(\mathscr{E}^{(\varepsilon')})$ *of the set* $\mathscr{E}^{(\varepsilon')}$

$$\lim_{\varepsilon' \downarrow 0} u_k^{\varepsilon'}(t,x) = 0 ,$$

uniformly in $(t,x) \in K$ *and* $1 \leqq k \leqq n$.
(iii) $\mathscr{E}^{(\varepsilon')} \subset [(\mathscr{E}^{(\varepsilon')})]$, *where* [D] *means the closure of the set D. If* $(t,x) \in \mathscr{E}^{(\varepsilon')}$, *then* $(t-h, x) \in \left(\mathscr{E}^{(\varepsilon')}\right)$ *for* $0 < h < t$.

Proof. The first statement follows immediately from Lemma 3.4. To prove the second statement note that K can be covered by a finite number of cones $D^A_{t_k,x_k}$ with vertices $(t_k, x_k) \in \left(\mathscr{E}^{(\varepsilon')}\right) \setminus K$. The uniformity follows from the uniformity of the bound in Lemma 3.4. The last statement follows from (i) and (ii). □

Denote

$$M = \{(t,x) \colon t > 0,\ x \in R^r, V_1(t,x) = 0\} ;$$

(M) means the interior of M.

Lemma 3.6 *Let K be compact, $K \subset (M)$. Then for any $k = 1, 2, \ldots, n$.*

$$\lim_{\varepsilon \downarrow 0} \varepsilon \ln u_k^\varepsilon (t, x) = 0$$

uniformly in $(t, x) \in K$.

Proof. The proof of this lemma is similar to the proof of Lemma 4 of [F3]. Assume that for a point $(t, x) \in (M)$ and for some $k = 1, \ldots, n$ there exists a sequence $\varepsilon' \downarrow 0$ such that $\lim_{\varepsilon' \downarrow 0} \varepsilon' \ln u_k^{\varepsilon'}(t, x) = -\beta < 0$. Then $\lim_{\varepsilon' \downarrow 0} u_k^{\varepsilon'}(t, x) = 0$, that is, $(t, x) \in \mathscr{E}^{(\varepsilon')}$. Without loss of generality we can assume that $(t, x) \in \left(\mathscr{E}^{(\varepsilon')}\right)$. If this is not true, one can take a point $(t - h, x)$ with small enough $h > 0$. This new point belongs to $\left(\mathscr{E}^{(\varepsilon')}\right)$ due to Lemma 3.5(iii) and belongs to (M) since (M) is open.

Define the stopping time τ corresponding to the complement of the set $\left(\mathscr{E}^{(\varepsilon')}\right)$:

$$\tau = \tau(t, \varphi) = \min\{s : (t - s, \varphi_s) \notin \left(\mathscr{E}^{(\varepsilon')}\right)\} .$$

It is clear that $\tau \leqq t$ a.s.

Since $(t, x) \in M$

$$\sup \left\{ \int_0^\tau \zeta(\varphi_s, \dot{\varphi}_s)\, ds \colon \varphi_0 = x, \varphi_t \in G_0 \right\} \geqq 0 ,$$

where G_0 is the support of $\sum_{i=1}^n g_i(x)$. Therefore for any $\delta > 0$ there exists $\varphi_s = \varphi_s^\delta$, $0 \leqq s \leqq t$, $\varphi_0 = x$, $\varphi_t \in G_0$, such that

$$R_{0,\tau}(\varphi) = \int_0^\tau \zeta(\varphi_s, \dot{\varphi}_s)\, ds \geqq -\frac{\delta}{4} ,$$

$$(t - s, \varphi_s) \in (\mathscr{E}^{(\varepsilon')}) \quad \text{for } 0 \leqq s < \tau(t, \varphi) ,$$

$$(t - \tau(t, \varphi), \varphi_{\tau(t,\varphi)}) \in \partial \mathscr{E}^{(\varepsilon')} .$$

Now we define a reconstruction of φ_s such that the new function $\bar{\varphi}_s$, $0 \leqq s \leqq \bar{T}$, spends most of the time in $\left(\mathscr{E}^{(\varepsilon')}\right)$, ends outside of the closure $\left[\mathscr{E}^{(\varepsilon')}\right]$ of $\mathscr{E}^{(\varepsilon')}$ and satisfies $R_{0,\bar{T}}(\bar{\varphi}) \geqq -\frac{\delta}{2}$.

