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A fragmentation process connected to Brownian motion

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Abstract. Let $(B_s, s \geq 0)$ be a standard Brownian motion and T_1 its first passage time at level 1. For every $t \geq 0$, we consider ladder time set $\mathscr{L}^{(t)}$ of the Brownian motion with drift t, $B_s^{(t)} = B_s + ts$, and the decreasing sequence $F(t) = (F_1(t), F_2(t), \ldots)$ of lengths of the intervals of the random partition of $[0, T_1]$ induced by $\mathscr{L}^{(t)}$. The main result of this work is that $(F(t), t \geq 0)$ is a fragmentation process, in the sense that for $0 \leq t < t'$, $F(t')$ is obtained from $F(t)$ by breaking randomly into pieces each component of $F(t)$ according to a law that only depends on the length of this component, and independently of the others. We identify the fragmentation law with the one that appears in the construction of the standard additive coalescent by Aldous and Pitman [3].

1. Introduction and main result

For $\ell \geq 0$, let $\mathbb{S}_\ell$ be the space of non-increasing numerical sequences $L = (\ell_1, \ell_2, \ldots)$ with $\sum L := \sum_1^\infty \ell_n = \ell$. We can think of ℓ as the length of an interval, and then of $L \in \mathbb{S}_\ell$ as the ranked sequence of the lengths of the subintervals resulting from some countable partition. Consider for every $\ell \geq 0$ a probability measure $\kappa(\ell)$ on $\mathbb{S}_\ell$ (of course $\kappa(0)$ must be the Dirac point mass at the sequence identical to 0). A fragmentation kernel κ on $\mathbb{S} = \bigcup_{\ell \geq 0} \mathbb{S}_\ell$ can be constructed from the family $(\kappa(\ell), \ell \geq 0)$ as follows. Given a sequence $L = (\ell_1, \ell_2, \ldots) \in \mathbb{S}$, we consider independent random variables $L_1, L_2, \ldots$ distributed according to the laws $\kappa(\ell_1)$, $\kappa(\ell_2), \ldots$, respectively. We then write $\kappa(L)$ for the distribution of the decreasing rearrangement of the elements of the sequences $L_1, L_2, \ldots$. We say that the family $(\kappa(\ell), \ell \geq 0)$ generates the fragmentation kernel $\kappa = (\kappa(L), L \in \mathbb{S})$. Note that a fragmentation kernel preserves the total length, in the sense that $\kappa(L)(\mathbb{S}_\ell) = 1$ for $\ell = \sum L$.

Call fragmentation process a time homogeneous Markov process with values in $\mathbb{S}$, whose transition semigroup is given by fragmentation kernels. Quite recently, Aldous and Pitman [3] derived a fragmentation process by logging the continuum random tree along its skeleton at the points of a certain independent Poisson process. This is connected by time-reversal to the so-called standard additive coalescent (cf.

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also Evans and Pitman [8], and Aldous [2] for a survey of that field). Rephrasing Theorem 4 in [3], the family of probability measures ($\theta_t(\ell)$, $\ell \geq 0$ and $t \geq 0$) that generates the fragmentation semigroup of Aldous and Pitman can be described as follows. For $t, \ell > 0$, let $\xi_1 > \xi_2 > \cdots$ be the atoms of a Poisson measure on $(0, \infty)$ with intensity $t\ell(2\pi x^3)^{-1/2}dx$, ranked in the decreasing order. Then we define $\theta_t(\ell)$ as the distribution of the sequence $\xi = (\xi_1, \xi_2, \ldots)$ conditionally on $\xi \in \mathbb{S}_\ell$, that is

$$\theta_t(\ell)(dL) = \mathbb{P}\left(\xi \in dL \mid \sum \xi = \ell\right), \qquad L \in \mathbb{S}_\ell \ .$$

We refer to Perman [10] and section 8.1 in Pitman and Yor [15] for more information about these conditional laws. We also mention that a different fragmentation process has been constructed by Pitman [13], again time-reversing a remarkable coalescent process, cf. Theorem 12 and Corollary 15 there.

Kingman [9] has observed that a regenerative set (i.e. the range of a subordinator) gives rise to an interesting random partition of $[0, \infty)$ which can be used to define a random discrete distribution. We refer to [10–12, 14, 15] and the references therein for some developments over the recent years. This suggests the possibility of representing certain fragmentation or coalescent processes using partitions of $[0, \infty)$ induced by a nested family of regenerative sets. We refer to [6] and [7] for applications of this idea, and also to [5] for a related work.

Here, we will construct for each $t \geq 0$ a regenerative set from the path of a Brownian motion and observe that the induced partitions of $[0, \infty)$ get finer and finer as t increases. More precisely, we will show that the partition at t' is obtained from that at $t < t'$ by breaking randomly into pieces each component of the latter, independently of the other parts, and according to a distribution that only depends of the length of this component. In other words, we will construct a fragmentation process which is naturally related to the Brownian motion.

