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Excessive kernels and Revuz measures*

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Abstract. We consider a proper submarkovian resolvent of kernels on a Lusin measurable space and a given excessive measure ξ. With every quasi bounded excessive function we associate an excessive kernel and the corresponding Revuz measure. Every finite measure charging no ξ–polar set is such a Revuz measure, provided the hypothesis (B) of Hunt holds. Under a weak duality hypothesis, we prove the Revuz formula and characterize the quasi boundedness and the regularity in terms of Revuz measures. We improve results of Azéma [2] and Getoor and Sharpe [20] for the natural additive functionals of a Borel right process.

1. Introduction

Let $\mathscr{U} = (U_\alpha)_{\alpha>0}$ be a submarkovian resolvent of kernels on a Lusin measurable space $(X, \mathscr{B})$. We suppose that $\mathscr{U}$ is proper and that the set $\mathscr{E}_\mathscr{U}$ of all $\mathscr{B}$–measurable $\mathscr{U}$–excessive functions on X which are finite $\mathscr{U}$–almost everywhere contains the positive constant functions, is min–stable and generates $\mathscr{B}$. Note that the resolvent of a transient Borel right process verifies the above conditions.

The purpose of this paper is to give a new approach for the Revuz measures, improving results of Getoor and Sharpe [20] and Azéma [2].

If $\mathscr{U}$ possesses a reference measure m then there exists a $\mathscr{B}$–measurable subset X_o of X such that $X \setminus X_o$ is semipolar and such that sufficiently many excessive functions s may be described as the *Green potentials* of measures ν_s on X_o given by

$$\nu_s(t) = [t, s]$$

where t runs in the set of all coexcessive functions and $[\ ,\ ]$ denotes the usual duality generated by m, between the excessive and coexcessive functions. In fact in this case (see [10]) there exists a Green function $(x, y) \mapsto G(x, y)$ on $X_o \times X_o$ such that

$$s = \int G(\cdot, y) d\nu_s(y) = V_s 1$$

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where V_s is the *excessive kernel* on X_o defined by $V_s f = \int G(\cdot, y) f(y) d\nu_s(y)$. The above relation between s and ν_s becomes the *Revuz formula*

$$[t, s] = [1, V_s t]$$

and ν_s is the so called *Revuz measure of* s.

If $\mathscr{U}$ does not possess a reference measure then for every $\mathscr{U}$–excessive measure ξ and each ξ–natural potential s (in the sense of [20]), the Revuz measure ν_s^ξ is defined by

$$\nu_s^\xi(f) = \lim_{t \to 0} t^{-1} P^\xi [\int_{(0,t]} f(\mathsf{X}_{u-}) d\mathsf{A}_u]$$

or equivalently (cf. [24] and [11])

$$(*) \qquad \nu_s^\xi(f) = L(\xi, V_s f)$$

where $V_s f(x) = P^x [\int_0^\infty f(\mathsf{X}_{u-}) d\mathsf{A}_u]$, $\mathsf{A} = (\mathsf{A}_t)_{t \geq 0}$ is the natural additive functional whose potential function is s and L is the energy functional (see [24], [20] and [2]). This construction demands that the process possesses left limits in X. We remark that s will be ξ–natural potential if and only if it is ξ–quasi bounded. For each natural additive functional (having the potential function s), V_s is the unique kernel on X with $V_s 1 = s$, Vf being $\mathscr{U}$–excessive if $f \geq 0$, and such that $B^G V_s(1_G) = V_s(1_G)$ for every Ray open subset G of X (cf. [23] and [11]). Also s will be ξ–regular if and only if the associated additive functional is continuous or equivalently $B^M V_s(1_M) = V_s(1_M)$ for all $M \in \mathscr{B}$.

In this paper we construct the Revuz measure ν_s^ξ, without assuming the existence of the left limits for the process, using the above formula $(*)$ where the excessive kernel V_s is obtained directly from s, by potential theoretic techniques on excessive functions. We show (Theorem 4.4) that if s is ξ–quasi bounded then ν_s^ξ charges no ξ–polar set. Conversely, there exists a ξ–semipolar set H such that every finite measure on $X \setminus H$, charging no ξ–polar set is the Revuz measure of a ξ–quasi bounded excessive function. If the hypothesis (B) of Hunt holds then the exceptional set H disappears. This is an improvement of a result of Azéma [2], obtained under restrictive assumtions. Moreover, s will be ξ–regular if and only if its Revuz measure ν_s^ξ charges no ξ–semipolar set and s is uniquely determined ξ–quasi everywere by ν_s^ξ (Theorem 4.3). Note that every finite measure on X, charging no ξ–semipolar set is of the above form ν_s^ξ with ξ–regular s. Such type of results have been previously obtained in [23], [2], [25], [15] and [16], in various contexts.

Under a weak duality hypothesis similar to that of Getoor and Sharpe [20], we are able to obtain (Theorem 4.5) the corresponding Revuz formula

$$L(t \cdot \xi, s) = L(\xi, V_s t)$$

for each coexcessive function t (see [25], [20] and [11]). We underline that we do not assume that X is sufficiently large to support the both direct and dual processes. Using this formula we show (Theorem 4.7 and Theorem 4.9) that an excessive function is ξ–quasi bounded (resp. ξ–regular) if and only if its Revuz measure charges no ξ–copolar set (resp. ξ–cosemipolar set). Particularly we get that the ξ–cosemipolar sets are ξ–semipolar and if the hypothesis (B) of Hunt holds, then the ξ–copolar sets are ξ–polar; compare with [20].

2. Preliminaries

Throughout the paper, $\mathcal{U}$ will be a resolvent as in Introduction.

Recall that the set X is called **semisaturated** with respect to $\mathcal{U}$ if each $\mathcal{U}$–excessive measure dominated by a potential is also a potential. This property is equivalent with the existence of a Borel right process on X having $\mathcal{U}$ as associated resolvent. The **fine topology** is the topology on X generated by $\mathcal{E}_{\mathcal{U}}$. We denote by $Exc_{\mathcal{U}}$ the set of all $\mathcal{U}$–excessive measures on X (see e.g. [12]). Also, if $\xi \in Exc_{\mathcal{U}}$ then we denote by $\mathcal{E}^{\xi}_{\mathcal{U}}$ the set of all $\mathcal{B}$–measurable $\mathcal{U}$–excessive functions which are finite ξ–almost everywhere (ξ–a.e.) and by Exc_{ξ} the set of those $\mathcal{U}$–excessive measures which are absolutely continuous with respect to ξ.

If $M \subset X$ and s is a $\mathcal{U}$–excessive function on X (i.e. s is universally measurable and $\alpha U_\alpha s \nearrow s$ as $\alpha \nearrow \infty$) then the **réduite of s on M** is the function $R^M s$ on X defined by

$$R^M s := \inf\{t/t\ \mathcal{U}\text{–excessive}, s \le t \text{ on } M\} .$$

If moreover $M \in \mathcal{B}$ then $R^M s$ is universally measurable (cf. [4]) and we denote by $B^M s$ its $\mathcal{U}$–excessive regularization.

Let θ be a measure on X. We say that a set $M \in \mathcal{B}$ is θ–**polar** if $\theta(B^M 1) = 0$. An arbitrary subset of X is called θ–polar if it is a subset of a $\mathcal{B}$–measurable θ–polar set. A property is said to hold θ–**quasi everywhere** (θ–q.e.) if the set where it does not hold is θ–polar and θ–negligible.

Recall that a set $M \in \mathcal{B}$ is **thin** at a point $x \in X$ if there exists $s \in \mathcal{E}_{\mathcal{U}}$ such that $B^M s(x) < s(x)$. An arbitrary subset of X is called thin at x if it is a subset of a $\mathcal{B}$–measurable set which is thin at x. A subset of X is said to be **totally thin** if it is thin at each point of X. A **semipolar** set is a countable union of totally thin sets. A set $A \in \mathcal{B}$ is termed θ–**semipolar** if it is of the form $A = A_o \cup A_1$ where A_o, $A_1 \in \mathcal{B}$ with A_o θ–polar and A_1 semipolar.

Recall now some considerations concerning the Ray compactification. Since the initial kernel U of the resolvent $\mathcal{U} = (U_\alpha)_{\alpha>0}$ is proper, there exists a bounded submarkovian resolvent $\mathcal{V} = (V_\alpha)_{\alpha>0}$ on X such that $\mathcal{E}_{\mathcal{U}} = \mathcal{E}_{\mathcal{V}}$. A **Ray cone** will be a subcone $\mathcal{R}$ of the bounded $\mathcal{U}$–excessive functions which is min–stable, separable in the uniform norm, generates the σ–algebra $\mathcal{B}$ and moreover $1 \in \mathcal{R}$, $V((\mathcal{R} - \mathcal{R})_+) \subset \mathcal{R}$, $V_\alpha(\mathcal{R}) \subset \mathcal{R}$, $\alpha > 0$. A **Ray topology** is the topology on X generated by a Ray cone. We consider the **Ray compactification** Y of X with respect to $\mathcal{R}$. Since $(X, \mathcal{B})$ is a Lusin measurable space, it follows that X is a Borel subset of Y and $\mathcal{B}(Y)|_X = \mathcal{B}$, where $\mathcal{B}(Y)$ denotes the σ–algebra of all Borel subsets of Y.

Let $u \in \mathcal{E}_{\mathcal{U}}$, $u > 0$, and $\xi \in Exc_{\mathcal{U}}$. A function $s \in \mathcal{E}_{\mathcal{U}}$ is called (u, ξ)–**quasi bounded** if there exists a sequence (s_n) in $\mathcal{E}_{\mathcal{U}}$ such that $s = \sum_n s_n$ ξ–q.e. and $s_n \leq u$ ξ–q.e. for all n.

A function $s \in \mathcal{E}_{\mathcal{U}}$ is called u–**quasi bounded** if there exists a sequence (s_n) in $\mathcal{E}_{\mathcal{U}}$ such that $s = \sum_n s_n$ and $s_n \leq u$ for all n. We say simply **quasi bounded** instead of 1–quasi bounded. If we denote by $E^{\xi}(\mathcal{E}_{\mathcal{U}})$ the factor set of $\mathcal{E}^{\xi}_{\mathcal{U}}$ by the equivalence relation

$$s \sim t \iff (s = t \quad \xi\text{–q.e.})$$

then $E^{\xi}(\mathcal{E}_{\mathcal{U}})$ becomes an H–cone with respect to the order relation

$$\widetilde{s} \leq \widetilde{t} \quad \text{iff} \quad s \leq t \quad \xi\text{–q.e.}$$

where $\widetilde{s}$ denotes the equivalence class of $s \in \mathcal{E}^{\xi}_{\mathcal{U}}$.

From [3] and [6] it follows that a function $s \in \mathcal{E}^{\xi}_{\mathcal{U}}$ will be (u, ξ)–quasi bounded if and only if $\bigwedge_n R(s - nu) = 0$ ξ–q.e. or equivalently $\bigwedge_n R(\widetilde{s} - n\widetilde{u}) = 0$. Also, $s \in \mathcal{E}_{\mathcal{U}}$ will be u–quasi bounded if and only if $\bigwedge_n R(s - nu) = 0$. Note that $s \in \mathcal{E}_{\mathcal{U}}$ will be (u, ξ)–quasi bounded, if and only if there exists $s' \in \mathcal{E}_{\mathcal{U}}$ such that $s' = s$ ξ–q.e. and s' is u–quasi bounded. Moreover s' may be chosen specifically dominated by s.