Let λ_1, λ_2 be small positive numbers, $T = \tau(t, \varphi)$. Define

$$\bar{\varphi}_s = \varphi^{\lambda_1, \lambda_2} = \begin{cases} x & \text{for } s \in [0, \lambda_1] , \\ \varphi_{(s-\lambda_1)(T-\lambda_1)/(T-2\lambda_1)} & \text{for } s \in [\lambda_1, T - \lambda_1] , \\ \varphi_{T-\lambda_1+(s-T+\lambda_1)/(1-\lambda_2)} & \text{for } s \in [T - \lambda_1, T - \lambda_1\lambda_2] , \end{cases}$$

The function $\bar{\varphi}_s$ is defined for $s \in [0, T - \lambda_1\lambda_2]$, $\bar{\varphi}_{\bar{T}} = \varphi_T = z$, where $\bar{T} = T - \lambda_1\lambda_2$.

According to Lemma 2.2(i) one can choose the numbers $\lambda_1, \lambda_2 > 0$ so small that

$$(3.25)\qquad \int_0^{\bar{T}} \zeta(\bar{\varphi}_s, \dot{\bar{\varphi}}_s)\, ds \geqq -\frac{\delta}{2}, \qquad \int_{\bar{T}-2(\lambda_1+\lambda_2)}^{T} \zeta(\bar{\varphi}_s, \dot{\bar{\varphi}}_s)\, ds < \frac{\delta}{8}, \qquad (\lambda_1+\lambda_2)(1+\bar{\beta}) < \frac{\delta}{8},$$

$\bar{\beta}$ is as in (A2). Note that since $(s-\lambda_1)(T-\lambda_1)(T-2\lambda_1)^{-1} \leqq s$, for $s \leqq T-\lambda_1$,

$$(t-s, \bar{\varphi}_s) \in \left(\mathscr{E}^{(\varepsilon')}\right), \quad 0 \leqq s \leqq T-\lambda_1 .$$

Since $(t-T, z) \notin (\mathscr{E}^{(\varepsilon')})$ and $\bar{T} < T$ we conclude from Lemma 3.4 that $(t-\bar{T}, z) \notin [\mathscr{E}(\varepsilon')]$. Thus there exists a subsequence $\{\varepsilon''\}$ of $\{\varepsilon'\}$ and $\alpha_0, \kappa > 0$ such that

$$(3.26)\qquad \lim_{\varepsilon''\downarrow 0} u_k^{\varepsilon''}(t-\bar{T}, x) \geqq \kappa > 0, \quad 1 \leqq k \leqq n,\ |x-z| < \alpha_0 .$$

Let $\tilde{c}_{ij}(x) = c_{ij}(x) - \gamma, 0 < \gamma < \frac{\beta}{2}$. Denote $\tilde{\zeta}(x,q)$ the function defined by (1.5) for $\{c_{ij}(x)\}_1^n$ replaced by $\{\tilde{c}_{ij}(x)\}_1^n$. Let $\gamma > 0$ be so small that if $\sup_{x\in R^r,\, 1\leqq i,j\leqq n} |c_{ij}(x) - \tilde{c}_{ij}(x)| < \gamma$, then

$$(3.27)\qquad \left| \int_0^{\bar{T}} \tilde{\zeta}(\bar{\varphi}_s, \dot{\bar{\varphi}}_s)\, ds - \int_0^{\bar{T}} \zeta(\bar{\varphi}_s, \dot{\bar{\varphi}}_s)\, ds \right| < \frac{\delta}{8} .$$

Such $\gamma > 0$ exists according to Lemma 2.2(ii). Let $B' = B'(\gamma)$ be chosen as in condition (A3), and $\alpha \in (0, \alpha_0)$ be so small that

$$(3.28)\qquad G_\alpha = \{(t-s, x)\colon\ 0 \leqq s \leqq t - 2\lambda_1,\ |x - \bar{\varphi}_s| \leqq 2\alpha\} \subset \left(\mathscr{E}^{(\varepsilon')}\right), \qquad \sum_{i,j} |\tilde{c}_{ij}(x) - \tilde{c}_{ij}(y)| < \frac{\delta}{8t} \quad \text{for } |x-y| < \alpha .$$

Consider the Markov process $(\tilde{X}_t^\varepsilon, \tilde{v}_t^\varepsilon)$ in the state space $R^r \times \{1, \ldots, n\}$ corresponding to the generator $\tilde{A}$,

$$\tilde{A}h(x,k) = L_k^\varepsilon h(x,k) + \frac{1}{\varepsilon} \sum_{j=1}^n \tilde{c}_{kj}(x)(h(x,j) - h(x,k)) .$$

Denote

$$\sigma_1^\varepsilon = \min\{s\colon\ (t-s, \tilde{X}_s^\varepsilon) \notin G_\alpha\} ,$$
$$\sigma_2^\varepsilon = \min\{s\colon\ u_{\tilde{v}_s^\varepsilon}^\varepsilon(t-s, \tilde{X}_s^\varepsilon) = \tfrac{1}{2}\min(\kappa, B')\} ,$$
$$\sigma^\varepsilon = \min(\sigma_1^\varepsilon, \sigma_2^\varepsilon) .$$

From now on we will write superscript ε in place of ε''. Since $G_\alpha \subset \left(\mathscr{E}^{(\varepsilon')}\right)$ and $u^\varepsilon(t-\bar{T}, x) > \frac{\kappa}{2}$ for $|x-z| < \alpha$ and ε small enough,

$$(3.29)\qquad P_{x,k}\{\bar{T} > \sigma_2^\varepsilon > (T - 2\lambda_1) \text{ or } \sigma_1^\varepsilon < \sigma_2^\varepsilon\} = 1 .$$

for ε small enough, $|x-z| < \alpha$.