To give a precise description, let $B = (B_s, s \geq 0)$ be a standard linear Brownian motion and

$$T_x = \inf\{s \geq 0 : B_s > x\}, \qquad x \geq 0$$

its first passage process. For every $t \geq 0$, we consider the Brownian motion with constant drift t and its supremum process,

$$B_s^{(t)} = B_s + ts, \quad S_s^{(t)} = \sup_{0 \leq u \leq s} B_u^{(t)}, \qquad s \geq 0.$$

The ladder time set $\mathcal{L}^{(t)}$ of $B^{(t)}$ is defined as the set of times when $B^{(t)}$ coincides with its supremum, i.e.

$$\mathcal{L}^{(t)} = \left\{s \geq 0 : S_s^{(t)} = B_s^{(t)}\right\}.$$

It is well-known that $\mathcal{L}^{(t)}$ is a.s. a random closed set with zero Lebesgue measure. The simple observation that for every fixed $0 \leq t < t'$, the process $s \to B_s^{(t')} - B_s^{(t)} = (t'-t)s$ is monotone increasing entails the embedding $\mathcal{L}^{(t)} \subseteq \mathcal{L}^{(t')}$. As a consequence, the partition of $[0, T_1]$ induced by $\mathcal{L}^{(t')}$ is finer than that induced by $\mathcal{L}^{(t)}$. We write

$$F(t) = (F_1(t), F_2(t), \ldots)$$

for the sequence of fragments at t, that is the lengths of the open intervals in the canonical decomposition of $[0, T_1] \backslash \mathscr{L}^{(t)}$, arranged according to the decreasing order.

We will show that $(F(t), t \geq 0)$ is a fragmentation process. To give a first description of its semigroup, we construct a family

$$(\varphi_t(\ell), t \geq 0 \text{ and } \ell \geq 0)$$

of probability measures on $\mathbb{S}$, called the *fragmentation laws*, as follows. For every $\ell > 0$, consider a process $\epsilon = (\epsilon(s), 0 \leq s \leq \ell)$ having the law of the positive Brownian excursion with duration ℓ. For every $t \geq 0$, introduce the ladder time set of $(st - \epsilon(s), 0 \leq s \leq \ell)$

$$\mathscr{L}^{(t)}(\epsilon) = \left\{ s \in [0, \ell] : st - \epsilon(s) = \sup_{0 \leq u \leq s} (ut - \epsilon(u)) \right\} \tag{1}$$

and write $\varphi_t(\ell)$ for the distribution on $\mathbb{S}_\ell$ of the sequence of the lengths of the open intervals in the canonical decomposition of $[0, \ell] \backslash \mathscr{L}^{(t)}(\epsilon)$ arranged in the decreasing order. Of course, $\varphi_0(\ell)$ is just the Dirac point mass at $(\ell, 0, \ldots)$. Then we write φ_t for the fragmentation kernel generated by $(\varphi_t(\ell), \ell \geq 0)$.

We now state the main result of this work. Recall that θ_t stands for the fragmentation kernel of Aldous and Pitman which has been described above.

Theorem 1. (i) $(F(t), t \geq 0)$ *is a fragmentation process with semigroup* $(\varphi_t, t \geq 0)$. *(ii) The fragmentation semigroups* φ_t *and* θ_t *are the same.*

The first part of Theorem 1 will be proved in the next section; the argument combines a variation of Skorohod's lemma and some features on the excursions of a Brownian motion with drift away from its supremum. It should be noted that the Markov property of $(F(t), t \geq 0)$ is essentially straightforward. The second part will be proved in Section 3. Despite the known connections between the continuum random tree and the normalized Brownian excursion (see [1] and the references therein), the construction of the fragmentation process by Aldous and Pitman and the present one look much different, and it is not clear *a priori* why they should yield the same semigroup. We shall first give an analytic expression for the fragmentation laws $\varphi_t(\ell)$, using special properties of Lévy processes with no positive jumps. Then we check that the fragmentation laws $\theta_t(\ell)$ can be given by the same expression. The final section is devoted to the following interesting identity. Consider the fragmentation process on $\mathbb{S}_1$ induced as above by a standard Brownian excursion (i.e. with unit duration), and let U be an independent variable uniformly distributed on $[0, 1]$. Write λ_t for the length of the left-most interval resulting from the fragmentation of $[0, 1]$ at time t, and λ_t^* for the length of the interval that contains U. Then the processes $(\lambda_t, t \geq 0)$ and $\left(\lambda_t^*, t \geq 0\right)$ have the same law.

2. The fragmentation property

2.1. A Skorohod-type formula

As a first step, we observe that for every $0 \le t < t'$, the supremum process $S^{(t)}$ can be expressed as a simple functional of $S^{(t')}$.

Lemma 2. *For every* $0 \le t < t'$, *we have*

$$S_s^{(t)} = \sup_{0\le u\le s} \left(S_u^{(t')} - u(t'-t) \right), \qquad s \ge 0\,.$$

Proof. The proof is an adaptation of the classical argument of Skorohod's lemma; see e.g. [16] on page 239. We first note that the obvious inequality

$$B_u^{(t)} = B_u^{(t')} - u(t'-t) \le S_u^{(t')} - u(t'-t)$$

yields

$$S_s^{(t)} \le \sup_{0\le u\le s} \left(S_u^{(t')} - u(t'-t) \right)\,.$$

Conversely, consider an arbitrary time $s \ge 0$ at which $\sup_{0\le u\le \cdot} \left(S_u^{(t')} - u(t'-t) \right)$ increases. Plainly, it must be an instant at which $S_\cdot^{(t')}$ increases, and therefore $S_s^{(t')} = B_s^{(t')}$. It follows that

$$\sup_{0\le u\le s} \left(S_u^{(t')} - u(t'-t) \right) = B_s^{(t')} - s(t'-t) = B_s^{(t)}\,,$$

which completes the proof of our claim. □

We next present a useful consequence of the formula of Lemma 2. For each $t \ge 0$ fixed, let $\mathcal{G}_t$ stand for the $\mathbb{P}$-completed sigma-field generated by the supremum process $S^{(t)}$.