A function $s \in \mathcal{E}^{\xi}_{\mathcal{U}}$ is termed ξ–**quasi bounded** if it is (u, ξ)–quasi bounded for all $u \in \mathcal{E}_{\mathcal{U}}$, $u > 0$. We denote by $Q_{bd}(\mathcal{E}_{\mathcal{U}}, \xi)$ the set of all $s \in \mathcal{E}^{\xi}_{\mathcal{U}}$ which are ξ–quasi bounded and by $Q_{bd}(\mathcal{E}_{\mathcal{U}})$ the set of all $s \in \mathcal{E}_{\mathcal{U}}$ which are u–quasi bounded for all $u \in \mathcal{E}_{\mathcal{U}}$, $u > 0$. From the above consideration we get

$$Q_{bd}(\mathcal{E}_{\mathcal{U}}) = \bigcap_{\xi \in Exc_{\mathcal{U}}} Q_{bd}(\mathcal{E}_{\mathcal{U}}, \xi) \ .$$

We remark that if $u \in Q_{bd}(\mathcal{E}_{\mathcal{U}})$, $u > 0$, then $Q_{bd}(\mathcal{E}_{\mathcal{U}})$ coincides with the set of all functions $s \in \mathcal{E}_{\mathcal{U}}$ which are u–quasi bounded. Also $Q_{bd}(\mathcal{E}_{\mathcal{U}}, \xi)$ coincides with the set of all $s \in \mathcal{E}^{\xi}_{\mathcal{U}}$ which are (u, ξ)–quasi bounded.

Remark. Suppose that $\mathcal{U}$ is the resolvent of a Borel right process on X. Recall that $s \in \mathcal{E}_{\mathcal{U}}$ is called ξ–**natural potential** (ξ is a given $\mathcal{U}$–excessive measure) if for every increasing sequence (T_n) of stopping times with $T_n \nearrow \zeta$ we have $P^x[s(\mathsf{X}_{T_n})] \searrow 0$ ξ–a.e. Let f_o be a $\mathcal{B}$–measurable functions on X, $0 < f_o \leq 1$, such that Uf_o is bounded. If we define the finely open set $D_n := [s < nUf_o]$ then the sequence of stopping times $(T_{X \setminus D_n})$ has the property $T_{X \setminus D_n} \nearrow \zeta$. On the other hand it is easy to see that the function s will be ξ–quasi bounded if and only if

$$B^{X \setminus D_n} s \searrow 0 \quad \xi\text{–a.e.}$$

By straightforward calculation it follows that Uf_o is ξ–natural potential for all ξ. From the above considerations we conclude that s will be ξ–natural potential if and only if it is ξ–quasi bounded. In particular we deduce that s will be ξ–natural potential if and only if for every increasing sequence (D_n) of $\mathscr{B}$–measurable finely open sets with $T_{X\setminus D_n} \nearrow \zeta$ we have $B^{X\setminus D_n} s \searrow 0$ ξ–a.e. This result is a version of Theorem 3.3 in [20].

Let $\xi \in Exc_{\mathscr{U}}$. A function $s \in \mathscr{E}_{\mathscr{U}}^{\xi}$ is called ξ–**regular** if for every increasing sequence (s_n) in $\mathscr{E}_{\mathscr{U}}^{\xi}$ with $\sup_n s_n \geq s$ ξ–q.e. we have $\bigwedge_n R(s - s_n) = 0$ ξ–q.e.

The function $s \in \mathscr{E}_{\mathscr{U}}$ is termed **regular** if it is ξ–regular for all $\xi \in Exc_{\mathscr{U}}$. We denote by $\mathscr{E}_{\mathscr{U}}^{\xi,r}$ the set of all ξ–regular functions from $\mathscr{E}_{\mathscr{U}}^{\xi}$ and by $\mathscr{E}_{\mathscr{U}}^{r}$ the set of all regular functions from $\mathscr{E}_{\mathscr{U}}$.

Obviously every ξ–regular function $s \in \mathscr{E}_{\mathscr{U}}^{\xi}$ belongs to $Q_{bd}(\mathscr{E}_{\mathscr{U}}, \xi)$. Particularly, each regular function $s \in \mathscr{E}_{\mathscr{U}}$ belongs to $Q_{bd}(\mathscr{E}_{\mathscr{U}})$.

Note that there exists $u \in \mathscr{E}_{\mathscr{U}}$, $u > 0$ which is regular. More precisely if f is a positive $\mathscr{B}$–measurable function on X such that $0 < f \leq 1$ and $Uf \in \mathscr{E}_{\mathscr{U}}$ then Uf is regular.

Proposition 2.1. *For all $\xi \in Exc_{\mathscr{U}}$ and every ξ–regular $\mathscr{U}$–excessive function s there exists a sequence (s_n) of regular $\mathscr{U}$–excessive functions such that $\sum_n s_n \prec s$ and $s = \sum_n s_n$ ξ–q.e.* ($\prec$ denotes the specific order in the $\mathscr{U}$–excessive functions.)

Proof. Firstly suppose that $s \in \mathscr{E}_{\mathscr{U}}^{\xi,r} \cap Q_{bd}(\mathscr{E}_{\mathscr{U}}, \xi)$. We construct inductively the sequences $(s'_n)_{n\geq 0}$ and $(r_n)_{n\geq 0}$ in $\mathscr{E}_{\mathscr{U}}$ as follows:

$$s'_0 = s,\ r_0 = 0, \quad r_{n+1} = \bigwedge_m R(s'_n - mU_m s'_n), \quad s'_{n+1} = s'_n - r_{n+1}\ .$$

Since $r_{n+1} \prec s'_n$, we have indeed $s'_{n+1} \in \mathscr{E}_{\mathscr{U}}$ and $s = \sum_{k=0}^{\infty} r_k + s'$ where $s' := \curlywedge\{s'_n\ /\ n \in \mathbb{N}\}$ ($\curlywedge$, $\curlyvee$ are the lattice operations with respect to the specific order in $\mathscr{E}_{\mathscr{U}}$). Because $s \in \mathscr{E}_{\mathscr{U}}^{\xi,r}$ we get $r_k = 0$ ξ–q.e. We show that s' is regular. This fact follows from $\bigwedge_m R(s' - mU_m s') \leq r_k$ for all k and consequently $\bigwedge_m R(s' - mU_m s') = 0$. By Proposition 2.1 in [7] we conclude that s' is regular.

Let now $s \in \mathscr{E}_{\mathscr{U}}^{\xi,r}$ be arbitrary and $u \in Q_{bd}(\mathscr{E}_{\mathscr{U}}, \xi)$, $u > 0$.

If for all $n \geq 0$ we set $\rho_n = R(s - s \wedge nu)$ then $\rho_n \prec s$ and there exists $t_n \in \mathscr{E}_{\mathscr{U}}$ such that $s = t_n + \rho_n$ and $t_n \leq s \wedge nu$. Since s is ξ–regular it follows that t_n is also ξ–regular and therefore, by the above considerations there exists $t'_n \in \mathscr{E}_{\mathscr{U}}^{r}$ such that $t'_n \prec t_n$ and $t_n = t'_n$ ξ–q.e. We put $s_0 = t'_0$ and $s_n = \curlyvee\{t'_k/0 \leq k \leq n\} - \curlyvee\{t'_k/0 \leq k \leq n-1\}$ for $n \geq 1$. We have $s_n \in \mathscr{E}_{\mathscr{U}}^{r}$, $\sum_n s_n \prec s$ and from $\bigwedge_n \rho_n = 0$ ξ–q.e. we get $s = \sum_n s_n$ ξ–q.e.

Proposition 2.2. *Suppose that X is semisaturated with respect to $\mathscr{U}$. If $\xi \in Exc_{\mathscr{U}}$ and $s \in \mathscr{E}_{\mathscr{U}}$ then the following assertions are equivalent*:

i) The function s is ξ–regular.

ii) For every sequence (μ_n) of positive measures on X such that $\mu_n \circ U \searrow \mu \circ U$, $\inf_n \mu_n(s) < \infty$ and μ_n charging no ξ–polar set, we have $\inf_n \mu_n(s) = \mu(s)$.

iii) For every increasing sequence (D_n) of finely open $\mathscr{B}$–measurable subsets of X we have $\inf_n B^{X\setminus D_n} s(x) = \mu_x(s)$ ξ–a.e. (in x) where for all $x \in X$, μ_x denotes the measure on X such that $\varepsilon_x \circ B^{X\setminus D_n} U \searrow \mu_x \circ U$.

iv) For each potential $\theta \circ U \in Exc_{\mathscr{U}}$ such that $\theta \circ U \ll \xi$, $\xi \ll \theta \circ U$, $\theta(s) < \infty$ and for every increasing sequence (D_n) of finely open $\mathscr{B}$–measurable subsets of X we have

$$\inf_n \theta(B^{X\setminus D_n} s) = \widetilde{\theta}(s)$$

where $\widetilde{\theta}$ is the measure on X defined by $\widetilde{\theta} = \int \mu_x d\theta(x)$.

Proof. The equivalence $i) \Longleftrightarrow ii)$ follows from [4] and [7]. The implications $ii) \Longrightarrow iii)$ and $iii) \Longrightarrow iv)$ are immediate.
$iv) \Longrightarrow i)$. Let (s_n) be an increasing sequence in $\mathscr{E}_{\mathscr{U}}$ such that $\sup_n s_n = s$ ξ–a.e. Let $r := \bigwedge_n R(s - s_n)$ and for all $\alpha > 1$ let $D_n := X \setminus \overline{[\alpha(s - s_n) - r > 0]}^f$, where for a subset M of X we have denoted by $\overline{M}^f$ its fine closure. As in the proof of Theorem 2.3 in [4] it follows that $B^{X\setminus D_n} r = r$ for all n. From

$$r(x) = B^{X\setminus D_n} r(x) \le \alpha B^{X\setminus D_n}(s - s_n)(x) \le \alpha[B^{X\setminus D_n} s(x) - \mu_x(s_n)]$$

we deduce $\theta(r) + \alpha\widetilde{\theta}(s) \le \inf_n \theta(B^{X\setminus D_n} s)$, $\theta(r) \le 0$, $r = 0$ ξ–q.e., completing the proof.

Remark. Suppose that $\mathscr{U}$ is the resolvent of a Borel right process. Then:
$a)$ From the above equivalence $i) \Longleftrightarrow ii)$ and Proposition 2.5 in [17] it follows that $s \in \mathscr{E}_{\mathscr{U}}$, $s < \infty$, will be ξ–regular if and only if there exists a continuous additive functional whose associated potential is equal s ξ–q.e.
$b)$ The assertions $iii)$ and $iv)$ in Proposition 2.2 are versions of the characterization of the regularity in terms of hitting times (see (3.4) in [20]) instead of arbitrary stopping times.

The following two results are essentially well known (see e.g [9], ch. VI, (3.6)).