Using the Feynman–Kac formula one can write:

$$u_k^\varepsilon(t,x) = E_{x,k} u^\varepsilon_{\tilde{v}^\varepsilon_{\sigma^\varepsilon}}(t-\sigma^\varepsilon, \tilde{X}^\varepsilon_{\sigma^\varepsilon}) \exp\left\{\frac{1}{\varepsilon}\int_0^{\sigma^\varepsilon} \tilde{c}_{\tilde{v}^\varepsilon_s}(\tilde{X}^\varepsilon_s)\,ds\right\}$$

$$+ \frac{1}{\varepsilon} E_{x,k} \int_0^{\sigma^\varepsilon} \left[F_{v^\varepsilon_s}(\tilde{X}^\varepsilon_s, u^\varepsilon(t-s,\tilde{X}^\varepsilon_s)) - \sum_{j=1}^n \tilde{c}_{\tilde{v}^\varepsilon_s j}(\tilde{X}^\varepsilon_s) u^\varepsilon_j(t-s,\tilde{X}^\varepsilon_s)\right]$$

$$\times \exp\left\{\frac{1}{\varepsilon}\int_0^{s} \tilde{c}_{\tilde{v}^\varepsilon_s}(X^\varepsilon_s)\,ds\right\} ds\,, \tag{3.30}$$

where $\tilde{c}_k(x) = \sum_{j=1}^n \tilde{c}_{kj}(x)$. Since $F_k(x,u) - \Sigma_j \tilde{c}_{kj}(x) u_j \geqq 0$ for $u_j \leqq B',\ 1 \leqq j \leqq n$, the second term in the righthand side of (3.30) is positive. Taking into account the definition of σ_2^ε we derive from (3.30):

$$u_k^\varepsilon(t,x) \geqq \tfrac{1}{2}\min(\kappa, B') E_{x,k} \chi_{\sigma_1^\varepsilon > \sigma_2^\varepsilon} \exp\left\{\frac{1}{\varepsilon}\int_0^{\sigma_2^\varepsilon} \tilde{c}_{\tilde{v}^\varepsilon_s}(\tilde{X}^\varepsilon_s)\,ds\right\}, \tag{3.31}$$

where $\chi_{\sigma_1^\varepsilon > \sigma_2^\varepsilon}$ is the indicator function of the set $\{\sigma_1^\varepsilon > \sigma_2^\varepsilon\}$. It follows from (3.29) and (3.31) that

$$u_k^\varepsilon(t,x) \geqq \tfrac{1}{2}\min(\kappa, B') E_{x,k} \chi_{\sigma_1^\varepsilon > \bar{T}} \exp\left\{\frac{1}{\varepsilon}\int_0^{T-2\lambda_1} \tilde{c}_{\tilde{v}^\varepsilon_s}(\tilde{X}^\varepsilon_s)\,ds\right\},$$

and because of (3.25) and (3.28)

$$u_k^\varepsilon(t,x) \geqq e^{-\delta/2\varepsilon} E_{x,k} \chi_{\sigma_1^\varepsilon > T-2\lambda_1} \exp\left\{\frac{1}{\varepsilon}\int_0^{T-2\lambda_1} \tilde{c}_{\tilde{v}^\varepsilon_s}(\tilde{X}^\varepsilon_s)\,ds - \delta\right\}, \tag{3.32}$$

for $\varepsilon > 0$ small enough.

One can derive from the lower bound for probabilities of large deviations for process $(\tilde{X}^\varepsilon_s, \tilde{v}^\varepsilon_s)$ that

$$\varepsilon \ln E_{x,k} \chi_{\sigma_1^\varepsilon > T-2\lambda_1} \exp\left\{\frac{1}{\varepsilon}\int_0^{T-2\lambda_1} \tilde{c}_{\tilde{v}^\varepsilon_s}(\tilde{X}^\varepsilon_s)\,ds\right\} \geqq \int_0^{T-2\lambda_1} \tilde{\zeta}(\bar{\varphi}_s, \dot{\bar{\varphi}}_s)\,ds - \delta \tag{3.33}$$

if ε is small enough. From (3.25), (3.27), (3.32) and (3.33) we conclude that there exists $\varepsilon_0 > 0$ such that

$$u_k^\varepsilon(t,x) \geqq e^{-10\delta/\varepsilon}, \quad \varepsilon < \varepsilon_0\,.$$