Corollary 3. $(\mathcal{G}_t, t \ge 0)$ *is a filtration, and* $(F(t), t \ge 0)$ *is* $(\mathcal{G}_t)$*-adapted.*

Proof. It is plain from Lemma 2 that the process $S^{(t)}$ is measurable with respect to $\mathcal{G}_{t'}$ when $t < t'$, so $(\mathcal{G}_t, t \ge 0)$ is a filtration. Clearly T_1 is $\mathcal{G}_t$-measurable (since it is obviously $\mathcal{G}_0$-measurable); and the same holds for the ladder time set $\mathcal{L}^{(t)}$, because it can be expressed as the support of the Stieltjes measure $dS^{(t)}$. We conclude that $(F(t), t \ge 0)$ is $(\mathcal{G}_t)$-adapted. □

2.2. On the excursions of a reflected Brownian motion with drift

We denote the process of first passage times of the Brownian motion with drift t by

$$T_x^{(t)} = \inf\left\{ s \ge 0 : S_s^{(t)} > x \right\}, \qquad x \ge 0\,.$$

Note that in particular $T^{(0)} = T$.

For every $x > 0$, we write $\ell_{1,x}^{(t)} > \ell_{2,x}^{(t)} > \cdots$ for the durations of the excursions away from 0 accomplished by the reflected process $S^{(t)} - B^{(t)}$ on the time interval $[0, T_x^{(t)}]$, arranged according to the decreasing order. Alternatively, $\ell_{1,x}^{(t)} > \ell_{2,x}^{(t)} > \cdots$ can also be viewed as the decreasing sequence of the amplitude of the jumps of the first passage process $T^{(t)}$ on the interval $[0, x]$. Note that this sequence is measurable with respect to $\mathcal{G}_t$. For every integer $k \geq 1$, we denote by $\epsilon_{k,x}^{(t)}$ the corresponding excursion with duration $\ell_{k,x}^{(t)}$, that is, if $y \in [0, x]$ is such that $\ell_{k,x}^{(t)} = T_y^{(t)} - T_{y-}^{(t)}$, then

$$\epsilon_{k,x}^{(t)}(s) = y - B^{(t)}\left(T_{y-}^{(t)} + s\right), \quad s \in [0, \ell_{k,x}^{(t)}].$$

In order to state the main technical result of this subsection, it is convenient to denote by $\mathtt{exc}$ the Itô measure of positive Brownian excursions, and for every $\ell > 0$, by $\mathtt{exc}(\ell)$ the law of the positive Brownian excursion with duration ℓ, that is that of the bridge of a 3-dimensional Bessel process with length ℓ, starting and ending at 0 (cf. Theorem XII.4.2 in [16]).

Lemma 4. *Let X be a $\mathcal{G}_t$-measurable random variable with values in $(0, \infty)$. Then conditionally on $\mathcal{G}_t$, the excursions $\epsilon_{1,X}^{(t)}, \epsilon_{2,X}^{(t)}, \ldots$ form a sequence of independent processes with distributions $\mathtt{exc}(\ell_{1,X}^{(t)})$, $\mathtt{exc}(\ell_{2,X}^{(t)})$, ..., respectively.*

Proof. Let us denote by $\mathtt{exc}^{(t)}$ the Itô measure of the excursions of $S^{(t)} - B^{(t)}$ away from 0. It is easily verified by a Girsanov's transformation that $\mathtt{exc}^{(t)}$ is absolutely continuous with respect to $\mathtt{exc}$, with density $\mathrm{e}^{-t^2\zeta/2}$, where ζ stands for the duration of a generic excursion. As a consequence, the excursion measures $\mathtt{exc}^{(t)}$ have all the same conditional laws given the duration, i.e. $\mathtt{exc}^{(t)}(\ell) = \mathtt{exc}(\ell)$ in the obvious notation.

We then first suppose that $X = x$ is deterministic and recall that $S^{(t)}$ is the local time at 0 for the reflected process $S^{(t)} - B^{(t)}$. The statement is then plain from excursion theory. The extension to the case when X is a simple (i.e. that can only take countably many values) $\mathcal{G}_t$-measurable random variable is immediate. A standard argument based on approximating a positive and finite random variable by a decreasing sequence of simple random variables and using right-continuity completes the proof. □

Next, write $\epsilon_1^{(t)}, \epsilon_2^{(t)}, \ldots$ for the sequence of excursions away from 0 accomplished by $S^{(t)} - B^{(t)}$ on the time interval $[0, T_1]$, arranged according to the decreasing order of the lengths. Note that the sequence of the lengths is precisely $F(t)$.

