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# On stationary renewal reward processes where most rewards are zero

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**Abstract.** We consider a stationary version of a renewal reward process, i.e., a renewal process where a random variable called a reward is associated with each renewal. The rewards are nonnegative and I.I.D., but each reward may depend on the distance to the next renewal. We give an explicit bound for the total variation distance between the distribution of the accumulated reward over the interval $(0, L]$ and a compound Poisson distribution. The bound depends in its simplest form only on the first two joint moments of $T$ and $Y$ (or $I\{Y > 0\}$), where $T$ is the distance between successive renewals and $Y$ is the reward. If $T$ and $Y$ are independent, and $LE(Y)$ (or $LP(Y > 0)$) is bounded or $Y$ binary valued, then the bound is $O(E(Y))$ as $E(Y) \to 0$ (or $O(P(Y > 0))$ as $P(Y > 0) \to 0$). To prove our result we generalize a Poisson approximation theorem for point processes by Barbour and Brown, derived using Stein's method and Palm theory, to the case of compound Poisson approximation, and combine this theorem with suitable couplings.

## 1. Introduction

In this paper we are concerned with some properties of *renewal reward* processes. By a renewal process (more exactly: a Palm version of a renewal process) we mean a simple point process $\xi$ on $R$ or $Z$ such that $P(\xi(\{0\}) = 1) = 1$ and such that the distances between successive *renewals* (i.e., points of $\xi$) are I.I.D. A renewal reward process is constructed by associating with each renewal a random variable, called a *reward*. We here assume that the rewards are nonnegative and I.I.D., and that each reward may depend on the distance to the next renewal. We also assume that the distances between successive renewals have finite means, so that there exists a stationary version of the renewal reward process.

Renewal reward processes are interesting for their connections with thinned point processes and with regenerative processes. Clearly, a renewal reward process with binary valued rewards can be regarded as a dependently thinned re-

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newal process. A regenerative process by definition contains an embedded renewal process, and it may in many situations be convenient to associate with each renewal a reward, which depends only on the cycle immediately following that renewal.

The quantity which is studied in this paper is the *accumulated reward* of a stationary renewal reward process over a bounded interval. More precisely, we derive Poisson or compound Poisson approximations with explicit total variation distance error bounds for the distribution of this quantity in the case when $E(Y)$ or $P(Y > 0)$ is small, where $Y$ is the reward. Such results are relevant in particular when we are dealing with occurrences of a "rare" event in a stationary regenerative process. E.g., we could be interested in a Poisson approximation for the number of cycles where the "rare" event occurs, a compound Poisson approximation for the total number of occurrences, or an exponential approximation for the time until the first occurrence.

Needless to say, a number of limit theorems in this field are today well-known. The first important result was obtained by Rényi (1956) for the case of independent thinning (i.e., binary valued rewards which are independent of the renewal process): the independently thinned renewal process converges weakly to a Poisson point process as $E(Y) \to 0$ after a change of time scale. This has subsequently been generalized to a larger class of point processes; see Kallenberg (1975). For dependent thinnings, some results can be deduced from theorems relating weak convergence of point processes to weak convergence of point process compensators; see e.g. Brown (1983).

Concerning total variation distance bounds, a number of so-called *compensator* bounds have been derived for the total variation distance between the distributions of point processes. These depend on how close (in some sense) the compensators of the point processes are to each other; see e.g. Brown (1983) or Kabanov and Liptser (1983). In Barbour and Brown (1992a), *Stein's method* is used to construct a different kind of bound (Theorem 3.1) for the total variation distance between the distribution of the number of points of a point process in a relatively compact set and a Poisson distribution. For this bound it is required that suitable couplings can be found between the point process and the corresponding Palm processes. In the same paper Stein's method is also used to derive a compensator bound, while in Barbour and Brown (1992b) it is used to produce bounds for the total variation distance between the distribution of a point process and that of a Poisson point process, which do not depend on compensators.

The main results in the present paper are the following. The total variation distance between the distribution of the accumulated reward of a stationary renewal reward process over the interval $(0, L]$, and a suitable compound Poisson distribution, is, if rewards are integer valued, bounded by $C(E(TY)/E(T) + E(T^2)E(Y)/E(T)^2)$, where $T$ is the distance between successive renewals and $C$ is an explicit constant which is bounded if the quantity $LE(Y)$ is bounded. If $LE(Y)$ is large then $C$ is also large, unless an additional condition is satisfied. If rewards are binary valued, then the approximating distribution is Poisson with mean $LE(Y)$, and $C \leq 3$ regardless of $LE(Y)$. If rewards are not integer valued, the bound is $C(E(TI\{Y > 0\})/E(T) + E(T^2)P(Y > 0)/E(T)^2)$, where $C$ is an explicit constant which is bounded if the quantity $LP(Y > 0)$ is bounded, but large if this quantity is large. To prove our results, we first generalize Theorem 3.1 in Barbour and Brown (1992a) to the case of compound Poisson approximation. This generalization is of some interest in its own right. We then apply this result to renewal reward processes, using couplings which generalize those used in the proof of Theorem 4.3 in Erhardsson (1999).

The paper is organized as follows. Section 2 contains some notation, as well as some definitions and relevant properties of point processes. In Section 3 the above mentioned generalization of the theorem by Barbour and Brown is given. In Section 4 renewal reward processes are defined, and some lemmas (not all new) about expectation measures and Palm kernels for such processes are collected. In Section 5 the main results are stated and proven. Finally, in Section 6 we consider some examples: independent rewards, rewards indicating a long distance to the next renewal, and the time spent in a "rare" set by a finite state Markov process in discrete or continuous time.

## 2. Preliminaries

We here give some notational conventions, definitions and well-known properties of point processes which will be used below. For more details, see Sections 1.1–2, 2.1, 10.1 and 15.7 in Kallenberg (1983) and Section 3.1 in Rolski (1981).

*Spaces and functions*. Let $S$ be a topological space. As a measure space, we always assume $S$ to be equipped with the Borel $\sigma$-algebra, denoted by $\mathscr{B}_S$. For any random element $X$ in $S$ we denote by $\mathscr{L}(X)$ the distribution of $X$. We denote by $\mathscr{B}_S^{\mathrm{b}}$ the sets in $\mathscr{B}_S$ which are relatively compact, by $\mathscr{F}_S$ the space of measurable functions $S \to R$, and by $\mathscr{F}_S^+$ the space of measurable functions $S \to R_+$. We use the following notation for sets of numbers: $R =$ the real numbers, $Z =$ the integers, $R_+ = [0, \infty)$, $R'_+ = (0, \infty)$, $R_- = (-\infty, 0]$, $R'_- = (-\infty, 0)$, $Z_+ = \{0, 1, 2, ...\}$, $Z'_+ = \{1, 2, ...\}$, $Z_- = \{..., -2, -1, 0\}$ and $Z'_- = \{..., -2, -1\}$. We denote by $S^Z$ the space of all functions $f : Z \to S$. If $S$ is a complete separable metric space, then so is $S^Z$. A random element in $S^Z$ is called an ($S$-valued) *random sequence*.