Since δ is an arbitrary positive number, the last inequality contradicts to the assumption that $\lim_{\varepsilon'\downarrow 0} \varepsilon' \ln u_k^{\varepsilon'}(t,x) = -\beta < 0$, and thus

$$\varliminf_{\varepsilon\downarrow 0} \varepsilon \ln u_k^\varepsilon(t,x) \geqq 0\,. \tag{3.34}$$

On the other hand, it follows from Lemma 3.1(i) that

$$\overline{\lim_{\varepsilon\downarrow 0}}\, \varepsilon \ln u_k^\varepsilon(t,x) \leqq 0 . \tag{3.35}$$

We derive from (3.34) and (3.35) that

$$\lim_{\varepsilon\to 0} \varepsilon \ln u_k^\varepsilon(t,x) = 0, \quad (t,x) \in (M) .$$

To finish the proof of the lemma we need to show that the convergence is uniform in $(t,x) \in K$. The compact K can be covered by a finite number of cones $K_{t_i,x_i}^{A/2}$, $i = 1,\dots,N$, with the vertices $(t_i,x_i) \in (M)\backslash K$, where the constant A is defined as in Lemma 3.4. It was proved that $\lim_{\varepsilon\downarrow 0} \varepsilon \ln u_k^\varepsilon(t_i,x_i) = 0$ for $1 \leqq i \leqq N$ and $1 \leqq k \leqq n$. Now the uniformity of the convergence follows from Lemma 3.4. □

Proof of Theorem 1.1. The first statement follows from Lemmas 2.4 and 3.2. The second statement follows from Lemmas 2.4, 3.4, and 3.6.

4 Geometric description of wave fronts: Some examples

Suppose the space R^r is provided with a Riemannian metric $ds^2 = \sum_{i,j=1}^r a_{ij}(x)\, dx^i\, dx^j$.

We say that domains $G_t \subset R^r$, $t \geqq 0$, grow according to the Huygens principle with a velocity field $v(x,p)$, $x, p \in R^r$, if

$$G_{t_1} = \left\{ y \in R^r : \inf_{\varphi_0 \in G_{t_0}, \varphi_1 = y} \int_0^1 \frac{\sqrt{\Sigma a_{ij}(\varphi_s)\dot\varphi_s^i \dot\varphi_s^j}}{v(\varphi_s, \dot\varphi_s)/|\dot\varphi_s|}\, ds < t_1 - t_0 \right\}$$

for any $0 \leqq t_0 \leqq t_1 < \infty$. The infimum here is taken over all smooth φ_s, $0 \leqq s \leqq 1$, with values in R^r, connecting points of G_{t_0} and $y \in R^r$.

It is well known that many asymptotic problems for hyperbolic differential equations describing wave processes lead to a Huygens principle. It was proved in [F1] that the asymptotic behavior of the solution of problem (1.2), (1.4) as $\varepsilon \downarrow 0$ for a single equation ($n = 1$) also can be described by a Huygens principle if $c(x) = \partial F_1(x,u)/\partial u|_{u=0} = c$ independent of $x \in R^r$. Namely, it was shown that domains $G_t = \{x : V^*(t,x) > 0\}$ grow according to the Huygens principle, and the corresponding velocity field $v(x,p)$ is homogeneous and isotropic if calculated in Riemannian metric with $(a_{ij}(x)) = (a^{ij}(x))^{-1}$, where $(a^{ij}(x))$ is the diffusion matrix; $v(x,p) = \sqrt{2c}$ in this metric.

If $c(x) = \partial F(x,u)/\partial u|_{u=0}$ depends on x, then there exists no universal (independent of the initial data) Huygens principle, describing the motion of the interface between areas where $u^\varepsilon(t,x)$ tends to zero as $\varepsilon \downarrow 0$ and where $\underline{\lim}_{\varepsilon\to 0}\, u^\varepsilon(t,x) > 0$. Moreover, the motion of the interface (wave front) can be non-Markovian: for a given position of the interface at time s, the future motion can depend on the behavior of the front before time s (see example 2 in [F2]).

In the case of systems it was noticed in [F2] that if all the operators L_k^ε are the same: $L_k^\varepsilon = \frac{\varepsilon}{2}\sum_{i,j=1}^n a^{ij}(x)\partial^2/\partial x^i \partial x^j$ and $c_{kj}(x) = c_{kj}$ are positive constants, then the front also propagates according to the Huygens principle. In the Riemannian metric with $(a_{ij}(x)) = (a^{ij}(x))^{-1}$ the velocity is equal to $\sqrt{2\lambda}$, where λ is the principle eigenvalue of the matrix (c_{ij}).