Corollary 5. *For every fixed $t \geq 0$, conditionally on $\mathcal{G}_t$, the excursions $\epsilon_1^{(t)}, \epsilon_2^{(t)}, \ldots$ form a sequence of independent processes with respective distributions* $\mathtt{exc}(F_1(t))$, $\mathtt{exc}(F_2(t))$, ...

Proof. Recall that T_1 is $\mathcal{G}_t$-measurable, and so the same holds for the variable $X = S_{T_1}^{(t)} = 1 + tT_1$. On the other hand, we have with probability one that $T_X^{(t)} = T_1$, which implies $\epsilon^{(t)}_{k,X} = \epsilon^{(t)}_k$ for every integer $k \geq 1$. Lemma 4 thus completes the proof. □

2.3. Proof of Theorem 1(i)

Corollary 5 makes Theorem 1(i) straightforward. More precisely, for a fixed integer $n \geq 1$, let $[g_n, d_n] \subseteq [0, T_1]$ be the interval with length $F_n(t)$ corresponding to the excursion $\epsilon^{(t)}_n$. Then

$$B^{(t)}_u = S^{(t)}_{g_n} - \epsilon^{(t)}_n(u - g_n), \quad u \in [g_n, d_n],$$

and it follows that for any $t' > t$

$$B^{(t')}_u = S^{(t')}_{g_n} + (u - g_n)(t' - t) - \epsilon^{(t)}_n(u - g_n), \quad u \in [g_n, d_n]$$

(recall that g_n is a ladder time for $B^{(t')}$). It is now plain that the restriction of the ladder time set of $B^{(t')}$ to $[g_n, d_n]$ coincides with the ladder time set of the process

$$\left(s(t' - t) - \epsilon^{(t)}_n(s), 0 \leq s \leq F_n(t)\right)$$

shifted by g_n. We thus see from Corollary 5 that the conditional distribution of $F(t')$ given $\mathscr{G}_t$ is $\varphi_{t'-t}(F(t))$, where the transition kernel φ_s has been defined in the first section.

3. Identification of the fragmentation laws

3.1. Preliminaries

Throughout this section, $t > 0$ is a fixed real number. We first develop some material on the first passage process $T^{(t)}$. To that end, recall that $T^{(t)} = \left(T^{(t)}_x, x \geq 0\right)$ is a pure jump subordinator, and more precisely its Lévy-Itô decomposition is given by

$$T^{(t)}_x = \sum_{0 \leq y \leq x} \Delta^{(t)}_y,$$

where the jump process $\Delta^{(t)} = \left(\Delta^{(t)}_y, y \geq 0\right)$ is a Poisson point process on $(0, \infty)$ with characteristic measure

$$\Lambda^{(t)}(ds) = \frac{1}{\sqrt{2\pi s^3}} \exp\left\{-st^2/2\right\} ds \quad s > 0,$$

i.e. $\Lambda^{(t)}$ is the Lévy measure of $T^{(t)}$.

We next introduce some notation related to $\mathscr{O} \subseteq [0, \infty)$, a generic open neighborhood of 0 (since $t > 0$ is fixed throughout this section, it is omitted in subsequent notation such as $T^{\mathscr{O}}$ or $\Phi^{\mathscr{O}}$ for the sake of simplicity). Distinguishing the jumps of

$\Delta^{(t)}$ with values in $\mathcal{O}$ and in $\mathcal{O}^{\mathrm{c}} = [0,\infty)\backslash\mathcal{O}$ yields the decomposition of $T^{(t)}$ as the sum of two independent subordinators, $T^{\mathcal{O}}$ and $T^{\mathcal{O}^{\mathrm{c}}}$, where

$$T_x^{\mathcal{O}} \;=\; \sum_{0\le y\le x} \Delta_y^{(t)}\,\mathbf{1}_{\{\Delta_y^{(t)}\in\mathcal{O}\}}\,, \qquad x\ge 0$$

and $T^{\mathcal{O}^{\mathrm{c}}} = T^{(t)} - T^{\mathcal{O}}$.

The process $\left(x - tT_x^{\mathcal{O}}, x\ge 0\right)$ is the difference between a drift and a subordinator; it is a Lévy process with no positive jumps and its Laplace exponent $\psi^{\mathcal{O}}$ is obtained by the Lévy-Khintchine formula:

$$\psi^{\mathcal{O}}(q) \;=\; q - \int_{\mathcal{O}} \left(1-\mathrm{e}^{-qts}\right)\Lambda^{(t)}(ds)\,;$$

(see section VII.1 in [4] and recall that the Lévy measure of $T^{\mathcal{O}}$ is $\mathbf{1}_{\mathcal{O}}\Lambda^{(t)}$). Because $\mathbb{E}(T_1^{\mathcal{O}}) \le \mathbb{E}(T_1^{(t)}) = 1/t$, this Lévy process has a nonnegative mean, i.e. the right-derivative of $\psi^{\mathcal{O}}$ at 0 is nonnegative. This ensures that the convex function $\psi^{\mathcal{O}} : [0,\infty) \to [0,\infty)$ is strictly increasing, and thus a bijection. We denote by $\Phi^{\mathcal{O}} : [0,\infty)\to[0,\infty)$ its inverse function, i.e.