Lemma 2.3. *Let $\mathscr{N}$ be a set of $\mathscr{B}$–measurable subsets of X such that if $(M_n) \subset \mathscr{N}$ then $\bigcup_n M_n \in \mathscr{N}$. Then for every σ–finite measure μ on X there exists a unique decomposition $\mu = \mu' + \mu''$ where μ' is carried by a set from $\mathscr{N}$ and μ'' charges no set from $\mathscr{N}$.*

Corollary 2.4. *Let ξ be a $\mathscr{U}$–excessive measure and μ be a σ–finite measure on X. Then μ can be written uniquely in the form $\mu = \mu' + \mu''$ where μ' is carried by a ξ–semipolar (resp. ξ–polar) $\mathscr{B}$–measurable set and μ'' charges no ξ–semipolar (resp. ξ–polar) set.*

Lemma 2.5. *Let $\mathscr{M}$ be a set of positive finite measures on $(X, \mathscr{B})$ such that if $\nu \in \mathscr{M}$ then $\theta \in \mathscr{M}$ for every positive measure θ with $\theta \leq \nu$. Let further μ be a finite positive measure on X such that if $F \in \mathscr{B}$ then $\nu(F) = 0$ for all $\nu \in \mathscr{M}$ if and only if $\mu(F) = 0$. Then there exists a sequence (μ_n) in $\mathscr{M}$ such that $\mu = \sum_n \mu_n$.*

Proof. We consider the set $\mathscr{F}_\mu$ of all $\mathscr{B}$-measurable functions $f : X \longrightarrow [0, 1]$ such that there exists a sequence (ν_n) in $\mathscr{M}$ with $f \cdot \mu = \sum_n \nu_n$. It is easy to see that for every sequence (f_n) in $\mathscr{F}_\mu$ the function $\sup_n f_n$ belongs also to $\mathscr{F}_\mu$. Hence there exists $f_o \in \mathscr{F}_\mu$ such that $f_o \geq f$ μ-a.e. for all $f \in \mathscr{F}_\mu$. On the other hand we have $0 \leq f_o + \inf(1 - f_o, f_o) \leq 1$, $f_o + \inf(1 - f_o, f_o) \in \mathscr{F}_\mu$. Consequently we get $\inf(1 - f_o, f_o) = 0$ μ-a.e. or equivalently there exists a set $F \in \mathscr{B}$ with $f_o = 1_F$ μ-a.e. From $(1_{X \setminus F} \cdot \mu) \wedge \nu = 0$ for all $\nu \in \mathscr{M}$ it follows that $\nu(X \setminus F) = 0$ for every $\nu \in \mathscr{M}$ and therefore $\mu(X \setminus F) = 0$ i.e. $f_o \cdot \mu = \mu$.

3. Excessive kernels

In this section we consider a fixed Ray cone $\mathscr{R}$ and the Ray compactification Y of X with respect to $\mathscr{R}$.

Definition. *If $s \in \mathscr{E}_{\mathscr{U}}$ we denote by $\mathsf{carr}\, s$ the set of all points $y \in Y$ such that $B^{X \setminus V} s \neq s$ for each open neighbourhood V of y.*

It follows immediately that $\mathsf{carr}\ s$ is a closed subset of Y such that for all $s, t \in \mathscr{E}_{\mathscr{U}}$ we have $\mathsf{carr}\ R(s - t) \subset \overline{[s > t]}$ and $\mathsf{carr}\ s \subset \mathsf{carr}\ t$ whenever $s \prec t$.

Definition. *We say that a function $s \in \mathscr{E}_{\mathscr{U}}$ is* ***of potential type on*** *Y if*

$$t \in \mathscr{E}_{\mathscr{U}}, \quad t \prec s, \quad \mathsf{carr}\ t = \emptyset \Longrightarrow t = 0 \ .$$

We denote by $\mathscr{P}_{\mathscr{U}}(Y)$ the set of all functions which are of potential type on Y.

Theorem 3.1. *Every bounded function $s \in \mathscr{E}_{\mathscr{U}}$ is of potential type on Y. The set $\mathscr{P}_{\mathscr{U}}(Y)$ is a convex subcone of $\mathscr{E}_{\mathscr{U}}$, solid in $\mathscr{E}_{\mathscr{U}}$ with respect to the specific order. If $s \in \mathscr{E}_{\mathscr{U}}$ is such that $s = \sum_n s_n$ where $(s_n) \subset \mathscr{E}_{\mathscr{U}}$ then $s \in \mathscr{P}_{\mathscr{U}}(Y)$ if and only if $s_n \in \mathscr{P}_{\mathscr{U}}(Y)$ for all n, in which case we have*

$$\mathsf{carr}\, s = \overline{\bigcup_n \mathsf{carr}\, s_n} \ .$$

Moreover for all $s \in \mathscr{P}_{\mathscr{U}}(Y)$, $\mathsf{carr}\, s$ is the smallest closed subset K of Y such that $B^{G \cap X} s = s$ for every open neighbourhood G of K.

Proof. The fact that each bounded $s \in \mathcal{E}_{\mathcal{U}}$ belongs to $\mathcal{P}_{\mathcal{U}}(Y)$ is a result of Wittmann (cf. [27]). The proof for the other assertions in the theorem is standard and similar with the case when $\mathcal{E}_{\mathcal{U}}$ is an H–cone of functions on the set X (see [10]).

Remark. Every quasi bounded function from $\mathcal{E}_{\mathcal{U}}$ is of potential type on Y.

Definition. *A function $s \in \mathcal{E}_{\mathcal{U}}$ is called **of potential type on** X if*

$$t \in \mathcal{E}_{\mathcal{U}}, \quad t \prec s, \quad \operatorname{carr} t \cap X = \emptyset \Longrightarrow t = 0 \ .$$

The set of all functions of potential type on X is denoted by $\mathcal{P}_{\mathcal{U}}(X)$. Obviously $\mathcal{P}_{\mathcal{U}}(X)$ is a σ–band in $\mathcal{E}_{\mathcal{U}}$ and we have $\mathcal{P}_{\mathcal{U}}(X) \subset \mathcal{P}_{\mathcal{U}}(Y)$. Also, it is easy to see that if $s \in \mathcal{P}_{\mathcal{U}}(Y)$ then $s \in \mathcal{P}_{\mathcal{U}}(X)$ if and only if $\operatorname{carr} t = \overline{\operatorname{carr} t \cap X}$ for all $t \prec s$.

Proposition 3.2. *Let $s \in \mathcal{P}_{\mathcal{U}}(X)$. Then $s \in Q_{bd}(\mathcal{E}_{\mathcal{U}})$ if and only if s is quasi bounded.*

Proof. Suppose that $s \leq 1$. We may assume also that $U((\mathcal{R} - \mathcal{R})_+) \subset \mathcal{R}$. If we set $r_n := R(s - nU1)$ then we have $r_n \prec s$ and $s_n := s - r_n \leq nU1$. Therefore $s_n \in Q_{bd}(\mathcal{E}_{\mathcal{U}})$. On the other hand $\operatorname{carr} r_n \subset \overline{[s > nU1]} \subset [1 \geq n\widetilde{U1}]$ (where $\widetilde{U1}$ denotes the continuous extension of the function $U1 \in \mathcal{R}$ to Y) and $s = \curlywedge\{r_n/n \in \mathbb{N}\} + \curlyvee\{s_n/n \in \mathbb{N}\}$. Since $\curlyvee\{s_n/n \in \mathbb{N}\} \in Q_{bd}(\mathcal{E}_{\mathcal{U}})$ and $\operatorname{carr} r \cap X = \emptyset$ where $r := \curlywedge\{r_n/n \in \mathbb{N}\} \in \mathcal{P}_{\mathcal{U}}(X)$ then we get $r = 0$ and consequently $s = \curlyvee\{s_n/n \in \mathbb{N}\}$.

Definition. *If $s \in \mathcal{E}_{\mathcal{U}}$ then we denote by $\operatorname{carr}_f s$ the set of all points $x \in X$ such that $B^{X \setminus V} s \neq s$ for each finely open neighbourhood V of x, $V \in \mathcal{B}$.*

It is easy to see that $\operatorname{carr}_f s$ is finely closed and for all $s, t \in \mathcal{E}_{\mathcal{U}}$ we have $\operatorname{carr}_f R(s - t) \subset \overline{[s > t]}^f$. If $s \prec t$ then $\operatorname{carr}_f s \subset \operatorname{carr}_f t$. Obviously we have $\operatorname{carr}_f s \subset \operatorname{carr} s \cap X$.

We denote by $\mathcal{P}^f_{\mathcal{U}}(X)$ the set of all $s \in \mathcal{E}_{\mathcal{U}}$ such that

$$t \in \mathcal{E}_{\mathcal{U}}, \quad t \prec s, \quad \operatorname{carr}_f t = \emptyset \Longrightarrow t = 0 \ .$$

We have $\mathcal{P}^f_{\mathcal{U}}(X) \subset \mathcal{P}_{\mathcal{U}}(X)$.

The set X is called **nearly saturated** with respect to $\mathcal{U}$ if every $\mathcal{U}$–excessive measure on X which is a quasi continuous element from Exc (see [4]) is a potential. Recall that an element $\xi \in Exc_{\mathcal{U}}$ is termed **quasi continuous** if for each sequence (ξ_n) in $Exc_{\mathcal{U}}$ with $\xi_n \nearrow \xi$ we have $R(\xi - \xi_n) \searrow 0$. It is known (cf. [4]) that if $\xi = \mu \circ U \in Pot$ then ξ will be quasi continuous if and only if μ charges no semipolar subset of X.

Remark. a) There exists a Lusin measurable space $(X_1, \mathcal{B}_1)$ such that X is a $\mathcal{B}_1$–measurable subset of X_1 and a submarkovian resolvent $\mathcal{U}' = (U'_\alpha)_{\alpha > 0}$ on X_1 such that $X_1 \setminus X$ is $\mathcal{U}'$–negligible, $U'_\alpha f|_X = U_\alpha(f|_X)$ for all $\alpha > 0$ and every positive $\mathcal{B}_1$–measurable function f on X_1 and such that every $\xi \in Exc_{\mathcal{U}'} = Exc_{\mathcal{U}}$ with $(\xi, 1) \leq 1$ is a potential on X_1 (see [5]). The set X_1 is uniquely determined and called the **saturation of** X. Particularly X_1 is semisaturated.
b) One can show that X is nearly saturated if and only if the set $X_1 \setminus X$ is semipolar (with respect to $\mathcal{U}'$). Consequently if X is semisaturated then it is nearly saturated.

Theorem 3.3. *Suppose that X is nearly saturated. Then every regular $\mathcal{U}$–excessive function belongs to $\mathcal{P}^f_{\mathcal{U}}(X)$. For all $s \in \mathcal{P}^f_{\mathcal{U}}(X)$ the set $\mathsf{carr}_f s$ is the smallest finely closed subset F of X such that $R^F s = s$. The set $\mathcal{P}^f_{\mathcal{U}}(X)$ is a σ–band in $\mathcal{E}_{\mathcal{U}}$ and for all $s \in \mathcal{P}^f_{\mathcal{U}}(X)$ and all $(s_n) \subset \mathcal{E}_{\mathcal{U}}$ such that $s = \sum_n s_n$ we have*

$$\mathsf{carr}_f s = \overline{\bigcup_n \mathsf{carr}_f s_n}^f .$$

Proof. The first assertion follows from [7]. The rest of the proof is standard.

We denote by $\mathcal{F}(Y)$ (resp. $\mathcal{F}(X)$) the set of all positive numerical Borel measurable (resp. $\mathcal{B}$–measurable) functions on Y (resp. on X).

Definition. *A kernel $V : \mathcal{F}(Y) \longrightarrow \mathcal{F}(X)$ is called **natural excessive** if*

i) Vf is $\mathcal{U}$–excessive for every bounded function $f \in \mathcal{F}(Y)$.

ii) There exists $f_o \in \mathcal{F}(Y), 0 < f_o \leq 1$ such that $Vf_o \in \mathcal{E}_{\mathcal{U}}$ and $B^{G\cap X} V(f_o 1_G) = V(f_o 1_G)$ for each open subset G of Y.