In a number of asymptotic problems for RDE's (see [F1] chap. 7, [F3]) the motion of the front can be described by a Huygens principle, but the velocity field has the simplest form not in a Riemannian but in a Finsler metric (The Finsler metric is a generalization of the Riemannian metric, when the unit spheres in the tangent spaces are not ellipsoids but any convex sets (see [R]). If the unit spheres are the same at all points $x \in R^r$, such a metric is called the Minkovskii metric.)

We present here four examples to demonstrate some possible motion of wave fronts in geometric terms. Some new effects are pointed out that are possible in a system of RDE but not in a single equation.

1. As it was mentioned above, for the case $n = 1$ (no transmutations) if $c(x) = c$ is a constant, then wave fronts propagate according to a Huygens principle related to the diffusion matrix $(a_1^{ij}(x))$ and the constant c. In what follows we show by example that an analogy for $n \geqq 2$ does not hold. In our example $n = 2$, $c_{km}(x) = c_{km}$ for $k, m \in (1, 2)$, $c_{11} + c_{12} = c = c_{21} + c_{22}$, $r = 1$ and assumptions (A1)–(A3) are satisfied. We show that the Huygens principle does not hold.

Let $\sigma[A]$ denote the maximal eigenvalue of matrix A and let θ be a fixed positive number. Define

$$\beta = \sup_{p \in R} \left\{ p - \sigma \begin{bmatrix} \frac{1}{2}p^2 - \theta & \theta \\ \theta & p^2 - \theta \end{bmatrix} \right\} . \tag{4.1}$$

Note that

$$\sup_{p \in R} \left\{ p - \sigma \begin{bmatrix} \frac{p^2}{4\beta} - \theta & \theta \\ \theta & \frac{p^2}{4\beta} - \theta \end{bmatrix} \right\} = \sup_{p \in R} \left\{ p - \frac{p^2}{4\beta} \right\} = \beta . \tag{4.2}$$

Consider the system of RDE with small parameter ε.

$$\frac{\partial u_1^\varepsilon}{\partial t} = \frac{1}{2}\, \varepsilon a_1(x) \frac{\partial^2 u_1^\varepsilon}{2\partial x^2} + \varepsilon^{-1}[\beta u_1^\varepsilon(1 - u_1^\varepsilon) + \theta(u_2^\varepsilon - u_1^\varepsilon)] ,$$

$$\frac{\partial u_2^\varepsilon}{\partial t} = \frac{1}{2}\, \varepsilon a_2(x) \frac{\partial^2 u_2^\varepsilon}{2\partial x^2} + \varepsilon^{-1}[\beta u_2^\varepsilon(1 - u_2^\varepsilon) + \theta(u_1^\varepsilon - u_2^\varepsilon)], \quad t > 0,\ x \in R ,$$

$$u_1^\varepsilon(0, x) = u_2^\varepsilon(0, x) = \chi^-(x) ,$$

where the functions $a_1(x)$ and $a_2(x)$ are continuously differentiable positive functions such that

$$a_1(x) = \begin{cases} 1, & x \leqq 1, \\ \frac{1}{2\beta}, & x \geqq 2, \end{cases} \qquad a_2(x) = \begin{cases} 2, & x \leqq 1 , \\ \frac{1}{2\beta}, & x \geqq 2 . \end{cases}$$

Note that assumption (A1) is satisfied with $B = 1$. Assumptions (A2) and (A3) are satisfied since $c_{km}(x) = c_{km}$ are constants and $u(1-u) \leqq u$, $u \in [0,1]$. Let us define

$$\sigma_x(p) = \sigma \begin{bmatrix} \frac{a_1(x)p^2}{2} + \beta - \theta & \theta \\ \theta & \frac{a_2(x)p^2}{2} + \beta - \theta \end{bmatrix}$$

$$= \beta - \theta + \frac{(a_1(x) + a_2(x))p^2}{4} + \left[\theta^2 + \frac{(a_1(x) + a_2(x))^2 p^4}{8}\right]^{1/2} .$$

It can be checked that

$$\frac{d\sigma_1}{dp}(p) \neq \frac{d\sigma_2}{dp}(p) , \tag{4.4}$$

when $\sigma_1(p) = \sigma_2(p)$ and $p > 0$.