$$\psi^{\mathcal{O}}\circ\Phi^{\mathcal{O}} \;=\; \mathrm{Id}\,.$$

If we write

$$\tau_s^{\mathcal{O}} \;=\; \inf\left\{x\ge 0 : x - tT_x^{\mathcal{O}} > s\right\}, \quad s\ge 0$$

for the first passage process of $x - tT_x^{\mathcal{O}}$, it is well-known that $\left(\tau_s^{\mathcal{O}}, s\ge 0\right)$ is again a subordinator with Laplace exponent $\Phi^{\mathcal{O}}$, i.e.

$$\mathbb{E}(\exp\left\{-q\tau_s^{\mathcal{O}}\right\}) \;=\; \exp\{-s\Phi^{\mathcal{O}}(q)\}, \quad q\ge 0\,, \tag{2}$$

see for instance Theorem VII.1 in [4].

We now end up this subsection by presenting an expression in terms of the one-dimensional distribution of $T^{\mathcal{O}}$ for the implicit function $\Phi^{\mathcal{O}}$. In this direction, we point out that the characteristic exponent $\lambda \to \kappa^{\mathcal{O}}(\lambda) = -\psi^{\mathcal{O}}(-\mathrm{i}\lambda) - \mathrm{i}\lambda$ of the subordinator $T^{\mathcal{O}}$ fulfills

$$\lim_{|\lambda|\to\infty} |\lambda|^{-1/2}|\Re\kappa^{\mathcal{O}}(\lambda)| \;=\; c > 0$$

(this is readily seen from the Lévy-Khintchine formula, as the Lévy measure of $T^{\mathcal{O}}$ coincides with $\Lambda^{(t)}$ on a neighborhood of 0). It follows from the Riemann-Lebesgue theorem that for every $s>0$ the distribution of $T_s^{\mathcal{O}}$ has a $\mathcal{C}^\infty$ density (see for instance Exercise I.4 in [4]) that we denote by

$$p_s^{\mathcal{O}}(x) \;=\; \mathbb{P}\left(T_s^{\mathcal{O}}\in dx\right)/dx, \quad s>0 \text{ and } x\ge 0\,.$$

Lemma 6. *The function $\Phi^{\mathcal{O}}$ is given by the Lévy-Khintchine formula*

$$\Phi^{\mathcal{O}}(q) = q + \int_0^\infty \left(1 - \mathrm{e}^{-qs}\right)(ts)^{-1} p_s^{\mathcal{O}}(s/t)\, ds, \quad q \geq 0\,.$$

Proof. On the one hand, it is plain that the drift coefficient of $\Phi^{\mathcal{O}}$ is given by

$$\lim_{q\to\infty} \Phi^{\mathcal{O}}(q)/q = \lim_{q\to\infty} q/\psi^{\mathcal{O}}(q) = 1\,.$$

On the other hand, the Lévy measure of the subordinator $\tau^{\mathcal{O}}$ is the vague limit as $\varepsilon \to 0+$ of the measures $\varepsilon^{-1}\mathbb{P}(\tau_\varepsilon^{\mathcal{O}} \in ds)$ on $(0, \infty)$; see e.g. Exercise I.1 in [4]. By Corollary VII.3 in [4], the latter can be expressed as $s^{-1}\mathbb{P}\left(s - tT_s^{\mathcal{O}} \in dx\right) ds/dx$ for $x = \varepsilon$. Since

$$\mathbb{P}\left(s - tT_s^{\mathcal{O}} \in dx\right) = t^{-1} p_s^{\mathcal{O}}((s-x)/t)\, dx,$$

this entails our claim. □

3.2. An analytic expression for the fragmentation laws $\varphi_t(\ell)$

The purpose of this subsection is to present an expression for the fragmentation laws in terms of quantities introduced in the preceding subsection. The starting point lies on the following description of the fragmentation $F(t)$ in terms of the subordinator $T^{(t)}$.

Lemma 7. *For every $t > 0$, $F(t)$ coincides with the family of the jumps accomplished by $T^{(t)}$ before $S_{T_1}^{(t)}$, $\left(\Delta_y^{(t)}, 0 \leq y \leq S_{T_1}^{(t)}\right)$, and arranged in the decreasing order. Moreover we have*

$$S_{T_1}^{(t)} = \inf\left\{x \geq 0 : x - tT_x^{(t)} = 1\right\}\,.$$

Proof. The first assertion is obvious; so let us check the formula for $S_{T_1}^{(t)}$. We deduce from Lemma 2 that

$$T_1 = \inf\left\{s \geq 0 : S_s^{(t)} - ts = 1\right\},$$

which entails

$$S_{T_1}^{(t)} = \inf\left\{S_s^{(t)} : S_s^{(t)} - ts = 1\right\},$$

and the substitution $x = S_s^{(t)}$, $s = T_x^{(t)}$ in the right-hand term yields our claim. □

Next, recall $\mathcal{O} \subseteq [0, \infty)$ is an arbitrary open neighborhood of 0, and write $\mathbb{S}(\mathcal{O})$ for the subset of $\mathbb{S}$ consisting of the sequences taking only values in $\mathcal{O}$. We first point out that any probability measure on $\mathbb{S}$ that is supported by the subspace of *strictly decreasing* sequences is completely characterized by the masses assigned to the $\mathbb{S}(\mathcal{O})$'s.