*The space $\mathscr{N}(S)$*. Let $S$ be a locally compact second countable Hausdorff topological space. We denote by $\mathscr{N}(S)$ the space of *counting measures* (i.e., integer valued Radon measures) on $S$. $\mathscr{N}(S)$ is a complete separable metric space in the vague topology. A random element $\xi$ in $\mathscr{N}(S)$ is called a *point process* (on $S$). The *expectation measure* of $\xi \in \mathscr{N}(S)$ is the measure $E(\xi(\cdot))$. If $\xi \in \mathscr{N}(S)$ has a $\sigma$-finite expectation measure $\mu$, then there exists a probability kernel $Q : S \to \mathscr{N}(S)$ called the *Palm kernel* with the following defining property:

$$E\left(\int_S f(x)d\xi(x)I\{\xi \in A\}\right) = \int_S Q(x, A)f(x)d\mu(x) \quad \forall A \in \mathscr{B}_{\mathscr{N}(S)},\ f \in \mathscr{F}_S^+.$$

A point process $\xi^x$ satisfying $P(\xi^x \in \cdot) = Q(x, \cdot)$ for some $x \in S$ is called a *Palm process* at $x$ for $\xi$. A counting measure $\nu$ satisfying $\nu(\{x\}) \leq 1\ \forall x \in S$ is called a *simple counting measure*, and a point process $\xi$ such that $P(\xi(\{x\}) \leq 1 \forall x \in S) = 1$ is called a *simple point process*.

*The spaces* $\mathscr{N}(R \times K)$ *and* $\mathscr{N}(Z \times K)$. Let $K$ be a locally compact second countable Hausdorff topological space. A random element $\xi$ in $\mathscr{N}(R \times K)$ such that $P(\xi(A \times K) < \infty \ \forall A \in \mathscr{B}_R^{\mathrm{b}}) = 1$ and $P(\xi(\{x\} \times K) \leq 1 \ \forall x \in R) = 1$ we call a *marked simple point process* (on $R \times K$). For each $R \times K$-valued random sequence $(X, Y)$ such that $P(... < X_{-1} < X_0 \leq 0 < X_1 < ...) = 1$ and $P(\mathrm{card}\{i \in Z; (X_i, Y_i) \in A \times K\} < \infty \ \forall A \in \mathscr{B}_S^{\mathrm{b}}) = 1$, there exists on the same probability space as $(X, Y)$ a marked simple point process $\xi$ on $R \times K$ such that $P(\xi(\cdot) = \mathrm{card}\{i \in Z; (X_i, Y_i) \in \cdot\}) = 1$. We call $\xi$ the marked simple point process *generated by* $(X, Y)$. Conversely, for each marked simple point process $\xi$ on $R \times K$ such that $P(\xi(R_+ \times K) = \infty) = P(\xi(R_- \times K) = \infty) = 1$ there exists on the same probability space as $\xi$ an $R \times K$-valued random sequence $(X, Y)$ such that $P(... < X_{-1} < X_0 \leq 0 < X_1 < ...) = 1$ and $P(\xi(\cdot) = \mathrm{card}\{i \in Z; (X_i, Y_i) \in \cdot\}) = 1$. We call $(X, Y)$ the *coordinates of the points of* $\xi$. We define the shift operator $\theta : R \times \mathscr{N}(R \times K) \to \mathscr{N}(R \times K)$ by $\theta_t(\nu)(\cdot) := \nu(\{(x + t, y); (x, y) \in \cdot\}) \ \forall t \in R, \nu \in \mathscr{N}(R \times K)$. $\theta$ is measurable and, for each $t \in R$, a bijection from $\mathscr{N}(R \times K)$ onto itself. Analogous properties hold for marked simple point processes on $Z \times K$.

*Total variation distance*. For any two probability measures $\nu_1$ and $\nu_2$ on any measurable space $(S, \mathscr{F})$ we define the *total variation distance* $d_{TV}(\nu_1, \nu_2)$ by:

$$d_{TV}(\nu_1, \nu_2) := \sup_{A \in \mathscr{F}} |\nu_1(A) - \nu_2(A)|.$$

*Coupling.* A pair of random variables $(X, Y)$ defined on the same probability space is called a *coupling* of two probability distributions $\nu_1$ and $\nu_2$ if $\mathscr{L}(X) = \nu_1$ and $\mathscr{L}(Y) = \nu_2$.

## 3. A total variation distance bound

**Definition 3.1.** (Cf. Section A.19 in Aldous (1989).) A nonnegative random variable $W$ is said to have a *compound Poisson* distribution $\mathrm{POIS}(\nu)$, where $\nu$ is a measure on $R'_+$ such that $\int_0^\infty (1 \wedge x) d\nu(x) < \infty$, if the Laplace transform of $W$ is $E(e^{-sW}) = \exp(-\int_{R'_+} (1 - e^{-sx}) d\nu(x)) \quad \forall s \in R'_+$. If $\nu$ is finite, then $\mathrm{POIS}(\nu) = \mathscr{L}(\sum_{i=1}^M T_i)$, where the variables $\{T_i; i \in Z'_+\}$ and $M$ are independent, $\mathscr{L}(T_i) = \nu/\nu(R'_+) \quad \forall i \in Z'_+$, and $M \sim \mathrm{Po}(\nu(R'_+))$. Here, $\mathscr{L}(T_1)$ is called the *compounding* distribution.

**Theorem 3.1.** *Let $S$ be a locally compact second countable Hausdorff topological space, and let $\xi$ be a point process on $S \times R_+$ with $\sigma$-finite expectation measure $\mu$. For each $(x, y) \in S \times R_+$, let $\xi^{(x,y)}$ be a Palm process at $(x, y)$ for $\xi$, and assume that $\xi^{(x,y)}$ is defined on the same probability space as $\xi$. Let $A \in \mathscr{B}_S^b$, and define $\mu_A(\cdot) := \mu(A \times \cdot)$ and $\mu'_A(\cdot) := \mu_A(\cdot \cap R'_+)$. Assume that $\mu'_A$ is finite. Define $\phi_A : \mathscr{N}(S \times R_+) \to R_+$ by $\phi_A(\nu) := \int_{A \times R'_+} y d\nu(x, y) \quad \forall \nu \in \mathscr{N}(S \times R_+)$. If* $\mathrm{supp}(\mu'_A) = \{1\}$, *then:*

$$d_{TV}\big(\mathscr{L}(\phi_A(\xi)), \mathrm{Po}(\mu_A(1))\big) \\ \le \frac{1-e^{-\mu_A(1)}}{\mu_A(1)} \int_{A\times\{1\}} E\big(|\phi_A(\xi) - \phi_A(\xi^{(x,y)}) + y|\big) d\mu(x,y). \quad (3.1)$$

*If* $\mathrm{supp}(\mu'_A) \subset Z'_+$*, then:*

$$d_{TV}\big(\mathscr{L}(\phi_A(\xi)), \mathrm{POIS}(\mu'_A)\big) \\ \le H(\mu'_A) \int_{A\times Z'_+} yE\big(|\phi_A(\xi) - \phi_A(\xi^{(x,y)}) + y|\big) d\mu(x,y), \quad (3.2)$$