Also recall from (1.5) the definition of $\zeta(x,q)$:

$$\zeta_x(q) \equiv \zeta(x,q) = -\sup_{p \in R}[qp - \sigma_x(p)] \tag{4.5}$$

We claim

$$\frac{d\zeta_1}{dq}(1) \neq \frac{d\zeta_2}{dq}(1) . \tag{4.6}$$

This can be proved as follows. If (4.6) is not true, then we have

$$\frac{d\zeta_1}{dq}(1) = \frac{d\zeta_2}{dq}(1) ,$$

$$\zeta_1(1) = \zeta_2(1) ,$$

where the last equality follows from (4.1) and (4.2). Let p_x be the maximizer in (4.5) for $q = 1$, that is, p_x be such that

$$\frac{d\sigma_x}{dp}(p_x) = 1, \quad x \in R . \tag{4.7}$$

It then follows from the properties of the Legendre transform that

$$p_1 = p_2 = p_* > 0 ,$$

$$\sigma_1(p_*) - p_* = \sigma_2(p_*) - p_*, \text{ thus } \sigma_1(p_*) = \sigma_2(p_*)$$

From (4.4) one can obtain

$$\frac{d\sigma_1}{dp}(p_*) \neq \frac{d\sigma_2}{dp}(p_*) ,$$

which contradicts (4.7).

Next we shall show that if

$$\frac{d\zeta_1}{dq}(1) > \frac{d\zeta_2}{dq}(1) ,$$

then the Huygens principle does not hold for the RDE system (4.3). It follows from (4.1) and (4.2) that

$$\zeta(x,q) = 0 \quad \text{for } x \in (-\infty, 1] \bigcup [2, \infty), \ |q| = 1 ,$$

Therefore, should the Huygens principle hold, the position $x = 1$ would become excited at time 1 (it means that $u_k^\varepsilon(t,1)$ tends to 0 as $\varepsilon \downarrow 0$ for $t < 1$ and tends to 1 for $t > 1$). The position $x = 2$ would become excited at time $1 + \tau$, τ being a certain positive number and the position $x = 3$ at time $1 + \tau + 1 = \tau + 2$. We shall show that $V(T,3) = 0$ for some T less than $\tau + 2$, thus the Huygens principle can NOT hold. The idea is to look at $\varphi, \varphi_0 = 3$ such that $|\dot{\varphi}_s| = 1/(1+\delta) < 1$ when $\varphi_s \geqq 2$ (thus $\zeta(\varphi_s, \dot{\varphi_s}) > 0$), $|\dot{\varphi}_s| = 1/(1-\delta)$ when $\varphi_s \leqq 1$ (thus $\zeta(\varphi_s, \dot{\varphi}_s) < 0$) and $\zeta(\varphi_s, \dot{\varphi}_s) = 0$ when $1 < \varphi_s < 2$. Notice that $\varphi_{1+\delta} = 2$, $\varphi_{1+\delta+r} = 1$, $\varphi_{2+r} = \varphi_{1+\delta+r+1-\delta} = 0$ and take into consideration that $\zeta_2(1) = 0 = \zeta_1(1)$, then

$$\begin{aligned}\int_0^{2+r} \zeta(\varphi_s, \dot{\varphi}_s)\, ds &= (1+\delta)\zeta_2\left(\frac{1}{1+\delta}\right) + \tau \cdot 0 + (1-\delta)\zeta_1\left(\frac{1}{1-\delta}\right) \\ &= \left[\frac{d\zeta_1}{dq}(1) - \frac{d\zeta_2}{dq}(1)\right]\delta + o(\delta) \quad \text{as } \delta \to 0 ,\end{aligned}$$

The positivity of $[(d\zeta_1/dq)(1) - (d\zeta_2/dq)(1)]$ guarantees the existence of $T < \tau + 2$ such that $V(T,3) = 0$ for some $\delta > 0$.

Now, if the difference $[(d\zeta_1/dq)(1) - (d\zeta_2/dq)(1)]$ is not positive, in view of (4.6), it must be negative. Then the Huygens principle does not hold for the new system of the same form as (4.3) but with the functions $a_1(x), a_2(x)$ replaced by $a_1(3-x)$, $a_2(3-x)$ correspondingly. This can be proved by the same arguments.

Let $\tau(a,b)$ be the first time the position $x = b$, $b > a$, is excited when the initial data is the indicator function of $(-\infty, a]$. This example actually shows that

$$\tau(0,1) + \tau(1,2) + \tau(2,3) \neq \tau(0,3) .$$

Thus the excited region at the present does not determine the motion of wave front in the future. We refer to this behavior as a non-Markovian law, which is a stronger statement than that no Huygens principle can hold.