Lemma 8. *Let $\mathbb{S}' \subseteq \mathbb{S}$ be the subset of strictly decreasing sequences, and consider two probability measures on $\mathbb{S}'$, μ_1 and μ_2, such that*

$$\mu_1(\mathbb{S}(\mathscr{O})) = \mu_2(\mathbb{S}(\mathscr{O})) \quad \textit{for every } \mathscr{O} \subseteq [0, \infty) \textit{ open neighborhood of } 0.$$

Then $\mu_1 = \mu_2$.

Proof. Fix an arbitrary $\ell > 0$ and consider the space $\mathscr{C}$ of closed subsets of $[0, \ell]$ endowed with the Hausdorff distance; in particular $\mathscr{C}$ is a compact metric space. There is a canonical application $\mathbb{S}([0, \ell]) \to \mathscr{C}$ that maps a sequence $L = (\ell_1, \ell_2, \ldots)$ to its closed range $\{\ell_i, i = 1, 2, \ldots\} \cup \{0\}$.

Let M_1 and M_2 be the image measures by this mapping of the restriction of μ_1 and μ_2 to $\mathbb{S}([0, \ell])$. For every open set $\mathscr{O} \subseteq [0, \ell]$, denote by $\mathscr{C}(\mathscr{O})$ the class of closed sets included in $\mathscr{O}$. It is known that the family $(\mathscr{C}(\mathscr{O}), \mathscr{O}$ open set) generates the topology on $\mathscr{C}$ induced by the Hausdorff distance. The hypothesis

$$M_1(\mathscr{C}(\mathscr{O})) = \mu_1(\mathbb{S}(\mathscr{O})) = \mu_2(\mathbb{S}(\mathscr{O})) = M_2(\mathscr{C}(\mathscr{O}))$$

entails by a monotone class theorem that $M_1 = M_2$.

The restriction of the canonical map $\mathbb{S}' \cap \mathbb{S}([0, \ell]) \to \mathscr{C}$ is one-to-one, and since ℓ can be chosen arbitrarily large, the identity $M_1 = M_2$ yields that $\mu_1 = \mu_2$. $\square$

We stress that with probability one, the sequence $F(t)$ has no multiple points (because the Lévy measure $\Lambda^{(t)}$ has no atoms) and thus takes values in $\mathbb{S}'$. Lemma 8 applies and ensures that the distribution of the fragmentation process evaluated at the fixed time $t > 0$ is characterized by the following (the same remark is relevant to the fragmentation laws $\varphi_t(\ell)$ and the next Corollary 10).

Lemma 9. *For every $a \geq 0$, we have*

$$\mathbb{E}\left(\mathrm{e}^{-aT_1}, F(t) \in \mathbb{S}(\mathscr{O})\right) = \exp\left\{-\Phi^{\mathscr{O}}(\Lambda^{(t)}(\mathscr{O}^{\mathrm{c}}) + a/t) + a/t\right\}.$$

Proof. We deduce from Lemma 7 that the event $\{F(t) \in \mathbb{S}(\mathscr{O})\}$ occurs if and only if

$$\tau_1^{\mathscr{O}} := \inf\left\{x \geq 0 : x - tT_x^{\mathscr{O}} = 1\right\} < \inf\left\{x \geq 0 : \Delta_x^{(t)} \notin \mathscr{O}\right\}. \tag{3}$$

Note also that then $\tau_1^{\mathscr{O}} = S_{T_1}^{(t)}$ and hence

$$T_1 = T^{(t)}_{\tau_1^{\mathscr{O}}} = T^{\mathscr{O}}_{\tau_1^{\mathscr{O}}} = \left(\tau_1^{\mathscr{O}} - 1\right)/t.$$

The variable defined by the right-hand side of (3) is the first jump-time of $T^{\mathscr{O}^{\mathrm{c}}}$; it is independent of $\tau_1^{\mathscr{O}}$ and has an exponential law with parameter $\Lambda^{(t)}(\mathscr{O}^{\mathrm{c}}) < \infty$ (the finiteness follows from the assumption that $\mathscr{O}$ is a neighborhood of 0). We deduce that

$$\mathbb{E}\left(\mathrm{e}^{-aT_1}, F(t) \in \mathbb{S}(\mathscr{O})\right) = \mathrm{e}^{a/t}\,\mathbb{E}\left(\mathrm{e}^{-a\tau_1^{\mathscr{O}}/t}, F(t) \in \mathbb{S}(\mathscr{O})\right) = \mathrm{e}^{a/t}\,\mathbb{E}\left(\mathrm{e}^{-q\tau_1^{\mathscr{O}}}\right)$$

for $q = \Lambda^{(t)}(\mathscr{O}^{\mathrm{c}}) + a/t$. Our claim thus follows from (2). $\square$

It is now easy to deduce the following characterization of the fragmentation laws $\varphi_t(\ell)$, which is the main result of this subsection.