*where* $H(\mu'_A) := (\mu_A(1)^{-1} \wedge 1)\exp(\mu_A(Z'_+))$*, unless* $\{k\mu_A(k); k \in Z'_+\}$ *is monotonically decreasing towards 0, in which case*

$$H(\mu'_A) := \frac{1}{\mu_A(1) - 2\mu_A(2)} \\ \times \left(\frac{1}{4(\mu_A(1) - 2\mu_A(2))} + \log^+ 2(\mu_A(1) - 2\mu_A(2))\right) \wedge 1.$$

*If* $\mathrm{supp}(\mu'_A) \subset R'_+$ *(the general case), then:*

$$d_{TV}\big(\mathscr{L}(\phi_A(\xi)), \mathrm{POIS}(\mu'_A)\big) \\ \le \exp(\mu_A(R'_+)) \int_{A\times R'_+} P\big(|\phi_A(\xi) - \phi_A(\xi^{(x,y)}) + y| > 0\big) d\mu(x,y). \quad (3.3)$$

*Proof.* This theorem generalizes both Theorem 3.1 in Barbour and Brown (1992a) and Theorem 4.1 in Erhardsson (1999). Consider for each $B \in \mathscr{B}_{R_+}$ the following so-called *Stein equation*:

$$xf_B(x) - \int_{R'_+} yf_B(x+y)d\mu'_A(y) = I_B(x) - P(W \in B) \quad \forall x \in R_+,$$

where $W \sim \mathrm{POIS}(\mu'_A)$. According to Theorem 1 in Barbour, Chen and Loh (1992) there exists a unique solution $f_B \in \mathscr{F}_{R'_+}$ of this equation satisfying the condition $\sup_{x\in R'_+} |xf_B(x)| < \infty$; $f_B(0)$ can be chosen arbitrarily. Hence, if we can find a bound for $|E(\phi_A(\xi)f_B(\phi_A(\xi)) - \int_{R'_+} yf_B(\phi_A(\xi)+y)d\mu'_A(y))|$ for each $B \in \mathscr{B}_{R_+}$, then we will also get a bound for the total variation distance $d_{TV}(\mathscr{L}(\phi_A(\xi)), \mathrm{POIS}(\mu'_A))$, an idea which is known as *Stein's method*. To do this we imitate the proof of Theorem 3.1 in Barbour and Brown (1992a). We have:

$$E\big(\phi_A(\xi)f_B(\phi_A(\xi))\big) = E\left(\int_{A\times R'_+} yf_B(\phi_A(\xi))d\xi(x,y)\right) \\ = E\left(\int_{A\times R'_+} yf_B(\phi_A(\xi - \delta_{(x,y)}) + y)d\xi(x,y)\right) \\ = \int_{A\times R'_+} yE\big(f_B(\phi_A(\xi^{(x,y)} - \delta_{(x,y)}) + y)\big)d\mu(x,y),$$

where the last equality follows from Lemma 10.2 in Kallenberg (1983). Hence,

$$\left|E\left(\phi_A(\xi)f_B(\phi_A(\xi)) - \int_{R'_+} yf_B(\phi_A(\xi)+y)d\mu'_A(y)\right)\right|$$
$$= |\int_{A\times R'_+} yE\big(f_B(\phi_A(\xi^{(x,y)}-\delta_{(x,y)})+y) - f_B(\phi_A(\xi)+y)\big)d\mu(x,y)|$$
$$\leq \int_{A\times R'_+} yE\big(|f_B(\phi_A(\xi^{(x,y)}-\delta_{(x,y)})+y) - f_B(\phi_A(\xi)+y)|\big)d\mu(x,y).$$

(3.1) now follows from the fact that if $\text{supp}(\mu'_A) = \{1\}$, then from Lemma 1.1.1 in Barbour, Holst and Janson (1992),

$$\sup_{B\subset Z_+}\sup_{k\in Z'_+}|f_B(k+1) - f_B(k)| \leq \frac{1-e^{-\mu_A(1)}}{\mu_A(1)}.$$

Similarly, (3.2) follows since if $\text{supp}(\mu'_A) \subset Z'_+$, then from Theorem 5 in Barbour, Chen and Loh (1992),

$$\sup_{B\subset Z_+}\sup_{k\in Z'_+}|f_B(k+1) - f_B(k)| \leq H(\mu'_A).$$

Finally, (3.3) follows since if $\text{supp}(\mu'_A) \subset R'_+$, then from Theorem 2 in Barbour, Chen and Loh (1992),

$$\sup_{B\in\mathscr{B}_{R_+}}\sup_{u\geq v\geq 0} v|f_B(u) - f_B(v)| \leq \exp(\mu_A(R'_+)). \qquad \square$$

**Remark 3.1.** Finding better bounds for the quantity $\sup_{B\subset Z_+}\sup_{k\in Z'_+}|f_B(k+1) - f_B(k)|$ than $H(\mu'_A)$ in the case when $\text{supp}(\mu'_A) \subset Z'_+$ is currently a very active research area. Some results in this direction are given in Barbour and Utev (1999), but in order to apply these in the present context the preceding proof must be somewhat modified. If attention is restricted to bounding the *Kolmogorov distance* $d_K(\mathscr{L}(\phi_A(\xi)), \text{POIS}(\mu'_A))$, this can be done in a manner similar to the proof of Theorem 3.1 if a bound for the quantity $\sup_{B\in\{[r,\infty);r\in Z_+\}}\sup_{k\in Z'_+}|f_B(k+1) - f_B(k)|$ is available. Some results can be found in Barbour and Utev (1998).

## 4. Renewal reward processes

**Definition 4.1.** By a *stationary renewal reward process* in continuous or discrete time, we mean a marked simple point process $\xi$ on $R\times R_+$ or $Z\times R_+$ respectively, defined in the following way. Let $(T^o, Y^o)$ be an I.I.D. random sequence, $R'_+\times R_+$-valued or $Z'_+\times R_+$-valued respectively, for which $E(T^o_0) < \infty$. Define the random sequence $X^o$ by:

$$X^o_t := \begin{cases} \sum_{i=0}^{t-1} T^o_i, & \text{if } t\geq 1; \\ 0, & \text{if } t = 0; \\ -\sum_{i=-1}^{t} T^o_i, & \text{if } t\leq -1\,. \end{cases}$$

Let $\xi^o$ be the marked simple point process generated by $(X^o, Y^o)$. We call $\xi^o$ a Palm version of a renewal reward process (with respect to marks from $R_+$). The distribution of $\xi$ is now given by the *Palm inversion formula* (a part of *Ryll-Nardzewski's*, aka *Slivnyak's*, theorem). In the continuous time case:

$$E(g(\xi)) = \frac{E\left(\int_0^{T_0^o} g(\theta_t(\xi^o))dt\right)}{E(T_0^o)} \qquad \forall g \in \mathscr{F}^+_{\mathscr{N}(R\times R_+)}, \tag{4.1}$$

and in the discrete time case:

$$E(g(\xi)) = \frac{E\left(\sum_{i=0}^{T_0^o-1} g(\theta_i(\xi^o))\right)}{E(T_0^o)} \quad \forall g \in \mathscr{F}^+_{\mathscr{N}(Z\times R_+)}. \tag{4.2}$$

In what follows, in the context of a particular stationary renewal reward process $\xi$, $(T^o, Y^o)$ and $X^o$ will always refer to the random sequences defined above. $(X, Y)$ will refer to the coordinates of the points of $\xi$; cf. Section 2.