If $V : \mathcal{F}(Y) \longrightarrow \mathcal{F}(X)$ is a natural excessive kernel such that $V(1_{Y\setminus X}) = 0$ then we say that V is a **natural excessive kernel on** X. Consequently, a natural excessive kernel on X is precisely a kernel on X, $V : \mathcal{F}(X) \longrightarrow \mathcal{F}(X)$, such that Vf is $\mathcal{U}$–excessive for every bounded function $f \in \mathcal{F}(X)$ and there exists $f_o \in \mathcal{F}(X)$, $0 < f_o \leq 1$ such that $Vf_o \in \mathcal{E}_{\mathcal{U}}$ and $B^G V(f_o 1_G) = V(f_o 1_G)$ for each Ray open subset G of X.

Remark. *a*) If V is a natural excessive kernel on X and $g \in \mathcal{F}(X)$ is finite then the kernel $g\cdot V$ on X defined by $(g\cdot V)(f) := V(gf)$, $f \in \mathcal{F}(X)$, is also a natural excessive kernel on X.

b) If (V_n) is a sequence of natural excessive kernels on X then $\sum_n V_n$ is also a natural excessive kernel on X provided that there exists $f_o \in \mathcal{F}(X)$, $0 < f_o \leq 1$ such that $\sum_n V_n f_o \in \mathcal{E}_{\mathcal{U}}$.

c) If $\mathcal{U}$ is the resolvent of a special standard process on X (see e.g. [19] or [26]) and $\mathsf{A} = (\mathsf{A}_t)_{t\geq 0}$ is an additive functional then its potential kernel U_{A} ($U_{\mathsf{A}} f(x) := P^x[\int_0^\infty f \circ \mathsf{X}_t d\mathsf{A}_t]$, assumed to be proper) will be a natural excessive kernel (with respect to the Ray topology associated with the process) if and only if the discontinuities of A are disjoint from those of the process. Such a functional was termed **of class (U)** by Meyer [23] and **natural** by Blumenthal and Getoor [9]. If A is general then it was noted in [11] that the kernel U^-_{A} on X given by

$$U^-_{\mathsf{A}} f(x) := P^x[\int_0^\infty f \circ \mathsf{X}_{t-} d\mathsf{A}_t]$$

is a natural excessive kernel provided it is proper. (Obviously $U^-_{\mathsf{A}} = U_{\mathsf{A}}$ if A is natural.) Note that the natural excessive kernels have been considered also by Garcia

Alvarez [18], in connection with the problem of characterization of the potential kernels of the additive functionals.

Another method to obtain natural excessive kernels was given by Azéma in [1] and essentially, it will be used in the following theorem.

Theorem 3.4. *For each $s \in \mathscr{P}_{\mathscr{U}}(Y)$ there exists a unique natural excessive kernel $V_s : \mathscr{F}(Y) \longrightarrow \mathscr{F}(X)$ such that $V_s 1 = s$. Moreover we have*

$$V_s(1_K) = \curlywedge\{s \curlywedge B^{G\cap X} s \ / \ G \text{ open}, G \supset K\}$$

for every compact subset K of Y. V_s is a natural excessive kernel on X if and only if $s \in \mathscr{P}_{\mathscr{U}}(X)$.

Proof. For each compact subset K of Y we put $V_s(1_K) := \curlywedge\{s \curlywedge B^{G\cap X} s \ / G$ open, $G \supset K\}$. The existence of $V_s(1_K)$ follows from the fact that there exists a countable fundamental system of open neighbourhoods of K.

Following [1] one can show that for each two compact subsets K_1, K_2 of Y we have

$$V_s(1_{K_1 \cup K_2}) + V_s(1_{K_1 \cap K_2}) = V_s(1_{K_1}) + V_s(1_{K_2}) \ .$$

By the definition, it follows that for all $x \in X$ with $s(x) < \infty$ and every sequence (K_n) such that $K_n \searrow K$ we have $V_s(1_{K_n})(x) \searrow V_s(1_K)(x)$. Therefore for each $x \in X$ with $s(x) < \infty$ there exists a measure φ_x on Y such that $\varphi_x(K) = V_s(1_K)(x)$ for every compact subset K of Y. Using standard arguments of monotone class, it follows that for each $M \in \mathscr{B}(Y)$ there exists a unique excessive function $V_s(1_M) \in \mathscr{E}_{\mathscr{U}}$ such that $V_s(1_M)(x) = \varphi_x(M)$ for all $x \in [s < \infty]$. Obviously the map $M \longmapsto V_s(1_M)(x)$ is a measure on Y for all $x \in X$ and $V_s 1 = s$. If G is an open subset of Y and K is a compact subset of G then $V_s(1_K) \prec B^{G\cap X} s$. Therefore $B^{G\cap X} V_s(1_K) = V_s(1_K)$ and consequently $B^{G\cap X} V_s(1_G) = V_s(1_G)$.

Let now $V : \mathscr{F}(Y) \longrightarrow \mathscr{F}(X)$ be a natural excessive kernel such that $V1 = s$. For each compact subset K of Y and every open neighbourhood G of K we have $V(1_K) \prec V(1_G)$. As a consequence, since $B^{G\cap X} V(1_G) = V(1_G)$, we get $B^{G\cap X} V(1_K) = V(1_K)$. From $V(1_K) \prec s$ it follows $V(1_K) \prec s \curlywedge B^{G\cap X} s$ and consequently $V(1_K) \prec V_s(1_K)$. Hence $V(1_M) \prec V_s(1_M)$ for all $M \in \mathscr{B}(Y)$. Since $V(1_M) + V(1_{Y\setminus M}) = V1 = s = V_s 1$ we obtain $V(1_M) = V_s(1_M)$ on $[s < \infty]$ and we conclude that $V(1_M) = V_s(1_M)$ for all $M \in \mathscr{B}(Y)$.

Suppose now that V_s is a natural excessive kernel on X and let $t \in \mathscr{E}_{\mathscr{U}}$ be such that $t \prec s$ and $\mathsf{carr}\, t \cap X = \emptyset$. From $V_s(1_{\mathsf{carr}\, t}) = 0$ it follows $0 = \curlywedge\{s \curlywedge B^{G\cap X} s \ / \ G$ open , $G \supset \mathsf{carr}\, t\}$. Since for each set G, $G \supset \mathsf{carr}\, t$, we have $t = t \curlywedge B^{G\cap X} t \prec s \curlywedge B^{G\cap X} s$, we get $t = 0$. Hence $s \in \mathscr{P}_{\mathscr{U}}(X)$. Conversely, if $s \in \mathscr{P}_{\mathscr{U}}(X)$ and K is a compact subset of $Y \setminus X$ then $\mathsf{carr}\, V_s(1_K) \subset K$ and $V_s(1_K) \prec s$. Therefore $V_s(1_K) = 0$ and consequently V_s is a natural excessive kernel on X.

Proposition 3.5. *Let V be a natural excessive kernel on X. Then the following assertions are equivalent.*

i) There exists $f_o \in \mathscr{F}(X)$, $0 < f_o \le 1$, such that $V f_o$ is quasi bounded.

ii) If $f \in \mathscr{F}(X)$ is such that $Vf \in \mathscr{E}_{\mathscr{U}}$ then Vf is quasi bounded.

iii) There exists a sequence (s_n) of quasi bounded $\mathscr{U}$–excessive functions such that $V = \sum_n V_{s_n}$.

Definition. *A natural excessive kernel V on X is called* ***quasi bounded excessive kernel*** *if it satisfies one of the equivalent conditions from Proposition 3.5. Note that every proper natural excessive kernel is a quasi bounded excessive kernel.*

Proposition 3.6. *The following assertions are equivalent for a natural excessive kernel V on X.*

i) There exists $f_o \in \mathscr{F}(X), 0 < f_o \leq 1$, such that Vf_o is a regular $\mathscr{U}$–excessive function.

ii) For every $f \in \mathscr{F}(X)$ such that $Vf \in \mathscr{E}_{\mathscr{U}}$ it follows that Vf is regular.

iii) There exists a sequence (s_n) of regular $\mathscr{U}$–excessive functions such that $V = \sum_n V_{s_n}$.

Definition. *A natural excessive kernel V on X is called* ***regular excessive kernel*** *if it satisfies one of the equivalent conditions from Proposition 3.6.*

Recall that a kernel V on X is termed **subordinated to** $\mathscr{E}_{\mathscr{U}}$ if for each $s \in \mathscr{E}_{\mathscr{U}}$ and $f, g \in \mathscr{F}(X)$ we have $Vf \leq Vg + s$ whenever $Vf \leq Vg + s$ on the set $[f > 0]$.

Theorem 3.7. *Let V be a natural excessive kernel on X. Then V is subordinated to $\mathscr{E}_{\mathscr{U}}$ if and only if it is a regular excessive kernel. Particulary $\mathscr{P}^f_{\mathscr{U}}(X)$ coincides with the set of all regular $\mathscr{U}$–excessive functions if and only if X is nearly saturated.*

Proof. Let $f_o \in \mathscr{F}(X)$, $0 < f_o \leq 1$, be such that $Vf_o \in \mathscr{E}_{\mathscr{U}}$. If V is regular then Vf_o is regular and therefore (cf. [7]) there exists a natural excessive kernel W on X such that $W1 = Vf_o$ and such that W is subordinated to $\mathscr{E}_{\mathscr{U}}$. From the uniqueness of the natural excessive kernel W on X with $W1 \in \mathscr{P}_{\mathscr{U}}(X)$ we get $W = f_o \cdot V$ and consequently V is also subordinated to $\mathscr{E}_{\mathscr{U}}$. Conversely, suppose that V is subordinated to $\mathscr{E}_{\mathscr{U}}$. If $(s_n) \subset \mathscr{E}_{\mathscr{U}}$, $s_n \nearrow Vf_o$ and $M_n := [\varepsilon + s_n > Vf_o] \cap [Vf_o < \infty]$, then $V(f_o 1_{M_n}) \nearrow Vf_o$ and $R(Vf_o - s_n) \leq \varepsilon + V(f_o 1_{X \setminus M_n})$. Consequently we get $\bigwedge_n R(Vf_o - s_n) = 0$. If $s \in \mathscr{P}^f_{\mathscr{U}}(X)$ then the natural excessive kernel $V_s : \mathscr{F}(X) \longrightarrow \mathscr{F}(X)$ is subordinated to $\mathscr{E}_{\mathscr{U}}$ and therefore $s = V_s 1$ is regular.

Remark. Every regular excessive kernel is proper.