2. Suppose that the RDE system (1.2) is space-homogeneous, that is,

$$a_k^{ij}(x) = a_l^{ij} \quad \text{and} \quad c_{km}(x) = c_{km} \text{ are constants}$$

for $i,j \in \{1,\ldots,r\}$ and $k,m \in \{1,\ldots,n\}$. Suppose also that assumptions (A1)–(A3) are satisfied. Then, in (1.5), $\zeta(x,q) = \zeta(q)$, and from the concavity of ζ

it follows that

$$V(t,x) = \sup \left\{ \min_{0 \leqq a \leqq t} \int_0^a \zeta(\dot{\varphi}_s)\, ds \colon \ \varphi_s = x + s\,\frac{y-x}{t}, \ \ y \in G_0 \right\}$$

$$= \sup_{y \in G_0} \min \left(0, \ t\zeta\left(\frac{y-x}{t} \right) \right)$$

$$= t \min \left(0, \sup_{y \in G_0} \zeta \left(\frac{y-x}{t} \right) \right) \ .$$

This result was obtained in [BES] under assumptions on the vector field similar to (A1)–(A3). A special case was considered in [F1, F2]. It implies that the wave front propagates according to the Huygens principles, and its speed is 1 with respect to the Minkovskii metric associated with the unit ball H:

$$H = \{ q \in R^r \colon \ \zeta(q) \geqq 0 \} \ .$$

In the particular case of $n = 1$ this metric is always Riemannian [F1]. For $n = 2$ this metric is in general not Riemannian.

3. Consider the RDE system like (1.2) but with two parameters, ε and θ,

$$\text{(4.8)} \qquad \frac{\partial u_k^{\varepsilon,\theta}(t,x)}{\partial t} L_k^\varepsilon u_k^{\varepsilon,\theta} + \frac{1}{\varepsilon} F_k^\theta(u^{\varepsilon,\theta}), \quad t > 0, \ x \in R^r, \ 1 \leqq k \leqq n \ ,$$

where $F_k^\theta(u) = f_k(u_k) + \theta \sum_{m \neq k} c_{km}(u_m - u_k), c_{km}$ are positive numbers and L_k^ε are independent of x. We assume $f_k(v)$ to be of KPP type, i.e., $f_k(v) < 0$ for $v \in (-\infty, 0) \cup (1, \infty)$, $f_k(v) > 0$ for $v \in (0,1)$, and $f_k'(0) = \sup_{v>0} f_k(v)/v$. Assumptions (A1)–(A3) are satisfied for each $\theta > 0$. The RDE system (4.8) is space-homogeneous. Let $\zeta^\theta(q)$, $\Lambda^\theta(q)$ be as in (1.5) (now depending on θ and not on x). It follows from example 2, that the wave front, formed as $\varepsilon \to 0$, propagates according to the Huygens principle with speed 1 with respect to the Minkovskii metric with unit ball H^θ

$$H^\theta = \{ q \in R^r \colon \ \zeta^\theta(q) \geqq 0 \} \ .$$

Denote by conv$[w(\,\cdot\,)]$ the largest convex function which is no greater than the function $w \colon R^r \to R$. Denote by $\gamma = (\gamma_1, \ldots, \gamma_n)$ the invariant distribution associated with transmutation intensities c_{km}, $k \neq m$, i.e.,

$$\gamma_k > 0, \qquad \sum_{k=1}^n \gamma_k = 1 \ ,$$

$$\left(\sum_{k \neq m} \gamma_k c_{km} \right) - \gamma_m \left(\sum_{l \neq m} c_{ml} \right) = 0 \quad \text{for } m \in \{1, \ldots, n\} \ .$$

The action function $L^\theta(\beta)$, $\beta \in R^n$ for the occupation time Z_t^ε, $t = 1$, as $\varepsilon \to 0$ is strictly positive except at $\beta = \gamma$ where it is 0. A simple scaling argument shows that, in fact, $L^\theta = \theta L^1$. This follows, for example, from Sect. 4 of Chap. 7

in [FW]. Moreover, the functions L^θ, Λ^θ are conjugate with respect to the Legendre transform. This implies that

$$\Lambda^\theta(p) = \sup_{\beta \in R^n} \left[\sum_{k=1}^n \beta_k (p \dot{a}_k p/2 + f_k'(0)) - \theta L^1(\beta) \right] . \tag{4.9}$$

Thus,

$$\lim_{\theta \to \infty} \Lambda^\theta(p) = \sum_{k=1}^n \gamma_k (p \dot{a}_k p/2 + f_k'(0)) .$$

Note also that

$$\lim_{\theta \to 0} \Lambda^0(p) = \Lambda^\theta(p) = \max_k (p \dot{a}_k p/2 + f_k'(0)) .$$

By the last two identities and straightforward calculation we see that ζ^θ has the following limits as $\theta \downarrow 0$ and as $\theta \uparrow \infty$.

$$\lim_{\theta \to 0} \zeta^\theta(q) = -\sup_{p \in R^r} \left\{ q \cdot p - \max_{1 \leqq k \leqq n} (p \cdot a_k p/2 + f_k'(0)) \right\}$$

$$= -\text{conv} \left[\min_{1 \leqq k \leqq n} (q \cdot a_k^{-1} q/2 - f_k'(0)) \right] ,$$

$$\lim_{\theta \to \infty} \zeta^\theta(q) = - \left[q \cdot \left(\sum_{k=1}^n \gamma_k a_k \right)^{-1} q/2 - \sum_{k=1}^n \gamma_k f'(0) \right] .$$