In the remainder of this section we give three lemmas which will be used below. We do not doubt that at least the first two of them can be found in the literature, but since we do not know the exact locations, we provide proofs.

**Lemma 4.1.** *Let $\xi$ be a stationary renewal reward process on $R \times R_+$ or $Z \times R_+$. Then $\xi$ has expectation measure $\mu = E(T_0^o)^{-1}\ell \times \mu_Y$, where $\ell$ is the Lebesgue measure or the counting measure respectively, and $\mu_Y = \mathscr{L}(Y_0^o)$.*

*Proof.* We consider only the continuous time case. $\mu$ is locally finite, since the stationarity of $\xi$ implies that $E(\xi([-x, x] \times R_+)) = 2xE(T_0^o)^{-1} < \infty \ \forall x \in R$. The fact that $\mu = E(T_0^o)^{-1}\ell \times \mu_Y$ for some locally finite measure $\mu_Y$ on $R_+$ follows from Proposition 10.5.I in Daley and Vere-Jones (1988), and that $\mu_Y = \mathscr{L}(Y_0^o)$ can be seen as follows. Clearly, $\mu_Y([0, y]) = E(\xi((-\delta, 0] \times [0, y]))E(T_0^o)/\delta \ \ \forall y \in R_+, \delta \in R'_+$, and (4.1) implies that

$$\begin{aligned}
&\frac{E\big((T_0^o \wedge \delta)I\{Y_0^o \le y\}\big)}{E(T_0^o)} \\
&\le E\big(\xi((-\delta, 0] \times [0, y])\big) \\
&\le \frac{\delta E\big(I\{Y_0^o \le y\} + \xi^o((-\delta, 0) \times [0, y])\big)}{E(T_0^o)} \qquad \forall y \in R_+, \delta \in R'_+.
\end{aligned}$$

Now let $\delta \to 0$ and use dominated convergence. □

**Lemma 4.2.** *Let $\xi$ be a stationary renewal reward process on $R \times R_+$ with expectation measure $\mu = E(T_0^o)^{-1}\ell \times \mu_Y$. For each $y \in R_+$, define the random sequence $(X^{(0,y)}, Y^{(0,y)})$ as follows: let $(X_{i+1}^{(0,y)} - X_i^{(0,y)}, Y_i^{(0,y)}) := (T_i^o, Y_i^o) \ \forall i \in Z \setminus \{0\}$, let $(X_0^{(0,y)}, Y_0^{(0,y)}) := (0, y)$, and let $X_1^{(0,y)}$ be independent of $(T^o, Y^o)$ such that $P(X_1^{(0,y)} \in \cdot)$ is a version of the regular conditional distribution of $T_0^o$ given*

*$Y_0^{\circ} = y$. For each $(x, y) \in R \times R_+$, let $\xi^{(0,y)}$ be the marked simple point process generated by $(X^{(0,y)}, Y^{(0,y)})$, and define $\xi^{(x,y)} := \theta_{-x}(\xi^{(0,y)})$. Then the Palm kernel of $\xi$ can be chosen as $Q'$, where $Q'((x, y), \cdot) := P(\xi^{(x,y)} \in \cdot)$ for each $(x, y) \in R \times R_+$. The analogous result holds for a stationary renewal reward process on $Z \times R_+$.*

*Proof.* We consider only the continuous time case. As mentioned in Section 2, the Palm kernel $Q : R \times R_+ \to \mathscr{N}(R \times R_+)$ is a probability kernel which satisfies the identity

$$E\left(\int_{R\times R_+} f(x, y)d\xi(x, y)I\{\xi \in A\}\right) = \int_{R\times R_+} Q((x, y), A)f(x, y)d\mu(x, y) \tag{4.3}$$

for each $A \in \mathscr{B}_{\mathscr{N}(R\times R_+)}$ and $f \in \mathscr{F}^+_{R\times R_+}$. We will show that $Q'$ is a probability kernel which satisfies (4.3) for each $A \in \mathscr{B}_{\mathscr{N}(R\times R_+)}$ and each $f$ in the class of indicator functions $\{I_{B\times C}(\cdot); B \in \mathscr{B}_R, C \in \mathscr{B}_{R_+}\}$. Dynkin's $\pi$-$\lambda$-theorem and monotone convergence then imply that $Q'$ satisfies (4.3) for each $A \in \mathscr{B}_{\mathscr{N}(R\times R_+)}$ and $f \in \mathscr{F}^+_{R\times R_+}$.

It is clear from the definition of $\xi^{(x,y)}$ that $Q'((x, y), \cdot)$ is a probability measure on $\mathscr{N}(R \times R_+)$ for each $(x, y) \in R \times R_+$. We must show that $Q'(\cdot, A)$ is a measurable function on $R \times R_+$ for each $A \in \mathscr{B}_{\mathscr{N}(R\times R_+)}$. From Dynkin's $\pi$-$\lambda$-theorem it follows that we need only consider sets $A$ in the $\pi$-system $\mathscr{H}$ generated by sets of the type $\{\nu \in \mathscr{N}(R \times R_+); \nu(B) = k\}$, where $B \in \mathscr{B}_{R\times R_+}$ and $k \in Z_+$. If $A := \cap_{i=1}^n \{\nu \in \mathscr{N}(R \times R_+); \nu(B_i) = k_i\} \in \mathscr{H}$, then $\{\xi^{(x,y)} \in A\}$ can be written as a countable disjoint union in the following way:

$$\begin{aligned}\{\xi^{(x,y)} \in A\} &= \cup_{D\in\Gamma} \cap_{i=1}^n \big(\cap_{j\in D_i}\{(x + X_j^{(0,y)}, Y_j^{(0,y)}) \in B_i\} \\ &\qquad \cap \cap_{j\notin D_i} \{(x + X_j^{(0,y)}, Y_j^{(0,y)}) \notin B_i\}\big) \\ &= \cup_{D\in\Gamma} \cap_{j\in Z} \{(x + X_j^{(0,y)}, Y_j^{(0,y)}) \in C_j^D\},\end{aligned}$$

where $\Gamma = \Gamma_1 \times ... \times \Gamma_n$, $\Gamma_i$ is the set of all $k_i$-tuples of distinct integers for each $i \in \{1, ..., n\}$, and the sets $\{C_j^D \in \mathscr{B}_{R\times R_+}; j \in Z\}$ depend only on $D$ and $A$. Consider for each $j \in Z$ the function $F_j^D : R \times R_+ \times R'_+ \times (R'_+ \times R_+)^{Z\setminus\{0\}} \to \{0, 1\}$, defined by:

$$F_j^D(x, y, z, (u, v)) := \begin{cases} I\left\{\left(x + \sum_{i=-1}^{j} u_i, v_j\right) \in C_j^D\right\}, & \text{if } j \le -1; \\ I\{(x, y) \in C_j^D\}, & \text{if } j = 0; \\ I\left\{\left(x + z + \sum_{i=1}^{j-1} u_i, v_j\right) \in C_j^D\right\}, & \text{if } j \ge 1\,. \end{cases}$$