Lemma 3.8. *Let $s \in \mathscr{P}_{\mathscr{U}}(Y)$, $t \in \mathscr{E}_{\mathscr{U}}$ and (s_n) be a sequence in $\mathscr{P}_{\mathscr{U}}(Y)$ with $s_n \leq t$ for all n, such that $s_n \longrightarrow s$. Then for each positive lower semicontinous function f on Y we have $V_s f(x) \leq \liminf_{n\to\infty} V_{s_n} f(x)$ in every point $x \in X$ with $t(x) < \infty$.*

Proof. We take a compact subset K of Y, an open neighbourhood G of K and we show that $V_s(1_K)(x) \leq \liminf_{n\to\infty} V_{s_n}(1_G)(x)$ for all $x \in [t < \infty]$. If x is such a point then the set $\{s_n \ / \ n \in \mathbb{N}\}$ is relatively weakly compact in $S^f_{\varepsilon_x}$. Recall that (see [8] and [13]) if θ is a finite measure on X then S^f_θ denotes the set of all θ-supermedian functionals which are finite on θ and $\mathscr{M}^f_\theta$ is the set of all positive mesures μ on X such that $\mu \circ U \leq \alpha\theta \circ U$ for a suitable positive number α. If $F \in S^f_\theta$ then, arguing as in the proof of Theorem 10 in [13], it follows that the set $\{G \in S^f_\theta \;/\; G \leq F\}$ is sequentially compact with respect to the weak topology $\sigma(S^f_\theta, \mathscr{M}^f_\theta)$. Hence there exists a subsequence (s_{k_n}) of (s_n) such that $(V_{s_{k_n}}(1_G))$ and $(V_{s_{k_n}}(1_{Y\setminus G}))$ are weakly convergent to t' and respectively t'' and

$$\liminf_{n\to\infty} V_{s_n}(1_G)(x) = \lim_{n\to\infty} V_{s_{k_n}}(1_G)(x) \;\; .$$

From $V_{s_{k_n}}(1_G) + V_{s_{k_n}}(1_{Y\setminus G}) = V_{s_{k_n}}1 = s_{k_n}$ we get $t' + t'' = s\ \varepsilon_x$–q.e. Let us denote by Γ an open subset of Y with $K \cap \overline{\Gamma} = \emptyset$ and $\Gamma \subset \overline{Y \setminus G}$. Since $B^{\Gamma\cap X}V_{s_{k_n}}(1_{Y\setminus G}) = V_{s_{k_n}}(1_{Y\setminus G})$ it follows that $B^{\Gamma\cap X}t'' = t''\ \varepsilon_x$–q.e. Hence $V_s(1_K) \prec t' + B^{\Gamma\cap X}t''\ \varepsilon_x$–q.e. Therefore, by [7] and from $\mathsf{carr}\, u \subset K \cap \overline{\Gamma} = \emptyset$ where $u := V_s(1_K) \curlywedge B^{\Gamma\cap X}t'' \prec s$, we obtain $V_s(1_K) \prec t'\varepsilon_x$–q.e. and consequently $V_s(1_K)(x) \leq \liminf_{n\to\infty} V_{s_n}(1_G)(x)$.

Since for each open subset G of Y we have

$$V_s(1_G)(x) = \sup\{V_s(1_K)(x) \;/\; K \subset G,\ K \text{ compact}\}$$

we deduce that $V_s(1_G)(x) \leq \liminf_{n\to\infty} V_{s_n}(1_G)(x)$. If f is a positive lower semicontinuous function on Y and $x \in [t < \infty]$ then, using Fatou Lemma we conclude that

$$\begin{aligned} V_s f(x) &= \int_0^\infty V_s(1_{[f>\alpha]})(x)d\alpha \leq \liminf_{n\to\infty} \int_0^\infty V_{s_n}(1_{[f>\alpha]})(x)d\alpha \\ &= \liminf_{n\to\infty} V_{s_n} f(x) \;\; . \end{aligned}$$

Theorem 3.9. *Let $s \in \mathscr{P}_{\mathscr{U}}(X)$, $t \in \mathscr{E}_{\mathscr{U}}$ and (s_n) be a sequence in $\mathscr{P}_{\mathscr{U}}(X)$ such that $s_n \leq t$ for all n and $s_n \longrightarrow s$. Then for each positive, bounded continuous function f on X and all $x \in X$ with $t(x) < \infty$ we have $V_s f(x) = \lim_{n\to\infty} V_{s_n} f(x)$.*

Proof. Obviously we may suppose that $f \leq 1$. We denote by $\overline{f}$ the lower semicontinuous extension of f to Y. Since $V_s(1_{Y\setminus X}) = 0$, $V_{s_n}(1_{Y\setminus X}) = 0$, by Lemma 3.8 we have

$$V_s f(x) = V_s \overline{f}(x) \leq \liminf_{n\to\infty} V_{s_n} \overline{f}(x) = \liminf_{n\to\infty} V_{s_n} f(x)$$

for all $x \in [t < \infty]$. We have also $V_s(1-f)(x) \leq \liminf_{n\to\infty} V_{s_n}(1-f)(x)$. From $V_s f + V_s(1-f) = V_s 1 = s$, $V_{s_n} f + V_{s_n}(1-f) = V_{s_n} 1 = s_n$ we conclude that $V_s f(x) = \lim_{n\to\infty} V_{s_n} f(x)$ in all point $x \in [t < \infty]$, completing the proof.

Corollary 3.10. *For all $s \in \mathscr{P}_{\mathscr{U}}(X)$, every sequence (g_n) of positive, bounded $\mathscr{B}$–measurable functions on X such that $Ug_n \nearrow s$ and each positive bounded real continuous function f on X we have $V_s f(x) = \lim_{n\to\infty} U(fg_n)(x)$ in all point $x \in X$ with $s(x) < \infty$.*

Let $\mathscr{T}$ be a topology on X such that $(X, \mathscr{T})$ is a Lusin topological space having $\mathscr{B}$ as its σ–algebra of Borel sets. Let further ξ be a $\mathscr{U}$–excessive measure on X and $p_o \in Q_{bd}(\mathscr{E}_{\mathscr{U}}, \xi)$, $p_o > 0$. A numerical function f on X is called ξ–**quasi continuous** (with respect to $\mathscr{T}$ and $\mathscr{E}_{\mathscr{U}}$) if there exists a decreasing sequence (G_n) of $\mathscr{T}$–open subsets of X such that $f|_{X \setminus G_n}$ is $\mathscr{T}$–continuous for all n and such that $\inf_n B^{G_n} p_o = 0$ ξ–a.e.

Remark. a) The ξ–quasi continuity does not depend on p_o. Indeed, if $q_o \in Q_{bd}(\mathscr{E}_{\mathscr{U}}, \xi)$, $q_o > 0$, then it is sufficient to show that $\inf_n B^{G_n} q_o = 0$ ξ–a.e. for each sequence (G_n) as above. Since q_o is ξ–quasi bounded we may find a sequence $(q_n)_{n \geq 1} \subset \mathscr{E}_{\mathscr{U}}$ such that $q_o = \sum_{n \geq 1} q_n$ ξ–q.e. and $q_n \leq p_o$. For all $k \in \mathbb{N}$ we get $B^{G_n} q_o \leq k B^{G_n} p_o + \sum_{n \geq k+1} q_n$ ξ–a.e. and consequently $\inf_n B^{G_n} q_o \leq \sum_{n \geq k+1} q_n$, $\inf_n B^{G_n} q_o = 0$ ξ–a.e.
b) If θ is a finite positive measure on X with $\theta(p_o) < \infty$ and such that the ξ–polar sets coincide with the sets which are θ–polar and θ–negligible (always such a measure exists), then an increasing sequence (G_n) of $\mathscr{T}$–open sets verifies the condition $\inf_n B^{G_n} p_o = 0$ ξ–a.e. if and only if $\inf_n c_\theta^{p_o}(G_n) = 0$ where $c_\theta^{p_o}$ is the 'capacity' on X given by $c_\theta^{p_o}(M) := \theta(R^M p_o)$ for all $M \in \mathscr{B}$.
c) In many cases, in the definition of ξ–quasi continuity, the constant function 1 is prefered as p_o even if it is not ξ–quasi bounded. The reason for is the following: If p_o is finite and $\mathscr{T}$–continuous and $\mathscr{U}'$ is the resolvent of kernels on $(X, \mathscr{B})$ having the initial kernel U/p_o then $\mathscr{E}_{\mathscr{U}'} = \mathscr{E}_{\mathscr{U}}/p_o$ and $Exc_{\mathscr{U}'} = Exc_{\mathscr{U}}$. Since $p_o \in Q_{bd}(\mathscr{E}_{\mathscr{U}}, \xi)$ it follows that $1 \in Q_{bd}(\mathscr{E}_{\mathscr{U}'}, \xi)$. A numerical function f on X will be ξ–quasi continuous with respect to $\mathscr{T}$ and $\mathscr{E}_{\mathscr{U}}$ if and only if it is ξ–quasi continuous with respect to $\mathscr{T}$ and $\mathscr{E}_{\mathscr{U}'}$.

Proposition 3.11. *Let $\mathscr{T}$ be a topology on X as above. We suppose in addition that*:

- α) *for each positive $\mathscr{B}$–measurable function f on X, Uf is ξ–quasi continuous with respect to $\mathscr{T}$ and $\mathscr{E}_{\mathscr{U}}$, provided that $Uf \in \mathscr{E}_{\mathscr{U}}$;*
- β) *there exists an increasing sequence (K_n) of $\mathscr{T}$–compact subsets of X such that $\inf_n B^{X \setminus K_n} p_o = 0$ ξ–a.e.*

Then the ξ–quasi continuity with respect to $\mathscr{T}$ and with respect to every Ray topology on X generated by a sequence $(Uf_n) \subset \mathscr{E}_{\mathscr{U}}$ are the same. Moreover for every such a Ray topology $\mathscr{T}_o$ on X, the sequence (K_n) from condition β) may be chosen such that $\mathscr{T}_o|_{K_n} = \mathscr{T}|_{K_n}$ for all n.

Proof. Let $(Uf_n) \subset \mathscr{E}_{\mathscr{U}}$ (with $\mathscr{B}$–measurable f_n, $0 \leq f_n \leq 1$) be a sequence which generates the topology $\mathscr{T}_o$. By hypothesis Uf_n is ξ–quasi continuous with respect to $\mathscr{T}$ and therefore there exists an increasing sequence (L_n) of $\mathscr{T}$–compact subsets of X with $L_n \subset K_n$, $\inf_n B^{X \setminus L_n} p_o = 0$ and such that $Uf_m|_{L_n}$ is $\mathscr{T}$–continuous for all m. We get $\mathscr{T}_o|_{L_n} \subset \mathscr{T}|_{L_n}$ and consequently $\mathscr{T}_o|_{L_n} = \mathscr{T}|_{L_n}$. Note that if (G_n)

is a decreasing sequence of either $\mathcal{T}$–open or $\mathcal{T}_o$–open sets then $(G_n \cup (X \setminus L_n))$ is a decreasing sequence in $\mathcal{T}$ and $\mathcal{T}_o$. In addition we have $\inf_n B^{G_n} p_o = 0$ if and only if $\inf_n B^{G_n \cup (X \setminus L_n)} p_o = 0$. From the above considerations it is easy to see that the ξ–quasi continuity with respect to $\mathcal{T}$ and $\mathcal{T}_o$ are the same, completing the proof.

Corollary 3.12. *Suppose that $\mathcal{U}$ is the resolvent of a Borel right process on $(X, \mathcal{T})$. If the topology $\mathcal{T}$ satisfies the above two conditions $\alpha)$ and $\beta)$ then the process is special ξ-standard; see [20]. (For the situation when the converse holds see [21] and [22]).*

In the sequel, the topology $\mathcal{T}$ on X will be a Ray topology and we say simply 'ξ–quasi continuous' instead of 'ξ–quasi continous with respect to $\mathcal{T}$ and $\mathcal{E}_{\mathcal{U}}$'.

Note that each regular $\mathcal{U}$–excessive function is ξ–quasi continuous for all $\xi \in Exc_{\mathcal{U}}$ (see [8]). Particularly, if f is a positive $\mathcal{B}$–measurable function on X then Uf is ξ–quasi continuous whenever it is finite ξ–a.e.