Corollary 10. *In the preceding notation, we have for every $a > 0$*

$$\int_0^\infty \left(1 - \mathrm{e}^{-a\ell}\right) \left(2\pi\ell^3\right)^{-1/2} \varphi_t(\ell)\,(\mathbb{S}(\mathscr{O}))\; d\ell$$
$$= \Phi^{\mathscr{O}}(\Lambda^{(t)}(\mathscr{O}^{\mathrm{c}}) + a/t) - \Phi^{\mathscr{O}}(\Lambda^{(t)}(\mathscr{O}^{\mathrm{c}})) - a/t\,.$$

Proof. Plainly, $F(t)$ only takes values in $\mathscr{O}$ if and only if the same holds for all the fragmentations at t resulting from the components of $F(0)$. By the Markov property of Theorem 1(i), conditionally on $F(0) = (\ell_1, \ell_2, \ldots)$, these fragmentations are independent with laws $\varphi_t(\ell_1)$, $\varphi_t(\ell_2)\ldots$. As $T_1 = \ell_1 + \ell_2 + \ldots$, we thus have

$$\mathbb{E}\left(\mathrm{e}^{-aT_1}, F(t) \in \mathbb{S}(\mathscr{O}) \mid F(0) = (\ell_1, \ell_2, \ldots)\right) = \prod_{n=1,2,\ldots} \mathrm{e}^{-a\ell_n}\varphi_t(\ell_n)(\mathbb{S}(\mathscr{O}))\,.$$

Since $F(0)$ has the law of the atoms of a Poisson measure on $(0, \infty)$ with intensity $(2\pi\ell^3)^{-1/2}d\ell$, we get by a classical formula for Poisson clouds

$$\begin{aligned}\mathbb{E}\left(\mathrm{e}^{-aT_1}, F(t) \in \mathbb{S}(\mathscr{O})\right) &= \mathbb{E}\left(\mathbb{E}\left(\mathrm{e}^{-aT_1}, F(t) \in \mathbb{S}(\mathscr{O}) \mid F(0)\right)\right)\\ &= \exp\left\{-\int_0^\infty \left(1 - \mathrm{e}^{-a\ell}\varphi_t(\ell)(\mathbb{S}(\mathscr{O}))\right)(2\pi\ell^3)^{-1/2}d\ell\right\}\,.\end{aligned}$$

Applying Lemma 9, we thus get

$$\int_0^\infty \left(1 - \mathrm{e}^{-a\ell}\varphi_t(\ell)(\mathbb{S}(\mathscr{O}))\right)(2\pi\ell^3)^{-1/2}d\ell = \Phi^{\mathscr{O}}(\Lambda^{(t)}(\mathscr{O}^{\mathrm{c}}) + a/t) - a/t\,,$$

which readily entails our claim. □

We mention that Corollary 10 can also be deduced from Lemma 9 by a standard argument of excursion theory (in a Poisson point process, the instant of the first point in a given set is independent of this first point and has an exponential distribution whose parameter is the characteristic measure of this set).

3.3. *Proof of Theorem 1(ii)*

The aim of this subsection is to identify the fragmentation kernels θ_t and φ_t. Recall first from Theorem 4 in [3] that for $\ell > 0$, the probability measure $\theta_t(\ell)$ on $\mathbb{S}_\ell$ can be defined as the ranked sequence of the jumps made by the stable(1/2) subordinator T before time $t\ell$, conditionally on $T_{t\ell} = \ell$. By Girsanov theorem, under the equivalent probability measure

$$d\mathbb{P}^{(t)} := \exp\left\{-\frac{t^2}{2}T_{t\ell} + t^2\ell\right\} d\mathbb{P}\,,$$

the process $(T_s, 0 \le s \le t\ell)$ has the same law as the process $\left(T_s^{(t)}, 0 \le s \le t\ell\right)$ under $\mathbb{P}$. It follows that for every open set $\mathscr{O} \subseteq [0, \infty)$

$$\theta_t(\ell)(\mathbb{S}(\mathcal{O})) \;=\; \mathbb{P}\left(\Delta_s^{(t)} \in \mathcal{O}, s \le t\ell \mid T_{t\ell}^{(t)} = \ell\right) .$$

Recall the decomposition $T^{(t)} = T^{\mathcal{O}} + T^{\mathcal{O}^c}$ as the sum of two independent subordinators, that the density of the law of $T_{t\ell}^{(t)}$ is

$$\frac{t\ell}{\sqrt{2\pi x^3}} \exp\left\{-\frac{t^2\ell^2}{2x}\right\} \exp\left\{-\frac{t^2 x}{2} + t^2\ell\right\}, \quad x > 0 ,$$

and that $p_{t\ell}^{\mathcal{O}}(\cdot)$ stands for the $\mathscr{C}^\infty$ version of the density of the law of $T_{t\ell}^{\mathcal{O}}$. We see that the former quantity can be expressed as

$$\begin{aligned}
\theta_t(\ell)(\mathbb{S}(\mathcal{O})) \;&=\; \frac{\mathbb{P}\left(T_{t\ell}^{\mathcal{O}^c} = 0 \text{ and } T_{t\ell}^{\mathcal{O}} \in d\ell\right)}{\mathbb{P}\left(T_{t\ell}^{(t)} \in d\ell\right)} \\
&=\; p_{t\ell}^{\mathcal{O}}(\ell) \exp\left\{-t\ell\Lambda^{(t)}(\mathcal{O}^c)\right\} \frac{\sqrt{2\pi\ell}}{t} .
\end{aligned}$$