Clearly $F_j^D$ is measurable, implying that also $F := \sum_{D\in\Gamma} \prod_{j\in Z} F_j^D$ is measurable. Since it holds that

$$I\{\xi^{(x,y)} \in A\} = F(x, y, X_1^{(0,y)}, (T^{\circ}, Y^{\circ})),$$

we get, using also that $X_1^{(0,y)}$ and $(T^o, Y^o)$ are independent,

$$\begin{aligned} &P(\xi^{(x,y)} \in A) \\ &\quad = E(F(x, y, X_1^{(0,y)}, (T^o, Y^o))) \\ &\quad = E(E(F(x, y, X_1^{(0,y)}, (T^o, Y^o))|x, y, X_1^{(0,y)})) \\ &\quad = \int_{R\times R_+\times R'_+} E(F(x', y', z', (T^o, Y^o)))d(\delta_x \times \delta_y \times \nu_{X_1}^{(0,y)})(x', y', z'), \end{aligned}$$

where $\nu_{X_1}^{(0,y)} = \mathscr{L}(X_1^{(0,y)})$. But it is easily shown using the $\pi$-$\lambda$-theorem and monotone convergence that the right-hand side of this expression defines a measurable function on $R \times R_+$.

We next show that $Q'$ satisfies (4.3) for each $A \in \mathscr{B}_{\mathscr{N}(R\times R_+)}$ and each $f \in \{I_{B\times C}(\cdot); B \in \mathscr{B}_R, C \in \mathscr{B}_{R_+}\}$. Fix $A \in \mathscr{B}_{\mathscr{N}(R\times R_+)}$, $B \in \mathscr{B}_R$, and $C \in \mathscr{B}_{R_+}$ such that $\mu_Y(C) > 0$ (if $\mu_Y(C) = 0$ then $Q'$ satisfies (4.3) trivially). Calculations similar to those in the first part of this proof give:

$$\begin{aligned} &\int_{R\times R_+} Q'((x, y), A)I_{B\times C}(x, y)d\mu(x, y) \\ &\quad = E(T_0^o)^{-1} \int_{R\times R_+} P(\xi^{(x,y)} \in A)I_B(x)I_C(y)d(\ell \times \mu_Y)(x, y) \\ &\quad = E(T_0^o)^{-1} \int_{R\times R_+} P(\theta_{-x}(\xi^{(0,y)}) \in A)I_B(x)I_C(y)d(\ell \times \mu_Y)(x, y) \\ &\quad = E(T_0^o)^{-1} \int_R I_B(x)P(\theta_{-x}(\xi^o) \in A, Y_0^o \in C)dx \\ &\quad = E(T_0^o)^{-1}\mu_Y(C) \int_R I_B(x)P(\theta_{-x}(\xi^o) \in A|Y_0^o \in C)dx. \end{aligned}$$

Proposition 3.7 in Rolski (1981) tells us that the distribution $P(\xi^o \in \cdot|Y_0^o \in C)$ is a Palm distribution with respect to marks from $C$. We can therefore apply *Mecke's* theorem, see Proposition 3.5 in Rolski (1981), which gives:

$$E(T_0^o)^{-1}\mu_Y(C) \int_R I_B(x)P(\theta_{-x}(\xi^o) \in A|Y_0^o \in C)dx = E(\xi(B \times C)I\{\xi \in A\}).$$

□

**Lemma 4.3.** *If $\xi$ is a stationary renewal reward process on $R \times R_+$, then, for each $z, b \in R'_+$ such that $z < b$,*

$$E\left(\int_{[0,z)\times R'_+} v d\xi^o(u, v)\right) \le z\frac{E(Y_0^o)}{E(T_0^o)} + \frac{E((T_0^o)^2)E(Y_0^o)}{E(T_0^o)^2};$$

$$E\left(\int_{(-z,0)\times R'_+} v d\xi^o(u, v)\right) \le z\frac{E(Y_0^o)}{E(T_0^o)} + \frac{E((T_0^o)^2)E(Y_0^o)}{E(T_0^o)^2};$$

$$E\left(\int_{(-b,-b+z]\times R'_+} v d\xi^o(u, v)\right) \le z\frac{E(Y_0^o)}{E(T_0^o)} + \frac{E((T_0^o)^2)E(Y_0^o)}{E(T_0^o)^2} + \alpha\frac{E(T_0^o Y_0^o)}{E(T_0^o)},$$

*where* $\alpha = 1$, *unless the rewards are independent of the renewal process, in which case* $\alpha = 0$. *If* $\xi$ *is a stationary renewal reward process on* $Z \times R_+$, *then, for each* $z, b \in Z'_+$ *such that* $z < b$, *the same assertions hold with* $E((T_0^{\text{o}})^2)$ *replaced by* $E(T_0^{\text{o}}(T_0^{\text{o}} - 1))$; *furthermore, for each* $b \in Z$ *and* $z \in Z'_+$,

$$E\left(\int_{(b,b+z]\times R'_+} v d\xi^{\text{o}}(u,v)\right) \le zE(Y_0^{\text{o}}).$$

*Proof.* In both continuous and discrete time, it holds that

$$\begin{aligned} E\left(\int_{[0,z)\times R'_+} v d\xi^{\text{o}}(u,v)\right) &= \sum_{j=0}^{\infty} E\big(Y_j^{\text{o}} I\{X_j^{\text{o}} < z\}\big) \\ &= E(Y_0^{\text{o}}) \sum_{j=0}^{\infty} P(X_j^{\text{o}} < z) \\ &= E(Y_0^{\text{o}}) E\big(\xi^{\text{o}}([0,z) \times R_+)\big). \end{aligned}$$

Hence, an application of Lorden's renewal inequality, see Corollary 1 in Lorden (1970), proves the first assertion. Moreover,

$$\begin{aligned} E\left(\int_{(-z,0)\times R'_+} v d\xi^{\text{o}}(u,v)\right) &= \sum_{j=1}^{\infty} E\big(Y_{-j}^{\text{o}} I\{X_{-j}^{\text{o}} > -z\}\big) \\ &\le E(Y_0^{\text{o}}) \sum_{j=1}^{\infty} P(X_{-j+1}^{\text{o}} > -z) \\ &= E(Y_0^{\text{o}}) E\big(\xi^{\text{o}}((-z,0] \times R_+)\big), \end{aligned}$$

which together with Lorden's renewal inequality proves the second assertion. Also,

$$\begin{aligned} (b-z)\frac{E(Y_0^{\text{o}})}{E(T_0^{\text{o}})} = E\left(\int_{(-b+z,0)\times R'_+} v d\xi(u,v)\right) &\le \sum_{j=1}^{\infty} E\big(Y_{-j} I\{X_{-j} > -b+z\}\big) \\ &+E(Y_0) \\ &= \sum_{j=1}^{\infty} E\big(Y_{-j} I\{X_{-j} - X_0 > -b+z+|X_0|\}\big) + E(Y_0) \\ &\le E\left(\int_{(-b+z,0)\times R'_+} v d\xi^{\text{o}}(u,v)\right) + E(Y_0), \end{aligned}$$

which together with (4.1) or (4.2) and the second assertion proves the third assertion. The case when the rewards are independent of the renewal process again follows from Lorden's renewal inequality. The last assertion in the discrete time case is an immediate consequence of Lemma 4.1, (4.2), and Theorem 2.1 in Rolski (1981), which says that $P(\xi^{\text{o}} \in \cdot) = P(\xi \in \cdot | \xi(\{0\} \times R_+) = 1)$. □