Theorem 3.13. *Let $s \in \mathcal{P}_{\mathcal{U}}(X)$, (s_n) be a sequence in $\mathcal{P}_{\mathcal{U}}(X)$ and $t \in \mathcal{E}_{\mathcal{U}}$ be a ξ–quasi bounded excessive function such that $s_n \le t$ for all n and $s_n \longrightarrow s$. Then for every positive bounded ξ–quasi continuous function f on X we have*

$$V_s f = \lim_{n \to \infty} V_{s_n} f \quad \xi\text{–q.e.}$$

Proof. Suppose that $f \le 1$. Let (G_n) be a decreasing sequence of open subsets of X such that $f|_{X \setminus G_n}$ is continuous and such that $\inf_n B^{G_n} p_o = 0$ ξ–q.e., where p_o is ξ–quasi bounded, $p_o > 0$. If we denote by f_n a positive continuous extension of $f|_{X \setminus G_n}$ to X such that $f_n \le 1$ then, by Theorem 3.9 we have $V_s f_n = \lim_{m \to \infty} V_{s_m} f_n$ on $[t < \infty]$. Also

$$|V_s f - V_s f_n| \le 2V_s(1_{G_n}) \le 2B^{G_n} s \le 2B^{G_n} t,$$
$$|V_{s_m} f - V_{s_m} f_n| \le 2V_{s_m}(1_{G_n}) \le 2B^{G_n} t \ .$$

Since $\inf_n B^{G_n} p_o = 0$ ξ–q.e. and t is (p_o, ξ)–quasi bounded we deduce that $\inf_n B^{G_n} t = 0$ ξ–q.e. and therefore $V_s f = \lim_{n \to \infty} V_{s_n} f$ ξ–q.e.

4. Revuz measures

In this section we fix a $\mathcal{U}$–excessive measure ξ.

Definition. *Let V be a natural excessive kernel on X. The positive measure on X defined by*

$$\nu_V^\xi(M) := L(\xi, V(1_M)), \quad M \in \mathcal{B}$$

*is called the **Revuz measure** of V (with respect to ξ). If $s \in \mathcal{P}_{\mathcal{U}}(X)$ then the Revuz measure $\nu_{V_s}^\xi$ is called the **Revuz measure** of s and will be denoted by ν_s^ξ.*

Remark. a) For each sequence (V_n) of natural excessive kernels on X such that $\sum_n V_n$ is a natural excessive kernel we have $\nu^{\xi}_{\sum_n V_n} = \sum_n \nu^{\xi}_{V_n}$.
b) If V is a natural excessive kernel on X and $f \in \mathscr{F}(X)$ is finite then $\nu^{\xi}_{f\cdot V} = f\cdot\nu^{\xi}_{V}$.
c) If V is a natural excessive kernel on X and there exists $f_o \in \mathscr{F}(X)$, $0 < f_o \leq 1$, with $Vf_o \in \mathscr{P}_{\mathscr{U}}(X)$ then $\nu^{\xi}_{V} = \frac{1}{f_o}\cdot\nu^{\xi}_{Vf_o}$.

Proposition 4.1. *If $s, t \in \mathscr{P}_{\mathscr{U}}(X)$ then $s = t$ ξ–q.e. if and only if $\nu^{\eta}_{s} = \nu^{\eta}_{t}$ for all $\eta \in Exc_{\xi}$. Moreover, if (s_n) and (t_n) are two sequences in $\mathscr{P}_{\mathscr{U}}(X)$ such that $\sum_n s_n \in \mathscr{E}^{\xi}_{\mathscr{U}}$ and $\sum_n s_n = \sum_n t_n$ ξ–q.e. then we have $\sum_n \nu^{\xi}_{s_n} = \sum_n \nu^{\xi}_{t_n}$.*

Proof. We have $s = s\curlywedge t + s'$, $t = s\curlywedge t + t'$, where $s', t' \in \mathscr{P}_{\mathscr{U}}(X)$ and $s' = t' = 0$ ξ–q.e. Since $\nu^{\xi}_{s'}(X) = L(\xi, s') = 0$ and analogously $\nu^{\xi}_{t'} = 0$, we get $\nu^{\xi}_{s} = \nu^{\xi}_{s\curlywedge t} + \nu^{\xi}_{s'} = \nu^{\xi}_{s\curlywedge t} = \nu^{\xi}_{t}$. From $\sum_n s_n = \sum_n t_n$ ξ–q.e. it follows that there exists a sequence (u_n) in $\mathscr{E}^{\xi}_{\mathscr{U}}$ such that $u_n = 0$ ξ–q.e. and $\sum_{n\leq k} s_n \prec \sum_n t_n + u_k$. Hence there exists a sequence (t'_n) in $\mathscr{P}_{\mathscr{U}}(X)$ such that $t'_n \prec t_n$ and a sequence $(u'_n) \subset \mathscr{E}_{\mathscr{U}} \cap \mathscr{P}_{\mathscr{U}}(X)$, $u'_n = 0$ ξ–q.e. with $\sum_{n\leq k} s_n = \sum_n t'_n + u'_k$. Therefore $\sum_{n\leq k} \nu^{\xi}_{s_n} = \nu^{\xi}_{\sum_{n\leq k} s_n} = \sum_n \nu^{\xi}_{t'_n} \leq \sum_n \nu^{\xi}_{t_n}$.

Definition. *We denote by $\mathscr{P}_{\mathscr{U},\xi}(X)$ the set of all $s \in \mathscr{E}^{\xi}_{\mathscr{U}}$ for which there exists a sequence (s_n) in $\mathscr{P}_{\mathscr{U}}(X)$ such that $s = \sum_n s_n$ ξ–q.e.*

By Proposition 4.1, for each $s \in \mathscr{P}_{\mathscr{U},\xi}(X)$ the measure $\nu^{\xi}_{s} := \sum_n \nu^{\xi}_{s_n}$ on X, where $s = \sum_n s_n$ ξ–q.e., $s_n \in \mathscr{P}_{\mathscr{U}}(X)$, is well defined and called the *Revuz measure* of s.

Remark. a) The set $\mathscr{P}_{\mathscr{U},\xi}(X)$ is a σ–band in $\mathscr{E}^{\xi}_{\mathscr{U}}$ and for every sequence (s_n) in $\mathscr{P}_{\mathscr{U},\xi}(X)$ such that $\sum_n s_n =: s \in \mathscr{P}_{\mathscr{U},\xi}(X)$ we have $\nu^{\xi}_{s} = \sum_n \nu^{\xi}_{s_n}$.
b) If $\eta_1, \eta_2 \in Exc_{\xi}$, $s \in \mathscr{P}_{\mathscr{U},\xi}(X)$ and $\eta_1 \leq \eta_2$ (resp. $\eta_1 \ll \eta_2$) then $\nu^{\eta_1}_{s} \leq \nu^{\eta_2}_{s}$ (resp. $\nu^{\eta_1}_{s} \ll \nu^{\eta_2}_{s}$).
c) For each $s \in \mathscr{P}_{\mathscr{U},\xi}(X)$ there exists $\eta \in Exc_{\xi}$ with $\eta \ll \xi$, $\xi \ll \eta$, such that ν^{η}_{s} is finite .
d) If $\eta_n \nearrow \eta$ in Exc_{ξ} then $\nu^{\eta_n}_{s} \nearrow \nu^{\eta}_{s}$. Particularly the Revuz measure ν^{ξ}_{s} of each $s \in \mathscr{P}_{\mathscr{U},\xi}(X)$ is s–finite (i.e. a countable sum of finite measures).

Definition. *A function $s \in \mathscr{E}^{\xi}_{\mathscr{U}}$ is called ξ–**potential** on X if for each increasing sequence (G_n) of Ray open subsets of X such that $\bigcup_n G_n = X$ we have $\inf_n B^{X\setminus G_n} s = 0$ ξ–a.e.*

It is easy to see that the set of all ξ–potentials on X is a σ–band in $\mathscr{E}^{\xi}_{\mathscr{U}}$ which is solid in $\mathscr{E}^{\xi}_{\mathscr{U}}$ with respect to the pointwise order relation.

Proposition 4.2. *Every $s \in \mathscr{P}_{\mathscr{U}}(Y)$ dominated by a ξ–potential belongs to $\mathscr{P}_{\mathscr{U},\xi}(X)$. Particularly, if there exists a ξ–potential on X which is strictly positive then $Q_{bd}(\mathscr{E}_{\mathscr{U}}, \xi) \subset \mathscr{P}_{\mathscr{U},\xi}(X)$ and $\mathscr{P}_{\mathscr{U},\xi}(X)$ is solid in $\mathscr{E}^{\xi}_{\mathscr{U}}$ with respect to the pointwise order relation.*

Proof. If V_s denotes the natural excessive kernel associated with s then it is sufficient to show that $V_s(1_{Y\setminus X}) = 0$ ξ–q.e. Let K be a compact subset of $Y \setminus X$ and (F_n) be a decreasing sequence of closed neighbourhoods of K such that $K = \bigcap_n F_n$. Since by hypothesis s is a ξ–potential on X and $X \setminus F_n \nearrow X$ we get $\inf_n B^{F_n \cap X} s = 0$ ξ–q.e. From $V_s(1_K) \prec B^{\overset{\circ}{F}_n \cap X} s$ for all n, we conclude that $V_s(1_K) = 0$ ξ–q.e. Note that if $s \in \mathscr{E}^{\xi}_{\mathscr{U}}$ then there exist $s_o \in Q_{bd}(\mathscr{E}_{\mathscr{U}}, \xi)$ and $s_1 \in \mathscr{E}^{\xi}_{\mathscr{U}}$ such that $s = s_o + s_1$ ξ–q.e. and $\widetilde{s_1}$ is subtractible in $E^{\xi}(\mathscr{E}_{\mathscr{U}})$ (see [6]). If $s \le t \in \mathscr{P}_{\mathscr{U},\xi}(X)$ then $\widetilde{s_1} \prec \widetilde{t}$ and therefore $s_1 \in \mathscr{P}_{\mathscr{U},\xi}(X)$.

Remark. If there exists a strictly positive ξ–potential on X then it is easy to see that the set $X_1 \setminus X$ is ξ–polar, where X_1 denotes the saturation of X.

Theorem 4.3. *The following assertions hold.*

i) A subset $M \in \mathscr{B}$ will be ξ–semipolar if and only if $\nu^{\xi}_s(M) = 0$ for all $s \in \mathscr{P}_{\mathscr{U},\xi}(X) \cap \mathscr{E}^{\xi,r}_{\mathscr{U}}$ (or only for all $s \in \mathscr{P}_{\mathscr{U}}(X) \cap \mathscr{E}^{r}_{\mathscr{U}}$).

ii) The Revuz measure ν^{ξ}_s of each $s \in \mathscr{P}_{\mathscr{U},\xi}(X) \cap \mathscr{E}^{\xi,r}_{\mathscr{U}}$ is σ–finite. Moreover there exists an increasing sequence (G_n) of finely open $\mathscr{B}$–measurable subsets of X such that $\nu^{\xi}_s(G_n) < \infty$ and $R^{X\setminus G_n} p \searrow 0$ on $[s < \infty]$ (and therefore $\inf_n R^{X\setminus G_n} p = 0$ ξ–q.e.) for all $p \in Q_{bd}(\mathscr{E}_{\mathscr{U}}, \xi)$.

iii) For every finite measure ν on X, charging no ξ–semipolar set there exists a ξ–regular function $s \in \mathscr{P}_{\mathscr{U},\xi}(X)$ such that $\nu = \nu^{\xi}_s$. If $s, t \in \mathscr{P}_{\mathscr{U},\xi}(X) \cap \mathscr{E}^{\xi,r}_{\mathscr{U}}$ and $\nu^{\xi}_s = \nu^{\xi}_t$ then $s = t$ ξ–q.e.