The order of the lim and sup can be exchanged because for each q the sup occurs in a bounded set of p (depending on q, but not on θ) and because the principal eigenvalue depends on the entries of the matrix continuously. Correspondingly, the set H^θ has the limits

$$H^0 \equiv \lim_{\theta \to 0} H^\theta = \left\{ q : \text{conv} \left[\min_{1 \leqq k \leqq n} (q \cdot a_k^{-1} q/2 - f_k'(0)) \right] \leqq 0 \right\} ,$$

$$H^\infty \equiv \lim_{\theta \to \infty} H^\theta = \left\{ q : \cdot \left(\sum_{k=1}^n \gamma_k a_k \right)^{-1} q/2 \leqq \sum_{k=1}^n \gamma_k f_k'(0) \right\} .$$

Note that H^∞ corresponds to a Riemannian metric while H^θ, $\infty > \theta \geqq 0$ are in general not Riemannian. Comparing this result with the $n = 1$ case one finds that frequent transmutation ($\theta \to \infty$) makes the wave front behave like coming from a single RDE with average diffusion and multiplication coefficients.

4. Consider (4.8) with one space variable. It is instructive to compare the speed v^θ of wave front for the system with the speeds of decoupled single RDE's as $\theta = 0$. The latter speeds are

$$v_k = \sqrt{2 a_k f_k'(0)}, \quad 1 \leqq k \leqq n .$$

The speeds should be considered with respect to the Euclidean metric. The following simple bounds of v^θ are readily checked

$$\sqrt{2\left(\max_k a_k\right)\left(\max_k f_k'(0)\right)} \geqq \inf_{p>0}\max_k\left(\frac{a_k p}{2}+\frac{f_k'(0)}{p}\right) \geqq v^\theta$$

$$\geqq \sqrt{2(\Sigma\gamma_k a_k)(\Sigma\gamma_k f_k'(0))} \geqq \sum_{k=1}^n \gamma_k v_k \;. \tag{4.10}$$

The first inequality is obtained by replacing $a_k, f_k'(0)$ with their maximum over k. The second inequality follows from the simple fact that

$$\Lambda^\theta(p) \leqq \max_k\,[a_k p^2/2 + f_k'(0)]$$

and a simple calculation. The third inequality holds once we show that

$$\Lambda^\theta(p) \geqq \sum_{k=1}^n \gamma_k[a_k p^2/2 + f_k'(0)]\,,$$

which is obtained using $\beta = 0$ in (4.9). The last inequality follows from the fact that the function $(a,b)\to(ab)^{1/2}, a,b>0$ is concave. As the coupling intensity θ vanishes, the lower bound in strengthened (see (4.9)) into

$$\lim_{\theta\downarrow 0} v^\theta \geqq \max\left\{\max_k v_k, \sqrt{2(\Sigma\gamma_k a_k)(\Sigma\gamma_k f_k'(0))}\right\}\;.$$

The following RDE system demonstrates that the speed of system can be arbitrarily larger than that of each decoupled RDE,

$$\partial u_1^\varepsilon(t,x)/\partial t = \frac{\varepsilon}{2}\frac{\partial^2 u_1^\varepsilon}{\partial x^2} + \frac{1}{\varepsilon}\,[f_1(u_1^\varepsilon) + (u_2^\varepsilon - u_1^\varepsilon)]\,,$$

$$\partial u_2^\varepsilon(t,x)/\partial t = \frac{\varepsilon\rho}{2}\frac{\partial^2 u_2^\varepsilon}{\partial x^2} + \frac{1}{\varepsilon\rho}\,[f_2(u_2^\varepsilon) + (u_1^\varepsilon - u_2^\varepsilon)], \quad t>0,\ x\in R\,,$$

$$u_k^\varepsilon(0,x) = \chi-(x), \quad k=1,2,\ x\in R\,,$$

where f_k are of KPP type and $f_1'(0)=1,\ f_2'(0)=1/\rho$. According to (4.10) the speed v satisfies

$$v \geqq \sqrt{2\left(\sum\gamma_k a_k\right)\left(\sum\gamma_k f_k'(0)\right)} = \sqrt{\frac{1+\rho}{2}\left(1+\frac{1}{p}\right)}\,.$$

The righthand side tends to ∞ as $\rho\to\infty$ or $\rho\to 0$ while the speed $\sqrt{2a_k f_k'(0)}$ of each decoupled equation is $\sqrt{2}$ for $k=1,2$ (compare with [F3], Sect. 4). The increase of the speed of the front in the system, roughly speaking, is due to the fact, that the particles can use one type for multiplication and the other type for motion.

Acknowledgements. We thank the referees for their useful comments.

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