## 5. Poisson and compound Poisson approximation

**Theorem 5.1.** *Let $\xi$ be a stationary renewal reward process on $R \times R_+$ with expectation measure $\mu = E(T_0^o)^{-1}\ell \times \mu_Y$, where $\ell$ is Lebesgue measure and $\mu_Y := \mathscr{L}(Y_0^o)$, and define $\mu'_Y(\cdot) := \mu_Y(\cdot \cap R'_+)$. Let $L \in R'_+$. If $\operatorname{supp}(\mu'_Y) = \{1\}$, then:*

$$d_{TV}\left(\mathscr{L}\left(\int_{(0,L]\times\{1\}} d\xi(x,y)\right), \mathrm{Po}\left(L\frac{E(Y_0^o)}{E(T_0^o)}\right)\right)$$
$$\leq \left(1-\exp\left(-L\frac{E(Y_0^o)}{E(T_0^o)}\right)\right)\left(C_1\frac{E(T_0^oY_0^o)}{E(T_0^o)} + C_2\frac{E((T_0^o)^2)E(Y_0^o)}{E(T_0^o)^2}\right). \quad (5.1)$$

*If $\operatorname{supp}(\mu'_Y) \subset Z'_+$, then:*

$$d_{TV}\left(\mathscr{L}\left(\int_{(0,L]\times Z'_+} y\,d\xi(x,y)\right), \mathrm{POIS}\left(L\frac{\mu'_Y}{E(T_0^o)}\right)\right)$$
$$\leq H\left(L\frac{\mu'_Y}{E(T_0^o)}\right)L\frac{E(Y_0^o)}{E(T_0^o)}\left(C_1\frac{E(T_0^oY_0^o)}{E(T_0^o)} + C_2\frac{E((T_0^o)^2)E(Y_0^o)}{E(T_0^o)^2}\right), \quad (5.2)$$

*where $H(E(T_0^o)^{-1}L\mu'_Y) := ((E(T_0^o)^{-1}L\mu_Y(1))^{-1} \wedge 1)\exp(E(T_0^o)^{-1}L\mu_Y(Z'_+))$, unless $\{k\mu_Y(k); k \in Z'_+\}$ is monotonically decreasing towards 0, in which case*

$$H\left(L\frac{\mu^+}{E(T_0^o)}\right) := \frac{1}{\Delta_Y(1)}\left(\frac{1}{4\Delta_Y(1)} + \log^+ 2\Delta_Y(1)\right) \wedge 1,$$

*where $\Delta_Y(1) := E(T_0^o)^{-1}L(\mu_Y(1) - 2\mu_Y(2))$. If $\operatorname{supp}(\mu'_Y) \subset R'_+$, then:*

$$d_{TV}\left(\mathscr{L}\left(\int_{(0,L]\times R'_+} y\,d\xi(x,y)\right), \mathrm{POIS}\left(L\frac{\mu'_Y}{E(T_0^o)}\right)\right)$$
$$\leq \exp\left(L\frac{\mu_Y(R'_+)}{E(T_0^o)}\right)L\frac{\mu_Y(R'_+)}{E(T_0^o)}$$
$$\times\left(C_1\frac{E(T_0^oI\{Y_0^o > 0\})}{E(T_0^o)} + C_2\frac{E((T_0^o)^2)\mu_Y(R'_+)}{E(T_0^o)^2}\right). \quad (5.3)$$

*Throughout, $C_1 = C_2 = 3$ unless the rewards are independent of the renewal process, in which case $C_1 = 2$ and $C_2 = 3$. If $\xi$ is a stationary renewal reward process on $Z \times R_+$, and if $L \in Z'_+$, then the same assertions hold with $E((T_0^o)^2)$ replaced by $E(T_0^o(T_0^o - 1))$, but in this case we may also take $C_1 = 2$ and $C_2 = E(T_0^o)$.*

*Proof.* We consider the continuous time case, pointing out the modifications needed in the discrete time case. In order to apply Theorem 3.1, we need to find suitable couplings of $\mathscr{L}(\xi)$ and $\mathscr{L}(\xi^{(x,y)})$ for $\mu$-a.e. $(x, y) \in R \times R_+$. We shall construct couplings which generalize those used in the proof of Theorem 4.3 in Erhardsson (1999). Let $(\Omega, \mathscr{G}, P)$ be a probability space which contains the stationary renewal reward process $\xi$, and hence the random sequence $(X, Y)$ (the coordinates of the points of $\xi$). Let the probability space also contain a collection of random variables $\{X_1^{(0,y)}; y \in R_+\}$ which are independent of $\xi$ and for which $P(X_1^{(0,y)} \in \cdot)$ is a version of the regular conditional distribution of $T_0^o$ given $Y_0^o = y$. Define the $R \times R_+$-valued random sequence $(X^{(0,y)}, Y^{(0,y)})$ in the following way:

$$\left(X_i^{(0,y)}, Y_i^{(0,y)}\right) := \begin{cases} \left(X_1^{(0,y)} + X_i - X_1, Y_i\right), & \text{if } i \geq 1 \text{ and } X_0 < 0; \\ \left(X_1^{(0,y)} + X_{i-1} - X_0, Y_{i-1}\right), & \text{if } i \geq 1 \text{ and } X_0 = 0; \\ (0, y), & \text{if } i = 0; \\ (X_i - X_0, Y_i), & \text{if } i \leq -1\,. \end{cases}$$

(Of course, $P(X_0 = 0) = 0$ in the continuous time case, but not in the discrete time case.) Let $\xi^{(0,y)}$ denote the marked simple point process generated by $(X^{(0,y)}, Y^{(0,y)})$. From Lemma 4.2 it follows that $\xi^{(x,y)} := \theta_{-x}(\xi^{(0,y)})$ can be chosen as the Palm process at $(x, y)$ for each $(x, y) \in R \times R_+$. We want to find a bound for the quantity within the expectation on the right-hand side of (3.1) and (3.2), for $\mu$-a.e. $(x, y) \in (0, L] \times R_+$. We note that the stationarity of $\xi$ implies that $\mathscr{L}(\theta_{-x}(\xi)) = \mathscr{L}(\xi)\ \forall x \in R$, and examining the proof of Theorem 3.1 we see that we can replace $\xi$ with $\theta_{-x}(\xi)$ within the expectation. Furthermore, the definition of the shift operator implies that:

$$\begin{aligned} &|\phi_{(0,L]}(\theta_{-x}(\xi)) - \phi_{(0,L]}(\theta_{-x}(\xi^{(0,y)})) + y| \\ &\quad = |\phi_{(-x,L-x]}(\xi) - \phi_{(-x,L-x]}(\xi^{(0,y)}) + y| \quad \forall (x, y) \in (0, L] \times R_+. \end{aligned}$$