Proof. i) Let $s \in \mathscr{P}_{\mathscr{U}}(X) \cap \mathscr{E}^{r}_{\mathscr{U}}$ and let $M \in \mathscr{B}$ be a ξ–semipolar set. Since by Theorem 3.7 the kernel V_s is subordinated to $\mathscr{E}_{\mathscr{U}}$, it follows that (cf. [7]) $V_s(1_M) = 0$ ξ–q.e. and therefore $\nu^{\xi}_s(M) = 0$. Conversely, suppose that $M \in \mathscr{B}$ is such that $\nu^{\xi}_s(M) = 0$ for all $s \in \mathscr{P}_{\mathscr{U}}(X) \cap \mathscr{E}^{r}_{\mathscr{U}}$. If M is not ξ–semipolar then by [7] there exists $s \in \mathscr{E}^{r}_{\mathscr{U}}$ with $\operatorname{carr}_f s \subset M$ and $\xi(s) \neq 0$. Hence $s \in \mathscr{P}_{\mathscr{U}}(X) \cap \mathscr{E}^{r}_{\mathscr{U}}$ and we have $V_s(1_M) = s$. Therefore $\nu^{\xi}_s(M) = L(\xi, s) = 0$ which is a contradiction.

ii) Let $s \in \mathscr{P}_{\mathscr{U},\xi}(X) \cap \mathscr{E}^{\xi,r}_{\mathscr{U}}$ and let $f_o \in \mathscr{F}(X)$, $0 < f_o \le 1$ be such that Uf_o is bounded and $\xi(f_o) < \infty$. If we put $G_n = [s < nUf_o]$, since $s \in Q_{bd}(\mathscr{E}_{\mathscr{U}}, \xi)$ we get $\bigwedge_n R(s - nUf_o) = 0$ ξ–q.e. Obviously G_n is finely open and we have $V_s(1_{G_n}) \le nUf_o$ i.e. $\nu^{\xi}_s(G_n) \le L(\xi, nUf_o) < \infty$. We have also $R^{X\setminus G_n} Uf_o \le \frac{1}{n} s$ for all n and consequently $\inf_n R^{X\setminus G_n} Uf_o = 0$ on $[s < \infty]$. Obviously the set $X \setminus \bigcup_n G_n$ is ξ–polar and ν^{ξ}_s–negligible and therefore the measure ν^{ξ}_s is σ–finite.

$iii)$ Let ν be a finite measure on X charging no ξ–semipolar set. The existence of a ξ–regular function $s \in \mathcal{P}_{\mathcal{U},\xi}(X)$ such that $\nu = \nu_s^\xi$ follows from [2], Théorème 3.4 (1), and from the fact that the potential operator associated with a continuous additive functional is a regular excessive kernel. Note in addition that each $\mathcal{U}$–excessive function is equal ξ– a.e. with a $\mathcal{B}$–measurable $\mathcal{U}$–excessive function.

Let now V_1, V_2 be two regular excessive kernels on X having the same Revuz measure. We show that for all $f \in \mathcal{F}(X)$ we have $V_1 f = V_2 f$ ξ–q.e. Let $f_o \in \mathcal{F}(X)$, $0 < f_o \le 1$ be such that $V_1 f_o, V_2 f_o \in \mathcal{E}_{\mathcal{U}}$. Because V_1 and V_2 satisfy the complete maximum principle we may suppose that $V_1 f_o$ and $V_2 f_o$ are bounded functions. We define $u = V_1 f_o + V_2 f_o$ and let $W = V_u$. Since W is the initial kernel of a submarkovian resolvent $\mathcal{W}$ on X such that each $\mathcal{U}$–excessive function is $\mathcal{W}$–supermedian, we deduce that there exists $f_1, f_2 \in \mathcal{F}(X)$, $0 \le f_1, f_2 \le 1$, such that $V_1 f_o = W f_1$ and $V_2 f_o = W f_2$. Consequently we get $f_1 \cdot W = f_0 \cdot V_1$ and $f_2 \cdot W = f_0 \cdot V_2$. If we set $g_1 = f_1 - f_1 \wedge f_2$, $g_2 = f_2 - f_1 \wedge f_2$ then $\nu^\xi_{f_1 \cdot W} = \nu^\xi_{f_2 \cdot W}$ and $\nu^\xi_{g_1 \cdot W} = \nu^\xi_{g_2 \cdot W}$. Therefore, since $g_1 g_2 = 0$, we obtain $\nu^\xi_{g_1 \cdot W} = \nu^\xi_{g_2 \cdot W} = 0$, $g_1 \cdot W(1) = g_2 \cdot W(1) = 0$ ξ–q.e. Hence $f_o \cdot V_1(f) = f_1 \cdot W(f) = f_2 \cdot W(f) = f_o \cdot V_2(f)$ ξ–q.e. for all $f \in \mathcal{F}(X)$ and we conclude that $V_1 f = V_2 f$ ξ–q.e.

Remark. $a)$ Assertion $ii)$ in Theorem 4.3 has been proved also by Revuz [25] and Fitzsimmons and Getoor [16].
$b)$ A σ–finite measure on X charges no ξ–semipolar set if and only if it is the Revuz measure of a regular excessive kernel on X. If V, W are two regular excessive kernels on X having the same Revuz measure then for all $f \in \mathcal{F}(X)$ we have $Vf = Wf$ ξ–q.e.
$c)$ One can prove the assertion $iii)$ from Theorem 4.3 by a different analytical method, using in addition the fact that each $\mathcal{U}$-excessive measure dominated by ξ possesses a "finely continuous" density (cf. [14] and [15]). These techniques will be developed in a forthcoming paper.

Recall that the **hypothesis (B) of Hunt** holds for the Ray topology on X if for each Ray open set G and every subset M of G, $M \in \mathcal{B}$, we have $B^G B^M s = B^M s$ for all $s \in \mathcal{E}_{\mathcal{U}}$.

Theorem 4.4. *The following assertions hold.*

i) If $s \in \mathcal{P}_{\mathcal{U},\xi}(X)$ is ξ–quasi bounded then its Revuz measure is s–finite and charges no ξ–polar set.

ii) Suppose that the hypothesis (B) of Hunt holds for the Ray topology. If $M \in \mathcal{B}$ is such that $\nu_s^\xi(M) = 0$ for all ξ–quasi bounded $s \in \mathcal{P}_{\mathcal{U},\xi}(X)$ with $\nu_s^\xi(X) < \infty$ then M is ξ–polar.

iii) If the hypothesis (B) of Hunt holds for the Ray topology then for each finite measure ν on X, charging no ξ–polar set there exists a ξ–quasi bounded function $s \in \mathcal{P}_{\mathcal{U},\xi}(X)$ such that $\nu = \nu_s^\xi$.

Proof. $i)$ Let $f_o \in \mathcal{F}(X)$, $0 < f_o \le 1$, be such that $U f_o \le 1$ and $\xi(f_o)$ is finite. Since s is ξ–quasi bounded there exists a sequence (s_n) in $\mathcal{E}_{\mathcal{U}}$ such that

$s_n \le Uf_o$ and $s = \sum_n s_n$ ξ–q.e. Consequently $\nu_s^\xi = \sum_n \nu_{s_n}^\xi$ and $\nu_{s_n}^\xi = L(\xi, s_n) \le L(\xi, Uf_o) < \infty$. If K is a ξ–polar compact subset of X then

$$V_{s_n}(1_K) \le \bigwedge\{B^G s_n / \ G \supset K, G \text{ open }\} \le \bigwedge\{B^G 1/ \ G \supset K, G \text{ open}\} = B^K 1$$

and therefore $\nu_{s_n}^\xi(K) = L(\xi, V_{s_n}(1_K)) \le L(\xi, B^K 1) = 0$. Since the measure $\nu_{s_n}^\xi$ is finite, we deduce that $\nu_{s_n}^\xi(M) = 0$ for all n and each ξ–polar subset M of X. We conclude that $\nu_s^\xi(M) = 0$.

ii) Let $M \in \mathscr{B}$ be such that $\nu_s^\xi(M) = 0$ for all $s \in \mathscr{P}_{\mathscr{U},\xi}(X) \cap Q_{bd}(\mathscr{E}_{\mathscr{U}}, \xi)$ with $\nu_s^\xi(X) < \infty$. We may suppose M is Ray compact and we consider $s_o = B^M p_o$ where $p_o \in Q_{bd}(\mathscr{E}_{\mathscr{U}}, \xi)$, $p_0 > 0$. Obviously $s_o \in Q_{bd}(\mathscr{E}_{\mathscr{U}}, \xi)$ and by the hypothesis (B) of Hunt for the Ray topology we have $V_{s_o}(1_M) = s_o$. On the other hand by hypothesis $V_{s_o}(1_M) = 0$ ξ–q.e. and therefore M is ξ–polar.

iii) Let ν be a finite measure on X charging no ξ–polar set. By Corollary 2.4, ν may be written in the form $\nu = \nu' + \nu''$, where ν' charges no ξ–semipolar set and ν'' is carried by a ξ–semipolar set $A \in \mathscr{B}$. From assertion *iii*) in Theorem 4.3 there exists a ξ–regular function $s \in \mathscr{P}_{\mathscr{U},\xi}(X)$ such that $\nu' = \nu_s^\xi$. Further let us denote by μ the Dellacherie mesure associated with ξ and the ξ–semipolar set A, i.e. a subset of A will be ξ–polar if and only if it is μ negligible. From the above assertions *i*) and *ii*) it follows that a $\mathscr{B}$–measurable set M is ξ–polar if and only if $\nu_s^\xi(M) = 0$ for all $s \in \mathscr{P}_{\mathscr{U},\xi}(X) \cap Q_{bd}(\mathscr{E}_{\mathscr{U}}, \xi)$ with $\nu_s^\xi(X) < \infty$. By Lemma 2.5 applied on the measurable space $(A, \mathscr{B}|_A)$ for the set $\mathscr{M} = \{1_A \cdot \nu_s^\xi /\ s \in \mathscr{P}_{\mathscr{U},\xi}(X) \cap Q_{bd}(\mathscr{E}_{\mathscr{U}}, \xi),\ \nu_s^\xi(X) < \infty\}$, and for the measure μ, there exists a sequnce $(s_n) \subset \mathscr{P}_{\mathscr{U},\xi}(X) \cap Q_{bd}(\mathscr{E}_{\mathscr{U}}, \xi)$ such that $\mu = \sum_n \nu_{s_n}^\xi$. Since the measure ν'' is absolutely continuous with respect to μ there exists a second sequnce $(t_n) \subset \mathscr{P}_{\mathscr{U},\xi}(X) \cap Q_{bd}(\mathscr{E}_{\mathscr{U}}, \xi)$ such that $\nu'' = \sum_n \nu_{t_n}^\xi$. If we put $t = \sum_n t_n$ then from $L(\xi, t) = \sum_n \nu_{t_n}^\xi(X) < \infty$ it follows that t is finite ξ-a.e. We conclude that $t \in \mathscr{P}_{\mathscr{U},\xi}(X) \cap Q_{bd}(\mathscr{E}_{\mathscr{U}}, \xi)$ and $\nu'' = \nu_t^\xi$.

Remark. *a*) Assertion *ii*) in Theorem 4.4 has been proved by Revuz [25] for standard processes satisfying the hypothesis (L) of Meyer.
b) By a different approach, a result similar to assertion *iii*) in the above theorem has been obtained by Azéma [2] under more restrictive assumptions: $\mathscr{U}$ being the resolvent of a Hunt process, satisfying also the hypotheses (L) of Meyer, (B) of Hunt and (CMF).
c) If the hypothesis (B) of Hunt holds then a σ–finite measure on X charges no ξ–polar set if and only if it is the Revuz measure of a proper natural (and therefore quasi bounded) excessive kernel on X.
d) In fact in Theorem 4.4, instead of the hypothesis (B) of Hunt we only need the following weaker **hypothesis (B) of Hunt with respect to the measure** ξ: for

every Ray open set G and each subset M of G, $M \in \mathscr{B}$, we have $B^G B^M s = B^M s$ ξ–q.e. for all $s \in \mathscr{E}_{\mathscr{U}}$.
e) Using Corollary 2.4 one can show that always there exists a ξ–semipolar set $H \in \mathscr{B}$ such that the hypothesis (B) of Hunt with respect to the measure ξ holds on $X \setminus H$ (i.e. for every Ray open set G and each subset M of $G \cap (X \setminus H)$, $M \in \mathscr{B}$, we have $B^G B^M s = B^M s$ ξ–q.e. for all $s \in \mathscr{E}_{\mathscr{U}}$). Particularly assertion iii) in Theorem 4.4 holds for every finite measure carried by $X \setminus H$.