An application of Lemma 6 at the second identity below entails that for every $a > 0$

$$\begin{aligned}
&\int_0^\infty (1 - \mathrm{e}^{-a\ell})\theta_t(\ell)(\mathbb{S}(\mathcal{O})) \left(2\pi\ell^3\right)^{-1/2} d\ell \\
&\quad = \int_0^\infty (1 - \mathrm{e}^{-a\ell}) \exp\left\{-t\ell\Lambda^{(t)}(\mathcal{O}^c)\right\} p_{t\ell}^{\mathcal{O}}(\ell) \frac{d\ell}{t\ell} \\
&\quad = \Phi^{\mathcal{O}}(\Lambda^{(t)}(\mathcal{O}^c) + a/t) - \Phi^{\mathcal{O}}(\Lambda^{(t)}(\mathcal{O}^c)) - a/t .
\end{aligned}$$

The comparison with Corollary 10 now shows that $\theta_t(\ell)(\mathbb{S}(\mathcal{O})) = \varphi_t(\ell)(\mathbb{S}(\mathcal{O}))$ for almost every $\ell > 0$. We then see from Lemma 8 that the laws $\theta_t(\ell)$ and $\varphi_t(\ell)$ coincide for almost every $\ell > 0$. A standard argument based on the semigroup and the scaling properties enables us to remove the word 'almost' in the last sentence, and thus we have established the identity of the fragmentation semigroups. □

4. The left-most and the size biased picked fragments

Our purpose in this section is to present an interesting identity arising in the study of the fragmentation process induced by the standard Brownian excursion. So let ϵ stand for a Brownian excursion with unit duration, and $\mathscr{L}^{(t)}(\epsilon)$ for the ladder time set given by (1). Because $\lim_{s\to 0+} \epsilon(s)/s = \infty$ a.s., 0 is an isolated point of the ladder time set, and this incites us to introduce

$$\lambda_t \;=\; \inf\{s > 0 : st = \epsilon(s)\} ,$$

which can be viewed as the length of the left-most fragment in the partition at t. We also introduce an independent variable U which is uniformly distributed on $[0, 1]$; and we write λ_t^* for the length of the open interval that contains U in the canonical decomposition of $(0, 1)\backslash\mathscr{L}^{(t)}(\epsilon)$. In other words,

$$\lambda_t^* \;=\; d^{(t)}(U) - g^{(t)}(U)$$

where for $u \in (0,1)$,

$$d^{(t)}(u) = \inf\left\{s \geq u : s \in \mathscr{L}^{(t)}(\epsilon)\right\} \text{ and } g^{(t)}(u) = \sup\left\{s \leq u : s \in \mathscr{L}^{(t)}(\epsilon)\right\}$$

are the first points in $\mathscr{L}^{(t)}(\epsilon)$ to the right of u and to the left of u, respectively.

Recall $T = (T_x, x \geq 0)$ is the first passage process for the Brownian motion with zero drift. We now state the following identity.

Proposition 11. *The processes* $(\lambda_t, t \geq 0)$, $\left(\lambda_t^*, t \geq 0\right)$ *and* $(1/(1+T_t), t \geq 0)$ *have the same distribution. They are Markov processes on* $[0,1]$ *with semigroup given by*

$$\rho_t(x, dy) \;=\; t\sqrt{\frac{x^3}{2\pi y(x-y)^3}}\,\exp\left\{-\frac{xyt^2}{2(x-y)}\right\} dy, \quad 0 < y < x\,.$$

Proof. Here is a slick proof due to Jim Pitman. Since $(\varphi_t = \theta_t, t \geq 0)$ is the fragmentation semigroup introduced by Aldous and Pitman [3] by logging the continuum random tree, the process $\left(\lambda_t^*, t \geq 0\right)$ is identical in distribution to the process of the mass of the tree component containing a random leaf picked according to the mass measure of the continuum random tree, in the fragmentation process of Aldous and Pitman. By Theorem 6 in [3], we get the identity in distribution

$$\left(\lambda_t^*, t \geq 0\right) \;\stackrel{\text{(d)}}{=}\; (1/(1+T_t), t \geq 0)$$

Next, recall that if $\beta = (\beta(s), s \geq 0)$ is a 3-dimensional Brownian motion started at 0, then the process $\left(s\beta(\frac{1-s}{s}), 0 < s \leq 1\right)$ is a version of the standard 3-dimensional Brownian bridge; see e.g. Exercise I.3.10 in [16]. This entails that if $R = |\beta|$ denotes a 3-dimensional Bessel process started from 0, then a version of the standard Brownian excursion is given by

$$\epsilon(s) \;=\; sR\left(\frac{1-s}{s}\right), \quad 0 < s \leq 1\,.$$

Working with this version, it is immediate that for all $t \geq 0$

$$\lambda_t \;=\; \frac{1}{1+L_t}\,,$$

where $L_x = \sup\{s \geq 0 : R_s = x\}$ stands for the last-passage time of R at level $x \geq 0$. As the processes $(L_x, x \geq 0)$ and $(T_x, x \geq 0)$ have the same law, the first claim of the statement is proven. The second follows readily. □

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