Bounding in a suitable way the sum of those rewards which do not cancel out in the difference, we get:

$$\begin{aligned} &|\phi_{(-x,L-x]}(\xi) - \phi_{(-x,L-x]}(\xi^{(0,y)}) + y| \\ &\quad \leq Y_0 + \int_{(L-x-X_1^{(0,y)}, L-x] \times R'_+} v d\xi(u, v) + \int_{(R'_- \cap (-x, -x+|X_0|]) \times R'_+} v d\xi^{(0,y)}(u, v) \\ &\qquad + \int_{(R'_+ \cap (L-x-X_1 I\{X_0<0\}, L-x]) \times R'_+} v d\xi^{(0,y)}(u, v). \end{aligned} \tag{5.4}$$

We calculate the expectations of the terms in 5.4 one by one, using (4.1). For the first term, we get:

$$E(Y_0) = \frac{E(T_0^o Y_0^o)}{E(T_0^o)}.$$

For the second term, Lemma 4.1 and the fact that $\xi$ is independent of $X_1^{(0,y)}$ gives:

$$E\left(E\left(\int_{(L-x-X_1^{(0,y)}, L-x] \times R'_+} v d\xi(u, v) | X_1^{(0,y)}\right)\right) = E(X_1^{(0,y)}) \frac{E(Y_0^o)}{E(T_0^o)},$$

and:

$$\int_{(0,L] \times R'_+} y E(X_1^{(0,y)}) d\mu(x, y) = L \frac{E(T_0^o Y_0^o)}{E(T_0^o)}.$$

For the third term, the fact that $\xi^{(0,y)}$ is independent of $X_0$ gives:

$$E\left(E\left(\int_{(R'_-\cap(-x,-x+|X_0|])\times R'_+} vd\xi^{(0,y)}(u,v)|X_0\right)\right)$$
$$= \int_{R'_+} E\left(\int_{(R'_-\cap(-x,-x+z])\times R'_+} vd\xi^{(0,y)}(u,v)\right)d\nu_0(z),$$

where $\nu_0 := \mathscr{L}(|X_0|)$. Here, in the continuous time case we get from Lemma 4.3 for each $z \in R'_+$:

$$E\left(\int_{(R'_-\cap(-x,-x+z])\times R'_+} vd\xi^{(0,y)}(u,v)\right) \leq z\frac{E(Y_0^o)}{E(T_0^o)} + \frac{E((T_0^o)^2)E(Y_0^o)}{E(T_0^o)^2}$$
$$+\alpha\frac{E(T_0^oY_0^o)}{E(T_0^o)},$$

implying that:

$$E\left(\int_{(R'_-\cap(-x,-x+|X_0|])\times R'_+} vd\xi^{(0,y)}(u,v)\right) \leq \frac{3E((T_0^o)^2)E(Y_0^o)}{2E(T_0^o)^2} + \alpha\frac{E(T_0^oY_0^o)}{E(T_0^o)}.$$

In discrete time the same result holds with $E((T_0^o)^2)$ replaced by $E(T_0^o(T_0^o-1))$, but in this case it also holds that

$$E\left(\int_{(R'_-\cap(-x,-x+|X_0|])\times R'_+} vd\xi^{(0,y)}(u,v)\right) \leq \frac{E(T_0^o(T_0^o-1))E(Y_0^o)}{2E(T_0^o)}.$$

For the fourth term, the fact that $\xi^{(0,y)}$ is independent of $X_1I\{X_0<0\}$ gives:

$$E\left(E\left(\int_{(R'_+\cap(L-x-X_1I\{X_0<0\},L-x])\times R'_+} vd\xi^{(0,y)}(u,v)|X_1I\{X_0<0\}\right)\right)$$
$$= \int_{R'_+} E\left(\int_{(R'_+\cap(L-x-z,L-x])\times R'_+} vd\xi^{(0,y)}(u,v)\right)d\nu_1(z),$$

where $\nu_1 := \mathscr{L}(X_1I\{X_0<0\})$. Here, in the continuous time case we get from Lemma 4.3 for each $z \in R'_+$:

$$E\left(\int_{(R'_+\cap(L-x-z,L-x])\times R'_+} vd\xi^{(0,y)}(u,v)\right) \leq z\frac{E(Y_0^o)}{E(T_0^o)} + \frac{E((T_0^o)^2)E(Y_0^o)}{E(T_0^o)^2},$$

implying that:

$$E\left(\int_{(R'_+\cap(L-x-X_1I\{X_0<0\},L-x])\times R'_+} vd\xi^{(0,y)}(u,v)\right) \leq \frac{3E((T_0^o)^2)E(Y_0^o)}{2E(T_0^o)^2}.$$

In discrete time the same result holds with $E((T_0^{\mathrm{o}})^2)$ replaced by $E(T_0^{\mathrm{o}}(T_0^{\mathrm{o}}-1))$, but it also holds that

$$E\left(\int_{(R'_+\cap(L-x-X_1 I\{X_0<0\},L-x])\times R'_+} v d\xi^{(0,y)}(u,v)\right) \leq \frac{E(T_0^{\mathrm{o}}(T_0^{\mathrm{o}}-1))E(Y_0^{\mathrm{o}})}{2E(T_0^{\mathrm{o}})}.$$

It remains to find a bound for the probability on the right-hand side of (3.3), for $\mu$-a.e. $(x,y)\in(0,L]\times R_+$. As before, we can replace $\xi$ with $\theta_{-x}(\xi)$ within the probability, and we get:

$$\begin{aligned} & I\{|\phi_{(-x,L-x]}(\xi)-\phi_{(-x,L-x]}(\xi^{(0,y)})+y|>0\} \\ & \leq I\{Y_0>0\}+\int_{(L-x-X_1^{(0,y)},L-x]\times R'_+} d\xi(u,v) \\ & \quad + \int_{(R'_-\cap(-x,-x+|X_0|])\times R'_+} d\xi^{(0,y)}(u,v) \\ & \quad + \int_{(R'_+\cap(L-x-X_1 I\{X_0<0\},L-x])\times R'_+} d\xi^{(0,y)}(u,v). \end{aligned} \tag{5.5}$$

However, it is easy to see that the marked simple point process generated by the random sequence $(X, I\{Y>0\})$ is also a stationary renewal reward process, and that (5.5) is the quantity corresponding to (5.4) for this process. Hence, the calculations for (5.4) can be repeated for (5.5) to give the desired result. □

## 6. Examples

We here rather briefly indicate how the error bounds given in Theorem 5.1 can be applied to specific examples: independent binary valued rewards, rewards indicating a long distance to the next renewal, the number of visits to a "rare" set by a stationary Markov chain on a finite state space, and the Lebesgue measure of the total time spent in a "rare" set by a Markov jump process on a finite state space.