In the sequel we suppose that there exists a second proper submarkovian resolvent $\widehat{\mathscr{U}} = (\widehat{U}_\alpha)_{\alpha>0}$ on $(X, \mathscr{B})$ which is in duality with the given resolvent $\mathscr{U}$ with respect to the measure ξ (i.e. $\int gU_\alpha f d\xi = \int f\widehat{U}_\alpha g d\xi$ for all $\alpha > 0$ and $f, g \in \mathscr{F}(X)$) and such that the function $\widehat{U}f$ is ξ–quasi continuous (with respect to the given Ray topology on X), for all $f \in \mathscr{F}(X)$ with $\widehat{U}f$ bounded.

Remark. In the probabilistic approach it is usually supposed that $\mathscr{U}$ and $\widehat{\mathscr{U}}$ come from two right processes on X.

Theorem 4.5. (Revuz) *For all bounded $t \in \mathscr{E}_{\widehat{\mathscr{U}}}$ and each ξ–quasi bounded $\mathscr{U}$–excessive function s on X, $s \in \mathscr{P}_{\mathscr{U}}(X)$, we have*

$$L(t\cdot\xi, s) = L(\xi, V_s t) \ .$$

Proof. Obviously it is sufficient to consider $t \in \mathscr{E}_{\widehat{\mathscr{U}}}$ of the form $t = \widehat{U}f$ where $f \in \mathscr{F}(X)$. Also we may assume that $s \in Q_{bd}(\mathscr{E}_{\mathscr{U}})$. Since $\widehat{U}f$ is supposed to be ξ–quasi continuous and $s = \lim_{n\to\infty} nU_n s = \lim_{n\to\infty} U(n(s-nU_n s))$, by Theorem 3.13 we obtain

$$L(\xi, V_s\widehat{U}f) = \lim_{n\to\infty} L(\xi, U(n(s-nU_n s)\widehat{U}f)) = \lim_{n\to\infty} \xi(n(s-nU_n s)\widehat{U}f)$$

$$= \lim_{n\to\infty} \xi(f\cdot nU_n s) = \lim_{n\to\infty} L(\widehat{U}f\cdot\xi, nU_n s) = L(\widehat{U}f\cdot\xi, s) \ .$$

Remark. a) Using the Revuz formula stated by Theorem 4.5, one can prove that every σ–finite measure on X charging no ξ–polar set is the Revuz measure of a proper natural excessive kernel on X, which is uniquely determined ξ–q.e. Moreover, under the above duality hypothesis, each natural excessive kernel is equal ξ-q.e. with a regular excessive kernel.
b) One can show that the above Revuz formula holds for every regular $\mathscr{U}$–excessive function s, without assuming any duality hypothesis.

Corollary 4.6. *If $s \in \mathscr{P}_{\mathscr{U},\xi}(X)$ is ξ–quasi bounded and $t \in \mathscr{E}_{\widehat{\mathscr{U}}}$ then $\nu_s^\xi(t) = L(t\cdot\xi, s)$.*

In the sequel we suppose in addition that the resolvent $\widehat{\mathscr{U}}$ is such that $\mathscr{E}_{\widehat{\mathscr{U}}}$ is min–stable, generates $\mathscr{B}$ and $1 \in \mathscr{E}_{\widehat{\mathscr{U}}}$. As usual, we mark with the prefix co the potential theoretical notions related to $\mathscr{E}_{\widehat{\mathscr{U}}}$, in order to distinguish them from the similar notions related to $\mathscr{E}_{\mathscr{U}}$.

Theorem 4.7. *If $s \in \mathscr{P}_{\mathscr{U},\xi}(X)$ is ξ–quasi bounded then its Revuz measure ν_s^ξ charges no ξ–copolar set and there exists $t \in \mathscr{E}_{\widehat{\mathscr{U}}}$, $t > 0$ ξ–a.e. such that $\nu_s^\xi(t) < \infty$. Moreover, for every $s_1, s_2 \in \mathscr{P}_{\mathscr{U},\xi}(X) \cap Q_{bd}(\mathscr{E}_{\mathscr{U}}, \xi)$ we have $s_1 \leq s_2$ ξ–q.e. if and only if $\nu_{s_1}^\xi(t) \leq \nu_{s_2}^\xi(t)$ for all $t \in \mathscr{E}_{\widehat{\mathscr{U}}}$.*

Conversely, suppose that ν is a positive measure on X which charges no ξ–copolar set and there exists $t \in \mathscr{E}_{\widehat{\mathscr{U}}}$, $t > 0$ ξ–a.e. such that $\nu(t) < \infty$. If there exists a ξ–potential on X which is strictly positive, then there exists $s \in \mathscr{P}_{\mathscr{U},\xi}(X)$ which is ξ–quasi bounded such that $\nu = \nu_s^\xi$.

Proof. Let $s \in \mathscr{P}_{\mathscr{U}}(X) \cap Q_{bd}(\mathscr{E}_{\mathscr{U}})$ be such that $L(\xi, s) < \infty$ and let $M \in \mathscr{B}$ be a ξ–copolar set. Then there exists a decreasing sequence (t_n) in $\mathscr{E}_{\widehat{\mathscr{U}}}$, $t_n \leq 1$ with $t_n \geq 1$ on M and $\inf_n t_n = 0$ ξ–a.e. It follows that $\nu_s^\xi(M) \leq \inf_n \nu_s^\xi(t_n) = \inf_n L(\xi, V_s t_n) = \inf_n L(t_n \cdot \xi, s)$. Since $s \in Q_{bd}(\mathscr{E}_{\mathscr{U}})$ and $L(t_n \cdot \xi, s) < \infty$ we deduce from [6] that $\inf_n L(t_n \cdot \xi, s) = 0$. Hence for all $s \in \mathscr{P}_{\mathscr{U},\xi}(X) \cap Q_{bd}(\mathscr{E}_{\mathscr{U}}, \xi)$ we get $\nu_s^\xi(M) = 0$.

If $s_1, s_2 \in \mathscr{P}_{\mathscr{U},\xi}(X) \cap Q_{bd}(\mathscr{E}_{\mathscr{U}}, \xi)$ then we have $s_1 \leq s_2$ ξ–q.e. if and only if $L(t \cdot \xi, s_1) \leq L(t \cdot \xi, s_2)$ for all $t \in \mathscr{E}_{\widehat{\mathscr{U}}}$. Therefore, by Corollary 4.6, we have $s_1 \leq s_2$ ξ–q.e. if and only if $\nu_{s_1}^\xi(t) \leq \nu_{s_2}^\xi(t)$ for all $t \in \mathscr{E}_{\widehat{\mathscr{U}}}$.

Let now ν be a positive measure on X which charges no ξ–copolar set and such that there exists $t \in \mathscr{E}_{\widehat{\mathscr{U}}}$, $t > 0$ ξ–a.e. with $\nu(t) < \infty$. The functional $t \longmapsto \nu(t)$ on $\mathscr{E}_{\widehat{\mathscr{U}}}$ is additive, increasing and continuous in order from below. Hence there exists $s \in \mathscr{E}_{\mathscr{U}}^\xi$ such that $\nu(t) = L(t \cdot \xi, s)$ for all $t \in \mathscr{E}_{\widehat{\mathscr{U}}}$. Let (t_n) be a decreasing sequence in $\mathscr{E}_{\widehat{\mathscr{U}}}$ such that $\inf_n t_n = 0$ ξ–a.e. and such that $\inf_n L(t_n \cdot \xi, s) < \infty$. Since the set $[\inf_n t_n > 0]$ is ξ–copolar, we deduce that $0 = \nu(\inf_n t_n) = \inf_n L(t_n \cdot \xi, s)$ and therefore, by [6], we get that $s \in Q_{bd}(\mathscr{E}_{\mathscr{U}}, \xi)$. If there exists a ξ–potential which is strictly positive, then by Proposition 4.2 it follows that $s \in \mathscr{P}_{\mathscr{U},\xi}(X)$. From $\nu_s^\xi(t) = L(t \cdot \xi, s)$ we conclude that $\nu_s^\xi(t) = \nu(t)$, for all $t \in \mathscr{E}_{\widehat{\mathscr{U}}}$ and consequently $\nu_s^\xi = \nu$.

Remark. The last assertion of Theorem 4.7 holds without assuming that there exists a strictly positive ξ–potential on X.

Corollary 4.8. *Suppose that the hypothesis* (*B*) *of Hunt with respect to ξ holds. Then the ξ–copolar sets are ξ–polar.*

Remark. By the remark following Theorem 4.5, in Corollary 4.8 it is not necessary to suppose explicitly that the hypothesis (B) of Hunt holds.

Theorem 4.9. *Let $s \in \mathscr{P}_{\mathscr{U},\xi}(X) \cap Q_{bd}(\mathscr{E}_{\mathscr{U}}, \xi)$. Then s is ξ–regular if and only if its Revuz measure charges no ξ–cosemipolar subset of X.*

Proof. By Theorem 4.5, for all $t \in \mathscr{E}_{\widehat{\mathscr{U}}}$ we have $L(t \cdot \xi, s) = L(\xi, V_s t) = \nu_s^\xi(t)$. From Theorem 2.3 in [4], s will be ξ–regular if and only if for each decreasing sequence (t_n) in $\mathscr{E}_{\widehat{\mathscr{U}}}$ and $t \in \mathscr{E}_{\widehat{\mathscr{U}}}$ such that $\bigwedge_n t_n = t$ ξ–a.e. and $\inf_n L(t_n \cdot \xi, s) < \infty$

we have $\inf_n L(t_n \cdot \xi, s) = L(t \cdot \xi, s)$ or equivalently $\inf_n \nu_s^\xi(t_n) = \nu_s^\xi(t)$. If ν_s^ξ charges no ξ–cosemipolar set then, since the set $[\inf_n t_n > t]$ is ξ–cosemipolar, we have $\inf_n \nu_s^\xi(t_n) = \nu_s^\xi(t)$ and consequently s is ξ–regular. Conversely, suppose that s is ξ–regular and let $M \in \mathscr{B}$ be a ξ–cosemipolar set. We have $M = M_o \cup \bigcup_{k \geq 1} M_k$ where M_o is ξ–copolar and M_k is totally cothin for all $k \geq 1$. By Theorem 4.7 we deduce that $\nu_s^\xi(M_o) = 0$ (since each ξ–regular excessive function belongs to $Q_{bd}(\mathscr{E}_{\mathscr{U}}, \xi)$). On the other hand for all $k \geq 1$ there exists a decreasing sequence (t_n^k) in $\mathscr{E}_{\widehat{\mathscr{U}}}$ such that $M_k \subset [\inf_n t_n^k > \bigwedge_n t_n^k]$ and such that $L(t_1^k \cdot \xi, s) < \infty$. We conclude that $\nu_s^\xi(M_k) = 0$ for all $k \geq 1$ and therefore $\nu_s^\xi(M) = 0$.

Corollary 4.10. *The ξ–cosemipolar sets are ξ–semipolar.*

Remark. By the remark following Theorem 4.5, one can show that in fact the ξ–semipolar and the ξ-cosemipolar sets coincide.

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