*Independent rewards.* For a stationary renewal reward process $\xi$ in continuous time such that $\mathrm{supp}(\mathscr{L}(Y_0^{\mathrm{o}}))=\{0,1\}$, for which the rewards are independent of the renewal process, 5.1 takes on the following appearance:

$$\begin{aligned} & d_{TV}\left(\mathscr{L}\left(\int_{(0,L]\times\{1\}} d\xi(x,y)\right), \mathrm{Po}\left(L\frac{E(Y_0^{\mathrm{o}})}{E(T_0^{\mathrm{o}})}\right)\right) \\ & \leq \left(1-\exp\left(-L\frac{E(Y_0^{\mathrm{o}})}{E(T_0^{\mathrm{o}})}\right)\right)\left(2+\frac{3E((T_0^{\mathrm{o}})^2)}{E(T_0^{\mathrm{o}})^2}\right)E(Y_0^{\mathrm{o}}). \end{aligned}$$

*Rewards indicating a long distance to the next renewal.* For a stationary renewal reward process $\xi$ in continuous time with the reward $Y_0^{\mathrm{o}}=I\{T_0^{\mathrm{o}}\geq z\}$, where $z\in R'_+$, 5.1 becomes:

$$d_{TV}\left(\mathscr{L}\left(\int_{(0,L]\times\{1\}} d\xi(x,y)\right), \mathrm{Po}\left(L\frac{P(T_0^{\mathrm{o}}\geq z)}{E(T_0^{\mathrm{o}})}\right)\right)$$

$$\leq \left(1 - \exp\left(-L\frac{P(T_0^{\rm o} \geq z)}{E(T_0^{\rm o})}\right)\right)$$
$$\times \left(\frac{3E(T_0^{\rm o} I\{T_0^{\rm o} \geq z\})}{E(T_0^{\rm o})} + \frac{3E((T_0^{\rm o})^2)}{E(T_0^{\rm o})^2} P(T_0^{\rm o} \geq z)\right).$$

For example if $T_0^{\rm o} \sim \exp(m)$, then $E(T_0^{\rm o}) = m$, $E((T_0^{\rm o})^2) = 2m^2$, $P(T_0^{\rm o} \geq z) = e^{-z/m}$, and $E(T_0^{\rm o} I\{T_0^{\rm o} \geq z\}) = (z+m)e^{-z/m}$, so the error bound reduces to $(1 - \exp(-(L/m)e^{-z/m}))(3z/m + 9)e^{-z/m}$.

*Number of visits to a "rare" set by a Markov chain.* (See Erhardsson (1999) for a more detailed, but also more restricted, account.) Let $\Phi$ be a stationary discrete time Markov chain on a finite state space $S$. Let $\xi$ be the embedded stationary renewal reward process for which the renewals are the times when the Markov chain visits a certain singleton $A$, and the rewards are the number of visits by $\Phi$ to a certain "rare" set $B$ before the next visit to $A$. In order to find the compound Poisson approximation error bound of Theorem 5.1, we need to calculate the quantities $E(T_0^{\rm o})$, $E(Y_0^{\rm o})$, $E(T_0^{\rm o}(T_0^{\rm o}-1))$ and $E(T_0^{\rm o} Y_0^{\rm o})$. To find the generating function of the approximating compound Poisson distribution, we must calculate $E(s^{Y_0^{\rm o}})\ \ \forall s \in (0,1)$. It is well-known, and follows from the Palm inversion formula for regenerative random sequences, that $E(T_0^{\rm o}) = 1/\mu(A)$ and $E(Y_0^{\rm o}) = \mu(B)/\mu(A)$, where $\mu$ is the stationary distribution of the Markov chain. It likewise follows from the Palm inversion formula that

$$\frac{E(T_0^{\rm o}(T_0^{\rm o}-1))}{2E(T_0^{\rm o})} = \int_{A^c} E(\tau_A|\Phi_0 = x)d\mu(x),$$

where $\tau_C := \min\{t \in Z'_+;\ \Phi_t \in C\}\ \ \forall C \subset S$, and that

$$\frac{E(T_0^{\rm o} Y_0^{\rm o})}{E(T_0^{\rm o})} = \int_B E(\tau_A|\Phi_0 = x)d\mu(x) + \int_B E(\tau_A^{\rm rev}|\Phi_0 = x)d\mu(x),$$

where $\tau_C^{\rm rev} := \min\{t \in Z'_+;\ \Phi_{-t} \in C\}\ \ \forall C \subset S$. Hence, the quantities needed can be calculated simply by solving linear equation systems (to find $\mu$, $E(\tau_A|\phi_0 = \cdot)$ and $E(\tau_A^{\rm rev}|\phi_0 = \cdot)$) with dimension less than or equal to card$(S)$. Similarly, the generating function $E(s^{Y_0^{\rm o}})\ \forall s \in (0,1)$ can be obtained by solving a linear equation system for each $s \in (0,1)$.

*Lebesgue measure of total time spent in a "rare" set by a Markov jump process.* Let $\Phi$ be a stationary continuous time Markov jump process on a finite state space $S$; for the exact definition of such a process, see Asmussen (1987). Let $\xi$ be the embedded stationary renewal reward process for which the renewals are the times when the Markov jump process *enters* a certain singleton $A$, and the rewards give the Lebesgue measure of the time spent by $\Phi$ in a certain "rare" set $B$ before the next time it enters $A$. To find the compound Poisson approximation error bound of Theorem 5.1, we need to calculate the quantities $E(T_0^{\rm o})$, $P(Y_0^{\rm o} > 0)$, $E((T_0^{\rm o})^2)$ and $E(T_0^{\rm o} I\{Y_0^{\rm o} > 0\})$. To find the Laplace transform of the approximating compound Poisson distribution, we must calculate $E(e^{-sY_0^{\rm o}})\ \ \forall s \in R'_+$. It is well-known that

$E(T_0^o) = E(\tau_{A^c}|\Phi_0 \in A)/\mu(A)$, where $\mu$ is the stationary distribution of $\Phi$ and $\tau_C := \min\{t \in R'_+; \Phi_t \in C\}\ \ \forall C \subset S$, and also that

$$P(Y_0^o > 0) = \int_{A^c} P(\tau_B < \tau_A|\Phi_0 = x) d\nu_A(x),$$

where $\nu_A(\cdot) := P(\Phi_{\tau_{A^c}} \in \cdot|\Phi_0 \in A)$. The Palm inversion formula gives:

$$\frac{E((T_0^o)^2)}{2E(T_0^o)} = \int_{A^c} E(\tau_A|\Phi_0 = x) d\mu(x) \\ +\mu(A)\left(E(\tau_{A^c}|\Phi_0 \in A) + \int_{A^c} E(\tau_A|\Phi_0 = x) d\nu_A(x)\right),$$

and:

$$\frac{E(T_0^o I\{Y_0^o > 0\})}{E(T_0^o)} \le \int_{A^c} P(\tau_B < \tau_A|\Phi_0 = x) d\mu(x) \\ + \int_{A^c} P(\tau_B^{\mathrm{rev}} < \tau_A^{\mathrm{rev}}|\Phi_0 = x) d\mu(x) \\ +\mu(A) \int_{A^c} P(\tau_B < \tau_A|\Phi_0 = x) d\nu_A(x),$$

where $\tau_C^{\mathrm{rev}} := \min\{t \in R'_+; \Phi_{-t} \in C\}\ \ \forall C \subset S$. Again, the quantities needed can be calculated by solving linear equation systems with dimension less than or equal to card$(S)$. The Laplace transform $E(e^{-sY_0^o})\ \ \forall s \in R'_+$ can be obtained by solving a linear equation system for each $s \in R'_+$.

## References

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